HAL open science

Muons and New Physics

Eduardo de Rafael

▶ **To cite this version:**

Eduardo de Rafael. Muons and New Physics. Inference: International Review of Science, 2021, 6 (3), 10.37282/991819.21.53 . hal-03596003

HAL Id: hal-03596003

https://hal.science/hal-03596003v1

Submitted on 4 Mar 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Muons and New Physics

Eduardo de Rafael

The muon is an elementary particle. It has the same electric charge and spin as the electron, but it is about two hundred times heavier. It was unexpected when observed in cosmic ray experiments in 1936.

"Who ordered that?" Isidor Rabi asked.[1]

Muons and electrons behave as if they are tiny electrically charged magnets. The magnitude of their magnetic moments is given by the ratio of the electric charge to twice their mass times a dimensionless parameter g—the gyromagnetic ratio. Paul Dirac's relativistic quantum mechanics equation fixes the value of g at 2 for both electrons and muons. The development of quantum electrodynamics imposed radiative corrections on the value of g. Its precise value is slightly different for electrons and muons, not precisely 2 in either case. Dirac's result is simply the first term in a perturbative expansion.[2]

The quantum field theory formulation of electrodynamics (QED) was developed after the Second World War by Richard Feynman, Julian Schwinger, and Sin-Itiro Tomonaga.[3] Results in QED are obtained by series expansion over powers of the dimensionless fine structure constant α. The measured reciprocal of α is 137.035999046(27), one of the very few quantities in nature whose value is known to ten decimal places.[4] Schwinger used QED to determine the lowest order radiative correction to Dirac's value for the gyromagnetic ratio of any charged lepton:

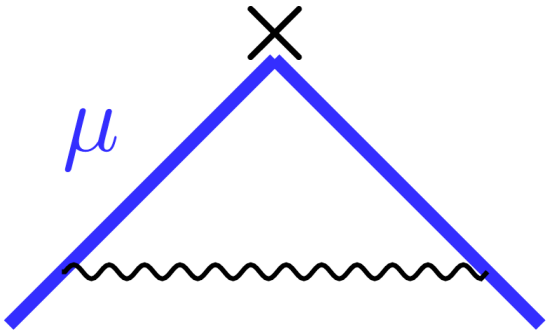
$$a \equiv \frac{1}{2}(g-2) = \frac{\alpha}{2\pi} \simeq 0.001166\ldots,$$

where a is a measure of the departure of g from its zeroth-order value of 2.[5]

The prediction was soon confirmed by a measurement of the electron anomaly a_e to a level of 4% precision.[6]

Schwinger's calculation represents the emission by a charged lepton in a magnetic field of an unobserved particle of light—a virtual photon—that is later reabsorbed by the lepton. The propagation of this virtual photon, illustrated by the Feynman diagram in Figure 1, generates a correction to the Dirac $g = 2$ result and hence an anomalous magnetic moment called the anomaly a.

Figure 1.



Feynman diagram corresponding to the Schwinger contribution in the equation above. The X in the muon μ (blue line) represents the external magnetic field. The black wavy line represents the propagating virtual photon.

Schwinger's result applies to any spin-½ charged particle or antiparticle; it is the same for the electron and the muon as well as the tau lepton, which has a higher mass, because g is a dimensionless number.[7] In his calculation, a massless photon is emitted prior to the particle's interaction with an external magnetic field and is absorbed afterward. No mass scale is involved other than that of the particle involved. Theorists have calculated corrections to Schwinger's one-loop order-α result, involving

additional loops and powers of α, as the precision of experimental measurements has increased. These corrections are now known in terms of exact mathematical quantities up to order $\left(\frac{\alpha}{\pi}\right)^3$, and at good numerical approximations up to $\left(\frac{\alpha}{\pi}\right)^5$, which is extraordinary. The analytic evaluation of the coefficients of this series in $\left(\frac{\alpha}{\pi}\right)$ powers involves higher and higher transcendental numbers, such as integral values of the Riemann zeta function, showing once more the beautiful relationship between mathematics and physics.

The values of the electron and the muon anomalies a_e and a_μ differ at the order of the $\left(\frac{\alpha}{\pi}\right)^2$ and higher power contributions because of quantum vacuum fluctuations induced by other particles. Quantum fluctuations induced by charged lepton–antilepton pairs are calculable using QED techniques.[8] Such fluctuations can be induced by all the particles of the electroweak theory of the Standard Model—charged leptons with their associated neutrinos, quarks and gluons, the heavy gauge W and Z bosons, as well as Higgs particles. They all contribute to the muon anomaly a_μ. These quantum fluctuations can be evaluated in the Standard Model because this theory,[9] when combined with the Higgs-particle mechanism,[10] is renormalizable.[11] Observables can be calculated without parameters other than those already present in the initial formulation of the theory.

