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# STAR SORTS, LELEK FANS, AND THE RECONSTRUCTION OF NON- $\aleph_0$ -CATEGORICAL THEORIES IN CONTINUOUS LOGIC

ITAI BEN YAACOV

ABSTRACT. We prove a reconstruction theorem valid for arbitrary theories in continuous (or classical) logic in a countable language, that is to say that we provide a complete bi-interpretation invariant for such theories, taking the form of an open Polish topological groupoid.

More explicitly, for every such theory  $T$  we construct a groupoid  $\mathbf{G}^*(T)$  that only depends on the bi-interpretation class of  $T$ , and conversely, we reconstruct from  $\mathbf{G}^*(T)$  a theory that is bi-interpretable with  $T$ . The basis of  $\mathbf{G}^*(T)$  (namely, the set of objects, when viewed as a category) is always homeomorphic to the Lelek fan.

We break the construction of the invariant into two steps. In the second step we construct a groupoid from any *reconstruction sort*, while in the first step such a sort is constructed. This allows us to place our result in a common framework with previously established ones, which only differ by their different choice of a reconstruction sort.

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## INTRODUCTION

This paper deals with what we have to refer to as *reconstruction theorems*. By this we mean a procedure that associates to a theory  $T$  (possibly under some hypotheses) a group-like object that is a complete bi-interpretation invariant for  $T$ . In other words, if  $T'$  is bi-interpretable with  $T$ , then we associate to it the same object (up to an appropriate notion of isomorphism), and conversely, the isomorphism class of this object determines the bi-interpretation class of  $T$ .

The best-known result of this kind is due to Coquand, and appears in Albrandt & Ziegler [AZ86]. It states that if  $T$  is an  $\aleph_0$ -categorical theory (in a countable language), then the topological group  $G(T) = \text{Aut}(M)$ , where  $M$  is the unique countable model, is such an invariant. This was originally proved for theories in classical (Boolean-valued) logic, and subsequently extended by Kaïchouh and the author [BK16] to continuous (real-valued) logic.

In [Ben] we proposed a reconstruction result that also covers some non- $\aleph_0$ -categorical theories, using a topological groupoid (rather than a group) as invariant. The result was presented in two times, first for classical logic and the for the more general continuous logic. This was done not only for the sake of presentation (do the more familiar case first), but mainly because of a fundamental difference between the two cases. In classical logic, we have a straightforward construction of a sort for “codes of models” (more about this later). In continuous logic, on the other hand, no such construction exists in general, and we were reduced to assuming that such a sort (satisfying appropriate axioms) existed, and was given to us. Worse still, we gave an example of a theory for which no such sort existed, and consequently, for which our reconstruction theorem was inapplicable.

In the present paper we seek to remedy this deficiency, proposing a reconstruction theorem that holds for all theories (in a countable language). This time, we work exclusively in continuous logic, keeping in mid that this contains classical logic as a special case.

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In [Section 1](#) we proved a few reminders regarding interpretable sorts, and more generally, interpretation and bi-interpretation, in a manner that allows us to avoid the rather tedious notions of interpretation schemes.

In [Section 2](#) we define a general notion of a *reconstruction sort*, namely of a sort  $D$  whose members (or most of them, at any rate) code models, allowing us to define a topological groupoid  $\mathbf{G}_D(T)$ , from which a theory  $T_D$ , bi-interpretable with  $T$ , can be recovered. In particular,  $\mathbf{G}_D(T)$  determines the bi-interpretation class of  $T$ . If we can associate such a sort to  $T$  in a manner that only depends on the bi-interpretation class of  $T$ , then  $\mathbf{G}_D(T)$  indeed becomes the desired complete bi-interpretation invariant. We point out, rather briefly, how previous reconstruction theorems fit in this general setting.

In [Section 3](#) and [Section 4](#) we define *star spaces* and then *star sorts*. The, by their very nature, require us to work in continuous (rather than classical) logic. In particular, we define a notion of a *universal star sort*, and show that if it exists, then it is unique up to definable bijection, and only depends on the bi-interpretation class of  $T$ .

In [Section 5](#) we use the star sort formalism to give a construction that is analogous to, though not a direct generalisation of, the construction of the reconstruction sort for classical theories in [[Ben](#)]. We then prove that the resulting sort is a universal star sort, so one always exists. Moreover, the construction is independent of the theory: we simply construct, for any countable language  $\mathcal{L}$ , a star sort  $D^*$  that is universal in any  $\mathcal{L}$ -theory, complete or incomplete.

We conclude in [Section 6](#), showing that the universal star sort must be a reconstruction sort, whence our most general reconstruction theorem: in a countable language, the groupoid  $\mathbf{G}_{D^*}(T)$  is a complete bi-interpretation invariant for  $T$ . We also show that the type-space of the sort  $D^*$ , relative to any complete theory  $T$  is the Lelek fan  $L$ . Finally, in case  $T$  does fall into one of the cases covered by previous results, we show that our last result can be viewed as some kind of generalisation. More precisely, using the Lelek fan, we can recover the reconstruction sort  $D^*$ , and therefore the corresponding groupoid  $\mathbf{G}_{D^*}(T)$ , from those given by the earlier results.

## 1. SORTS AND INTERPRETATIONS

As said in the introduction, we are going to work exclusively in continuous logic. We allow many-sorted logic. Some of the time we also require the language to be countable, which means in particular that the set of sorts is countable, although this will not be a requirement for the present section.

We are going to talk quite a bit about sorts and interpretations, so let us begin with a few reminders. By a *sort* we mean an interpretable sort in the sense of continuous logic, as discussed, for example, in [[BK16](#), [Ben](#)]. Sorts are obtained by closing the family of basic sorts (namely, sorts named in the language) by

- adding the constant sort  $[0, 1]$  (so it is always implicitly interpretable),
- countable product,
- quotient by a definable pseudo-distance (in a model that is not saturated, this may also require a passage to the completion), and
- definable subset.

Two sorts that admit a definable bijection are, for most intents and purposes (in particular, for those of the present paper) one and the same. Under this convention, any sort can be obtained from the basic sorts by applying each of the operations once, in the given order. By the same convention, we may also that a sort  $D$  (which may be a basic sort, or one that has already been obtained through some interpretation procedure) is *interpretable* in a family of sorts  $(E_i)$  if we can construct from this family  $(E_i)$  a sort  $D'$  that admits a definable bijection with  $D$ .

*Remark 1.1.* If  $D$  is interpretable in a family of sorts  $(E_i)$ , then, by definition, there exists a definable embedding of  $D$  into a quotient of a product of (powers of) the sorts  $E_i$ . The converse is also true. More precisely, for  $D$  to be interpretable in the family  $(E_i)$ , it is enough to assume that there exists a product  $E$  of sorts from this family, as well as a type-definable relation  $R \subseteq D \times E$  that associates to each  $x \in D$  one or more  $y \in E$ , and never the same  $y$  to distinct  $x$ .

Indeed, by compactness, for each  $\varepsilon > 0$  there exists a formula  $\varphi(x, y) \geq 0$  that vanishes on  $R$ , and  $\varphi(x, y) + \varphi(x', y) < 1$  implies  $d(x, x') < \varepsilon$ . Combining these, one can produce a single formula  $\varphi(x, y) \geq 0$  that vanishes on  $R$ , such that  $\varphi(x, y) + \varphi(x', y) < 2^{-n}$  implies  $d(x, x') < 2^{-n-1}$ . In particular, we may replace  $R$  with the zero-set of  $\varphi$ . In addition, if  $(x_0, y) \in R$  (i.e., if  $\varphi(x_0, y) = 0$ ), then  $\varphi(x, y) \geq d(x, x_0)$ , for all  $x \in D$ .

Define

$$\varphi'(x, y) = \inf_{x'} \left( d(x, x') + \varphi(x', y) \right), \quad d'(y_1, y_2) = \sup_x |\varphi'(x, y_1) - \varphi'(x, y_2)|.$$

Then  $\varphi'$  has the same zero-set  $R$  as  $\varphi$ , and in addition, if  $(x_0, y) \in R$ , then  $\varphi'(x, y) = d(x, x_0)$ . It follows that if  $(x_i, y_i) \in R$ , then  $d'(y_1, y_2) = d(x_1, x_2)$ . The formula  $d'$  defines a pseudo-distance on  $E$ , and there exists a definable isometric embedding  $\sigma: (D, d) \rightarrow (E, d')$ , where  $d'(\sigma(x), \bar{y}) = \varphi'(x, y)$ . In particular,  $\bar{y} = \sigma(x)$  if and only if  $(x, y) \in R$ . The image is a definable subset, as any definable image of a definable set.

*Remark 1.2.* Let

$$D_0 \xleftarrow{\pi_0} D_1 \xleftarrow{\pi_1} \dots$$

be an inverse system of sorts with surjective maps  $\pi_n: D_{n+1} \twoheadrightarrow D_n$ . Then the inverse limit  $D = \varprojlim D_n$  is again a sort.

Indeed, it is the zero-set of the formula

$$\varphi(x) = \sum_n 2^{-n} \wedge d(x_n, \pi_n x_{n+1}).$$

Let  $\varepsilon > 0$ , and choose  $N \in \mathbf{N}$  large enough depending on  $\varepsilon$ , and  $\delta > 0$  small enough depending on both. Let  $a \in \prod D_n$ , and assume that  $\varphi(a) < \delta$ . Since the maps are surjective, there exists  $b \in D$  such that  $b_N = a_N$ . This determines  $b_n$  for all  $n \leq N$ , and having chose  $\delta$  small enough, we have  $d(a_n, b_n)$  as small as desired for all  $n \leq N$ . Having chosen  $N$  large enough, this yields  $d(a, D) \leq d(a, b) < \varepsilon$ .

Consider two languages  $\mathcal{L} \subseteq \mathcal{L}'$ , where  $\mathcal{L}'$  is allowed to add not only symbols, but also sorts. If  $M'$  is an  $\mathcal{L}'$ -structure, and  $M$  is the  $\mathcal{L}$ -structure obtained by dropping the sorts and symbols not present in  $\mathcal{L}$ , then  $M$  is the  $\mathcal{L}$ -reduct of  $M'$  and  $M'$  is an  $\mathcal{L}'$ -expansion of  $M$ . If  $T'$  is an  $\mathcal{L}'$ -theory and  $T$  is the collection of  $\mathcal{L}$ -sentences in  $T'$ , then  $T$  is also the theory of all  $\mathcal{L}$ -reducts of models of  $T'$  (notice, however, that an arbitrary model of  $T$  need only admit an elementary extension that is a reduct of a model of  $T'$ ). In this situation we say that  $T$  is the  $\mathcal{L}$ -reduct of  $T'$  and that  $T'$  is an  $\mathcal{L}'$ -expansion of  $T$ .

One special case of an expansion is a *definitional expansion*, in which  $\mathcal{L}$  and  $\mathcal{L}'$  have the same sorts, and each new symbol of  $\mathcal{L}'$  admits an  $\mathcal{L}$ -definition in  $T'$ . In this case,  $T'$  is entirely determined by  $T$  together with these definitions. A more general case is that of an *interpretational expansion* of  $T$ , where  $T'$  identifies each new sort of  $\mathcal{L}'$  with an interpretable sort of  $T$ , and gives  $\mathcal{L}$ -definitions to all new symbols in  $\mathcal{L}'$  (for this to work we also require  $\mathcal{L}'$  to contain, in particular, those new symbols that allow  $T'$  to identify the new sorts with the corresponding interpretable ones). Again,  $T$ , together with the list of interpretations of the new sorts and definitions of the new symbols, determine  $T'$ . Moreover, unlike the general situation described in the previous paragraph, here every model of  $T$  expands to a model of  $T'$ .

**Definition 1.3.** Let  $T$  and  $T'$  be two theories, say in disjoint languages. We say that  $T'$  is *interpretable* in  $T$  if  $T'$  is a reduct of an interpretational expansion of  $T$ . The two theories are *bi-interpretable* if they admit a common interpretational expansion (which is stronger than just being each interpretable in the other).

A theory has the same sorts (up to a natural identification) as an interpretational expansions. Therefore, somewhat informally, we may say that two theories are bi-interpretable if and only if they have the same sorts.

## 2. GROUPOID CONSTRUCTIONS AND RECONSTRUCTION STRATEGIES

In this section we propose a general framework for “reconstruction theorems”. In a nutshell, to any sufficiently universal sort we associate a topological groupoid from which the theory of the said sort can be reconstructed. If the sort is uniquely determined by the bi-interpretation class of the theory (up to definable bijection), then the groupoid is a bi-interpretation invariant. Conversely, the full theory should be interpretable from the sort in question, so the groupoid determines the bi-interpretation class. Various previously known constructions fit in this framework, as well as the one towards which aims the present paper.

**Definition 2.1.** Let  $T$  be a theory,  $D$  a sort, and  $D^0 \subseteq D$  a type-definable subset. We say that  $D$  is a *reconstruction sort*, with *exceptional set*  $D^0$ , if the following holds:

- (i) *Coding models*: if  $M \models T$  and  $a \in D(M) \setminus D^0(M)$ , then there exists  $N \preceq M$  such that  $\text{dcl}(a) = \text{dcl}(N)$ . We then say that  $a$  codes  $N$ .
- (ii) *Density*: if  $M \models T$  is separable, then the set of  $a \in D(M) \setminus D^0(M)$  that code  $M$  is dense in  $D(M)$ .
- (iii) *Universality*: Every sort embeds definably in a quotient of  $D$ .

We may denote a reconstruction sort by  $D$  alone, considering  $D^0$  as implicitly given together with  $D$ .

The need for an exceptional set will become obvious later – for the time being, we are simply going to ensure that its presence does not cause any trouble.

