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# ON NÖRLUND SUMMATION AND ERGODIC THEORY, WITH APPLICATIONS TO POWER SERIES OF HILBERT CONTRACTIONS.

CHRISTOPHE CUNY AND MICHEL WEBER

ABSTRACT. We show that if  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  is a good weight for the dominated weighted ergodic theorem in  $L^p$ ,  $p > 1$ , then the Nörlund matrix  $N_{\mathbf{a}} = \{a_{i-j}/A_i\}_{0 \leq j \leq i}$ ,  $A_i = \sum_{k=0}^i |a_k|$  is bounded on  $\ell^p(\mathbb{N})$ . We study the regularity (convergence in norm, almost) of operators in ergodic theory: power series of Hilbert contractions, and power series  $\sum_{n \in \mathbb{N}} a_n P_n f$  of  $L^2$ -contractions, and establish similar tight relations with the Nörlund operator associated to the modulus coefficient sequence  $(|a_n|)_{n \in \mathbb{N}}$ .

## 1. Introduction

Let  $\mathbf{a} := (a_n)_{n \in \mathbb{N}}$  be a sequence of complex or real numbers (we take the convention  $\mathbb{N} = \{0, 1, 2, \dots\}$ ). We associate with  $\mathbf{a}$  an infinite matrix  $N_{\mathbf{a}} = (a_{ij})_{i,j \in \mathbb{N}}$ , called a Nörlund matrix, in the following way. For every  $i \geq 0$ , set  $A_i := \sum_{k=0}^i |a_k|$ . We then define

$$(1) \quad \begin{cases} a_{ij} := a_{i-j}/A_i & \text{if } 0 \leq j \leq i \text{ and } A_i > 0, \\ a_{ij} := 0 & \text{if } j > i \text{ or } A_i = 0. \end{cases}$$

Some authors consider instead  $A_i = \sum_{k=0}^i a_k$ , assuming then that it does not vanish.

Then  $N_{\mathbf{a}}$  induces naturally a (possibly unbounded) operator on  $\ell^p(\mathbb{N})$  for any  $p \geq 1$ . The matter of deciding whether this operator is bounded on some (or any)  $\ell^p(\mathbb{N})$  is far from being solved. As noted by Bennett [4], it seems, so far, that the best general known condition guaranteeing that  $N_{\mathbf{a}}$  is bounded on any  $\ell^p(\mathbb{N})$ ,  $p > 1$ , is that  $a_n = O(A_n/n)$ , see for instance Borwein and Cass [6]. That condition is realized when, for instance  $(a_n)_{n \in \mathbb{N}}$  is a non-increasing sequence of positive numbers.

Let  $c_{00} := \{(u_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \exists n_0 \in \mathbb{N} / u_n = 0, \forall n \geq n_0\}$ . Recall that the boundedness of  $N_{\mathbf{a}}$  on  $\ell^p$ ,  $p > 1$ , means that there exists  $C_p > 0$  such that, for any sequence  $(u_n)_{n \in \mathbb{N}} \in c_{00}$ ,

$$(2) \quad \sum_{i \geq 0} \left| \frac{1}{A_i} \sum_{j=0}^i a_{i-j} u_j \right|^p \leq C_p^p \sum_{i \geq 0} |u_i|^p.$$

Equivalently, we have the dual formulation: for any sequence  $(v_n)_{n \in \mathbb{N}} \in c_{00}$ ,  $q = p/(p-1)$ ,

$$(3) \quad \sum_{j \geq 0} \left| \sum_{i \geq j} a_{i-j} v_i / A_i \right|^q \leq C_p^q \sum_{j \geq 0} |v_j|^q.$$

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The latter is easily seen to be also equivalent to: for any sequence  $(v_n)_{n \in \mathbb{N}} \in c_{00}$ ,

$$(4) \quad \sum_{j \geq 0} \left| \sum_{i \geq j} a_{i-j} v_i \right|^q = \sum_{j \geq 0} \left| \sum_{i \geq 0} a_i v_{i+j} \right|^q \leq C_p^q \sum_{j \geq 0} |A_j v_j|^q.$$

Moreover, it follows from (2) that

$$(5) \quad \sum_{i \geq 0} \frac{|a_i|^p}{A_i^p} \leq C_p^p,$$

where  $a_i/A_i$  has to be interpreted as 0 when  $A_i = 0$ .

We show that Nörlund matrices are connected with two different topics from ergodic theory. We establish tight relations between regularity (convergence in norm, almost everywhere) of operators in ergodic theory (power series of Hilbert contractions, power series of  $L^2$ -contractions, dominated weighted ergodic theorems, and naturally associated Nörlund matrices. We obtain conditions ensuring norm convergence of power series of Hilbert contractions, and also almost everywhere convergence of power series  $\sum_{n \in \mathbb{N}} a_n P_n f$  of  $L^2$ -contractions. These conditions are expressed in terms of the Nörlund operator associated to the modulus coefficient sequence  $(a_n)_{n \in \mathbb{N}}$ .

## 2. Nörlund matrices and dominated weighted ergodic theorems

We first observe a connection between Nörlund matrices and dominated weighted ergodic theorems.

We say that a sequence  $(a_n)_{n \in \mathbb{N}}$ , of complex numbers, is good for the dominated weighted ergodic theorem in  $L^p$ ,  $p > 1$ , if there exists  $C > 0$  such that for every dynamical system  $(X, \Sigma, \nu, \tau)$ , writing  $A_n := \sum_{k=0}^n |a_k|$ , we have

$$(6) \quad \left\| \sup_{n \geq 0} \frac{1}{A_n} \left| \sum_{k=0}^n a_k f \circ \tau^k \right| \right\|_{L^p(\nu)} \leq C \|f\|_{L^p(\nu)} \quad \forall f \in L^p(\nu).$$

Here again we take the convention that  $\frac{1}{A_n} \left| \sum_{k=0}^n a_k f \circ \tau^k \right| = 0$  if  $A_n = 0$ .

The next lemma is well-known, it is in the spirit of the so-called Conze principle, see for instance [29, Th. 5.4.3]. It states a converse of Calderon's transference principle.