The contribution to a_μ from hadron quantum fluctuations—strongly interacting particles like pions and protons—is not straightforward. Calculation requires the underlying theory of the strong interactions, quantum chromodynamics (QCD), figuring at all energy scales and, in particular, at the low energies and long distances at which the fundamental quarks and gluon-gauge particles of QCD are confined and

condense into the observed hadronic particles. Theorists do not have as yet a full dynamical understanding of confinement. They resort either to phenomenological estimates, or to numerical simulations of the underlying dynamics of QCD theory made in discrete lattices of the space-time continuum. Implemented in some of the world's most powerful supercomputers, these simulations involve sophisticated techniques that have been developed under the name of lattice QCD (LQCD).[12] LQCD simulations in smaller and smaller space-time lattices of larger and larger volumes have already produced impressive results in hadron physics.[13]

Two dedicated experiments have provided precise measurements of the muon magnetic moment anomaly, one at the Brookhaven National Laboratory (BNL) the other at Fermilab (FNAL). These labs give the results

$$a_\mu^{\mathrm{BNL}} = 116{,}592{,}089(63) \times 10^{-11}$$

and

$$a_\mu^{\mathrm{FNAL}} = 116{,}592{,}040(54) \times 10^{-11}.$$ [14]

They agree with each other at the level of 0.6σ (0.6 standard deviations), and their combined result,

$$a_\mu(2021) = 116{,}592{,}040(41) \times 10^{-11} \text{ (0.35 ppm)},$$

is accurate to 0.35 parts per million (ppm).

The theoretical evaluation of the same observable in the Standard Model, if made at a comparable level of precision, involves many subtle details about the interaction dynamics of *all* Standard Model particles, as well as high precision measurements in atomic physics.

It requires a total knowledge of the relevant physics.

The failure to reproduce the experimental results would constitute strong evidence for physics lying beyond the Standard Model.

What do physicists know for sure about this confrontation?

Measurements of the Muon Anomalous Magnetic Moment

The clever experimental determination of the anomalous magnetic moment of the muon a_μ is based on three fundamental physical properties:

1. Muons produced by the decay of pions are longitudinally polarized: the positively charged muons in the FNAL experiment have their spins oriented opposite to their momenta.
2. The orbit frequency of a muon turning in a horizontal storage ring in the presence of a uniform vertical magnetic field differs from its spin precession by a factor precisely proportional to the muon anomalous magnetic moment a_μ that one wants to measure. This difference is called the anomalous precession frequency.
3. Muons, like pions, are unstable particles and have a lifetime at rest of about two microseconds. Their weak decay into an electron and two neutrinos violates parity, and this provides the trick to measuring the anomalous precession frequency, hence the muon anomalous magnetic moment.

The trick succeeds because the muon spins are strongly correlated with the decay electron momenta. This spin-energy correlation results in a modulation of the electron

decay energy spectrum that occurs at the rate of the anomalous precession frequency, and this is how the anomaly is measured.

In practice, one needs good statistics of a pion source. This is provided by the proton collisions emerging from a high energy accelerator, and for this reason, the experiments have been made at places such as CERN, BNL, and FNAL. The polarized muons from pion decays are then boosted in a storage ring, so as to sufficiently increase their lifetimes, in the presence of a very precise homogeneous vertical magnetic field and an inner ring of energy detectors measuring the electron decay from the stored muons. This approach was pioneered in a series of dedicated experiments at CERN between 1962 and 1968, which were further pursued by the E821 experiment at BNL and recently by the FNAL Muon $g - 2$ experiment, which is still underway.[15]

Vacuum Polarization Effects

Vacuum polarization is a characteristic phenomenon arising in the quantum formulation of electrodynamics. First discussed by Dirac and Werner Heisenberg, it occurs when a virtual photon propagating between two sources probes a vacuum fluctuation of electrically charged particle–antiparticle pairs.[16] These particles can be leptons or quark–antiquark pairs. Observable effects of vacuum polarization in atomic physics were first evaluated to lowest order in powers of the fine structure constant α by Robert Serber and Edwin Uehling in 1935.[17] In QED, Feynman, Schwinger, Tomonaga, and Dyson showed how to do systematic calculations of these effects in powers of the small coupling constant α and, in particular, how to evaluate the QED contributions to the lepton anomalies. The lepton vacuum polarization effects (lepton-VP) to a_μ induced by muon–antimuon pairs give mass-independent contributions, since g is massless and the only mass in this case is that of the muon. Those induced by tau–