**Proposition 2.2.** *Let  $T$  be a theory, say in a language  $\mathcal{L}$ , and let  $D$  be a reconstruction sort for  $T$ . Let  $\mathcal{L}_D$  be a language with a single sort, which we shall identify with  $D$ , and with a binary predicate symbol for each definable predicate on  $D \times D$  (or possibly restricting this to a dense family of such predicates). Let  $T_D$  be the  $\mathcal{L}_D$ -theory of  $D$  – namely, the theory of all  $D(M)$ , viewed naturally as an  $\mathcal{L}_D$ -structure, where  $M$  varies over models of  $T$ .*

*Then  $T_D$  is bi-interpretable with  $T$ .*

*Proof.* Consider  $T'$ , obtained from  $T$  by adjoining  $D$  as a new sort, and naming the full induced structure. It is, by definition, an interpretational expansion of  $T$ , and it will suffice to show that it is also an interpretational expansion of  $T_D$ .

For each sort  $E$  of  $T$ , choose some definable embedding of  $E$  in a quotient of  $D$ . As in [Remark 1.1](#), this means that there exists a pseudo-distance  $d_E$  on  $D$  such that  $E$  embeds definably and isometrically in the quotient metric space  $(D, d_E)$ . We may pretend that  $E$  is a subset of this quotient space, in which case the distance  $d_E(\cdot, E)$  is a definable unary predicate on  $D$ . Notice that both  $d_E$  (binary) and  $d_E(\cdot, E)$  (unary) are in  $\mathcal{L}_D$  (a unary predicate can be viewed as a binary one through the addition of a dummy variable).

In addition, any definable unary predicate on  $E$  extends (in more than one way) to a definable predicate on the quotient  $(D, d_E)$ , which in turn pulls back to a definable predicate on  $D$ , that is present in  $\mathcal{L}_D$ .

Consider now a finite product of sorts  $E = \prod_{i < n} E_i$ . We have already chosen embeddings as in the previous paragraph for each  $E_i$ , as well as for the product sort  $E$ . The projection map  $\pi_i: E \rightarrow E_i$  can be coded by a predicate on  $E \times E_i$ , namely

$$\Gamma_{\pi_i}(x, y) = d_{E_i}(x_i, y),$$

where  $\Gamma$  stands for “Graph”. As in the previous paragraph, this can be extended to a definable predicate on the quotient  $(D, d_E) \times (D, d_{E_i})$ , and then pulled back to  $D \times D$ , yielding a predicate that is named in  $\mathcal{L}_D$ . In other words, given our chosen interpretations of  $E_i$  and  $E$  in the sort  $D$ , the projection map  $\pi_i: E \rightarrow E_i$  is definable in  $\mathcal{L}_D$ . Consequently, there is a definable bijection between the interpretation of  $E$  and the product of interpretations of the  $E_i$ . Finally, any definable predicate on this product is the same thing as a unary predicate on  $E$ , which is definable in  $\mathcal{L}_D$  as we have already observed.

We conclude that not only can we interpret each sort of  $T$  in  $T_D$ , but we can also recover the full structure on these sorts. This is applicable in particular to the basic sorts of  $T$ , so  $T'$  is indeed an interpretational expansion of  $T_D$ . ■

**Definition 2.3.** Let  $T$  be a theory in a countable language, and  $D$  a reconstruction sort. We let  $S_{D \times D}(T)$  be the space of types of pairs of elements of  $D$ . We define the following two subsets of  $S_{D \times D}(T)$ :

$$\begin{aligned} \mathbf{G}_D^0(T) &= \{\text{tp}(a, a) : a \in D^0\}, \\ \mathbf{G}_D(T) &= \mathbf{G}_D^0(T) \cup \{\text{tp}(a, b) : a, b \in D \setminus D^0 \text{ \& \ } \text{dcl}(a) = \text{dcl}(b)\}, \end{aligned}$$

where  $a$  and  $b$  vary of all members of  $D$  (or  $D^0$ ) in models of  $T$ . We equip  $\mathbf{G}_D(T)$  with the induced topology, as well as with the following inversion law and partial composition law:

$$\text{tp}(a, b)^{-1} = \text{tp}(b, a), \quad \text{tp}(a, b) \cdot \text{tp}(b, c) = \text{tp}(a, c).$$

We also write  $\mathbf{B}_D(T)$  for  $S_D(T)$ , and identify  $\text{tp}(a) \in \mathbf{B}_D(T)$  with  $\text{tp}(a, a) \in \mathbf{G}_D(T)$ . This identifies  $\mathbf{B}_D^0(T) = S_{D^0}(T)$  with  $\mathbf{G}_D^0(T)$ .

Notice that the density hypothesis in [Definition 2.1](#) implies that  $\mathbf{G}_D(T)$  is dense in  $S_{D \times D}(T)$ .

**Convention 2.4.** We usually consider the theory  $T$  and the reconstruction sort  $D$  to be fixed and drop them from notation, so  $\mathbf{G} = \mathbf{G}_D(T)$ ,  $\mathbf{B} = \mathbf{B}_D(T)$ , and so on.

**Lemma 2.5.** *Let  $D$  be a reconstruction sort for  $T$ .*

- (i) *As defined above  $\mathbf{G} = \mathbf{G}_D(T)$  is a Polish open topological groupoid over  $\mathbf{B} = \mathbf{B}_D(T)$ .*
- (ii) *If  $g = \text{tp}(a, b) \in \mathbf{G}$ , then  $s(g) = \text{tp}(b) \in \mathbf{B}$  is its source, and  $t(g) = \text{tp}(a) \in \mathbf{B}$  its target.*
- (iii) *If  $d$  is a definable distance on  $D$ , then the family of sets*

$$U_r = \{\text{tp}(a, b) \in \mathbf{G} : d(a, b) < r\},$$

*for  $r > 0$ , forms a basis of open neighbourhoods for  $\mathbf{B}$  in  $\mathbf{G}$ .*

*Proof.* It is easy to check that  $\mathbf{G}$  is a topological groupoid over  $\mathbf{B}$ , with the stated source and target. Since the language is countable, the space  $S_{D \times D}(T)$  is compact metrisable and  $\text{dcl}(a) = \text{dcl}(b)$  is a  $G_\delta$  condition on  $\text{tp}(a, b)$ , so  $\mathbf{G}$  is Polish. Each set  $U_r$  is open and contains  $\mathbf{B}$ . On the other hand, if  $U$  is any open neighbourhood of  $\mathbf{B}$  in  $\mathbf{G}$ , then it must be of the form  $U_1 \cap \mathbf{G}$ , where  $U_1$  is an open neighbourhood of  $\mathbf{B}$  in  $S_{D \times D}(T)$ . Since  $\mathbf{B}$  is

defined there by the condition  $d(x, y) = 0$ , and by compactness,  $U_1$  must contain  $[d(x, y) < r]$  for some  $r > 0$ , so  $U$  contains  $U_r$ .

It is left to show that the target map  $t: \mathbf{G} \rightarrow \mathbf{B}$  is open. Since  $\mathbf{G}^0$  is compact, a neighbourhood of  $g \in \mathbf{G} \setminus \mathbf{G}^0$  may always be assumed to be disjoint from  $\mathbf{G}^0$ . We have a basis of open sets  $U \subseteq \mathbf{G} \setminus \mathbf{G}^0$  of the form  $[\varphi(x, y) > 0] \cap \mathbf{G}$ , where  $[\varphi(x, y) > 0] \subseteq S_{D \times D}(T)$  is the corresponding basic open set in the type space. We may further require that  $\varphi(x, y) = 0$  if either  $x$  or  $y$  belongs to  $D^0$ .

Given such  $U$ , let  $V = [\sup_y \varphi(x, y) > 0] \subseteq S_D(T)$ . Then  $V$  is open, and clearly  $t(U) \subseteq V$ . Conversely, assume that  $\text{tp}(a) \in V$ , where  $a \in D(M)$  for some  $M \models T$ . Then there exists  $b \in D$  such that  $\varphi(a, b) > 0$ . It follows that  $a \notin D^0$ , so we may even assume that  $a$  codes  $M$ . We may assume that  $b \in D(M)$ , and necessarily  $b \notin D^0$ . Now, by the density property and the uniform continuity of  $\varphi$ , we may assume that  $b$  also codes  $M$ , so  $\text{tp}(a, b) \in U$ . This proves that  $t(U) = V$ .

Now let  $g = \text{tp}(a, a) \in \mathbf{G}^0$ . We have a basis of neighbourhoods of  $g$  in  $\mathbf{G}$  consisting of sets of the form

$$U = [\varphi(x) > 0] \cap [d(x, y) < r] \cap \mathbf{G},$$

where  $\varphi(a) > 0$ . It is then easily checked that  $t(U) = [\varphi(x) > 0]$ , since we may always take  $y = x$  as witness.

This completes the proof.  $\blacksquare$

**Definition 2.6.** Let  $\mathbf{G}$  be a topological groupoid. Say that a function  $\varphi: \mathbf{G} \rightarrow \mathbf{R}$  is *uniformly continuous and continuous (UCC)* if it is continuous on  $\mathbf{G}$ , and in addition satisfies the following uniform continuity condition: for every  $\varepsilon > 0$  there exists an open neighbourhood  $U$  of the basis  $\mathbf{B}$  such that for every  $g \in \mathbf{G}$ ,

$$u, v \in U \implies |\varphi(g) - \varphi(ugv)| < \varepsilon \quad \text{provided that } ugv \text{ is defined.}$$

Notice that unlike the situation for groups, the uniform continuity condition does not imply continuity (it is very well possible that  $g_n \rightarrow h$  while  $h$  cannot be expressed as  $ug_nv$  for any  $n$  or  $u, v \in \mathbf{G}$ ).

**Proposition 2.7.** Assume that  $D$  is a reconstruction sort for  $T$ , and let  $\mathbf{G} = \mathbf{G}_D(T)$ . Let  $\varphi(x, y)$  be a formula on  $D \times D$ , and let  $\varphi_{\mathbf{G}}: \mathbf{G} \rightarrow \mathbf{R}$  be the naturally induced function

$$g = \text{tp}(a, b) \implies \varphi_{\mathbf{G}}(g) = \varphi(a, b).$$

Then the map  $\varphi \mapsto \varphi_{\mathbf{G}}$  defines a bijection between formulas on  $D \times D$ , up to equivalence, and UCC functions on  $\mathbf{G}$ .

*Proof.* Let us first check that if  $\varphi$  is a formula, then  $\varphi_{\mathbf{G}}$  is UCC. It is clearly continuous. The uniform continuity condition follows from the fact that  $\varphi$  is uniformly continuous in each argument, together with the fact that for any  $\delta > 0$  we may take choose  $U = [d(x, y) < \delta] \cap \mathbf{G}$ .

Conversely, assume that  $\psi: \mathbf{G} \rightarrow \mathbf{R}$  is UCC. By density, the function  $\psi$  admits at most one continuous extension to  $S_{D \times D}(T)$ , and we need to show that one such exists. In other words, given  $p \in S_{D \times D}(T)$  and  $\varepsilon > 0$ , it will suffice to find a neighbourhood  $p \in V \subseteq S_{D \times D}(T)$  such that  $\psi$  varies by less than  $\varepsilon$  on  $V \cap \mathbf{G}$ . If  $p \in \mathbf{G}$  this is easy, so we may assume that  $p \notin \mathbf{G}$ .

Let us fix  $\varepsilon > 0$  first. By uniform continuity of  $\psi$  and **Lemma 2.5(iii)**, there exists  $\delta > 0$  such that  $|\psi(g) - \psi(ugv)| < \varepsilon$  whenever  $g \in \mathbf{G}$ ,  $u, v \in [d(x, y) < \delta] \cap \mathbf{G}$ , and  $ugv$  is defined.

Given  $p = \text{tp}(a_0, b_0)$ , we may assume that  $a_0, b_0 \in D(M)$  for some separable model  $M$ . Since  $p \notin \mathbf{G}$ , we must have  $a_0 \neq b_0$ , and (possibly decreasing  $\delta$ ) we may assume that  $d(a_0, b_0) > 2\delta$ . By the density property, there exist  $a_1, b_1 \in D(M)$  that code  $M$ , with  $d(a_0, a_1) + d(b_0, b_1) < \delta$ , so  $d(a_1, b_1) > \delta$ . Let  $g_1 = \text{tp}(a_1, b_1) \in \mathbf{G}$ . By continuity, there exists an open neighbourhood  $g_1 \in V_1 \subseteq S_{D \times D}(T)$  such that  $|\psi(g_1) - \psi(h)| < \varepsilon$  for every  $h \in V_1 \cap \mathbf{G}$ . Possibly decreasing  $V_1$ , we may further assume that  $\text{tp}(a, b) \in V_1$  implies  $d(a, b) > \delta$ . We may even assume that  $V_1$  is of the form  $[\chi < \delta]$ , where  $\chi(x, y) \geq 0$  is a formula and  $\chi(a_1, b_1) = \chi(g_1) = 0$ . Define

$$\chi'(x, y) = \inf_{x', y'} [d(x, x') + d(y, y') + \psi(x', y')],$$

$$V = [\chi'(x, y) < \delta] \subseteq S_{D \times D}(T).$$

Then  $V$  is open,  $p \in V$ , and  $\text{tp}(a, b) \in V$  implies  $a \neq b$  (in other words,  $V \cap \mathbf{B} = \emptyset$ ).