**Lemma 1.** *Let  $(a_n)_{n \in \mathbb{N}}$  be good for the dominated weighted ergodic theorem in  $L^p$ ,  $p > 1$ . Then, with the best constant  $C > 0$  appearing in (6), we have for every  $(v_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ ,*

$$(7) \quad \sum_{i \in \mathbb{Z}} \left( \sup_{n \geq 0} \frac{1}{A_n} \left| \sum_{j=0}^n a_j v_{i+j} \right| \right)^p \leq C^p \sum_{i \in \mathbb{Z}} |v_i|^p.$$

**Proof.** Let  $(v_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ . Let  $N > M \geq 1$  be integers (one has in mind that  $N \gg M$ ). Take  $X = \{-N, -N+1, \dots, N-1, N\}$ ,  $\nu := \frac{1}{2N+1} \sum_{k=-N}^N \delta_k$  and  $\tau$  the transformation given by  $\theta(k) = k+1$  if  $k \neq N$  and  $\theta(N) = -N$ . Define  $f$  on  $X$ , by  $f(k) = v_k$  for every  $k \in X$ . By (6), we have

$$(8) \quad \begin{aligned} \frac{1}{2N+1} \sum_{i=-N}^{N+1-M} \left( \sup_{0 \leq m \leq M} \frac{1}{A_m} \left| \sum_{k=0}^m a_k v_{i+k} \right| \right)^p &\leq \left\| \sup_{n \geq 0} \frac{1}{A_n} \left| \sum_{k=0}^n a_k f \circ \tau^k \right| \right\|_{L^p(\nu)}^p \\ &\leq C^p \|f\|_{L^p(\nu)}^p = \frac{C^p}{2N+1} \sum_{i=-N}^N |v_i|^p. \end{aligned}$$

Multiplying (8) by  $2N + 1$ , letting first  $N \rightarrow +\infty$  and then  $M \rightarrow +\infty$ , we derive (7).  $\square$

**Remark.** Our proof is based on the use of the dominated weighted ergodic theorem on periodic systems (the rotations on  $\mathbb{Z}/(2N + 1)\mathbb{Z}$ ). To give a proof based on the dominated weighted ergodic theorem on a *single* (but ergodic and non-atomic) dynamical system, one could use Rohlin's lemma (see for instance Weber [29, p. 270] for a statement of the lemma).

We deduce the following.

**Proposition 2.** *Let  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  be a good weight for the dominated weighted ergodic theorem in  $L^p$ ,  $p > 1$ . Then, the Nörlund matrix  $N_{\mathbf{a}}$  is bounded on  $\ell^p(\mathbb{N})$ . Moreover, for every non-increasing sequence of nonnegative numbers  $(b_n)_{n \in \mathbb{N}}$ , writing  $\mathbf{c} := (a_n b_n)_{n \in \mathbb{N}}$ ,  $N_{\mathbf{c}}$  is bounded on  $\ell^p(\mathbb{N})$ .*

**Remark.** It is unclear whether " $N_{\mathbf{a}}$  bounded on  $\ell^p$ " implies " $N_{\mathbf{c}}$  bounded on  $\ell^p$ ", if one does not assume that  $\mathbf{a}$  is a good weight for the dominated ergodic theorem in  $L^p$ .

**Proof.** Let  $(u_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N})$ . Define  $(v_n) \in \ell^p(\mathbb{Z})$  as follows.  $v_n := u_{-n}$  if  $n \leq 0$  and  $v_n := 0$  if  $n > 0$ . Using (7) and for every  $i \geq 1$  the trivial estimate

$$\frac{1}{A_i} \left| \sum_{j=0}^i a_j v_{-i+j} \right| \leq \sup_{n \geq 0} \frac{1}{A_n} \left| \sum_{j=0}^n a_j v_{-i+j} \right|,$$

we infer that

$$\sum_{i \geq 0} \left( \frac{1}{A_i} \left| \sum_{j=0}^i a_j v_{-i+j} \right| \right)^p \leq C^p \sum_{i \in \mathbb{Z}} |v_i|^p = C^p \sum_{i \geq 0} |u_i|^p.$$

Using that  $v_{-i+j} = u_{i-j}$  when  $j \leq i$ , we derive that  $N_{\mathbf{a}}$  is bounded on  $\ell^p$ .

To prove the last assertion, one just has to notice that, using Abel summation,  $(c_n)_{n \in \mathbb{N}}$  is a good weight for the dominated weighted ergodic theorem.  $\square$

Of course, as one can see from the above proof, the fact that  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  be a good weight for the dominated weighted ergodic theorem in  $L^p$  is a much stronger statement than the fact that  $N_{\mathbf{a}}$  be bounded on  $\ell^p(\mathbb{N})$ . Hence, Proposition 2 should not be seen as a method to prove boundedness of some Nörlund matrices, but as a source of examples of Nörlund matrices, since there are many examples of sequences that are known to be good for the dominated weighted ergodic theorem. We provide some of them below. One may also consult the survey by Bellow and Losert [3] for dominated weighted ergodic theorems with bounded weights. More arithmetical sequences may be found in Cuny and Weber [17].

**Examples.** The following sequences  $(a_n)_{n \in \mathbb{N}}$  are good for the dominated weighted ergodic theorem in  $L^p$ , for every  $p > 1$ :

(i) (Bourgain and Wierdl, [7], [30]) Let  $\mathcal{P}$  be the set of prime numbers and take  $a_n := \delta_{\mathcal{P}}(n)$ , for every  $n \in \mathbb{N}$ .

(ii) (Bourgain, [9]) Let  $\mathcal{S}$  be the set of squares and take  $a_n := \delta_{\mathcal{S}}(n)$ , for every  $n \in \mathbb{N}$ .

(iii) (Cuny and Weber, [17]) Take  $a_0 = 0$  and for every  $n \in \mathbb{N}$  take  $a_n = d_n$ , the number of divisors of  $n$ .

We now give an example which does not work on every  $L^p$ ,  $p > 1$ . Let  $(X, \Sigma, \mu, \theta)$  be an ergodic dynamical system. Let  $g \in L^q(\mu)$ , for some  $1 < q \leq \infty$ .

(iv) (Bourgain, Demeter, Lacey, Tao and Thiele, [8], [21] and [19]) There exists  $\bar{X} \in \Sigma$  with  $\mu(\bar{X}) = 1$  such that for every  $x \in \bar{X}$ , setting  $(a_n)_{n \in \mathbb{N}} := (g \circ \theta^n(x))_{n \in \mathbb{N}}$ , for every ergodic dynamical system  $(X, \Sigma, \nu, \tau)$  and for every  $f \in L^p(\nu)$  where  $p > 1$  is such that  $1/p + 1/q < 3/2$ , one has

$$\left\| \sup_{n \geq 0} \frac{1}{n} \left| \sum_{k=0}^n a_k f \circ \tau^k \right| \right\|_{L^p(\nu)} \leq C \|f\|_{L^p(\nu)} \quad \forall f \in L^p(\nu)$$

Now, by the ergodic theorem, modifying  $\bar{X}$  if necessary, we may assume that for every  $x \in \bar{X}$ ,  $A_n(x)/n \xrightarrow[n \rightarrow \infty]{} \int_X |g(u)| \nu(du)$ . Hence, we infer that if  $g$  is not identically 0,  $(a_n)_{n \in \mathbb{N}}$  is a good weight for the dominated ergodic theorem in  $L^p$ .