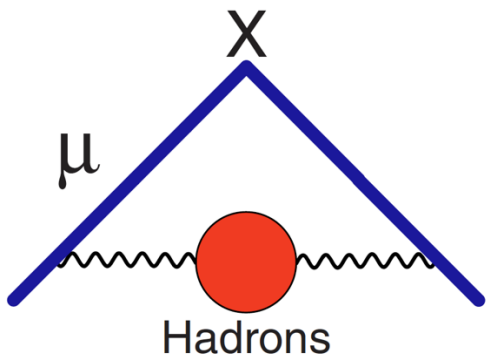
antitau pairs decouple by a factor $\frac{m_\mu^2}{M_\tau^2}$ because virtual photons coming from the external muon carry an average momentum too small to probe the heavy tau–antitau pairs from the vacuum. The lepton-VP effects induced by the vacuum fluctuations of electron–positron pairs are large because the average momentum carried by the virtual photon emitted by the external muon is large compared to the electron mass. This produces a $\log \frac{m_\mu}{m_e} \sim 5$ factor at the $\left(\frac{\alpha}{\pi}\right)^2$ order of the QED contribution and higher $\log \frac{m_\mu}{m_e}$ powers at higher orders. These give rise to large contributions to the total theoretical budget. Although the calculations are technically difficult, they have been done analytically and in some cases numerically to the needed accuracy, mostly by Toichiro Kinoshita and his collaborators.[18]

Vacuum polarization resulting from lepton–antilepton loops was first considered in the context of QED. However, every charged particle contributes to vacuum polarization. Hadronic vacuum polarization (HVP) is due to hadron–antihadron loops. Its evaluation is not straightforward because hadrons are subject to strong QCD forces, but its magnitude can easily be estimated: it appears first at the level of the $\left(\frac{\alpha}{\pi}\right)^2$ contributions because, as illustrated by the Feynman diagram in Figure 2, at least two virtual photon propagators, represented by the wavy lines, are needed to connect to the hadron–antihadron loop indicated by the disc. The virtual photons in this case carry an average momentum that is small compared to the masses of the hadronic particles that they can probe. Because of spin symmetry properties, the overall effect must be modulated by a mass factor m_μ^2, which, in order to make g dimensionless, must be normalized to another mass scale, which is expected to have the size of the mass of the lowest lying hadronic resonance to which the photons can couple. This is the ρ (770 MeV) particle. The overall m_μ^2 dependence makes the g factor of the muon more sensitive to hadronic

effects than the electron by a ratio $\frac{m_\mu^2}{m_e^2} \simeq 43{,}000$. The muon anomaly is much more sensitive to effects beyond those encoded in the Standard Model, and these may be contributing to vacuum polarization. This is the reason for the interest in a precise determination of a_μ.

The estimated order of magnitude, $\left(\frac{\alpha}{\pi}\right)^2 \frac{m_\mu^2}{M_\rho^2} \sim 10^{-7}$, of the HVP contribution seems rather small, but it is big enough when compared to the 0.35 ppm accuracy reached in recent experimental determinations of a_μ. It requires more attention than just an order of magnitude evaluation.

Figure 2.



Feynman diagram of the HVP contribution. The X in the muon line represents the external magnetic field.

The HVP contribution to a_μ is called a_μ(HVP). Its evaluation depends critically on the hadronic forces generated by QCD, whose effects cannot be perturbatively calculated, but which are described by an analytic function $\Pi_{\mathrm{HVP}}(q^2)$ of the energy-momentum squared of the virtual photon. This function encodes how photons interact via QCD with hadrons. The contribution to a_μ(HVP) due to hadron loops is given by q^2-range

dependence of $\Pi_{\mathrm{HVP}}(q^2)$. Cauchy's integral theorem shows how to evaluate $\Pi_{\mathrm{HVP}}(q^2)$ everywhere once its shape on the real axis is known from the hadronic threshold at $q^2 = 4m_\pi^2$, where m_π is the mass of a charged pion, to the spectral function region at infinity. It turns out that it is precisely over a large range of this spectral function region that $\Pi_{\mathrm{HVP}}(q^2)$ is accessible to experimental determination by means of the total cross section that emerges when electrons and positrons annihilate, thereby producing hadrons. Furthermore, the shape of the spectral function in the very high q^2 energy region, where there is no more experimental information, is well predicted by the asymptotic freedom property of QCD.[19]

The precise wording of a_μ(HVP), what is now called the dispersive HVP evaluation of a_μ, was first given by Claude Bouchiat and my thesis advisor Louis Michel in 1961.[20] In 1969, using the results from experiments at the Linear Accelerator in Orsay, a first quantitative evaluation was made at the level of 8% accuracy, which was sufficient for an overall comparison with the experimental value of a_μ.[21] With the advent of increasingly refined experiments on e^+e^- annihilations into hadrons, the precision of the HVP contribution to a_μ has improved considerably. The most recent determination, made by Michel Davier and collaborators from dedicated experiments at several facilities, quotes a precision level of 0.5%.[22] In spite of this accuracy, it still remains the one that at present has the largest error.