In order to conclude, consider any  $g_2 = \text{tp}(a_2, b_2) \in V \cap \mathbf{G}$ . Since  $a_2 \neq b_2$ , they cannot belong to the exceptional set, so both code some separable model  $N$ . By definition of  $V$ , there exist  $a_3, b_3 \in D(N)$  such that  $\chi(a_3, b_3) + d(a_2, a_3) + d(b_2, b_3) < \delta$ . By the density property, and uniform continuity of  $\chi$ , we may assume that  $a_3$  and  $b_3$  code  $N$  as well. Let  $g_3 = \text{tp}(a_3, b_3)$ ,  $u = \text{tp}(a_3, a_2)$ ,  $v = \text{tp}(b_2, b_3)$ . Then  $g_3 = ug_2v \in V_1$ , so

$$|\psi(g_2) - \psi(g_1)| \leq |\psi(g_2) - \psi(g_3)| + |\psi(g_3) - \psi(g_1)| \leq 2\varepsilon.$$

Therefore  $\psi$  varies by less than  $4\varepsilon$  on  $V \cap \mathbf{G}$ , which is good enough.  $\blacksquare$

**Corollary 2.8.** Every UCC function on  $\mathbf{G}_D(T)$  is bounded.

**Definition 2.9.** Let  $\mathbf{G}$  be a groupoid. A *semi-norm* on  $\mathbf{G}$  is a function  $\rho: \mathbf{G} \rightarrow \mathbf{R}^+$  that satisfies

- $\rho \upharpoonright_{\mathbf{B}} = 0$ , and
- $\rho(gh) \leq \rho(g) + \rho(h^{-1})$ , when defined.

It is a *norm* if  $\rho(g) = 0$  implies  $g \in \mathbf{B}$ .

A norm  $\rho$  is *compatible* with a topology on  $\mathbf{G}$  if it is continuous, and the sets

$$\{\rho < r\} = \{g \in \mathbf{G} : \rho(g) < r\},$$

for  $r > 0$ , form a basis of neighbourhoods for  $\mathbf{B}$ .

The axioms of a semi-norm imply that  $\rho(h) = \rho(h^{-1})$ .

**Corollary 2.10.** The correspondence of [Proposition 2.7](#) restricts to a one-to-one correspondence between definable distances  $d$  on  $D$  and compatible norms on  $\mathbf{G} = \mathbf{G}_D(T)$ .

*Proof.* Let  $d$  be a definable distance on  $D \times D$  and  $\rho_d$  the corresponding UCC function on  $\mathbf{G}$ . Then  $\rho_d$  is clearly a continuous norm, and it is a compatible norm by [Lemma 2.5\(iii\)](#).

The converse is more delicate. Let  $\rho$  be a compatible norm. Then it is continuous, and it is easy to see that every continuous semi-norm is UCC, so  $\rho = \varphi_{\mathbf{G}}$  (in the notations of [Proposition 2.7](#)) for some formula  $\varphi(x, y)$ . If  $a, b, c \in D$  all code the same separable model, then  $\varphi(a, a) = 0$  and  $\varphi(a, b) \leq \varphi(a, c) + \varphi(b, c)$ . The set of types of such triplets is dense in  $S_{D \times D \times D}(T)$ , by the density property, so the same holds throughout and  $\varphi$  defines a pseudo-distance.

It is left to show that  $\varphi$  defines a distance (and not merely a pseudo-distance). Let  $d$  be any definable distance on  $D$ , say the one distinguished in the language. We already know that  $\rho_d$  is a compatible norm. Therefore, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\{\rho < \delta\} \subseteq \{\rho_d < \varepsilon\}$ . As in the previous paragraph, this means that the (closed) condition  $\varphi(a, b) < \delta \implies d(a, b) \leq \varepsilon$  holds on a dense set of types, and therefore throughout. In particular, if  $\varphi(a, b) = 0$ , then  $a = b$ , and the proof is complete.  $\blacksquare$

Let  $T$  be a theory,  $D$  a reconstruction sort for  $T$ , and  $\mathbf{G} = \mathbf{G}_D(T)$ . Then from  $\mathbf{G}$ , given as a topological groupoid, we can essentially recover the language  $\mathcal{L}_D$  and the theory  $T_D$ , as follows.

- (i) We choose, arbitrarily, a compatible norm  $\rho$  on  $\mathbf{G}$  (which exists, by [Corollary 2.10](#)).
- (ii) We let  $\mathcal{L}_{\mathbf{G}}$  consists of a single sort, also named  $D$ , together with a binary predicate symbol  $P_{\psi}$  for each UCC function  $\psi$  on  $\mathbf{G}$ . We know that  $\psi$  is bounded ([Corollary 2.8](#)), and we impose the same bound on  $P_{\psi}$ . We also know that for every  $\varepsilon > 0$  there exists a neighbourhood  $U$  of  $\mathbf{B}$  such that  $h \in UgU$  implies  $|\psi(g) - \psi(h)| < \varepsilon$ , and since  $\rho$  is compatible, there exists  $\delta = \delta_{\psi}(\varepsilon) > 0$  such that the same holds when  $U = \{\rho < \delta\}$ . We then impose the corresponding modulus of uniform continuity on  $P_{\psi}$ , namely, requiring that

$$d(x, x') \vee d(y, y') < \delta_{\psi}(\varepsilon) \implies |P_{\psi}(x, y) - P_{\psi}(x', y')| \leq \varepsilon.$$

We also use the bound on  $\rho$  as bound on the distance predicate.

- (iii) Let us fix  $e \in \mathbf{B}$ , and consider the set

$$e\mathbf{G} = \{g \in \mathbf{G} : t_g = e\}.$$

If  $g, h \in e\mathbf{G}$ , then  $g^{-1}h$  is defined, and for any UCC  $\psi$  we let:

$$P_{\psi}(g, h) = \psi(g^{-1}h).$$

In particular,  $d(g, h) = P_{\rho}(g, h) = \rho(g^{-1}h)$  is a distance function on  $e\mathbf{G}$ .

Assume now that  $g', h' \in e\mathbf{G}$  as well, and  $d(g, g') \vee d(h, h') < \delta = \delta_{\psi}(\varepsilon)$ . Let  $u = g'^{-1}g$  and  $v = h^{-1}h'$ . Then  $g'^{-1}h' = ug^{-1}hv$ , and  $u, v \in \{\rho < \delta\}$ , so indeed

$$|P_{\psi}(g, h) - P_{\psi}(g', h')| \leq \varepsilon,$$

as required. The bounds are also respected, so  $e\mathbf{G}$ , equipped with the distance and interpretations of  $P_{\psi}$ , is an  $\mathcal{L}_{\mathbf{G}}$ -pre-structure, and its completion  $\widehat{e\mathbf{G}}$  is an  $\mathcal{L}_{\mathbf{G}}$ -structure.

- (iv) We define  $T_{\mathbf{G}}$  as the theory of the collection of all  $\mathcal{L}_{\mathbf{G}}$ -structures of this form:

$$T_{\mathbf{G}} = \text{Th}_{\mathcal{L}_{\mathbf{G}}}(\widehat{e\mathbf{G}} : e \in \mathbf{B}).$$

By “essentially recover”, we mean the following.

**Theorem 2.11.** *Let  $T$  be a theory,  $D$  a reconstruction sort for  $T$ , and  $\mathbf{G} = \mathbf{G}_D(T)$ . Let  $\mathcal{L}_{\mathbf{G}}$  and  $T_{\mathbf{G}}$  be constructed as in the preceding discussion. Then  $T_{\mathbf{G}}$  and  $T_D$  are one and the same, up to renaming the binary predicate symbols, and up to an arbitrary choice of the distance on the sort  $D$  (from among all definable distances).*

*In particular, this procedure allows us to recover from  $\mathbf{G}$  a theory  $T_{\mathbf{G}}$  that is bi-interpretable with  $T$ .*

*Proof.* By [Corollary 2.10](#), step (i) consists exactly of choosing a definable distance  $d$  on  $D$ , and the corresponding norm  $\rho = d_{\mathbf{G}}$ . This choice is irremediably arbitrary. By [Proposition 2.7](#), in step (ii) there is a natural bijection between symbols of  $\mathcal{L}_D$  (corresponding to formulas  $\varphi(x, y)$  on  $D \times D$ , up to equivalence) and symbols of  $\mathcal{L}_{\mathbf{G}}$ : to  $\varphi$  we associate the UCC function  $\psi_{\varphi} = \varphi_{\mathbf{G}}$ , to which in turn we associate the symbol  $P_{\psi_{\varphi}}$ .

Finally, let  $M \models T$  be separable, let  $a \in D(M)$  be a code for  $M$ , and let  $e = \text{tp}(a) \in \mathbf{B}$ . Let  $D(M)_1$  denote the set of  $b \in D(M)$  that also code  $M$ . If  $b \in D(M)_1$ , then  $g_b = \text{tp}(a, b) \in e\mathbf{G}$ . Moreover, if  $b, c \in D(M)_1$  and  $\varphi$  is a formula on  $D \times D$ , then  $\text{tp}(b, c) = g_b^{-1}g_c \in \mathbf{G}$ , so

$$\varphi(b, c) = \psi_{\varphi}(g_b^{-1}g_c) = P_{\psi_{\varphi}}(g_b, g_c).$$

In particular,  $d(b, c) = d(g_b, g_c)$  (where the first is the distance we chose on  $D$ , and the second the distance we defined on  $e\mathbf{G}$  in step (iii)). Thus, up to representing  $\varphi$  by the symbol  $P_{\psi_{\varphi}}$ , the map  $b \mapsto g_b$  defines an isomorphism of the  $\mathcal{L}_D$ -pre-structure  $D(M)_1$  with the  $\mathcal{L}_{\mathbf{G}}$ -pre-structure  $e\mathbf{G}$ . This extends to an isomorphism of the respective completions:  $D(M) \simeq \widehat{e\mathbf{G}}$ .

It follows that, up to this change of language (and choice of distance), the theory  $T_{\mathbf{G}}$  defined in step (iv) is the theory of all separable models of  $T_D$ . Since  $T$  is in a countable language,  $T_D$  is in a “separable language”, so it is equal to the theory of all its separable models.

By [Proposition 2.2](#),  $T$  is bi-interpretable with  $T_D$ , and therefore also with  $T_{\mathbf{G}}$ . ■

Having achieved this, we are ready to start producing reconstruction theorems: all we need now is a reconstruction sort that only depends (up to definable bijection) on the bi-interpretation class of  $T$ .

*Example 2.12.* Let  $T$  be an  $\aleph_0$ -categorical theory. Let  $M$  be its unique separable model, and let  $a$  be any sequence (possibly infinite, but countable), in any sort or sorts, such that  $\text{dcl}(a) = \text{dcl}(M)$  (for example, any dense sequence will do). Let  $D_{T,0}$  be the set of realisations of  $p = \text{tp}(a)$ . Since  $T$  is  $\aleph_0$ -categorical,  $D_{T,0}$  is a definable set, i.e., a sort. It is easy to check that it is a reconstruction sort (with no exceptional set).

If  $b$  is another code for  $M$ , and  $D'_{T,0}$  is the set of realisations of  $\text{tp}(b)$ , then  $\text{dcl}(a) = \text{dcl}(b)$  and  $\text{tp}(a, b)$  defines the graph of a definable bijection  $D_{T,0} \simeq D'_{T,0}$ . Therefore,  $D_{T,0}$  does not depend on the choice of  $a$ . Moreover, assume that  $T'$  is an interpretational expansion of  $T$ . Then it has a model  $M'$  that expands  $M$  accordingly. But then  $\text{dcl}(M') = \text{dcl}(M) = \text{dcl}(a)$  (as calculated when working in  $T'$ ), so  $D_{T',0} = D_{T,0}$ . It follows that  $D_{T,0}$  only depend on the bi-interpretation class of  $T$ .

Since  $S_{D_{T,0}}(T) = \{p\}$  is a singleton, the groupoid

$$G(T) = \mathbf{G}_{D_{T,0}}(T)$$

is in fact a group. It only depends on the bi-interpretation class of  $T$  (since  $D_{T,0}$  only depends on it) and by [Theorem 2.11](#), it is a complete bi-interpretation invariant for  $T$ .

We leave it to the reader to check that

$$G(T) \simeq \text{Aut}(M),$$

and that the reconstruction result is just a complicated restatement of those of [[AZ86](#), [BK16](#)].

*Example 2.13.* Let  $T$  be a theory in classical logic. In [[Ben](#)], using an arbitrary parameter  $\Phi$ , we gave an explicit construction of a set of infinite sequences  $D_{\Phi}$ . We showed that it is a definable set in the sense of continuous logic, and that its interpretation in models of  $T$  only depend on the bi-interpretation class of  $T$  (up to a definable bijection). It also follows from what we showed that it is a reconstruction sort (without exceptional set). Since it is unique, let us denote it by  $D_T$  (in fact, we could also just denote it by  $D$ : its construction only depends on the language, and then we simply restrict our consideration of it to models of  $T$ ). We then proved that the groupoid

$$\mathbf{G}(T) = \mathbf{G}_{D_T}(T)$$

is a complete bi-interpretation invariant for  $T$ . This is, again, a special case of [Theorem 2.11](#).

*Example 2.14.* Let  $T$  be a (complete) theory in continuous logic. In [[Ben](#)] we defined when a sort  $D_T$  is a *universal Skolem sort*, and proved that if such a sort exists, then it is unique, and only depends on the bi-interpretation class of  $T$  (in contrast with the previous example, here we do not have a general construction for such a sort,



let alone a uniform one, so it really does depend on  $T$ ). We proved that if  $T$  admits a universal Skolem sort, then

$$\mathbf{G}(T) = \mathbf{G}_{D_T}(T)$$

is a complete bi-interpretation invariant for  $T$ . This is, again, a special case of [Theorem 2.11](#).

*Remark 2.15.* [Example 2.14](#) encompasses the two previous examples in the following sense.