Let us notice that none of the above examples satisfies the previously mentioned criterium:  $\sup_{n \in \mathbb{N}} n|a_n|/A_n < \infty$ . The fact that the Nörlund matrix associated with the sequence  $(a_n)_{n \in \mathbb{N}}$  in example (ii) is bounded has been proved by Borwein [5].

### 3. Norm convergence of power series of Hilbert contractions

Let  $P$  be a contraction of a (real or complex) Hilbert space  $\mathcal{H}$ . Given a sequence of complex numbers  $(a_n)_{n \in \mathbb{N}}$  and  $f \in \mathcal{H}$ , we are interested in finding conditions involving  $(\|P^n f\|_{\mathcal{H}})_{n \in \mathbb{N}}$  sufficient for the norm convergence of  $\sum_{n \in \mathbb{N}} a_n P^n f$ .

An obvious condition is the following

$$(9) \quad \sum_{n \in \mathbb{N}} |a_n| \|P^n f\|_{\mathcal{H}} < \infty.$$

Sufficient conditions involving  $(\|f + \dots + P^n f\|_{\mathcal{H}})_{n \in \mathbb{N}}$  have been obtained when  $P$  is unitary (i.e.  $P^* = P^{-1}$ ) or, more generally, normal (i.e.  $PP^* = P^*P$ ), if moreover  $(a_n)_{n \in \mathbb{N}}$  is regular (at least nonnegative and nonincreasing). Let us mention the papers [23] and [13], see also [14] for some  $L^p$  versions.

Recall that, see for instance Nagy and Foias [25] (see also Schäffer [26] for an explicit matrix construction),  $P$  admits a unitary dilation, that is, there exist another Hilbert space  $\mathcal{K}$ , with  $\mathcal{H} \subset \mathcal{K}$ , and a unitary operator  $U$  on  $\mathcal{K}$  such that  $EU^n = P^n$  for every  $n \geq 1$ , where  $E$  is the orthogonal projection onto  $\mathcal{H}$ .

We start with some simple lemmas. The first one appears in Cuny and Lin [16], but we recall the short proof.

**Lemma 3.** *For every  $n \in \mathbb{N}$  and every  $\ell \geq 1$ , the spaces  $(U^{-n}P^n - U^{-n-1}P^{n+1})\mathcal{H}$  and  $U^{-n-\ell}P^{n+\ell}\mathcal{H}$  are orthogonal (in  $\mathcal{K}$ ).*

**Proof.** Let  $f, g \in \mathcal{H}$ . Let  $n \in \mathbb{N}$  and  $\ell \geq 1$ . We have

$$\begin{aligned} \langle (U^{-n}P^n - U^{-n-1}P^{n+1})f, U^{-n-\ell}P^{n+\ell}g \rangle_{\mathcal{K}} &= \langle U^\ell P^n f, P^{n+\ell}g \rangle_{\mathcal{K}} - \langle U^{\ell-1}P^{n+1}f, P^{n+\ell}g \rangle_{\mathcal{K}} \\ &= \langle P^{n+\ell}f, P^{n+\ell}g \rangle_{\mathcal{K}} - \langle P^{n+\ell}f, P^{n+\ell}g \rangle_{\mathcal{K}} = 0. \end{aligned}$$

□

**Lemma 4.** *Let  $f \in \mathcal{H}$  be such that  $\|P^m f\|_{\mathcal{H}} \rightarrow 0$  as  $m \rightarrow +\infty$ . Then, for every  $n \geq 1$ ,  $\|P^n f\|_{\mathcal{H}}^2 = \|P^n f\|_{\mathcal{K}}^2 = \sum_{k \geq n} \|U^{-k}P^k f - U^{-k-1}P^{k+1}f\|_{\mathcal{K}}^2$ . In particular, for any positive and non-decreasing sequence  $(b_n)_{n \geq 0}$ , the following are equivalent (setting  $b_{-1} = 0$ ).*

- (i)  $\sum_{n \in \mathbb{N}} (b_n - b_{n-1}) \|P^n f\|_{\mathcal{H}}^2 < \infty$ ;
- (ii)  $\sum_{n \in \mathbb{N}} b_n \|U^{-n}P^n f - U^{-n-1}P^{n+1}f\|_{\mathcal{K}}^2 < \infty$ .

**Remarks.** Notice that by Kronecker's lemma, if (i) holds, then  $\|P^n f\|_{\mathcal{H}}^2 \sum_{k=0}^n (b_{k+1} - b_k) = \|P^n f\|_{\mathcal{H}}^2 (b_{n+1} - b_0) \rightarrow 0$  as  $n \rightarrow +\infty$ . In particular, since  $(b_n)_{n \in \mathbb{N}}$  is non decreasing,  $\|P^n f\|_{\mathcal{H}} \rightarrow 0$ . Item (i) is satisfied if  $\sum_{n \geq 0} b_{2^{n+1}} \|P^{2^n} f\|_{\mathcal{H}}^2 < \infty$ .

**Proof.** Since  $\|P^n f\| \rightarrow 0$ , for every  $n \in \mathbb{N}$ , we have, (with convergence in  $\mathcal{K}$ )

$$(10) \quad P^n f = \sum_{k \geq 0} (U^{-k} P^{n+k} f - U^{-k-1} P^{n+k+1} f).$$

By the above lemma the terms of that series lie in orthogonal spaces. Hence,

$$\begin{aligned} \|P^n f\|_{\mathcal{K}}^2 &= \sum_{k \geq 0} \|U^{-k} P^{n+k} f - U^{-k-1} P^{n+k+1} f\|_{\mathcal{K}}^2 \\ &= \sum_{k \geq n} \|U^{-k} P^k f - U^{-k-1} P^{k+1} f\|_{\mathcal{K}}^2, \end{aligned}$$

where we used that  $U$  is unitary (and a change of variable) for the last identity. Then, the equivalence of (i) and (ii) follows by Fubini.  $\square$

Given a sequence of complex numbers  $(a_n)_{n \in \mathbb{N}}$ , consider the following conditions

$$(11) \quad \sum_{n \in \mathbb{N}} |a_n| \left( \sum_{k=0}^n |a_k| \right) \|P^n f\|_{\mathcal{H}}^2 < \infty,$$

$$(12) \quad \sum_{n \in \mathbb{N}} \left( \sum_{k=0}^n |a_k| \right)^2 \|U^{-n} P^n f - U^{-n-1} P^{n+1} f\|_{\mathcal{K}}^2 < \infty$$

By Lemma 4, when  $\|P^n f\|_{\mathcal{H}} \rightarrow 0$ , (11) and (12) are equivalent. Assume that (9) holds. Then, since  $(\|P^n f\|_{\mathcal{H}})_{n \in \mathbb{N}}$  is nonincreasing,  $\sup_{n \in \mathbb{N}} \|P^n f\| \sum_{k=0}^n |a_k| < \infty$  and (11) holds. Hence, (11) is always weaker than (9).