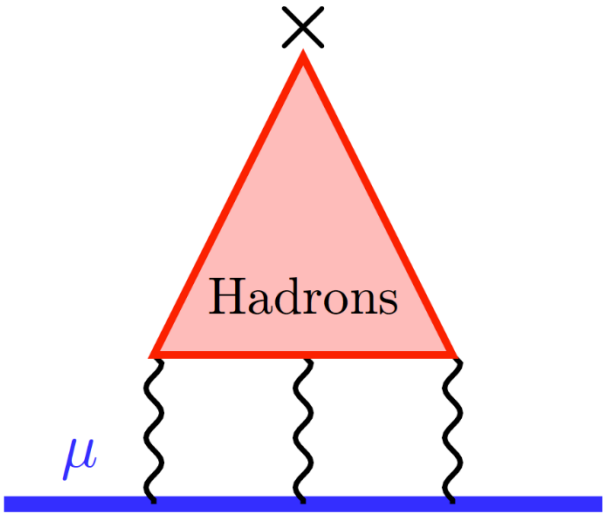
Alternative and increasingly accurate evaluations of the a_μ(HVP) contribution have been made during the last few years using LQCD techniques, and, in fact, the recent results published by the BMW collaboration are comparable to the dispersive evaluations.[23]

Light-by-Light Scattering Effects

The scattering of light off light (LbyLS) is another rare phenomenon that only happens at the quantum level and is predicted by QED. It has been recently detected at the Large Hadron Collider at CERN, but it also occurs due to the vacuum fluctuations induced by the effect of four photons. LbyLS contributes to a_μ at the order of $\left(\frac{\alpha}{\pi}\right)^3$ because, in the presence of an external magnetic field, as seen as one photon, an incoming external muon can emit and reabsorb three virtual photons that simultaneously—one + three—probe the vacuum fluctuation of leptons and/or hadrons. The Feynman diagram in Figure 3 illustrates this process; the red triangle represents either leptons or hadrons.

As for the lepton-VP effects discussed earlier, the LbyLS induced by muon loops yields mass-independent contributions to a_μ. These have been calculated to the needed accuracy either numerically or analytically.[24] Those arising from tau-lepton loops also decouple like those of lepton-VP and are very small. Those arising from electron loops turn out to be very important giving a $\pi^2 \log \frac{m_\mu}{m_e}$ factor at the leading $\left(\frac{\alpha}{\pi}\right)^3$ level.[25] This came as a big surprise in 1969 because in QED, the LbyLS does not have an intrinsic new coupling constant beyond its electric charge coupling e^4 to the four photons and, therefore, does not require renormalization. This is in contrast to the vacuum polarization calculation discussed earlier. The unexpectedly large LbyLS contribution served to eliminate a possible discrepancy between theory and experiment.[26]

Figure 3.



Feynman diagram of the hadronic LbyLS contribution.

Hadronic light-by-light scattering (HLbyL) is the corresponding effect induced by hadron loops, and its evaluation is not as straightforward as the one induced by leptons because QCD must be dealt with. Furthermore, in the HLbyL case, Cauchy's integral theorem is unavailing. Contributions of order $\left(\frac{\alpha}{\pi}\right)^3$ became relevant at the last BNL experiment:[27] the remarkable accuracy of the a_μ measurement indicated a deviation from theoretical predictions.[28] Theorists proposed several model estimates of the a_μ(HLbyL) contribution, which all agreed within the quoted errors. They found that the contribution came with an overall negative sign. Soon after the publication of the BNL result, this theoretical conclusion was questioned.[29] There is a QCD limit at which the a_μ(HLbyL) contribution is calculable. It is obtained in a model where the number of quark colors and the mass gap are allowed to increase without limit.[30] In that unphysical limit, the contribution to a_μ(LbyLS) turns out to be positive, implying that model calculations compatible with QCD must *also* give a positive result when

model parameters are extrapolated accordingly. They all failed the test! The authors eventually found a convention inconsistency in their calculations which, when corrected, indeed gave a positive contribution. The significant discrepancy between theory and experiment was, if not eliminated, then reduced.

Conflicting theoretical estimates of a_μ(LbyLS) created some confusion. At a workshop at the University of Glasgow in 2009, Lee Roberts suggested that three theorists of different groups get together and examine the discrepancies between models. He appointed Joachim Prades, Arkady Vainshtein, and me to do the job. Our result was $a_\mu(\text{LbyLS – Glasgow}) = (10.5 \pm 2.6) \times 10^{-10}$, the so-called Glasgow consensus.[31] There have been many evaluations of the a_μ(LbyLS) contribution since then, using ever more sophisticated models. They turn out to be consistent, within errors, with the Glasgow consensus.

The evaluation of the a_μ(LbyLS) contribution has also been undertaken by various LQCD collaborations. These are QCD first principle, model-independent evaluations, which little by little have reached a better level of accuracy than the Glasgow consensus.[32] For all that, no one believes that the a_μ(LbyLS) contribution can by itself explain the tension between theory and experiment.

