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# Identification of Blackwell Policies for Deterministic MDPs

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We consider the problem of the identification of Blackwell optimal policies for deterministic finite Markov Decision Processes (d-MDPs). Specifically, we are interested in algorithms that learn reward distributions by querying samples over time, that stop almost surely and return a Blackwell optimal policy with high probability. We provide a characterization of the class of MDPs over which such algorithms exist together with an algorithm identifying Blackwell optimal policies with arbitrarily high probability.

## 1 Blackwell Optimality & Identification Algorithms

**Blackwell optimality.** A *deterministic Markov Decision Process* (d-MDP)  $M$  is given by a state space  $\mathcal{S}$ , action space  $\mathcal{A}$  with reward distributions  $q(x, a) \in \mathcal{P}([0, 1])$  and degenerate transition distributions, that is,  $\forall(x, a, y) \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}, P(y|s, a) \in \{0, 1\}$ . In general, given  $x, y \in \mathcal{S}$ , there may be distinct actions  $a, b$  such that  $P(y|x, a) = P(y|x, b) = 1$  but up to a state-wise elimination of actions, transitions may be seen as edges of a graph. One can think of  $P(y|x, a) = 1$  as an edge  $(x, y)$  and write  $q(x, y)$  for  $q(x, a)$ . The set of edges  $\mathcal{E} := \{(x, y) \mid \exists a, P(y|x, a) = 1\}$  will be called *edge space*.

Upon choosing an edge  $(x, y)$  from a state  $x$ , the system changes state to  $y$  and produces a reward  $r \sim q(x, y)$ . A policy is an application  $\pi : \mathcal{S} \rightarrow \mathcal{E}$  that, from each state  $x \in \mathcal{S}$ , selects an outgoing edge  $\pi(x) \in \{(x, y) \in \mathcal{E} \mid y \in \mathcal{S}\}$ . Iterating a policy over time gives a sequence of state-reward pairs  $(x_t, r_t), t \geq 0$ . Policies are usually discriminated with respect to the reward they score either at discounted infinite horizon  $\nu_\beta^\pi(x_0) := \mathbb{E}_{x_0}^{M, \pi}[\sum_{t=0}^{\infty} \beta^t r_t]$  or at undiscounted infinite horizon  $g_\pi(x_0) := \mathbb{E}_{x_0}^{M, \pi}[\lim_{\frac{1}{T}} \sum_{t=0}^{T-1} r_t]$ . The discounted and undiscounted infinite settings are linked by the Laurent Serie Expansion

$$\nu_\beta^\pi(x) = \frac{g_\pi(x)}{1 - \beta} + h_\pi(x) + \sum_{n=1}^{\infty} h_\pi^{(n)}(x)(1 - \beta)^n \quad (1)$$

when  $\beta$  is close enough to 1, see [1]. It is known that as  $\beta \rightarrow 1$ , the class of discounted optimal policies stabilises onto a single class  $\Pi_\infty^*(M)$ , called *Blackwell optimal policies*. They are policies which maximize the whole vector  $(g_\pi(x), h_\pi(x), h_\pi^{(1)}(x), h_\pi^{(2)}(x), \dots)$  for the lexicographic order. Namely, they maximize the asymptotical average reward or *gain*  $g_\pi(x)$ , but also the transient rewards i.e. the *bias*  $h_\pi(x)$  and all higher order biases  $h_\pi^{(n)}(x)$ . Blackwell optimality is the last refinement of infinite horizon optimality that merges the discounted and undiscounted cases. When  $M$  is given, algorithms that compute Blackwell optimal policies are already known, see [1]. Our interest is to figure out if such policies can be *learned*.

**Probably Correct Identification Algorithms.** We are interested in the *identification* of Blackwell optimal policies in the *generative model* in a similar fashion as best-arm identification

algorithms for stochastic bandits [2]. By generative model, we mean that at each time step, the algorithm is allowed to sample any edge in edge space. An identification algorithm  $\mathcal{I}$  is made of three components:

- an *allocation rule* that chooses, according to past observations, the next edge  $(x_t, y_t)$  to be sampled;
- a *stopping rule*  $\tau_\delta$  to stop the learning phase of the algorithm;
- a *recommendation rule* to return a policy  $\pi_{\tau_\delta}^A$  at the end of the learning phase.