**Proposition 5.** *Let  $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$  be such that  $N_{|\mathbf{a}|}$  is bounded on  $\ell^2(\mathbb{N})$  where  $|\mathbf{a}| = (|a_n|)_{n \in \mathbb{N}}$ . Let  $f \in \mathcal{H}$  be such that either of conditions (11) or (12) hold. Then, the series  $\sum_{n \in \mathbb{N}} a_n P^n f$  converges in  $\mathcal{H}$ .*

**Proof.** Since  $N_{|\mathbf{a}|}$  is bounded on  $\ell^2(\mathbb{N})$ , then by (5) (with  $p = 2$ )

$$(13) \quad \sum_{n \in \mathbb{N}} \frac{a_n^2}{A_n^2} < \infty.$$

Let  $q > p \geq 1$  be integers and write  $V_{p,q} f := \sum_{k=p}^q a_k P^k f$ . For every  $n \in \mathbb{N}$ , let  $u_n := \|U^{-n} P^n f - U^{-n-1} P^{n+1} f\|_{\mathcal{K}}$  and  $v_n := A_n u_n$ , where  $A_n = \sum_{k=0}^n |a_k|$ . Finally, let  $\mathbf{v} := (v_n)_{n \in \mathbb{N}}$ . By Lemma 3 and using that  $U$  is unitary, we have

$$(14) \quad \|V_{p,q} f\|_{\mathcal{K}}^2 = \sum_{n \in \mathbb{N}} \|U^{-n} P^n V_{p,q} f - U^{-n-1} P^{n+1} V_{p,q} f\|_{\mathcal{K}}^2 \leq \sum_{n \in \mathbb{N}} \left( \sum_{k=p}^q |a_k| u_{n+k} \right)^2.$$

By Cauchy's criteria one has to prove that  $\|V_{p,q} f\|_{\mathcal{K}} \rightarrow 0$  as  $p, q \rightarrow +\infty$ . Using the Lebesgue dominated theorem for the counting measure on  $\mathbb{N}$ , it suffices to prove that

$$(15) \quad \sum_{k=p}^q |a_k| u_{n+k} \xrightarrow{p, q \rightarrow +\infty} 0,$$

and that

$$(16) \quad \sum_{n \in \mathbb{N}} \left( \sum_{k \geq 0} |a_k| u_{n+k} \right)^2 < \infty.$$

The convergence (15) follows from Cauchy-Schwarz combined with the assumed condition (12) and (13).

To prove (16), it suffices to notice that

$$\begin{aligned} \sum_{n \in \mathbb{N}} \left( \sum_{k \geq 0} |a_k| u_{n+k} \right)^2 &= \sum_{n \in \mathbb{N}} \left( \sum_{k \geq n} |a_{k-n}| u_k \right)^2 = \|N_{|\mathbf{a}|}^* \mathbf{v}\|_{\ell^2(\mathbb{N})}^2 \\ &\leq \|N_{|\mathbf{a}|}^*\|^2 \|\mathbf{v}\|_{\ell^2(\mathbb{N})}^2 = \|N_{|\mathbf{a}|}\|^2 \sum_{n \in \mathbb{N}} A_n^2 u_n^2, \end{aligned}$$

which is finite by (12).  $\square$

The proposition has been proved in [16] in the case where  $a_n = n^{-1/2}$ . An important case corresponds to the situation where  $a_n = 1$  for every  $n \in \mathbb{N}$ . Then, the proposition gives a sufficient condition (namely  $\sum_{n \in \mathbb{N}} n \|P^n f\|_{\mathcal{H}}^2 < \infty$ ) for  $f$  to be a coboundary (i.e.  $f = (I - P)g$  for some  $g \in \mathcal{H}$ ). This sufficient condition has been obtained independently by Volný [28] in the special case where  $P$  is a Markov operator on  $L^2(m)$ . His proof (which does not appeal to the notion of Nörlund matrices) is essentially the same, since the shift on the space of trajectories of the associated Markov chain plays the role of the unitary dilation.

**Proposition 6.** *Let  $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ . Assume that for every contraction  $P$  on a Hilbert space  $\mathcal{H}$  the following property holds : "If (11) holds for some  $f \in \mathcal{H}$ , then  $\sum_{n \in \mathbb{N}} a_n P^n f$  converges in  $\mathcal{H}$ ". Then,  $N_{\mathbf{a}}$  is bounded on  $\ell^2$ .*

**Proof.** Let  $P$  be a contraction on a Hilbert space  $\mathcal{H}$  satisfying the above property. Let  $\mathcal{L} := \{f \in \mathcal{H} : \sum_{n \in \mathbb{N}} |a_n| (\sum_{k=0}^n |a_k|) \|P^n f\|_{\mathcal{H}}^2 < \infty\}$ . Then,  $\mathcal{L}$  is a Hilbert space and we define an operator  $T$  on  $\mathcal{L}$ , by setting  $Tf = \sum_{n \in \mathbb{N}} a_n P^n f$  for every  $f \in \mathcal{L}$ . Then, by the Banach-Steinhaus theorem  $T$  is continuous. Hence, there exists  $C = C_{\mathcal{H}, P}$ , such that  $\|\sum_{n \in \mathbb{N}} a_n P^n f\|_{\mathcal{H}} \leq C (\sum_{n \in \mathbb{N}} |a_n| (\sum_{k=0}^n |a_k|) \|P^n f\|_{\mathcal{H}}^2)^{1/2} < \infty$ .

Let us prove the proposition. We give a probabilistic proof. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the probability space given by  $\Omega = \{-1, 1\}^{\mathbb{Z}}$ ,  $\mathcal{F}$  the product  $\sigma$ -algebra and  $\mathbb{P} = \mu^{\otimes \mathbb{Z}}$ , with  $\mu(0) = \mu(1) = 1/2$ . Let  $\theta$  be the shift on  $\Omega$  and  $(\varepsilon_n)_{n \in \mathbb{Z}}$  be the coordinate process. In particular,  $\varepsilon_{n+1} = \varepsilon_n \circ \theta$  and  $(\varepsilon_n)_{n \in \mathbb{Z}}$  is iid.