Present Status of the Theoretical Contributions

The following table summarizes the status of the various theoretical contributions to the muon anomaly in 10^{-11} units and prior to the FNAL result. Several comments are in order:

- The results in the third column of Table 1 show the level at which each contribution is effective.
- The a_μ(QED) results from the interactions of photons and leptons include the contributions from the calculated coefficients of the first five powers in $\frac{\alpha}{\pi}$. The result in the first line has been evaluated using the value of the fine-structure constant α determined from a cesium atom interferometry experiment:[33]

$$\alpha^{-1}(\mathrm{Cs}) = 137.035999046(27).$$

- The entry in the second line of the table makes use of the high-precision determination of the electron $g - 2$ by Gerald Gabrielse's group at Harvard,[34]

$$a_e(\mathrm{exp.}) = 1{,}159{,}652{,}180.73(28) \times 10^{-12},$$

yielding

$$\alpha^{-1}(a_e) = 137.0359991496(13)(14)(330).$$

The uncertainty of ±330 results from the error in the experimental determination of a_e, ±14 from the hadronic contribution to a_e and ±13 from the numerical evaluation of the $\left(\frac{\alpha}{\pi}\right)^5$ QED term.[35]

One can conclude from these results that the theoretical contribution from the QED interactions of photons and leptons to the muon $g - 2$ is well known.

- By $a_\mu(\mathrm{HVP})_{\mathrm{lowest\ order}}$, one means the HVP contribution from the Feynman graph in Figure 2. The third and fourth lines of Table 1 are the result of adding third and fourth leading orders hadronic contribution. These results have been

obtained using the dispersive HVP evaluation method described above near Figure 2.

- The contribution $a_\mu(\mathrm{EW})$ from the electroweak interactions of the Standard Model is small because it is proportional to $G_\mathrm{F} m_\mu^2$, where $G_\mathrm{F} \sim 10^{-5}\ \mathrm{GeV}^{-2}$ is the Fermi coupling constant that governs the strength of the weak interactions. Its evaluation is well understood in the Standard Model.[36]

Table 1.

Contribution	Comments	Results
$a_\mu(\mathrm{QED})$	Photons and leptons, with $\alpha(\mathrm{Cs})$	116,584,718.931(30)
$a_\mu(\mathrm{QED})$	Photons and leptons, with $\alpha(a_e)$	116,584,718.842(34)
$a_\mu(\mathrm{HVP})_{\mathrm{lowest\ order}}$	Experimental dispersive evaluations	6,931(40)
$a_\mu(\mathrm{HVP})_{\mathrm{total}}$	Experimental dispersive evaluations	6,845(40)
$a_\mu(\mathrm{EW})$	$W^\pm$, Z^0, and Higgs with leptons and quarks	153.6(1.0)
$\boldsymbol{a_\mu(\mathrm{HLbyL})_{\mathrm{total}}}$	Phenomenology and LQCD	92(18)

Theoretical contributions to the muon anomaly in units of 10^{-11}.

The sum of the contributions listed in the results column of Table 1 gives

$$a_\mu(\mathrm{Th.\,WP}) = 116{,}591{,}810(43) \times 10^{-11},$$

which is the consensus reported in the 2020 white paper (WP).[37] When compared to the experimental result $a_\mu(2021) = 116{,}592{,}040(41) \times 10^{-11}$ (0.35 ppm), the result indicates a significant 4.2σ difference—triggering in the literature, of course, speculations about whether new physics might be at work. Do notice that a discrepancy of 4.2σ is about twice the size of the electroweak contribution. The order of magnitude of what might be the contribution of new physics (NP) to the muon

anomaly are expected to be: $a_\mu(\mathrm{NP}) \sim a_\mu(\mathrm{EW}) \times \left(\frac{M_W}{M_{\mathrm{NP}}}\right)^2 \times couplings$ with $M_{\mathrm{NP}} \gg M_W$. This would require that the coupling be fine tuned.

Many NP models have been excluded for that reason.

The situation is confusing. The same day that the results of the FNAL muon $g-2$ collaboration were published,[38] *Nature* published a new result by the Budapest–Marseille–Wuppertal (BMW) LQCD collaboration. At issue was the lowest-order HVP contribution to the muon $g-2$. The result:[39]

$$a_\mu(\mathrm{HVP})_{\mathrm{BMW}} = 7{,}075(55) \times 10^{-11},$$

which reduces the total discrepancy in $a_\mu(2021) = 116{,}592{,}040(41) \times 10^{-11}$ (0.35 ppm) from 4.2σ to 1.6σ.

The discrepancy is still there, but it is not significant enough to suggest that new physics is at work.

The BMW result is under detailed examination by other LQCD collaborations. They are expected to produce their own results. If the disagreement between LQCD and the experimental dispersive evaluation of the HVP persists, researchers will have to find the explanation for that. Because they involve integrals of different quantities, comparison of the two methods is difficult but not impossible.