- If  $T$  is classical, then the sort  $D_T$  of [Example 2.13](#) is a universal Skolem sort, so [Example 2.13](#) is a special case of [Example 2.14](#).
- If  $T$  is  $\aleph_0$ -categorical, then  $D_T = D_{T,0} \times 2^{\mathbf{N}}$  is a universal Skolem sort, so

$$\mathbf{G}(T) \simeq 2^{\mathbf{N}} \times G(T) \times 2^{\mathbf{N}}, \quad \text{with groupoid law} \quad (\alpha, g, \beta) \cdot (\beta, h, \gamma) = (\alpha, gh, \gamma),$$

and conversely,

$$e \in \mathbf{B}(T) \quad \Longrightarrow \quad G(T) \simeq e\mathbf{G}(T)e.$$

Therefore, the reconstruction of [Example 2.12](#) may be viewed as a special case of that of [Example 2.14](#). In all of these cases, the basis  $S_{D_T}(T)$  is homeomorphic to the Cantor space  $2^{\mathbf{N}}$ .

However, in [\[Ben\]](#) we also gave an example of a continuous theory which does not admit a universal Skolem sort. In particular, the explicit construction of  $D_T$  as  $D_{\Phi}$  in the case of a classical theory simply does not extend, as is, to continuous logic. The rest of this article is dedicated to presenting a modified version of this construction, giving rise to a reconstruction sort that *does* have an exceptional set (a very simple one, consisting of a single point), allowing us to prove a reconstruction theorem for every first order theory in a countable language (in continuous logic, or classical one).

### 3. STAR SPACES

**Definition 3.1.** A *retraction set* is a set  $X$  equipped with an action of the multiplicative monoid  $[0, 1]$ . In particular,  $1 \cdot x = x$  for all  $x \in X$ , and  $\alpha(\beta x) = (\alpha\beta)x$  (so this is a little stronger than a homotopy).

It is a *star set* if  $0 \cdot x$  does not depend on  $X$ . We then denote this common value by  $0 \in X$ , and call it the *root* of  $X$ .

A *topological retraction (star) space* is one equipped with a topology making the action  $[0, 1] \times X \rightarrow X$  continuous.

A *metric retraction (star) space* is one equipped with a distance function satisfying  $d(\alpha x, \alpha y) \leq \alpha d(x, y)$ . If it is a star space, then we denote  $d(x, 0)$  by  $\|x\|$ .

Notice that a retraction set  $X$  can be fibred over  $0 \cdot X$ , with each fibre a star set.

*Example 3.2.* The real half line  $\mathbf{R}^+$  is naturally a topological and metric star space. The interval  $[0, 1]$  (or  $[0, r]$  for any  $r > 0$ ) is a compact topological and bounded metric star space.

*Example 3.3.* If  $X$  and  $Y$  are two star sets, then  $X \times Y$ , equipped with the diagonal action  $\alpha(x, y) = (\alpha x, \alpha y)$ , is again a star set. If both are metric star spaces, then equipping the product with the sum or maximum distance makes it a metric star space as well.

*Example 3.4.* Let  $X$  be a set, and equip  $[0, 1] \times X$  with the equivalence relation

$$(\alpha, x) \sim (\beta, y) \quad \Longleftrightarrow \quad (\alpha, x) = (\beta, y) \quad \text{or} \quad \alpha = \beta = 0.$$

The *cone* of  $X$  is the quotient space

$$*X = ([0, 1] \times X) / \sim.$$

A member of  $*X$  will be denoted  $[\alpha, x]$ . We equip it with the action  $\alpha \cdot [\beta, x] = [\alpha\beta, x]$ . This makes it a star set, with  $[0, x] = 0$  regardless of  $x$ .

We shall tend to identify  $x \in X$  with  $[1, x] \in *X$ , so  $[\alpha, x]$  may also be denoted by  $\alpha x$ .

When  $X$  is a compact Hausdorff space, the relation  $\sim$  is closed,  $*X$  is again compact and Hausdorff, and the identification  $X \subseteq *X$  is a topological embedding. When  $X$  is a bounded metric space, say  $\text{diam}(X) \leq 2$ , we propose to metrize  $*X$  by

$$(1) \quad d(\alpha x, \beta y) = |\alpha - \beta| + (\alpha \wedge \beta)d(x, y).$$

(We recall that  $\vee$  and  $\wedge$  are used as infix notation for maximum and minimum, respectively.) In particular, if either  $\alpha$  or  $\beta$  vanishes, then the right hand side does not depend on either  $x$  or  $y$ , so  $d$  is well defined.

The only property that is not entirely obvious is the triangle inequality, namely

$$(2) \quad |\alpha - \gamma| + (\alpha \wedge \gamma)d(x, z) \leq |\alpha - \beta| + (\alpha \wedge \beta)d(x, y) + |\beta - \gamma| + (\beta \wedge \gamma)d(y, z).$$

We may assume that  $\alpha \geq \gamma$ , so  $\alpha \wedge \gamma = \gamma$ . If  $\beta \geq \gamma$ , then (2) holds trivially since  $\alpha \wedge \beta \geq \gamma = \beta \wedge \gamma$ . If  $\beta \leq \gamma$ , then the right hand side evaluates to

$$(\alpha - \gamma) + 2(\gamma - \beta) + \beta d(x, y) + \beta d(y, z).$$

Applying the triangle inequality for  $X$  and the hypothesis that  $2 \geq d(x, z)$ , we obtain (2) in this case as well.

We conclude that  $(*X, d)$  is a metric space. The embedding  $X \subseteq *X$  is isometric, and  $\text{diam}(*X) = 1 \vee \text{diam}(X)$ . If  $X$  is complete, then so is  $*X$ .

A special instance of this is the cone of a singleton, which can be identified with the interval  $[0, 1]$  equipped with the natural star, topological or metric structures.

*Example 3.5.* More generally, let  $S$  be a star set,  $X$  an arbitrary set, and define

$$(s, x) \sim (t, y) \iff (s, x) = (t, y) \text{ or } s = t = 0, \\ S * X = (S \times X) / \sim.$$

As in the definition of a cone, a member of  $S * X$  will be denoted  $[s, x]$  or  $s * x$  (in analogy with the notation  $\alpha x$ ). We make  $S * X$  into a star set by defining  $\alpha \cdot (s * x) = (\alpha s) * x$ .

This indeed generalises the cone construction, with  $*X = [0, 1] * X$ .

When  $S$  and  $X$  are compact Hausdorff spaces, the relation  $\sim$  is closed, and  $S * X$  is again compact and Hausdorff. When  $S$  and  $X$  are bounded metric spaces, say  $\text{diam}(X) \leq 2$  and  $\|s\| \leq 1$  for all  $s \in S$ , we equip  $S * X$  with the distance function

$$d(s * x, t * y) = d(s, t) \vee d(\|s\|x, \|t\|y),$$

where the second distance is calculated in  $*X$ . Notice that  $\|s * x\| = \|s\|$ , and the distance functions on  $[0, 1] * X$  and  $*X$  agree.

**Definition 3.6.** Let  $X$  and  $Y$  be two retraction (star) spaces. A map  $f: X \rightarrow Y$  is *homogeneous* if  $f(\alpha x) = \alpha f(x)$ . It is *sub-homogeneous* if  $f(\alpha x) = \beta f(x)$  for some  $\beta \leq \alpha$ .

The latter will be mostly used when  $Y = \mathbf{R}^+$ , in which sub-homogeneity becomes  $f(\alpha x) \leq \alpha f(x)$ .

We may also equip a retraction space with a partial order defined by  $\alpha x \leq x$  whenever  $\alpha \in [0, 1]$ . This induces the usual partial order on  $\mathbf{R}^+$ , and sub-homogeneity can be stated as  $f(\alpha x) \leq \alpha f(x)$  for arbitrary maps between retraction spaces. Notice also that our definition of a metric retraction space  $X$  simply requires the distance function to be sub-homogeneous on  $X \times X$ .

#### 4. STAR SORTS

**Definition 4.1.** A *star sort* is a sort equipped with a definable structure of a metric star space. Star sorts will usually be denoted by  $D^*$ ,  $E^*$ , and so on.

**Definition 4.2.** Let  $D^*$  be a star sort and  $\varphi(u, y)$  a formula on  $D^* \times E$ . We say that  $\varphi$  is *sub-homogeneous* if it satisfies  $\alpha \varphi(u, y) \geq \varphi(\alpha u, y) \geq 0$ .

We may specify that it is sub-homogeneous in the variable  $u$ , especially if  $u$  is not the first variable. More generally, we may say that  $\varphi(u, v, \dots)$  is sub-homogeneous in  $(u, v)$  if  $\alpha \varphi(u, v, \dots) \geq \varphi(\alpha u, \alpha v, \dots) \geq 0$ , and similarly for any other tuple of variables.

If it is sub-homogeneous in the tuple of all its variables, we just say that  $\varphi$  is *jointly sub-homogeneous*.

*Example 4.3.* • If  $D$  is any sort (of diameter at most two), then the cone  $*D$ , equipped with the distance proposed in [Example 3.4](#), is a star sort. More generally, if  $D^*$  is a star sort and  $E$  an arbitrary sort, then  $D^* * E$ , as per [Example 3.5](#), is a star sort.

- Any finite product of star sorts, equipped with the diagonal action of  $[0, 1]$  and the maximum or sum distance, is again a star sort. Similarly, any countable product of star sorts, equipped with  $d(u, v) = \sum_n \frac{d_n(u_n, v_n)}{2^n \text{diam}(d_n)}$ , is again a star sort, and the same holds with supremum in place of sum.
- If  $D^*$  is a star sort and  $d'(u, v)$  a jointly sub-homogeneous definable pseudo-distance on  $D^*$ , then the quotient  $(D^*, d')$  can be equipped with an induced star structure, making it again a star sort.
- Let  $D^*$  be a star sort and  $E^* \subseteq D^*$  a definable subset. Then the distance  $d(u, E^*)$  is sub-homogeneous if and only if  $E^*$  is closed under multiplication by  $\alpha \in [0, 1]$ , in which case  $E^*$  is again a star sort.

Notice that  $\varphi(u, y)$  is sub-homogeneous in  $u$  if for every fixed parameter  $b$ , the formula  $\varphi(u, b)$  (in  $u$  alone) is sub-homogeneous.

For an alternate point of view, notice that a sub-homogeneous formula  $\varphi(u, y)$  does not depend on  $y$  when  $u = 0$ . It can therefore be viewed as a formula  $\varphi(u * y)$  in the sort  $D^* * E$  (see [Example 3.5](#)). Since  $\alpha(u * y) = (\alpha u) * y$ , a sub-homogeneous (in  $u$ ) formula  $\varphi(u, y)$  is the same thing as a sub-homogeneous formula  $\varphi(u * y)$  in a single variable from the sort  $D^* * E$ .

Similarly, a formula  $\varphi(u, v)$  on  $D^* \times E^*$  is jointly sub-homogeneous if and only if it is sub-homogeneous as a formula on the product star sort.

*Question 4.4.* We ordered the clauses of [Example 4.3](#) in order to reflect the three operations by which we construct sorts in general. Still, something more probably needs to be said regarding the construction of sub-homogeneous pseudo-distance functions. In the usual context of plain sorts (and plain pseudo-distances), to every formula  $\varphi(x, t)$  on  $D \times E$  we can associate a formula on  $D \times D$ , defined by

$$d_\varphi(x, y) = \sup_t |\varphi(x, t) - \varphi(y, t)|.$$

This is always a definable pseudo-distance on  $D$ . Moreover, in the case where  $E = D$  and  $\varphi$  already defines a pseudo-distance,  $d_\varphi$  agrees with  $\varphi$ .

Can something analogous be done in the present context as well?

The following essentially asserts that we can retract continuously (with Lipschitz constant one, even) all formulas into sub-homogeneous ones. The analogous result for a formula in several variables, with respect to joint sub-homogeneity in some of them, follows.

**Proposition 4.5.** *Let  $D^*$  be a star sort and  $\varphi(u, y) \geq 0$  a positive formula on  $D^* \times E$ . For  $k \in \mathbf{N}$ , define*

$$(\text{SH}_k \varphi)(u, y) = \inf_{u', \alpha} \left( \alpha \varphi(u', y) + kd(\alpha u', u) \right), \quad \text{where } u' \in D^*, \alpha \in [0, 1].$$

- (i) *For any  $\varphi \geq 0$  and  $k$ , the formula  $(\text{SH}_k \varphi)(u, y)$  is  $k$ -Lipschitz and sub-homogeneous in  $u$ , and  $\text{SH}_k \varphi \leq \varphi$ .*
- (ii) *For any two formulas  $\varphi, \psi \geq 0$  and  $r \geq 0$ , if  $\varphi \leq \psi + r$ , then  $\text{SH}_k \varphi \leq (\text{SH}_k \psi) + r$ . Consequently,  $\sup_u |(\text{SH}_k \varphi) - (\text{SH}_k \psi)| \leq \sup_u |\varphi - \psi|$ .*
- (iii) *If  $\varphi$  is sub-homogeneous, then  $(\text{SH}_k \varphi) \rightarrow \varphi$  uniformly, at a rate that only depends on the bound and uniform continuity modulus of  $\varphi$ .*

*Proof.* Clearly,  $(\text{SH}_k \varphi)(u, y)$  is  $k$ -Lipschitz in  $u$ . If  $(\text{SH}_k \varphi)(u, y) < r$  and  $\beta \in [0, 1]$ , then there exist  $u'$  and  $\alpha$  such that  $\alpha \varphi(u', y) + kd(\alpha u', u) < r$ . Then  $\alpha \beta \varphi(u', y) + d(\alpha \beta u', \beta u) < \beta r$ , showing that  $(\text{SH}_k \varphi)(\beta u) < \beta r$ . This proves sub-homogeneity. We also always have  $(\text{SH}_k \varphi)(u, y) \leq 1 \cdot \varphi(u, y) + d(1 \cdot u, u) = \varphi(u, y)$ .