Denote  $\mathcal{F}_0 := \sigma\{\varepsilon_i, i \leq 0\}$ . Set  $\mathcal{H} := L^2(\Omega, \mathcal{F}_0, \mathbb{P})$  and  $\mathcal{K} := L^2(\Omega, \mathcal{F}, \mathbb{P})$  and define two operators  $U$  and  $P$  on  $\mathcal{K}$  and  $\mathcal{H}$  respectively by  $Uf = f \circ \theta$  for every  $f \in \mathcal{K}$  and  $Pf = \mathbb{E}(f \circ \theta | \mathcal{F}_0)$  for every  $f \in \mathcal{H}$  (then  $P$  is a Markov operator). Clearly,  $U$  is a unitary dilation of  $P$ . Let  $(u_i)_{i \in \mathbb{N}} \in c_{00}$  and define  $f := \sum_{i \in \mathbb{N}} u_i \varepsilon_i \in \mathcal{H}$ . Assume moreover that  $\sum_{n \in \mathbb{N}} |a_n| (\sum_{k=0}^n |a_k|) \|P^n f\|_{\mathcal{H}}^2 < \infty$ , or equivalently (by Lemma 4),  $\sum_{n \in \mathbb{N}} (\sum_{k=0}^n |a_k|)^2 \|U^{-n} P^n f - U^{-n-1} P^{n+1} f\|_2^2$ . Notice that  $P^n f = \sum_{i \in \mathbb{N}} u_{i+n} \varepsilon_{-i}$  and that  $\|U^{-n} P^n f - U^{-n-1} P^{n+1} f\|_2^2 = u_n^2$ . Moreover,  $\|\sum_{n \in \mathbb{N}} a_n P^n f\|_{\mathcal{H}}^2 = \sum_{i \in \mathbb{N}} |\sum_{n \in \mathbb{N}} a_n u_{i+n}|^2$ . Hence,  $\sum_{i \in \mathbb{N}} |\sum_{n \in \mathbb{N}} a_n u_{i+n}|^2 \leq C^2 \sum_{n \in \mathbb{N}} b_n |u_n|^2$ , i.e. (4) holds with  $q = 2$ , and the proof is complete.  $\square$

We shall now prove that Proposition 5 cannot be improved.

**Definition 1.** *We say that a contraction  $P$  on  $\mathcal{H}$  is Ritt if  $\sup_{n \in \mathbb{N}} n \|P^n - P^{n+1}\| < \infty$ .*

**Proposition 7.** *Let  $P$  be a contraction on  $\mathcal{H}$ . For every  $0 \leq \alpha < 1$ , consider the following properties.*

- (i) *The series  $\sum_{n \in \mathbb{N}} (n+1)^{-\alpha} P^n f$  converges in  $\mathcal{H}$  ;*
- (ii)  *$\sum_{n \in \mathbb{N}} (n+1)^{1-2\alpha} \|P^n f\|_{\mathcal{H}}^2 < \infty$  .*

*Then, (ii)  $\Rightarrow$  (i). If moreover  $P$  is Ritt then (i)  $\Rightarrow$  (ii).*

**Remark.** By [11], when  $P$  is a positive operator on  $L^2(m)$  then (i) of the proposition implies that the series  $\sum_{n \in \mathbb{N}} (n+1)^{-\alpha} P^n f$  converges  $m$ -almost everywhere and the associated maximal function is in  $L^2$ . The fact that (i)  $\Rightarrow$  (ii) has been proved by Cohen, Cuny and Lin [10] using results from Arhancet and Le Merdy [2]) when  $\alpha \in (0, 1)$  and  $P$  is a positive Ritt contraction of some  $L^2(m)$  (there are also analogous results in  $L^p$  in [10]).

**Proof.** The fact that (ii)  $\Rightarrow$  (i) is a direct application of Proposition 5. Assume that  $P$  is a Ritt operator and that  $\sum_{n \in \mathbb{N}} \frac{P^n f}{(n+1)^\alpha}$  converges in  $\mathcal{H}$ .

We start with the case  $0 < \alpha < 1$ . By Proposition 4.6 of Cohen, Cuny and Lin [11] (see also their example (v) page 8), we have

$$\sum_{n \geq 0} \frac{\|Pf + \dots + P^{2^n} f\|_{\mathcal{H}}^2}{2^{2\alpha n}} < \infty,$$

Then, using (3) of Cohen, Cuny and Lin [10] combined with Lemma 13 below, we infer that  $\sum_{n \geq 0} 2^{(2-2\alpha)n} \|P^{2^n} f\|_{\mathcal{H}}^2 < \infty$ , which finishes the proof, in that case.

Assume now that  $\alpha = 0$ . Let  $g := \sum_{n \in \mathbb{N}} P^n f$ . Then,  $f = (I - P)g$ . Hence, by Theorem 8.1 of Le Merdy [24],

$$\sum_{n \in \mathbb{N}} n \|P^n f\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{N}} n \|P^n (I - P)g\|_{\mathcal{H}}^2 \leq \|g\|_{\mathcal{H}},$$

which is the desired result.  $\square$

#### 4. Almost everywhere convergence of power series of $L^2$ -contractions

Once norm convergence has been proven, one may wonder, in the case where  $\mathcal{H} = L^2(m)$ , whether almost everywhere convergence holds. As mentioned in the remark following Proposition 7, for "regular" sequences, if  $P$  is a positive contraction of  $L^2(m)$  then norm convergence implies almost everywhere convergence. However, as we shall see below (see Proposition 10), there is no such result for contractions that are not positive. Let us mention that the almost everywhere convergence of power series (for regular  $(a_n)_{n \in \mathbb{N}}$ ) for unitary or normal operators on  $L^2(m)$  has been proven under conditions involving  $(\|f + \dots + P^n f\|_{\mathcal{H}})_{n \in \mathbb{N}}$  in [23] and [13], see also [15] for  $L^p$ -versions.

**Theorem 8.** *Let  $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$  be such that  $N_{|\mathbf{a}|}$  is bounded on  $\ell^2(\mathbb{N})$  where  $|\mathbf{a}| = (|a_n|)_{n \in \mathbb{N}}$ . Let  $A_n := \sum_{k=0}^n |a_k|$ . Let  $P$  be a contraction on  $L^2(m)$ . Let  $f \in L^2(m)$  be such that*

$$(17) \quad \sum_{n \geq 1} (\log(n+1))^2 A_{2^{n+1}}^2 \|P^{2^n} f\|_{L^2(m)}^2 < \infty.$$

*Then, the series  $\sum_{n \in \mathbb{N}} a_n P^n f$  converges  $m$ -almost everywhere and*

$$\sup_{N \geq 1} \left| \sum_{n=0}^N a_n P^n f \right| \in L^2(m).$$



**Remark.** A sufficient condition for (17) is the following

$$(18) \quad \sum_{n \geq 1} (\log \log(n+3))^2 \frac{A_{4n}^2}{n+1} \|P^n f\|_{L^2(m)}^2 < \infty.$$

**Proof.** Let  $N \in \mathbb{N}$ . We have

$$\max_{2^N \leq n \leq 2^{N+1}-1} \left| \sum_{k=2^N}^n a_n P^k f \right| \leq \sum_{k=2^N}^{2^{N+1}-1} |a_k| |P^k f|.$$

Hence,

$$\sum_{N \in \mathbb{N}} \left\| \max_{2^N \leq n \leq 2^{N+1}-1} \left| \sum_{k=2^N}^n a_n P^k f \right| \right\|_{L^2(m)}^2 \leq \sum_{N \in \mathbb{N}} A_{2^{N+1}}^2 \|P^{2^N} f\|_{L^2(m)}^2 < \infty.$$

In particular, it suffices to prove that  $(\sum_{n=0}^{2^N} a_n P^n f)_{N \geq 0}$  converges and that  $\sup_{N \geq 0} |\sum_{n=0}^{2^N} a_n P^n f| \in L^2(m)$ .