As new statistics accumulate, the FNAL muon $g-2$ experiment is expected to reduce their error. There is also a new experiment at the Japan Proton Accelerator Research Complex in Tokai. This J-PARC experiment E34 will employ a new and different technique to measure the muon anomaly.[40] All this gives more time for the theorists to check and improve their calculations before the next awaited confrontation.

[1] See endnote 20 in Sheldon Lee Glashow, "The Standard Model," *Inference: International Review of Science* 4, no. 1 (2018), doi:10.37282/991819.18.15. Glashow's article is a good introduction to the physics discussed here.

[2] Glashow, "The Standard Model."

[3] Richard Feynman, "Nobel Lecture: The Development of the Space-Time View of Quantum Electrodynamics," December 11, 1965, *NobelPrize.org*; Julian Schwinger, "Nobel Lecture: Relativistic Quantum Field Theory," December 11, 1965, *NobelPrize.org*; and Sin-Itiro Tomonaga, "Nobel Lecture: Development of Quantum Electrodynamics," May 6, 1966, *NobelPrize.org*.

[4] The numbers in parenthesis, here and subsequently, indicate the experimental uncertainty of the final two decimal digits.

[5] Schwinger's $\frac{\alpha}{2\pi}$ is inscribed on the memorial marker near his grave in the Mount Auburn Cemetery in Cambridge, Massachusetts. Julian Schwinger, "On Quantum-Electrodynamics and the Magnetic Moment of the Electron," *Physical Review* 73, no. 4 (1948): 416–17, doi:10.1103/physrev.73.416; and Julian Schwinger, "Quantum Electrodynamics. III. The Electromagnetic Properties of the Electron—Radiative Corrections to Scattering," *Physical Review* 76, no. 6 (1949): 790–817, doi:10.1103/physrev.76.790.

[6] Polykarp Kusch and Henry Foley, "Precision Measurement of the Ratio of the Atomic '*g* Values' in the $^2P_{\frac{3}{2}}$ and $^2P_{\frac{1}{2}}$ States of Gallium," *Physical Review* 72 (1947): 1,256, doi:10.1103/PhysRev.72.1256.2; and Polykarp Kusch and Henry Foley, "The Magnetic Moment of the Electron," *Physical Review* 74, no. 3 (1948): 250–63, doi:10.1103/physrev.74.250.

[7] Negatively charged electrons, muons, and tau leptons, as well as their associated neutrinos, are called leptons. Their antiparticles are called antileptons.

[8] Freeman Dyson, "The Radiation Theories of Tomonaga, Schwinger, and Feynman," *Physical Review* 75, no. 3 (1949): 486–502, doi:10.1103/physrev.75.486. See also Feynman, "Nobel Lecture: Development of the Space-Time View"; Schwinger, "Nobel Lecture: Relativistic Quantum Field Theory"; and Tomonaga, "Nobel Lecture: Development of Quantum Electrodynamics."

[9] Sheldon Lee Glashow, "Nobel Lecture: Towards a Unified Theory—Threads in a Tapestry," December 8, 1979, *NobelPrize.org*; Abdus Salam, "Nobel Lecture: Gauge Unification of Fundamental Forces," December 8, 1979, *NobelPrize.org*; and Steven Weinberg, "Nobel Lecture: Conceptual Foundations of the Unified Theory of Weak and Electromagnetic Interactions," December 8, 1979, *NobelPrize.org*.

[10] Peter Higgs, "Nobel Lecture: Evading the Goldstone Theorem," December 8, 2013, *NobelPrize.org*; and François Englert, "Nobel Lecture: The BEH Mechanism and Its Scalar Boson," December 8, 2013, *NobelPrize.org*.

[11] Gerard 't Hooft, "Nobel Lecture: A Confrontation with Infinity," December 8, 1999, *NobelPrize.org*; and Martin Veltman, "Nobel Lecture: From Weak Interactions to Gravitation," December 8, 1999, *NobelPrize.org*.

[12] One year's work on one of these supercomputers is equivalent to 100,000 years on a laptop (information from Laurent Lellouch, the spokesman of the BMW collaboration).

[13] For example, the calculation of the proton mass from QCD first principles made by the Budapest–Marseille–Wuppertal Collaboration. Stephan Dürr et al., "Ab Initio Determination of Light Hadron Masses," *Science* 322, no. 5,905 (2008): 1,224–27, doi:10.1126/science.1163233.

[14] The numbers in parentheses indicate the error in the last two significant figures.