If  $\mathcal{M}$  is a class of MDP,  $\mathcal{I}$  is said to be  $\delta$ -PC on  $\mathcal{M}$  if when executed on any  $M \in \mathcal{M}$ , it returns a Blackwell optimal policy with probability at least  $1 - \delta$ . Recent works [3] have designed identification algorithms for the discounted setting. The undiscounted setting remained open.

## 2 An Identification Algorithm for Blackwell Optimality

The learning of Blackwell optimal policies is limited to a specific class of d-MDP that we denote  $\mathcal{M}$ , defined as the set of d-MDPs  $M$  such that :

- (H1)**  $M$  has a unique optimal cycle i.e. the cycle  $\mathcal{C}_* \subseteq \mathcal{E}$  that maximize its average expected reward  $g(\mathcal{C}_*) := \frac{1}{|\mathcal{C}_*|} \sum_{e \in \mathcal{C}_*} r(e)$  is unique.
- (H2)** Under H1, writing  $\mathcal{C}_*$  as the sequence of states  $u_0, u_1, \dots, u_{c-1}$ , we define the bias of a state  $u_i \in \mathcal{C}_*$  as  $h_*(u_i) := \frac{1}{c} \sum_{\ell=1}^c \sum_{k=0}^{\ell-1} [r(u_{i+k}, u_{i+k+1}) - g(\mathcal{C}_*)]$  where indices are taken modulo  $c$ . Then for all state  $x_0$ , there exists a unique path  $(x_0, x_1, \dots, x_k)$  to  $\mathcal{C}_*$  that maximizes  $h_*(x_k) + \sum_{i=0}^{k-1} [r(x_i, x_{i+1}) - g(\mathcal{C}_*)]$ ,  $x_k \in \mathcal{C}_*$ .

These two assumptions are minimal. Specifically, we can show that if  $\mathcal{M}'$  is a space of d-MDPs such that  $\mathcal{M}' \cap \mathcal{M}^{\text{G}} \neq \emptyset$ , there is no  $\frac{1}{4}S^{-A}$ -PC identification algorithm on  $\mathcal{M}'$ . Finally, we propose a  $\delta$ -PC identification algorithm on  $\mathcal{M}$  for any  $\delta > 0$ .

**Theorem 1** Consider the algorithm  $\mathcal{I}$  that samples edges uniformly with stopping time

$$\tau_\delta := \inf \left\{ t \geq A : \left( \frac{\frac{1}{2} \log\left(\frac{2At^2}{\delta}\right)}{\lfloor t/A \rfloor} \right)^{\frac{1}{2}} \leq \min \left\{ \frac{\Delta_0(\hat{M}_t)}{4S}, \frac{\Delta_{-1}(\hat{M}_t)}{2} \right\} \right\} \quad (2)$$

where  $\hat{M}_t$  is the MDP of empirical observations up to time  $t$  and  $\Delta_0(\hat{M}_t), \Delta_{-1}(\hat{M}_t)$  are MDP dependent parameters and that returns any  $\pi_{\tau_\delta}^{\mathcal{I}} \in \Pi_\infty^*(\hat{M}_{\tau_\delta})$ . Then  $\mathcal{I}$  is  $\delta$ -PC and stops almost surely. In addition, setting  $\Delta(M) := \min\{\frac{1}{8S}\Delta_0(M), \frac{1}{4}\Delta_{-1}(M)\}$ ,

$$\mathbb{P} \left\{ \tau_\delta \leq 3A \max \left\{ 1, \frac{1}{2} \log\left(\frac{2A}{\delta}\right) \Delta(M)^{-2}, 3A \Delta(M)^{-4} \right\} \right\} \geq 1 - \delta. \quad (3)$$

This uniform algorithm can be improved into a faster, non-uniform one. Moreover, these results can be generalized to MDPs with general probabilistic transitions.

## References

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