By (14), for every  $q \geq p$ , we have

$$(19) \quad \left\| \sum_{n=2^p}^{2^q-1} a_n P^n f \right\|_{L^2(m)}^2 \leq \sum_{n \in \mathbb{N}} \left( \sum_{k=2^p}^{2^q-1} |a_k| |u_{n+k}| \right)^2.$$

Set  $d(p, q) := \sum_{n \in \mathbb{N}} \left( \sum_{k=2^p}^{2^q-1} |a_k| |u_{n+k}| \right)^2$  and notice that  $d$  is super-additive in the following sense: for every  $m \geq l \geq k$ ,  $d(k, l) + d(l, m) \leq d(k, m)$ . By Proposition 2.2 of Cohen and Lin [12], there exists  $C > 0$ , such that for every  $n \geq 0$ ,

$$(20) \quad \left\| \max_{2^n \leq m \leq 2^{n+1}-1} \left| \sum_{k=2^{2^n}}^{2^m} a_k P^k f \right| \right\|_{L^2(m)}^2 \leq C(n+1)^2 d(2^n, 2^{n+1}-1).$$

Assume that

$$(21) \quad \sum_{n \geq 0} (n+1)^2 d(2^n, 2^{n+1}-1) < \infty.$$

Then, using (19) and Cauchy-Schwarz we see that

$$\left( \sum_{n \in \mathbb{N}} \left\| \sum_{k=2^{2^n}}^{2^{2^{n+1}}-1} a_k P^k f \right\|_{L^2(m)} \right)^2 \leq \sum_{n \in \mathbb{N}} \frac{1}{(n+1)^2} \sum_{n \in \mathbb{N}} (n+1)^2 d(2^n, 2^{n+1}-1).$$

In particular (by a version of the monotone convergence theorem),  $\sum_{n \in \mathbb{N}} \left| \sum_{k=2^{2^n}}^{2^{2^{n+1}}-1} a_k P^k f \right| < \infty$  almost everywhere and our series converges almost everywhere along the subsequence  $(2^{2^n})_{n \in \mathbb{N}}$ .

Combining (20) and (21) (and the monotone convergence theorem again), we infer that  $\sum_{n \in \mathbb{N}} \max_{2^{2^n} \leq m \leq 2^{2^{n+1}}-1} \left| \sum_{k=2^{2^n}}^m a_k P^k f \right|^2 < \infty$  almost everywhere, which yields the almost everywhere convergence of the series itself.

To finish the proof, it remains to prove (21). But, (21) reads

$$\sum_{n \in \mathbb{N}} \sum_{\ell \geq 0} (\ell+1)^2 \left( \sum_{k=2^{2^\ell}}^{2^{2^{\ell+1}}-1} |a_k| |u_{n+k}| \right)^2 < \infty.$$

Using that  $\|\cdot\|_{\ell^2} \leq \|\cdot\|_{\ell^1}$ , we infer that

$$\begin{aligned} \sum_{n \in \mathbb{N}} \sum_{\ell \geq 0} (\ell + 1)^2 \left( \sum_{k=2^{2^\ell}}^{2^{2^{\ell+1}}-1} |a_k| u_{n+k} \right)^2 &\leq \sum_{n \in \mathbb{N}} \left( \sum_{k \geq 0} (\log \log(k+3)) |a_k| u_{n+k} \right)^2 \\ &\leq \sum_{n \in \mathbb{N}} \left( \sum_{k \geq 0} (\log \log(n+k+3)) |a_k| u_{n+k} \right)^2. \end{aligned}$$

Then, proceeding as in the (end of the) proof of Proposition 5 we see that (21) holds provided that

$$\sum_{n \in \mathbb{N}} (\log \log(n+3))^2 A_n^2 u_n^2 < \infty,$$

which follows from (17) using that  $(A_n)_{n \in \mathbb{N}}$  is non-decreasing and that  $\sum_{k=2^n}^{2^{n+1}-1} u_k^2 \leq \|P^{2^n} f\|_{L^2(m)}^2$ .  
□

**Corollary 9.** *Let  $(X, \Sigma, \mu, \theta)$  be an ergodic dynamical system. Let  $g \in L^p(\mu)$  for some  $p > 1$ . There exists  $\bar{X} \in \Sigma$  with  $\mu(\bar{X}) = 1$  such that for every  $x \in \bar{X}$ , setting  $(a_n)_{n \in \mathbb{N}} := (g \circ \theta^n(x))_{n \in \mathbb{N}}$  the following holds: for every  $0 \leq \alpha < 1$ , every contraction  $P$  on  $L^2(m)$  and every  $f \in L^2(m)$  such that*

$$\sum_{n \in \mathbb{N}} (\log \log(n+3))^2 (n+1)^{1-2\alpha} \|P^n f\|_2^2 < \infty,$$

the sequence  $\sum_{n \in \mathbb{N}} \frac{a_n P^n f}{(n+1)^\alpha}$  converges  $m$ -almost everywhere and the associated maximal function is in  $L^2(m)$ .

**Proof.** Let  $(X, \Sigma, \mu, \theta)$  and let  $g \in L^p(\mu)$ . Let  $\bar{X}$  be the set appearing in the example (iv). Modifying  $\bar{X}$  if necessary we may assume that  $A_n = |a_0| + \dots + |a_n| \leq K(x)n$ , for some finite  $K(x) > 0$ . Then, for every  $x \in \bar{X}$ ,  $(g \circ \theta^n(x))_{n \in \mathbb{N}}$  is good for the dominated weighted ergodic theorem. Applying Proposition 2, we see that, with  $\mathbf{c} = (c_n)_{n \in \mathbb{N}} := ((n+1)^{-\alpha} f \circ \theta^n(x))_{n \in \mathbb{N}}$ ,  $N_{\mathbf{c}}$  is bounded on  $\ell^2$ . Set  $C_n := \sum_{k=0}^n |c_k|$  (we see  $C_n$  as a function on  $X$ ). By Theorem 8 (see the remark after the theorem), we are back to prove that  $\sum_{n \geq 1} (\log \log(n+1))^2 C_{4n}^2 \|P^n f\|_2^2 < \infty$ . But this follows our assumption (and an Abel summation) since  $A_n \leq K(x)n$ . □

We shall now prove that Corollary 9 (and hence Theorem 8) is sharp.