[15] See, e.g., Francis Farley, "Muon $g - 2$ and Tests of Relativity," in *60 Years of CERN Experiments and Discoveries*, ed. Herwig Schopper and Luigi Di Lella (Singapore: World Scientific, 2015), 371–96. Bennett et al., "Final Report of the E821 Muon Anomalous Magnetic Moment Measurement at BNL"; Babak Abi et al., "Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm," *Physical Review Letters* 126 (2021): 141801, doi:10.1103/PhysRevLett.126.141801; and Talal Albahri et al., "Measurement of the Anomalous Precession Frequency," *Physical Review D* 103 (2021): 072002, doi:10.1103/PhysRevD.103.072002.

[16] Paul Dirac, "Discussion of the Infinite Distribution of Electrons in the Theory of the Positron," *Mathematical Proceedings of the Cambridge Philosophical Society* 30 (1934): 150–63, doi:10.1017/s030500410001656x; and Werner Heisenberg, "Bemerkungen zur Diracschen Theorie des Positrons," *Zeitschrift für Physik* 90, no. 3–4 (1934): 209–31, doi:10.1007/bf01333516.

[17] Robert Serber, "Linear Modifications in the Maxwell Field Equations," *Physical Review* 48, no. 1 (1935): 49–54, doi:10.1103/physrev.48.49; and Edwin Uehling, "Polarization Effects in the Positron Theory," *Physical Review* 48, no. 1 (1935): 55–63, doi:10.1103/physrev.48.55.

[18] Tatsumi Aoyama, Toichiro Kinoshita, and Makiko Nio, "Theory of the Anomalous Magnetic Moment of the Electron," *Atoms* 7 (2019): 28, doi:10.3390/atoms7010028.

[19] At very high energies, the short-distance interactions between quarks and gluons in QCD become weaker and weaker and thus calculable. The quarks become less and less confined, and this is why one says that the QCD theory is asymptotically free. See David Gross, "Nobel Lecture: The Discovery of Asymptotic Freedom and the Emergence of QCD," December 8, 2004, *NobelPrize.org*; David Politzer, "Nobel Lecture: The Dilemma of Attribution,"

December 8, 2004, *NobelPrize.org*; and Frank Wilczek, "Nobel Lecture: Asymptotic Freedom: From Paradox to Paradigm," December 8, 2004, *NobelPrize.org*.

[20] I remember Claude Bouchiat discussing this on the blackboard with Louis Michel in Louis's office sixty years ago, but I could not understand a word of it at that time! Claude Bouchiat and Louis Michel, "La Résonance dans la diffusion méson π— méson π et le moment magnétique anormal du méson μ," *Journal de Physique et le Radium* 22, no. 2 (1961): 121–21, doi:10.1051/jphysrad:01961002202012101.

[21] Michel Gourdin and Eduardo de Rafael, "Hadronic Contributions to the Muon *g*-Factor," *Nuclear Physics B* 10 (1969): 667–74, doi:10.1016/0550-3213(69)90333-2. The 8% accuracy of this first estimate only reflected the uncertainty in the Orsay data used as an input. I was then a postdoc at CERN and gave a seminar there on my work with Gourdin. John Bell was in the audience and asked: "What do you know about the behavior of the spectral function at infinity?" I answered that I assumed that it must be like the one in QED for leptons. That did not satisfy John, for the good reason that there was no theory, like QCD nowadays, which answered that question. It gave me, however, the privilege to work with him on an effective field theory approach to HVP. John Bell and Eduardo de Rafael, "Hadronic Vacuum Polarization and g_μ–2," *Nuclear Physics B* 10, no. 4 (1969): 611–20, doi:10.1016/0550-3213(69)90250-8.

[22] Michel Davier et al., "A New Evaluation of the Hadronic Vacuum Polarisation Contributions to the Muon Anomalous Magnetic Moment and to $\alpha(m^2{}_Z)$," *European Physical Journal C* 80, no. 3 (2020): 241, doi:10.1140/epjc/s10052-020-7792-2; and Michel Davier et al., "Erratum to: A New Evaluation of the Hadronic Vacuum Polarisation Contributions to the Muon Anomalous Magnetic Moment and to $\alpha(m2Z)$," *European Physical Journal C* 80, no. 5 (2020): 410, doi:10.1140/epjc/s10052-020-7857-2. For a detailed description of the various HVP contribution evaluations, see Tatsumi Aoyama et al., "The Anomalous Magnetic Moment of the Muon in the Standard Model," *Physics Reports* 887 (2020): 1–166, doi:10.1016/j.physrep.2020.07.006.

[23] Szabolcs Borsanyi et al. (BMW Collaboration), "Leading Hadronic Contribution to the Muon Magnetic Moment from Lattice QCD," *Nature* 593, no. 7,857 (2021): 51–55, doi:10.1038/s41586-021-03418-1.

[24] For details, see Toichiro Kinoshita, "Lepton *g* – 2 from 1947 to Present," in *Advance Series on Directions in High Energy Physics, Vol. 20, Lepton Dipole Moments*, ed. B. Lee Roberts and William Marciano (Singapore: World Scientific Publishing, 2010). Aoyama et al., "Anomalous Magnetic Moment."