The second item is immediate.

For the third item, we assume that  $\varphi$  is sub-homogeneous, in which case

$$(\text{SH}_k \varphi)(u, y) = \inf_{u'} \left( \varphi(u', y) + kd(u', u) \right) \leq \varphi(u).$$

Say that  $|\varphi| \leq M$  and  $d(u, u') < \delta$  implies  $|\varphi(u, y) - \varphi(u', y)| < \varepsilon$ , and let  $k > 2M/\delta$ . If  $d(u', u) \geq \delta$ , then  $\varphi(u', y) + kd(u', u) \geq \varphi(u)$ , so such  $u'$  may be ignored. Restricting to those where  $d(u', u) < \delta$ , we see that  $(\text{SH}_k \varphi) \geq \varphi - \varepsilon$ .  $\blacksquare$

**Definition 4.6.** We say that a formula  $\varphi(x, y)$  is *witness-normalised* (in  $x$ , unless another variable is specified explicitly) if  $\inf_y \varphi = 0$  (equivalently, if  $\varphi \geq 0$  and  $\sup_x \inf_y \varphi = 0$ ).

More generally, for  $\varepsilon > 0$ , we say that  $\varphi(x, y)$  is  $\varepsilon$ -*witness-normalised* (in  $x$ ) if  $0 \leq \inf_y \varphi \leq \varepsilon$ .

Witness-normalised formulas are analogous to formulas  $\varphi(x, y)$  in classical logic for which  $\exists y \varphi$  is valid: in either case, we require that witnesses exist. If  $\varphi(x, y)$  is any formula, then  $\varphi(x, y) - \inf_z \varphi(x, z)$  is witness-normalised (we may say that it is *syntactically* witness normalised), where we subtract a “normalising” term.

By definition, a sub-homogeneous or a witness-normalised formula is positive. If  $\varphi$  is witness-normalised in any of its arguments and  $\varphi \geq \psi \geq 0$ , then so is  $\psi$ . This applies in particular to the formulas  $\text{SH}_k \varphi$  constructed in [Proposition 4.5](#), assuming  $\varphi$  is witness-normalised.

**Definition 4.7.** Let  $D^*$  and  $E^*$  be two star sorts. A *star correspondence* between  $D^*$  and  $E^*$  is a formula  $\varphi(u, v)$  on  $D^* \times E^*$  that is sub-homogeneous in  $(u, v)$  and witness-normalised in each of  $u$  and  $v$ .

Similarly, an  $\varepsilon$ -*star correspondence* is a jointly sub-homogeneous formula that is  $\varepsilon$ -witness-normalised in each argument.

*Remark 4.8.* If  $\varphi$  is  $\varepsilon$ -witness-normalised (in one of its variables), then  $\varphi' = \varphi \dot{-} \varepsilon$  is witness-normalised (in the same), and  $|\varphi - \varphi'| \leq \varepsilon$ . If  $\varphi$  is sub-homogeneous, then so is  $\varphi \dot{-} \varepsilon$ ,

Therefore, if  $\varphi$  is an  $\varepsilon$ -star correspondence, then  $\varphi' = \varphi \dot{-} \varepsilon$  is a star correspondence, and  $|\varphi - \varphi'| \leq \varepsilon$ .

Say that a definable map  $\sigma: D \rightarrow E$  is *densely surjective* if it is surjective in every sufficiently saturated model of the ambient theory, or equivalently, if  $\sigma$  has dense image in every model. Recall that a definable map  $\sigma: D^* \rightarrow E^*$  between star sorts is *homogeneous* if  $\sigma(\alpha u) = \alpha \sigma(u)$ .

Notice that a definable map  $\sigma: D^* \rightarrow E^*$  is homogeneous if and only if the formula  $d(\sigma u, v)$  is sub-homogeneous in  $(u, v)$ , if and only if it is homogeneous, and it is always witness-normalised in  $u$ . If  $\sigma$  is bijective, then it is homogeneous if and only if  $d(\sigma u, v)$  is a star correspondence, if and only if  $d(u, \sigma^{-1}v)$  is.

**Definition 4.9.** Say that a star sort  $D^*$  is *universal* (as a star sort) if for every star sort  $E^*$ , every star correspondence  $\varphi$  between  $D^*$  and  $E^*$ , and every  $\varepsilon > 0$ , there exists a  $1/2$ -star correspondence  $\psi$  such that, in addition, if  $\psi(u, v_i) < 1$  for  $i = 0, 1$ , then  $\varphi(u, v_i) < \varepsilon$  and  $d(v_0, v_1) < \varepsilon$ .

The choice of one and one half is quite arbitrary, and any two constants  $0 < r_1 < r_2$  would do just as well (in the proof of [Proposition 4.10\(i\)](#) below, replace  $2\psi \dot{-} 1$  with  $(\psi \dot{-} r_1)/(r_2 - r_1)$ ).

**Proposition 4.10.** *Let  $D^*$  and  $E^*$  be star sorts,  $\varphi(u, v)$  a star correspondence on  $D^* \times E^*$ , and  $\varepsilon > 0$ .*

- (i) *If  $D^*$  is a universal star sort, then there exists  $\psi$  as in [Definition 4.9](#) that is a star correspondence (rather than a mere  $\varepsilon$ -star correspondence).*
- (ii) *If  $D^*$  is a universal star sort, then there exists a densely surjective homogeneous definable map  $\sigma: D^* \rightarrow E^*$  such that  $\varphi(u, \sigma u) \leq \varepsilon$ .*
- (iii) *If both  $D^*$  and  $E^*$  are both universal star sorts, then the same can be achieved with  $\sigma$  bijective.*

*Proof.* For (i), let  $\psi$  be as in the conclusion of [Definition 4.9](#). Then  $2\psi \dot{-} 1$  will do.

For (ii), define a sequence of formulas  $\varphi_n(u, v)$  as follows. We start with  $\varphi_0 = \varphi$ , and we may assume that  $0 < \varepsilon < 1$ . Then, assuming that  $\varphi_n$  is a star correspondence, we find a star correspondence  $\varphi_{n+1}$  such that  $\varphi_{n+1}(u, v_i) < 1$  implies  $\varphi_n(u, v_i) \leq \varepsilon$  and  $d(v_0, v_1) < \varepsilon/2^n$ . Let  $X_n \subseteq D^* \times E^*$  be the (type-definable) set defined by  $\varphi_n \leq \varepsilon$  and  $X = \bigcap X_n$ . By hypothesis, for every  $u \in D^*$  and  $n$ , there exists  $v \in E^*$  such that  $(u, v) \in X_n$ . We also have  $X_{n+1} \subseteq X_n$ , so in a sufficiently saturated model there exists  $v \in E^*$  such that  $(u, v) \in X$ . By the second hypothesis on  $\varphi_n$ , such  $v$  is unique, so  $X$  is the graph of a definable map  $\sigma$  (and  $v$  belongs to any model that contains  $u$ ). By the same reasoning as above, for every  $v \in E^*$  there exists  $u \in D^*$  (not necessarily unique, so potentially only in a sufficiently saturated model) such that  $(u, v) \in X$ , so  $\sigma$  is densely surjective.

Assume now that  $v = \sigma u$ , i.e.,  $(u, v) \in X$ . Since each  $\varphi_n$  is sub-homogeneous,  $(\alpha u, \alpha v) \in X$  for every  $\alpha \in [0, 1]$ , i.e.,  $\alpha v = \sigma(\alpha u)$ , and  $\sigma$  is homogeneous. Finally, since  $\varphi_0 = \varphi$ , we have  $(u, \sigma u) \in X \subseteq X_0$ , so  $\varphi(u, \sigma u) \leq \varepsilon$ .

For (iii) we argue in the same fashion, exchanging the roles of  $D^*$  and  $E^*$  at odd steps. ■

In particular, if a universal star sort exists, then it is unique, up to a homogeneous definable bijection.

**Lemma 4.11.** *Let  $(D_n^*)$  be an inverse system of star sorts, where each  $\pi_n: D_{n+1}^* \rightarrow D_n^*$  is surjective and homogeneous.*

- (i) *The inverse limit  $D^* = \varprojlim D_n^*$  is a star sort, with the natural action  $\alpha(u_n) = (\alpha u_n)$  and the distance proposed in [Example 4.3](#).*
- (ii) *A star correspondence between  $D^*$  and  $E^*$  that factors through  $D_n^* \times E^*$  is the same thing as a star correspondence between  $D_n^*$  and  $E^*$ .*
- (iii) *In order for  $D^*$  to be a universal star sort, it is enough for it to satisfy the condition of [Definition 4.9](#) for star-correspondences  $\varphi$  that factor through  $D_n^* \times E^*$  for some  $n$ .*

*Proof.* The first two assertions are fairly evident. In what follows, we are going to identify a formula  $\varphi(u_n, v)$  on  $D_n^* \times E^*$  with the formula  $\varphi(\pi_n(u), v)$  on  $D^* \times E^*$ , which is essentially what the second point says.

For the last one, say that  $\varphi$  is a star correspondence between  $D^*$  and  $E^*$ , and let  $\varepsilon > 0$ . For  $n$  large enough we may find a formula  $\varphi_1(u_n, v)$  on  $D_n^* \times E^*$  such that  $\varphi \geq \varphi_1 \geq \varphi \dot{-} \varepsilon$  (with the identification proposed in the previous paragraph). Both  $\varphi$  and  $\varphi \dot{-} \varepsilon$  are jointly sub-homogeneous, so using the construction of [Proposition 4.5](#), for large enough  $k$  we have

$$\varphi \geq \text{SH}_k \varphi \geq \text{SH}_k \varphi_1 \geq \text{SH}_k(\varphi \dot{-} \varepsilon) \geq \varphi \dot{-} 2\varepsilon.$$

Since  $\varphi' = \text{SH}_k \varphi_1$  is jointly sub-homogeneous, it is a star correspondence, and it factors through  $D_n^* \times E^*$ . Assume now that  $\psi(u, v)$  exists, as per [Definition 4.9](#), for  $\varphi'$  and  $\varepsilon$ . In particular, if  $\psi(u, v) < 1$ , then  $\varphi'(u, v) < \varepsilon$ , so  $\varphi(u, v) < 3\varepsilon$ , which is good enough. ■

## 5. SORTS WITH WITNESSES

In this section, we provide an explicit construction of a universal star sort. We follow a path similar to the construction of  $D_\Phi$  in [Ben], seeking a sort that contains “all witnesses”.

Let us consider first the case of a single formula  $\varphi(x, y)$  on  $D \times E$ , which we assume to be witness-normalised (namely, such that  $\inf_y \varphi = 0$ , see Definition 4.6). The sort  $D$  is viewed as the sort of *parameters*, and  $E$  is the sort of potential *witnesses*. One may then wish to consider the set of “parameters with witnesses”, namely the collection of all pairs  $(x, y)$  such that  $\varphi(x, y) = 0$ , but this may be problematic for several reasons.

First of all, in a fixed (non-saturated) structure, for all  $a$  there exist  $b$  such that  $\varphi(a, b)$  is arbitrarily small, but not necessarily such that  $\varphi(a, b) = 0$ . This can be overcome by allowing an error, e.g., by considering all the solution set of  $\varphi(x, y) \leq \varepsilon$  for some  $\varepsilon > 0$ . In fact, it is enough to consider the solution set of  $\varphi(x, y) \leq 1$ : if we want a smaller error, we need only replace  $\varphi$  with  $\varphi/\varepsilon$ .

A second, and more serious issue, is that the resulting set(s) need not be definable. That is to say that it may happen that  $1 < \varphi(a, b) < 1 + \varepsilon$  for arbitrarily small  $\varepsilon > 0$  without there existing a pair  $(a', b')$  close to  $(a, b)$  such that  $\varphi(a', b') \leq 1$ . We can solve this by allowing a *variable error*, considering triplets  $(r, x, y)$  where  $r \in \mathbf{R}$  and  $\varphi(x, y) \leq r$ . Now, if  $\varphi(x, y) < r + \varepsilon$ , then the triplet  $(x, y, r)$  is very close to  $(x, y, r + \varepsilon)$ , which does belong to our set.

This may seem too easy, and raises some new issues. For example, if we allow errors greater than the bound for  $\varphi$ , then the condition  $\varphi(x, y) \leq r$  becomes vacuous. This is not, in fact, a real problem, since soon enough we are going to let  $\varphi$  vary (or more precisely, consider an infinite family of formulas simultaneously), and any finite bound  $r$  will be meaningful for *some* of the formulas under consideration. However, in order for the previous argument to work,  $r$  cannot be bounded (we must always be able to replace it with  $r + \varepsilon$ ). By compactness,  $r = +\infty$  must be allowed as well – and now there is no way around the fact that  $\varphi(x, y) \leq \infty$  is vacuous, regardless of  $\varphi$ .

We seem to be chasing our own tail, each time shovelling the difficulty underneath a different rug – indeed, a complete solution is impossible, or else we could construct a universal Skolem sort, which was shown in [Ben] to be impossible in general. What we propose here is a “second best”: allow infinite error, but use the formalism of star sorts to identify all instances with infinite error as the distinguished root element. Thus, at the root, all information regarding the (meaningless) witnesses will be lost, while every point outside the root will involve finite error, and therefore meaningful witnesses. Since we want the root to be at zero, rather than at infinity, we replace  $r \in [1, \infty]$  with  $\alpha = 1/r \in [0, 1]$ .

Let  $D^*$  be a star sort,  $E$  a sort. The set  $D^* * E = \{u * y : u \in D^*, y \in E\}$ , as per Example 3.5, is again a star sort, in which  $0 * y = 0$  regardless of  $y$ .