**Proposition 10.** *Let  $0 \leq \alpha < 1$ . There exists a contraction  $P$  on some  $L^2(m)$  and  $f \in L^2(m)$  such that, for every  $\varepsilon > 0$ ,  $\sum_{n \in \mathbb{N}} (\log \log(n+3))^{2-\varepsilon} (n+1)^{1-2\alpha} \|P^n f\|_2^2 < \infty$  and the series  $\sum_{n \in \mathbb{N}} (n+1)^{-\alpha} P^n(f)$  diverges  $m$ -almost everywhere.*

**Remarks.** The proof is related to some arguments of Gaposhkin [22] and makes use of a counterexample by Tandori in the theory of orthogonal series. The construction of the operator  $P$  is related to the construction of the operator used in the proof of Proposition 6. Actually, the operator  $P$  used in the proof is a one-sided shift, hence is a co-isometry which prevents it from being Ritt. This raises the question whether it is possible to find a Ritt contraction satisfying the conclusion of the Proposition.

**Proof.** Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be an orthonormal system on some  $L^2(m)$  that we shall specify later. We define an operator  $P$  on  $\overline{\text{Vect}\{\varepsilon_n : n \in \mathbb{N}\}}$  as follows. For every  $f = \sum_{n \in \mathbb{N}} c_n \varepsilon_n$  let  $Pf := \sum_{n \in \mathbb{N}} c_{n+1} \varepsilon_n$ . One may extend  $P$  to the whole  $L^2(m)$  as one pleases.

For every  $n \in \mathbb{N}$ , let  $c_n := \frac{1}{(n+1)^{3/2-\alpha} \sqrt{\log(n+2) \log \log(n+3)^3}}$  and define  $f$  as above.

We have

$$\sum_{n=0}^{2^N} (n+1)^{-\alpha} P^n f = \sum_{n=0}^{2^N} \sum_{k \geq 0} (n+1)^{-\alpha} c_{k+n} \varepsilon_k.$$

We first prove that

$$(22) \quad \sum_{n=0}^{2^N} \sum_{k \geq 2^{N+1}} (n+1)^{-\alpha} c_{k+n} \varepsilon_k := v_N \xrightarrow[N \rightarrow \infty]{} 0 \quad m\text{-a.e.}$$

We have

$$\|v_N\|_{L^2(m)}^2 = \sum_{k \geq 2^{N+1}} \left( \sum_{n=0}^{2^N} (n+1)^{-\alpha} c_{k+n} \right)^2 \leq C 2^{2N(1-\alpha)} \sum_{k \geq 2^{N+1}} c_k^2 \leq \frac{C'}{N(\log(N+1))^3}.$$

Hence,  $\sum_{N \in \mathbb{N}} \|v_N\|_{L^2(m)}^2 < \infty$  and (22) holds.

Next, we prove that

$$(23) \quad \sum_{k=0}^{2^N} \sum_{n \geq 2^{N+1}} (n+1)^{-\alpha} c_{k+n} \varepsilon_k := w_N \xrightarrow[N \rightarrow \infty]{} 0 \quad m\text{-a.e.}$$

We have

$$\|w_N\|_{L^2(m)}^2 = \sum_{k=0}^{2^N} \left( \sum_{n \geq 2^{N+1}} (n+1)^{-\alpha} c_{k+n} \right)^2 \leq C 2^N \left( \sum_{n \geq 2^{N+1}} (n+1)^{-\alpha} c_n \right)^2 \leq \frac{C'}{N(\log(N+1))^3}.$$

Hence,  $\sum_{N \in \mathbb{N}} \|w_N\|_{L^2(m)}^2 < \infty$  and (23) holds.

Combining those first results, we see that we are back to finding  $(\varepsilon_n)_{n \in \mathbb{N}}$  such that  $(\sum_{k=0}^{2^N} \sum_{n \geq 0} (n+1)^{-\alpha} c_{n+k} \varepsilon_k)_{N \in \mathbb{N}}$  diverges  $m$ -almost everywhere.

For every  $k \geq 1$ , define

$$\begin{aligned} \alpha_k &:= \int_0^\infty \frac{dx}{x^\alpha (x+k)^{3/2-\alpha} (\log(x+k) (\log \log(x+k+2))^3)^{1/2}} \\ &= k^{-1/2} \int_0^\infty \frac{du}{u^\alpha (u+1)^{3/2-\alpha} (\log(ku+k) (\log \log(ku+k+2))^3)^{1/2}}. \end{aligned}$$

Hence,  $(\sqrt{k} \alpha_k)_{k \geq 1}$  is non-increasing. Moreover, it is not hard to see that the series

$$\sum_{k \geq 0} \left( \sum_{n \geq 0} (n+1)^{-\alpha} c_{n+k} - \alpha_k \right) \varepsilon_k$$

converges  $m$ -almost everywhere.

Then, by a result of Tandori, see Theorem 2.9.1 page 143 of Alexits [1] (combined with Theorem 2.7.3 page 120) there exists an orthonormal system  $(\varepsilon_n)_{n \in \mathbb{N}}$  such that  $(\sum_{n=0}^{2^N} \alpha_n \varepsilon_n)_{N \in \mathbb{N}}$  diverges  $m$ -a.e., and the proof is complete.  $\square$

### 5. Extensions, problems

Recall that an operator  $T$  on  $\mathcal{H}$  is said to be similar to a contraction if there exists a continuous invertible operator  $V$  from  $\mathcal{H}$  onto  $\mathcal{H}$  such that  $\|VTV^{-1}\| \leq 1$ , i.e. such that  $VTV^{-1}$  be a contraction.

Clearly, all the results from section 3 extend to operators that are similar to a contraction. Now, when  $\mathcal{H} = L^2(m)$ , it can be checked that all the results from section 4 also hold for operators that are similar to a contraction, even though the operator  $V$  in the definition need not be positive.

The most general class of operators on  $\mathcal{H}$  to which one may hope to extend Proposition 7 is the class of power bounded operators. Recall that an operator  $P$  on  $\mathcal{H}$  is said to be power bounded if  $\sup_{n \in \mathbb{N}} \|P^n\| < \infty$ . However, we shall see that this extension is not possible, even if we ask the operator to be Ritt.