[25] Janis Aldins et al., "Photon-Photon Scattering Contribution to the Sixth-Order Magnetic Moment of the Muon," *Physical Review Letters* 23, no. 8 (1969): 441–43, doi:10.1103/physrevlett.23.441; and Janis Aldins et al., "Photon-Photon Scattering Contribution to the Sixth-Order Magnetic Moments of the Muon and Electron," *Physical Review D* 1, no. 8 (1970): 2,378–95, doi:10.1103/physrevd.1.2378.

[26] The unexpected $\pi^2 \log \frac{m_\mu}{m_e}$ factor emerges because of a particular singular mass behavior, specific to certain configurations which happen in the case of the LbyLS in the limit where the internal m_e mass becomes small compared to the external m_μ mass.

[27] Bennett et al., "Final Report."

[28] I remember a paper by illustrious theorists claiming that the beyond the Standard Model theory of supersymmetry fit the discrepancy like a glove fits a hand.

[29] Marc Knecht et al., "Hadronic Light-By-Light Scattering Contribution to the Muon $g-2$: An Effective Field Theory Approach," *Physical Review Letters* 88, no. 7 (2002): 071802, doi:10.1103/physrevlett.88.071802.

[30] The number of colors (Nc) denotes the charge-like degrees of freedom of the quarks. The mass gap is the one between the Goldstone-like pion mass m_π and the larger mass scale $\sim M_\rho$, where quarks and gluons condensate into hadrons. The large-Nc limit selects the effect of all the contributing hadronic resonances in the limit where their widths vanish. In this limit $a_\mu(\mathrm{HLbyL}) = \left(\frac{\alpha}{\pi}\right)^3 \mathrm{Nc}^2 \frac{\mathrm{m}_\mu^2}{48\pi^2 \mathrm{f}_\pi^2} \log^2 \frac{\mathrm{M}_\rho}{\mathrm{m}_\pi} \simeq 95 \times 10^{-11}$, where $f_\pi \simeq 92$ MeV is the coupling constant associated to the π^0 decay into two photons (Stephen Adler, "Axial-Vector Vertex in Spinor Electrodynamics," *Physical Review* 177, no. 5 [1969]: 2,426–38, doi:10.1103/physrev.177.2426; and John Bell and Roman Jackiw, "A PCAC Puzzle: $\pi^0 \to \gamma\gamma$ in the σ-Model," *Il Nuovo Cimento A* 60, no. 1 [1969]: 47–61, doi:10.1007/bf02823296). The first numerical check of this limit with a phenomenological model was made in Marc Knecht and Andreas Nyffeler, "Hadronic Light-by-Light Corrections to the Muon $g-2$: The Pion-Pole Contribution," *Physical Review D* 65, no. 7 (2002): 073034, doi:10.1103/physrevd.65.073034.

[31] Joachim Prades, Eduardo de Rafael, and Arkady Vainshtein, "The Hadronic Light-by-Light Scattering Contributions to the Muon and Electron Anomalous Magnetic Moments," in *Advance Series on Directions in High Energy Physics, Vol. 20, Lepton Dipole Moments*, ed. B. Lee Roberts and William Marciano (Singapore: World Scientific Publishing, 2010), 303–17, doi:10.1142/9789814271844_0009.

[32] Details can be found in Aoyama et al., "Anomalous Magnetic Moment," and more recently in En-Hung Chao et al., "Hadronic Light-by-Light Contribution to $(g-2)\mu$ from Lattice QCD: A Complete Calculation," arXiv:2104.02632v1[hep-lat] (2021).

[33] Richard Parker et al., "Measurement of the Fine-Structure Constant as a Test of the Standard Model," *Science* 360, no. 3,685 (2018): 191–95, doi:10.1126/science.aap7706.

[34] David Hanneke, Shannon Fogwell, and Gerald Gabrielse, "New Measurement of the Electron Magnetic Moment and the Fine Structure Constant," *Physical Review Letters* 100, no. 12 (2008): 120801, doi:10.1103/physrevlett.100.120801.

[35] Aoyama, Kinoshita, and Nio, "Theory of the Anomalous Magnetic Moment."

[36] See, e.g., Aoyama et al., "Anomalous Magnetic Moment."

[37] Aoyama et al., "Anomalous Magnetic Moment."

[38] Abi et al., "Measurement of the Positive Muon Anomalous Magnetic Moment"; and Albahri et al., "Measurement of the Anomalous Precession Frequency."

[39] Borsanyi et al., "Leading Hadronic Contribution."

[40] M. Abe et al., "A New Approach for Measuring the Muon Anomalous Magnetic Moment and Electric Dipole Moment," *Progress of Theoretical and Experimental Physics* 2,019, no. 5 (2019): O53C02, doi:10.1093/ptep/ptz030.