**Lemma 5.1.** *Let  $D^*$  be a star sort,  $E$  a sort, and let  $\varphi(u, y)$  a formula on  $D^* \times E$ , witness-normalised and sub-homogeneous in  $u$ . Then*

$$D_\varphi^* = \{u * y : u \in D^* \text{ and } \varphi(u, y) \leq 1\} \subseteq D^* * E$$

*is again a star sort, and the natural projection map  $D_\varphi^* \rightarrow D^*$ , sending  $u * y \mapsto u$ , is surjective.*

*Proof.* We may view  $\varphi$  as a formula on  $D^* * E$ , since, by sub-homogeneity,  $\varphi(0, y) = 0$  regardless of  $y$ . The set  $D_\varphi^*$  is the zero-set in  $D^* * E$  of the formula  $\varphi \div 1$ . Assume now that  $a * b \in D^* * E$  and  $\varphi(a, b) \div 1 < \delta$ . Then  $(1 - \delta)a * b \in D_\varphi^*$ , and it is as close as desired (given  $\delta$  small enough) to  $a * b$ . Therefore,  $D_\varphi^*$  is definable. Since  $\varphi$  is sub-homogeneous,  $D_\varphi^*$  is closed under multiplication by  $\alpha \in [0, 1]$  and is therefore a star sort. Since  $\varphi$  is witness-normalised, the projection is onto. ■

Let us iterate this construction. For this, notice that if  $D$  is a sort, then  $(*D) * E = *(D \times E)$ , identifying  $(\alpha x) * y = \alpha(x, y)$ . Therefore, if  $D^* \subseteq *D$  (with the induced star structure), then  $D^* * E \subseteq *(D \times E)$ .

**Definition 5.2.** Fix a sort  $D$ , as well as a sequence of formulas  $\Phi = (\varphi_n)$ , where each  $\varphi_n(x_{<n}, y)$  is a witness-normalised formula on  $D^n \times D$ . Since  $\Phi$  determines the sort  $D$ , we shall say that  $\Phi$  is a sequence on  $D$ . We then define

$$\begin{aligned} D_n^* &= \{\alpha x_{<n} : \alpha \varphi(x_{<k}, x_k) \leq 1 \text{ for all } k < n\} \subseteq *(D^n), \\ D_\Phi^* &= \{\alpha x : \alpha \varphi(x_{<n}, x_n) \leq 1 \text{ for all } n\} \subseteq *(D^{\mathbf{N}}). \end{aligned}$$

In other words,

$$D_0^* = [0, 1] = *(\text{singleton}), \quad D_{n+1}^* = (D_n^*)_{\varphi_n'}, \quad D_\Phi^* = \varprojlim D_n^*$$

where  $\varphi_n'(\alpha x_{<n}, y) = \alpha \varphi(x_{<n}, y)$ . By Lemma 5.1, each  $D_n^*$  is a star sort, and the natural projection  $D_{n+1}^* \rightarrow D_n^*$  is onto. By Remark 1.2,  $D_\Phi^* = \varprojlim D_n^*$  is also a sort, and therefore a star sort.

Notice that any formula in  $D_n^*$  can be viewed, implicitly, as a formula in  $D_k^*$  for any  $k \geq n$ , or even in  $D_\Phi^*$ , via the projections  $D_k^* \rightarrow D_n^*$  or  $D_\Phi^* \rightarrow D_n^*$  (this is, essentially, an addition of dummy variables). In what follows, variables in  $D_n^*$  will be denoted by  $u_n$  or  $\alpha x_{<n}$  (where  $x_{<n} \in D^n$ ), and similarly, variables in  $D_\Phi^*$  will be denoted by  $u$  or  $\alpha x$ .

**Definition 5.3.** We say that the sequence  $\Phi$  is *rich* if  $D$  admits a definable projection onto any countable product of basic sorts, and for every witness-normalised formula  $\varphi(x_{<n}, y)$  in  $D^n \times D$  and every  $\varepsilon > 0$  there exist arbitrarily big  $k \geq n$  such that  $|\varphi_k(x_{<k}, y) - \varphi(x_{<n}, y)| < \varepsilon$  (so  $\varphi$  is viewed as a formula in  $x_{<k}, y$  through the addition of dummy variables).

With our standing hypothesis that the language is countable, with countably many basic sorts, it is fairly easy to construct a rich sequence. In particular, for  $D$  we may take the (countable) product of all infinite countable powers of the basic sorts. We may even do this in a uniform manner, irrespective of the ambient theory  $T$ , by considering only formulas of the form  $\varphi(x_{<n}, y) - \inf_z \varphi(x_{<n}, z)$  (we may call such formulas *syntactically witness normalised*).

Let  $\Phi = (\varphi_n)$  (and  $D$ ) be fixed, with  $\Phi$  rich. We define a formula on  $D^n$  by

$$\rho_n(x_{<n}) = \frac{1}{1 \vee \bigvee_{k < n} \varphi_k(x_{<k}, x_k)}.$$

In other words,  $\rho_n(x_{<n})$  is the maximal  $\alpha \in [0, 1]$  such that  $\alpha x_{<n} \in D_n^*$ , or equivalently, such that  $x_{<n}$  can be extended to  $x$  with  $\alpha x \in D_\Phi^*$ .

**Lemma 5.4.** Let  $\Phi = (\varphi_n)$  be rich. Let  $E^*$  be another star sort,  $\psi(u_n, v)$  a star correspondence on  $D_\Phi^* \times E^*$  that factors through  $D_n^* \times E^*$ , and  $\varepsilon > 0$ . Then  $\psi$  factors through  $D_k^* \times E^*$  for every  $k \geq n$ , and for every large enough  $k$  the formula  $\psi_1^k(x_{<k}, v) = \psi(\rho_k(x_{<k})x_{<n}, v)$  is  $\varepsilon$ -witness-normalised in either argument.

*Proof.* If  $k \geq n$ , then  $\rho_k(x_{<k}) \leq \rho_n(x_{<n})$ , so  $\rho_k(x_{<k})x_{<n} \in D_n^*$ . Since  $\psi(u_n, v)$  is witness-normalised in  $u_n$ ,  $\psi_1^k(x_{<k}, v)$  is in  $x_{<k}$ . It is left to show that for  $k$  large enough, it is also  $\varepsilon$ -witness-normalised in  $v$ .

Our hypothesis regarding  $D$  implies, among other things, that there exists a surjective definable map  $\chi: D \rightarrow [0, 1]$  (namely, a surjective formula). Therefore, for a constant  $C$  that we shall choose later, there exists  $m \geq n$  such that  $C\chi(y) \geq \varphi_m(x_{<m}, y) \geq C\chi(y) - 1/C$ .

Assume that  $k > m$ . For every possible value of  $v \in E^*$ , which we consider as fixed, there exists  $\alpha x_{<n} \in D_n^*$  such that  $\psi(\alpha x_{<n}, v) < \varepsilon$ . We can always extend  $x_{<n}$  to  $x_{<m}$  in such a manner that  $\rho_m(x_{<m}) = \rho_n(x_{<n}) \geq \alpha$ , so  $\alpha x_{<m} \in D_m^*$ . We choose  $x_m$  so  $\chi(x_m) = (\alpha C \vee 1)^{-1}$ , and extend  $x_{\leq m}$  to  $x_{<k}$  so  $\rho_k(x_{<k}) = \rho_{m+1}(x_{\leq m})$ .

If  $\alpha C \geq 1$ , then  $1/\alpha \geq \varphi_m(x_{<m}, x_m) \geq 1/\alpha - 1/C$ , so  $\alpha \leq \rho_{m+1}(x_{\leq m}) \leq \alpha(1 - \alpha/C)^{-1}$ . Having chosen  $C$  large enough,  $\rho_k(x_{<k}) = \rho_{m+1}(x_{\leq m})$  is as close to  $\alpha$  as desired. If  $\alpha C < 1$ , then  $0 \leq \alpha \leq 1/C$  and  $0 < \rho_{k+1}(x_{\leq k}) \leq 1/(C - 1/C)$ , so the same conclusion holds.

Either way, having chosen  $C$  large enough,  $\psi_1^k(x_{<k}, v)$  is as close as desired to  $\psi(\alpha x_{<n}, v)$ , and in particular  $\psi_1^k(x_{<k}, v) < 2\varepsilon$ , which is good enough.  $\blacksquare$

Given our hypothesis regarding  $D$ , every sort can be expressed as a definable subset of a quotient of  $D$  by a pseudo-distance. Such a quotient will be denoted  $(D, \bar{d})$  (which includes an implicit step of identifying points at  $\bar{d}$ -distance zero).

**Convention 5.5.** From this point, and through the proof of [Lemma 5.7](#), we fix a star sort  $E^*$ . By the preceding remark, we may assume that  $(E^*, d_{E^*}) \subseteq (D, \bar{d})$  isometrically, where  $\bar{d}$  is a definable pseudo-distance on  $D$  which we also fix. In particular, the distance on  $E^*$  will also be denoted by  $\bar{d}$ . If  $y \in D$ , we denote its image in the quotient  $(D, \bar{d})$  by  $\bar{y}$ .

It is worthwhile to point out that if  $\alpha x \in D_\Phi^*$ , then for every  $k \in \mathbf{N}$  and  $\delta > 0$ ,

$$(3) \quad (\alpha\delta/2) \left( \varphi_k(x_{<k}, x_k) + 1 \right) = (\delta/2) \left( \alpha \varphi_k(x_{<k}, x_k) + \alpha \right) \leq \delta.$$

Given  $n \leq k$  and  $\delta > 0$ , let us define for  $\alpha x \in D_\Phi^*$ ,  $v \in E^*$  and  $y \in D$ :

$$\begin{aligned} \chi^n(\alpha x, y, v) &= \inf_{w \in E^*} \left[ \bar{d}(\alpha \rho_n(x_{<n})^{-1} w, v) + \alpha \bar{d}(\bar{y}, w) \right], \\ \chi^{n,k}(\alpha x, v) &= \chi^n(\alpha x, x_k, v) = \inf_{w \in E^*} \left[ \bar{d}(\alpha \rho_n(x_{<n})^{-1} w, v) + \alpha \bar{d}(\bar{x}_k, w) \right]. \end{aligned}$$

Let us explain this. First of all, since  $\alpha x \in D_\Phi^*$ , we must have  $\alpha \leq \rho_n(x_{<n})$ , so the expression  $\alpha \rho_n(x_{<n})^{-1} w$  makes sense. Also, if  $\alpha = 0$ , then  $\chi^n(\alpha x, y, v) = \|v\|$  does not depend on  $x$ , so this is well defined.

Now, let  $y \in D$  (possibly,  $y = x_k$  for some  $k \geq n$ , but this will happen later). We want  $v$  to be equal to  $\alpha\rho_n(x_{<n})^{-1}\bar{y}$ , and in particular, we want  $\bar{y}$  to belong to  $E^*$ . We may not multiply by  $\alpha\rho_n(x_{<n})^{-1}$  outside  $E^*$ , but we may quantify over  $E^*$ . Therefore, we ask for  $\bar{y}$  to be very close to some  $w \in E^*$ , and for  $\alpha\rho_n(x_{<n})^{-1}w$ , which always makes sense, to be close to  $v$ .

**Lemma 5.6.** *The formula  $\chi^{n,k}(u, v)$  has the following properties:*

- (i) *It is jointly sub-homogeneous in its arguments.*
- (ii) *For every  $n, \varepsilon > 0$  there exists  $\delta = \delta(n, \varepsilon) > 0$  such that, if  $\chi^n(u, y, v_i) \leq \delta$  for  $i = 0, 1$ , then  $\bar{d}(v_0, v_1) < \varepsilon$ . In particular, for ant  $k$ , if  $\chi^{n,k}(u, v_i) \leq \delta$  for  $i = 0, 1$ , then  $\bar{d}(v_0, v_1) < \varepsilon$ .*
- (iii) *Assuming that  $\varphi_k(x_{<k}, y) \geq 2\bar{d}(\bar{y}, E^*)/\delta - 1$ , the formula  $\chi^{n,k}(u, v)$  is  $\delta$ -witness-normalised in  $u$ .*

*Proof.* Item (i) is immediate (among other things, we use the fact that  $\bar{d}$  is sub-homogeneous on  $E^*$ ).

For (ii), assume that  $\chi^n(\alpha x, y, v_i) = 0$ . Then either  $\alpha = 0$ , in which case  $v_i = 0$ , or  $\alpha > 0$ , in which case we have  $\bar{y} \in E^*$  and  $v_i = \alpha\rho_n(x_{<n})^{-1}\bar{y}$ . Either way,  $v_0 = v_1$ , and in particular  $\bar{d}(v_0, v_1) < \varepsilon$ . The conclusion follows by compactness.

For (iii), let  $u = \alpha x \in D_{\Phi}^*$ . By (3) we have  $\alpha\bar{d}(\bar{x}_k, E^*) \leq \delta$ . Choose  $w \in E^*$  such that  $\alpha\bar{d}(\bar{x}_k, w) \leq \delta$ , and let  $v = \alpha\rho_n(x_{<n})^{-1}w$ . Then  $\chi^{n,k}(u, v) \leq \delta$ .  $\blacksquare$

**Lemma 5.7.** *Let  $\Phi = (\varphi_n)$  be rich. Let  $E^* \subseteq (D, \bar{d})$  be a star sort, as per [Convention 5.5](#),  $\psi(u, v)$  a star correspondence on  $D_{\Phi}^* \times E^*$ , and  $\varepsilon > 0$ . Then there exist  $n \leq k$  and  $\delta > 0$  such that  $\chi^{n,k}(u, v)$  is a  $\delta$ -star correspondence between  $D_{\Phi}^*$  and  $E^*$ , and in addition, if  $\chi^{n,k}(u, v_i) \leq 2\delta$  for  $i < 2$ , then  $\psi(u, v_i) \leq \varepsilon$  and  $\bar{d}(v_0, v_1) < \varepsilon$ .*

*Proof.* By [Lemma 4.11](#) and [Lemma 5.4](#), for some  $n$  (in fact, any  $n$  large enough), we may assume that  $\psi$  is a star correspondence that factors as  $\psi(u_n, v)$  through  $D_n^* \times E^*$ , and that  $\psi_1(x_{<n}, v) = \psi(\rho_n(x_{<n})x_{<n}, v)$  is  $\varepsilon$ -witness-normalised in either argument. In particular,  $\psi_1 \div \varepsilon$  is witness-normalised.