The next proposition is a reformulation of Proposition 8.2 of [24].

**Proposition 11.** *There exists a Ritt power bounded operator  $T$  on some Hilbert space  $\mathcal{H}$  such that, taking  $\alpha = 0$ , (ii) of Proposition 7 does not imply (i).*

**Proof.** Let  $T$  be the operator defined in Proposition 8.2 of Le Merdy [24]. Then,  $T$  is power bounded and Ritt and has no fixed point. Assume that for every  $f \in \mathcal{H}$ , the condition  $\sum_{n \in \mathbb{N}} n \|T^n f\|_{\mathcal{H}}^2 < \infty$  implies that  $\sum_{n \in \mathbb{N}} T^n f$  converges in  $\mathcal{H}$ . Then, arguing as in the proof of Proposition 6, there exists  $C > 0$  such that  $\|\sum_{n \in \mathbb{N}} T^n f\|_{\mathcal{H}} \leq C \sum_{n \in \mathbb{N}} n \|T^n f\|_{\mathcal{H}}^2$ , whenever the right-hand side converges.

It follows from the proof of Proposition 8.2 of [24] that for every  $g \in \mathcal{H}$ ,  $\sum_{n \in \mathbb{N}} n \|(I - T)T^n g\|_{\mathcal{H}}^2 < \infty$ . Hence, for every  $g \in \mathcal{H}$ , the series  $\sum_{n \in \mathbb{N}} T^n (I - T)g$  converges in  $\mathcal{H}$ , say to  $h$ . Then,  $\|h\|_{\mathcal{H}} \leq C \sum_{n \in \mathbb{N}} n \|(I - T)T^n g\|_{\mathcal{H}}^2$  and  $(I - T)h = (I - T)g$ . Since  $T$  has no fixed point, we infer that  $g = h$  and  $\|g\|_{\mathcal{H}} \leq C \sum_{n \in \mathbb{N}} n \|(I - T)T^n g\|_{\mathcal{H}}^2$  for every  $g \in \mathcal{H}$ . But it is proved in [24] that this cannot hold.  $\square$

We now give a result in the spirit of Theorem 8 which may be seen as an extension of Corollary 9 to the case where  $\alpha = 1$  but in the specific situation where  $a_n \equiv 1$ .

**Proposition 12.** *Let  $P$  be a contraction on  $L^2(m)$ . Let  $f \in L^2(m)$  be such that*

$$(24) \quad \sum_{n \in \mathbb{N}} \frac{\log(n+1)}{n+1} \|P^n f\|_2^2 < \infty.$$

*Then  $\sum_{n \in \mathbb{N}} \frac{P^n f}{n+1}$  converges in  $L^2(m)$ . If moreover*

$$\sum_{n \in \mathbb{N}} \frac{\log(n+1) (\log \log \log(n+9))^2}{n+1} \|P^n f\|_2^2 < \infty,$$

*then  $\sum_{n \in \mathbb{N}} \frac{P^n f}{n+1}$  converges  $m$ -almost everywhere and the associated maximal function is in  $L^2(m)$ .*

**Proof.** The norm convergence follows easily from Proposition 5. We now give the main argument for the proof of the almost everywhere convergence. Assume (24). We have

$$\left\| \sum_{k=2^{2^n}}^{2^{2^{n+1}}-1} \frac{|P^k f|}{k} \right\|_{L^2(m)}^2 \leq C 2^{2^n} \|P^{2^{2^n}} f\|_{L^2(m)}^2.$$

It is not hard to see that (24) implies that  $\sum_{n \geq 0} 2^{2n} \|P^{2^{2n}}\|_{L^2(m)}^2 < \infty$ . Hence it suffices to prove the desired convergence of the series along the sequence  $(2^{2^n})_{n \in \mathbb{N}}$ . Then the rest of the proof is similar to that of Theorem 8.  $\square$

A natural question is the following : does there exist analogous results to, say, Proposition 7 for contractions of  $L^p$  spaces? For instance, by Cohen, Cuny and Lin [10], if  $P$  is a positive Ritt contraction of some  $L^p$ ,  $1 < p \leq 2$ , then the condition  $\sum_{n \in \mathbb{N}} (n+1)^{1-p\alpha} \|P^n f\|_p^p < \infty$  is sufficient for the convergence in  $L^p$  of  $\sum_{n \in \mathbb{N}} (n+1)^{-\alpha} P^n f$  (and the a.e. convergence holds as well). The approach used in the present paper partially works for Markov operators. However, it does not seem to allow one to extend the results of [10] to Markov operators. It would be interesting either to prove that  $L^p$  extensions are possible or to find an example where it cannot.

#### APPENDIX A

We made use of the following lemma which is related to Lemma 2.7 of Peligrad and Utev [27].

**Lemma 13.** *Let  $(V_n)_{n \geq 1}$  be a nonnegative subadditive sequence (i.e.  $V_{n+m} \leq V_n + V_m$  for every  $m, n \geq 1$ ). Then, for every  $q \geq 1$  and every  $p > 1$ , there exists  $C > 0$  such that  $\sum_{n \geq 1} \frac{\max_{1 \leq i \leq n} V_n^q}{n^p} \leq C \sum_{n \geq 0} \frac{V_{2^n}^q}{2^{np}}$ .*

**Proof.** The proof basically follows the arguments to prove Lemma 4.1 of [27]. We start with the following basic (dyadic) decomposition. For every  $r \geq 0$ , and every  $2^r \leq n \leq 2^{r+1} - 1$ , we have, by an easy induction,

$$V_n \leq V_{n-2^r} + V_{2^r} \leq \sum_{k=0}^r V_{2^k}.$$

Hence,

$$\sum_{n \geq 1} \frac{\max_{1 \leq i \leq n} V_n^q}{n^p} \leq \sum_{r \geq 0} \sum_{n=2^r}^{2^{r+1}-1} \frac{\max_{1 \leq i \leq n} V_n^q}{n^p} \leq \sum_{r \geq 0} \frac{1}{2^{(p-1)r}} \left( \sum_{k=0}^r V_{2^k} \right)^q$$

When  $q = 1$  the result follows. Assume that  $q > 1$ . Let  $0 < \varepsilon < p - 1$ . Using Hölder's inequality (with  $1/q + 1/q' = 1$ ), we have

$$\sum_{n \geq 1} \frac{\max_{1 \leq i \leq n} V_n^q}{n^p} \leq \sum_{r \geq 0} \frac{1}{2^{(p-1)r}} \left( \sum_{j=0}^r 2^{k\varepsilon q'} \right)^{q/q'} \left( \sum_{k=0}^r 2^{-k\varepsilon} V_{2^k}^q \right) \leq C \sum_{n \geq 0} \frac{V_{2^n}^q}{2^{np}}.$$

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