We may extend  $\psi_1 \div \varepsilon$  to  $D^n \times (D, \bar{d})$ , obtaining a formula  $\psi_2(x_{<n}, y)$  on  $D^n \times D$ , which is uniformly  $\bar{d}$ -continuous in  $y$ . Since  $\psi_1 \geq 0$ , we may assume that  $\psi_2 \geq 0$ , and even that

$$\psi_2(x_{<n}, y) \geq \bar{d}(\bar{y}, E^*).$$

Let us choose  $\delta > 0$  small enough, based on choices made so far. Since  $\psi_2(x_{<n}, y)$  is witness-normalised in  $x_{<n}$  (choosing witnesses  $\bar{y} \in E^*$ ), there exists  $k \geq n$  such that  $|\varphi_k - 2\psi_2/\delta| \leq 1$ . By [Lemma 5.6](#), having chosen  $\delta$  small enough, the formula  $\chi^{n,k}(u, v)$  is jointly sub-homogeneous,  $\delta$ -witness-normalised in  $u$ , and  $\chi^{n,k}(u, v_i) \leq 2\delta$  implies  $\bar{d}(v_0, v_i) < \varepsilon$ . There are two more properties we need to check.

First, we need to check that  $\chi^{n,k}(u, v)$  is  $\delta$ -witness-normalised in  $v$ . Indeed, given  $v = \bar{y} \in E^*$ , we know that there exists a sequence  $x_{<n} \in D^n$  such that  $\psi_1(x_{<n}, v) \div \varepsilon = 0$ . Let  $\alpha = \rho_n(x_{<n})$ , so  $\alpha x_{<n} \in D_n^*$ , and extend the sequence  $x_{<n}$  to  $x_{<k}$  keeping  $\alpha x_{<k} \in D_k^*$ . We now choose  $x_k = y$ , so  $\psi_2(x_{<n}, x_k) = 0$  and  $\varphi_k(x_{<k}, x_k) \leq 1$ . Therefore,  $\alpha x_{\leq k} \in D_{k+1}^*$ , and we may complete the sequence to  $x \in D^N$  such that  $\alpha x \in D_{\Phi}^*$ . Then  $\chi^{n,k}(\alpha x, v) = 0$ , as witnessed by  $w = v$  (recalling that we chose  $\alpha = \rho_n(x_{<n})$ ).

Second, we need to check that, having chosen  $\delta$  appropriately,  $\chi^{n,k}(\alpha x, v) \leq 2\delta$  implies  $\psi(\alpha x, v) \leq \varepsilon$ . Indeed, following a path similar to the proof of [Lemma 5.6\(ii\)](#), assume that

$$\chi^n(\alpha x, y, v) = \alpha\psi_2(x_{<n}, y) = 0.$$

If  $\alpha = 0$ , then  $v = 0$  and  $\psi(\alpha x, v) = \psi(0, 0) = 0$ . If  $\alpha > 0$ , then  $\bar{y} \in E^*$ , and  $v = \alpha\rho_n(x_{<n})^{-1}\bar{y}$ , and  $\psi(\rho_n(x_{<n})x, \bar{y}) \div \varepsilon = \psi_2(x_{<n}, y) = 0$ . Since  $(\alpha x, v) = \alpha\rho_n(x_{<n})^{-1}(\rho_n(x_{<n})x, \bar{y})$ , it follows that  $\psi(\alpha x, v) \leq \varepsilon$  in this case as well. By compactness, for  $\delta$  small enough, if  $\chi^n(\alpha x, y, v) \leq 2\delta$  and  $\alpha\psi_2(x_{<n}, y) \leq \delta$ , then  $\psi(\alpha x, v) < 2\varepsilon$ . This last argument does not depend on  $k$ , so we may assume that  $\delta$  was chosen small enough to begin with. By (3), the inequality  $\alpha\psi_2(x_{<n}, x_k) \leq \delta$  is automatic when  $\alpha x \in D_{\Phi}^*$ . If, in addition, we assume that  $\chi^{n,k}(\alpha x, v) = \chi^n(\alpha x, x_k, v) \leq 2\delta$ , then  $\psi(\alpha x, v) < \varepsilon$ , completing the proof.  $\blacksquare$

**Theorem 5.8.** *Let  $\Phi$  be a rich sequence. Then  $D_{\Phi}^*$  is universal. In particular, a universal star sort exists.*

*Proof.* Immediate from [Lemma 5.7](#), using the formula  $2\chi^{n,k}/\delta$ .  $\blacksquare$

## 6. FURTHER PROPERTIES OF THE UNIVERSAL STAR SORT

In [Section 4](#) we showed that the universal star sort, if it exists, is unique up to a homogeneous definable bijection, and in [Section 5](#) we showed that one exists as  $D_{\Phi}^*$  for any rich sequence  $\Phi$ . Let us prove a few additional properties of this special sort.

**Convention 6.1.** From now on,  $D^*$  denotes any universal star sort. Since it is unique up to a homogeneous definable bijection, multiplication by  $\alpha \in [0, 1]$  is well defined regardless of the construction we choose for  $D^*$ . In particular, its root is well defined.

Notice that we can construct it as  $D_\Phi^*$  in a manner that only depends on the language (and not on  $T$ ): we obtain a universal star sort for  $T$  simply by restricting our consideration of this sort to models of  $T$ .

The uniqueness of  $D^*$  means that we may choose it to be  $D_\Phi^*$  for any rich  $\Phi$ , and in particular, that we are allowed some leverage in choosing a convenient sequence  $\Phi$ , as in the proof of the following result.

**Theorem 6.2.** *The universal star sort  $D^*$  is a reconstruction sort for  $T$  (see [Definition 2.1](#)), with the exceptional set being the root  $D^0 = \{0\}$ .*

*Proof.* We may assume that  $D_T^* = D_\Phi^*$  for some rich sequence  $\Phi$  on a sort  $D$ , as per [Definition 5.3](#). Let  $M \models T$  and  $\alpha a \in D_\Phi^*(M) \setminus \{0\}$ , and let  $N = \text{dcl}(\alpha a) \subseteq M$ , necessarily a closed set (if  $M$  is multi-sorted, closed in each sort separately). Then  $\alpha \neq 0$ , and  $N = \text{dcl}(a)$ . In order to show that  $N \preceq M$ , it will suffice to show that it satisfies the Tarski-Vaught criterion: for every formula  $\varphi(x, y)$ , where  $x$  is in the sort  $D^N$  and  $y$  in one of the basic sorts,

$$\inf_y \varphi(a, y) = \inf_{b \in N} \varphi(a, b),$$

where the truth values are calculated in  $M$ . Since  $D$  projects, by hypothesis, onto any basic sort, we replace  $\varphi$  with its pull-back and assume that it is a formula on  $D^N \times D$ . Replacing  $\varphi$  with  $\varphi(x, y) - \inf_z \varphi(x, z)$ , we may assume that  $\varphi$  is witness-normalised and the left hand side vanishes. Then it is enough to show that for every  $\varepsilon > 0$  there exists  $b \in N$  such that  $\varphi(a, b) < \varepsilon$ , and replacing  $\varphi$  with an appropriate multiple, it is enough to require  $\varphi(a, b) \leq 1 + 1/\alpha$ . Choosing  $n$  such that  $\varphi_n$  is a good-enough approximation of  $\varphi$ , it is enough to find  $b \in D(N)$  such that  $\varphi_n(a_{<n}, b) \leq 1/\alpha$ . For this,  $b = a_n$  will do. This proves the coding models property of [Definition 2.1](#).

For the density property, assume that  $M$  is separable, and let  $\alpha a \in D(M)$ . Assume first that  $\alpha > 0$ . We may freely assume that  $\varphi_k = 0$  infinitely often. Let us fix  $n_0$ , and define a sequence  $b \in D^N$  as follows.

- We start with  $b_{<n_0} = a_{<n_0}$ .
- Having chosen  $b_{<k}$  (for  $k \geq n_0$ ) such that  $\alpha b_{<k} \in D_k^*$ , we can always choose  $b_k \in D(M)$  so  $\alpha b_{\leq k} \in D_{k+1}^*$ .
- If  $\varphi_k = 0$ , then we may choose any  $b_k \in D(M)$  that we desire. Since this happens infinitely often, we may ensure that  $\text{dcl}(b) = M$ .

In the end,  $\alpha b \in D_\Phi^*$  and  $\text{dcl}(\alpha b) = \text{dcl}(b) = M$ , so  $\alpha b$  codes  $M$ . Taking  $n_0$  large enough,  $\alpha b$  is as close as desired to  $\alpha a$ .

This argument shows, in particular, that there exists  $\alpha a \in D(M)$  that codes  $M$ . Let  $\alpha_n = \alpha/2^n$ . Then  $\alpha_n a \in D(M)$  codes  $M$  for each  $n$ , and  $\alpha_n a \rightarrow 0$ , so the root can also be approximated by codes for  $M$ .

We are left with the universality property: given a sort  $E$ , we want to show that it embeds in a quotient of  $D_\Phi^*$ . The sort  $D_1^*$  is a quotient of  $D_\Phi^*$ . We may freely assume that  $\varphi_0 = 0$ , so  $D_1^* = *D$ . We may also assume that  $D$  projects onto  $E$ , giving rise to a projection of  $*D$  onto  $*E$ . Composing the two,  $D_\Phi^*$  projects onto  $*E$ , into which  $E$  embeds. ■

**Definition 6.3.** Let  $T$  be any theory in a countable language, and  $D^*$  its universal star sort. View it as a reconstruction sort, as per [Theorem 6.2](#), with exceptional set  $D^0 = \{0\}$ , and define the corresponding groupoid, as per [Definition 2.3](#):

$$\mathbf{G}^*(T) = \mathbf{G}_{D^*}(T).$$

We already know that this is an open Polish topological groupoid, with basis  $\mathbf{B}^*(T) \simeq \mathbf{S}_{D^*}(T)$ .

**Theorem 6.4.** *The groupoid  $\mathbf{G}^*(T)$  is a complete bi-interpretation invariant for the class of theories in countable languages.*

*Proof.* On the one hand, we have seen that  $D^*$ , and therefore  $\mathbf{G}^*(T)$ , only depends on the bi-interpretation class of  $T$ . Conversely, by [Theorem 2.11](#), a theory bi-interpretable with  $T$  (namely, the theory of the sort  $D^*$ , up to some arbitrary choices of definable distance and symbols for the language) can be recovered from  $\mathbf{G}^*(T)$ . ■

Our last task is to calculate the basis  $\mathbf{S}_{D^*}(T)$  explicitly, and show how [Theorem 6.4](#) extends previous results, in a style similar to that of [Remark 2.15](#).

Let us fix a rich sequence  $\Phi$  on a sort  $D$ , so we may take  $D^* = D_\Phi^*$ . We also fix a formula  $\chi(y)$  on  $D$  that is onto  $[0, 1]$ . Finally, we may assume that  $\varphi_n(x_{<n}, y) = n\chi(y)$  for infinitely many  $n$ .



Let  $X = S_{D^N}(T)$  and  $Y = S_{D_\Phi^*}(T)$ . We may identify  $S_{*D^N}(T)$  with  $*X$ , identifying  $\text{tp}(\alpha x)$  with  $\alpha \text{tp}(x)$  (here we need to assume that  $T$  is complete, so there exists a unique possible complete type for  $0 \in D_\Phi^*$ ). This identifies  $Y$  with a subset of  $*X$ , namely that of all  $\alpha p$  where  $p(x)$  implies that  $\alpha x \in D_\Phi^*$ , or equivalently, such that  $\alpha \varphi_n(p) \leq 1$  for all  $n$ .

For  $\alpha \in [0, 1]$ , let

$$X_\alpha = \{p \in X : \alpha p \in Y\}.$$

In particular,  $X_0 = X$ . Define  $\rho: X \rightarrow [0, 1]$  by

$$\rho(p) = \sup \{\alpha : \alpha p \in Y\} = \sup \{\alpha : p \in X_\alpha\}.$$

**Lemma 6.5.** *Let  $\alpha > 0$ . Then for every  $p \in X$  we have  $\alpha \leq \rho(p)$  if and only if  $p \in X_\alpha$ , and  $X_\alpha$  is compact, totally disconnected. In particular,  $\rho: X \rightarrow [0, 1]$  is upper semi-continuous.*

*Proof.* For the first assertion, it is enough to notice that by compactness, the supremum is attained, namely,  $p \in X_{\rho(p)}$ . It follows that the condition  $\rho(p) \geq \alpha$  is equivalent to  $p \in X_\alpha$ , so it is closed, and  $\rho$  is upper semi-continuous.

Assume that  $\alpha q_i \in Y$  and  $q_0 \neq q_1$ . Then for some finite  $n$ , there exists a formula  $\psi(x_{<n})$  that separates  $q_0$  from  $q_1$ , say  $\psi(q_i) = i$ . We may also find a  $[0, 1]$ -valued formula  $\chi(y)$  on  $D$  that attains (at least) the values 0 and 1.

By Urysohn's Lemma, there exists a formula  $\varphi(x_{<n}, y) \geq 0$  such that

$$\begin{aligned} |\psi(x_{<n}) + \chi(y) - 1| \geq 1/3 &\implies \varphi(x_{<n}, y) = 0, \\ |\psi(x_{<n}) + \chi(y) - 1| \leq 1/6 &\implies \varphi(x_{<n}, y) = 17/\alpha + 42. \end{aligned}$$

Since the formula  $\chi$  attains both 0 and 1, the formula  $\varphi(x_{<n}, y)$  is witness-normalised, so there exists  $k \geq n$  with  $|\varphi - \varphi_k| \leq 1$ .

Assume now that  $\alpha p \in Y$ . Then  $\varphi_k(x_{<k}, x_k)^p \leq 1/\alpha$ , so  $\varphi(x_{<n}, x_k)^p \leq 1/\alpha + 1 < 17/\alpha + 42$  and  $|\psi(x_{<n}) + \chi(x_k) - 1| > 1/6$ . This splits the set  $X_\alpha$  in two (cl)open sets, defined by  $\psi(x_{<n}) + \chi(x_k) > 7/6$  and  $\psi(x_{<n}) + \chi(x_k) < 5/6$ , respectively. Since  $\chi$  is  $[0, 1]$ -valued,  $q_0$  must belong to the latter and  $q_1$  to the former, so they can be separated in  $X_\alpha$  by clopen sets, completing the proof.  $\blacksquare$

**Lemma 6.6.** *The set  $X_{>0} = \{p \in X : \rho(p) > 0\} = \bigcup_{\alpha > 0} X_\alpha$  is totally disconnected, admitting a countable family of clopen sets  $(U_n : n \in \mathbf{N})$  that separates points.*

*Proof.* We may write  $X_{>0}$  as  $\bigcup_k X_{2^{-k}}$ . Each  $X_{2^{-k}}$  is compact, totally disconnected, and it is metrisable by countability of the language. Therefore, it admits a basis of clopen sets.

The inclusion  $X_{2^{-k}} \subseteq X_{2^{-k-1}}$  is a topological embedding of compact totally disconnected spaces. Therefore, if  $U \subseteq X_{2^{-k}}$  is clopen, then we may find a clopen  $U' \subseteq X_{2^{-k-1}}$  such that  $U' \cap X_{2^{-k}} = U$ . Proceeding in this fashion, we may find a clopen  $\bar{U} \subseteq X_{>0}$  such that  $\bar{U} \cap X_{2^{-k}} = U$ .

We can therefore produce a countable family of clopen sets  $(U_n : n \in \mathbf{N})$  in  $X_{>0}$  such that for each  $k$ ,  $(U_n \cap X_{2^{-k}} : n \in \mathbf{N})$  is a basis of clopen sets for  $X_{2^{-k}}$ , and in particular separates points. It follows that  $(U_n)$  separates points in  $X_{>0}$ .  $\blacksquare$

Given this family  $(U_n)$ , we may define a map  $\theta_0: X_{>0} \rightarrow 2^{\mathbf{N}}$ , where  $\theta_0(p)_n = 0$  if  $p \in U_n$  and  $\theta_0(p)_n = 1$  otherwise. It is continuous by definition, and injective since the sequence  $(U_n)$  separates points. If  $\alpha p \in Y$ , then either  $\alpha = 0$  or  $p \in X_{>0}$  (or possibly both), and we may define

$$\theta(\alpha p) = \alpha \theta_0(p) \in *2^{\mathbf{N}},$$

where  $\theta(0) = \theta(0 \cdot p) = 0$  regardless of  $p$ . It is clearly continuous at 0, and at every point of  $Y$  (since  $\theta_0$  is continuous). It is also injective on  $Y$ . Since  $Y$  is compact,  $\theta: Y \rightarrow *2^{\mathbf{N}}$  is a topological embedding.

**Lemma 6.7.** *The set of  $\rho(p)p$  for  $p \in X_{>0}$  is dense in  $Y$ .*

*Proof.* We already know that  $\rho(p)p \in Y$ . Assume now that  $U \subseteq Y$  is open and non-empty, so it must contain some point  $\alpha p$  with  $\alpha > 0$ .

We may assume that

$$U = \{\beta q \in Y : |\beta - \alpha| < \varepsilon, q \in V\},$$

where  $V$  is an open neighbourhood of  $p$  in  $X$ . The set  $V$  may be taken to be defined by a condition  $\psi > 0$ , where  $\psi(x_{<n})$  only involves finitely many variables. By hypothesis on  $\Phi$ , possibly increasing  $n$ , we may assume that  $\varphi_n(x_{<n}, y) = n\chi(y)$ , and we may further assume that  $\alpha > 1/n$ .

Choose a realisation  $a$  of  $p$ . Let  $b_{<n} = b_{<n}$  and choose  $b_n$  so  $\chi(b_n) = 1/n\alpha$ . Then  $\varphi_n(b_{<n}, b_n) = 1/\alpha$ , so  $\rho_{n+1}(b_{\leq n}) = \alpha$ , and we may extend  $b_{\leq n}$  to a sequence  $b$  such that  $\rho(x') = \alpha$ . In particular,  $q = \text{tp}(b) \in V \cap X_{>0}$  and  $\alpha q = \rho(q)q \in U$ . ■

Let us recall from Charatonik [Cha89] a few definitions and facts regarding fans. The *Cantor fan* is the space  $*2^{\mathbb{N}}$ . It is a connected compact metrisable topological space. More generally, a *fan*  $F$  is a connected compact space that embeds in the Cantor fan. An *endpoint* of  $F$  is a point  $x \in F$  such that  $F \setminus \{x\}$  is connected (or empty, in the extremely degenerate case where  $F$  is reduced to a single point). If the set of endpoints is dense in  $F$ , then  $F$  is a *Lelek fan*. By the main theorem of Charatonik [Cha89], the Lelek fan is unique up to homeomorphism.

**Proposition 6.8.** *Let  $T$  be a complete theory. Then  $S_{D^*}(T)$ , the type-space of the universal star sort  $D^*$  in  $T$ , is homeomorphic to the Lelek fan.*

*Proof.* By Lemma 6.5 to Lemma 6.7, the space  $S_{D^*}(T)$  is a Lelek fan. ■

This gives us a hint as to how to relate the universal star sort with previously known reconstruction sorts referred to in the examples of Section 2.

**Theorem 6.9.** *Assume  $T$  admits a universal Skolem sort  $D$  in the sense of [Ben], and let  $L$  denote the Lelek fan. Then  $L * D$  is a universal star sort.*

*Proof.* We may assume that  $L \subseteq *2^{\mathbb{N}}$ , and moreover, that for every non-empty open subset  $U \subseteq 2^{\mathbb{N}}$  there exists  $\alpha > 0$  and  $t \in U$  such that  $\alpha t \in L$  (otherwise, we may replace  $2^{\mathbb{N}}$  with the intersection of all clopen subsets for which this is true).

For each  $n \in \mathbb{N}$  there is a natural initial projection  $2^{\mathbb{N}} \rightarrow 2^n$ . This induces in turn a projection  $*2^{\mathbb{N}} \rightarrow *2^n$ . Let  $L_n \subseteq *2^n$  be the image of  $L$  under this projection, so  $L = \varprojlim L_n$ . Consequently,  $L * D = \varprojlim (L_n * D)$ .

Our previous hypothesis regarding  $L$  implies that the endpoints of  $L_n$  can be enumerated as  $\{\alpha_{it} : t \in 2^n\}$ . If  $m \geq n$ , then we have a natural projection  $L_m \rightarrow L_n$ . If  $t \in 2^n$ ,  $s \in 2^{m-n}$ , and  $ts \in 2^m$  is the concatenation, then  $\alpha_{ts}ts$  gets sent to  $\alpha_{it}t \in L_n$ , so  $\alpha_{ts} \leq \alpha_{it}$ , and  $\alpha_{ts} = \alpha_{it}$  for at least one  $s$ . For any  $\delta > 0$ , we may always choose  $m$  large enough such that for every  $t \in 2^n$ , the set  $\{\alpha_{ts} : s \in 2^{m-n}\}$  is  $\delta$ -dense in the interval  $[0, \alpha_{it}]$ .

Let  $\varphi(u, v)$  be a star correspondence between  $L_n * D$  and some other star sort  $E^*$ , and let  $\varepsilon > 0$ . Choose  $\delta > 0$  appropriately, and a corresponding  $m$  as in the previous paragraph. Define a formula on  $2^n \times 2^{m-n} \times D \times E^*$  by

$$\varphi'(ts, x, v) = \varphi(\alpha_{ts}t * x, v).$$

On the one hand, since  $\varphi$  is witness-normalised in the first argument,  $\varphi'$  is witness-normalised in  $(ts, x)$ . On the other hand, if  $v \in E^*$ , then there exist  $\alpha t \in L_n$  (so  $\alpha \leq \alpha_{it}$ ) and  $x \in D$  such that  $\varphi(\alpha t * x, v) = 0$ . Having chosen  $\delta$  small enough to begin with, and  $m$  large enough accordingly, we may now find  $s \in 2^{m-n}$  such that  $\alpha_{ts}$  is close to  $\alpha$ , sufficiently so that  $\varphi'(ts, x, v) = \varphi(\alpha_{ts}t * x, v) < \varepsilon$ . It follows that  $\varphi' \dot{-} \varepsilon$  is witness-normalised in either  $(ts, x)$  or  $v$ .

Let us now evoke a few black boxes from [Ben]. First,  $2^m \times D$  is again a universal Skolem sort (and therefore stands in definable bijection with  $D$ ). Second, since  $\varphi' \dot{-} \varepsilon$  is witness-normalised in either group of arguments, there exists a surjective definable function  $\sigma : 2^m \times D \rightarrow E^*$  that satisfies  $(\varphi' \dot{-} \varepsilon)(ts, x, \sigma(ts, x)) \leq \varepsilon$ , i.e.,  $\varphi'(ts, x, \sigma(ts, x)) \leq 2\varepsilon$ . Define on  $L_m * D \times E^*$  (keeping in mind that if  $\alpha ts \in L_m$ , then  $\alpha \leq \alpha_{ts}$ ):

$$\psi(\alpha ts * x, v) = d(v, \alpha \alpha_{ts}^{-1} \sigma(ts, x)).$$

This formula is jointly sub-homogeneous (since  $d$  is, on  $E^*$ ). It is also witness-normalised in  $\alpha ts * x$  (just choose  $v = \alpha \alpha_{ts}^{-1} \sigma(ts, x)$ ), and in  $v$  (since  $\sigma$  is surjective, and we may always choose  $\alpha = \alpha_{ts}$ ). By construction,  $\varphi(\alpha_{ts}t * x, \sigma(ts, x)) \leq 2\varepsilon$ , so multiplying all arguments by  $\alpha \alpha_{ts}$ :

$$\varphi(\alpha t * x, \alpha \alpha_{ts}^{-1} \sigma(ts, x)) \leq 2\varepsilon.$$

Therefore, if  $\psi(\alpha ts * x, v)$  is small enough,  $\varphi(\alpha t * x, v) \leq 3\varepsilon$ , and by definition, if  $\psi(\alpha ts * x, v_i)$  is small for  $i = 0, 1$ , then  $d(v_0, v_1)$  is small. Replacing  $\psi$  with a multiple, we may replace “small enough” with “smaller than one”, and now, by Lemma 4.11,  $L * D$  is a universal star sort. ■

**Corollary 6.10.** *Assume that  $T$  is  $\aleph_0$ -categorical. Fix a countable dense sequence  $a$  in a model, and let  $D_0$  be the set of all sequences of the same type. Then  $D_0$  is a definable set of sequences, i.e., a sort, and  $L * D_0$  is a universal star sort.*

*Proof.* In an  $\aleph_0$ -categorical theory, every type-definable set is definable. By [Ben, Proposition 4.17],  $2^{\mathbb{N}} \times D_0$  is a universal star sort. Now,  $L * 2^{\mathbb{N}} \subseteq (*2^{\mathbb{N}}) * 2^{\mathbb{N}} = *(2^{\mathbb{N}} \times 2^{\mathbb{N}})$  is easily checked to be a fan, whose set of endpoints is dense, so it is homeomorphic to  $L$ . Therefore

$$L * (2^{\mathbb{N}} \times D_0) = (L * 2^{\mathbb{N}}) * D_0 \simeq L * D_0.$$

By **Theorem 6.9**, this is a universal star sort. ■

Define  $L^{(2)} \subseteq L^2$  as the set of pairs  $(x, y)$  such that either both  $x = y = 0$ , or both are non-zero. This is a Polish, albeit non-compact, star space, with root  $(0, 0)$ . When  $\mathbf{G}$  is a topological groupoid, we may equip  $L^{(2)} * \mathbf{G}$  with the partial operation

$$[x, y, g] \cdot [y, z, h] = [x, z, gh],$$

so in particular  $0 \cdot 0 = 0$ .

**Corollary 6.11.** *Let  $T$  be a continuous theory admitting a universal Skolem sort  $D$ , and let  $\mathbf{G}(T) = \mathbf{G}_D(T)$ , as in **Example 2.14**. Then  $\mathbf{G}^*(T) \simeq L^{(2)} * \mathbf{G}(T)$ . If  $T$  is  $\aleph_0$ -categorical, and  $G(T)$  is the automorphism group of its unique separable model, then  $\mathbf{G}^*(T) \simeq L^{(2)} * \mathbf{G}(T)$ .*

*Proof.* Just put the identities  $D^* = L * D$  and  $D^* = L * D_0$  through the groupoid construction. ■

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