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Computationally efficient guaranteed cost control design for clustered networks. ⋆

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Abstract

We consider a clustered network where connections inside the cluster are dense and between clusters are sparse. This leads us to a classical decoupling into fast (intra-cluster) and slow (inter-cluster) dynamics. Our objective is to provide a computationally efficient method to design control strategies that guarantee a certain bound on the cost for each cluster. Basically, we design a composite synchronizing controller with two terms: one responsible for the intra-cluster synchronization and the other achieving the synchronization between clusters. The first one does not require much computational effort since an analytic expression describes it. The second term is designed through a satisfaction equilibrium approach. In other words, the internal (fast) and external (slow) controllers are independently designed, and they ensure a guaranteed satisfactory cost for each cluster. Moreover, we show that the internal control affects the cluster cost only for a short time period. Finally, numerical simulations illustrate the theoretical results.

Key words: Synchronization; Distributed Control; Networked System; Singular Perturbation; Time-Scale Modeling; Clustered Network

1 Introduction

Due to its application in various domains such as power systems [11], wireless sensor networks [23], social networks [30], and biology [4], analysis and control of network synchronization have received significant attention in the literature. A particular case of the networked system is *clustered network* where the network is divided into distinct groups (*clusters*) and the communication inside these groups are dense while the communication between these groups is sparse, see e.g. [20]. Networks with such properties also appear in various disciplines, such as energy systems [26], physics [2], [28], biological systems [13], social networks [7], [12] etc.

A majority of the publication on clustered networks propose an analysis of networks in consensus framework, see, e.g. [9], [8], [22]; while the problem of control design is less common in such a setting. A particular setup for synchronizing clustered networks using two time-scale is considered in [5], [25]. In [5], the authors expressed the consensus problem in terms of the synchronization problem and proposed a computationally efficient control design strategy using time-scale separation. A distributed two time-scale consensus algorithm is presented in [25] with an explicit formula for the convergence rate. However, none of the previously mentioned works consider the problem where control objective has a cost optimization requirement in addition to synchronization. On the one hand, these requirements are timely, and on the other, induce a high computational load, preventing the design of (sub-)optimal controllers in a centralized manner.

A major problem related to the synchronization of large-scale networks is the computational load associated with the design of effective controllers. The cost related to the synchronization is either considered to be global or not considered at all in most of the existing literature, for example in [15], [6]. In [15], the authors propose an energy-aware controller to minimize a global cost consisting of communication and controller parts. The control design with optimal global cost in the framework of multi-agent systems is presented in [6]. The computational effort required is very high, and the problem is NP-hard due to the information structure imposed by the graph.

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A computationally efficient decentralized control design approach is presented with global cost guarantees in [3]; however, the assumption of the same gain for all the agents in the network is quite restrictive. This obstacle in [3] was removed in [29] with individual cost guarantees for each agent. The proposed strategy works well with small-scale networks; however, the computational effort required to obtain the gain is huge for large-scale networks. In this paper, we aim to address this problem and provide an effective control design strategy for large-scale networks that reduces the computational effort while satisfying the performance guarantees.

One methodology to address the synchronization of the large-scale networks is by model reduction, which is based on *Singular Perturbation Theory* (*SPT*) that exploits the time-scale properties of clustered networks. The objective is to decrease the size of the system state while approximating its overall dynamic behavior. To the best of our knowledge, the first time-scale analysis of the networked system dates back to the 1980s, see [10], [11] . In these publications, the consensus problem for linear systems was analyzed using a singular perturbation approach, assuming that graphs were undirected. Later on, these results were extended in [8] to nonlinear networks. Furthermore, the results from [10], [11] for the case time-varying directed graphs are extended in [22]. In the synchronization framework, singular perturbation analysis of heterogeneous network with fixed topology is presented in [21], and for the time-varying case in [1].

In this paper, we consider the problem of a distributed controller design for a clustered network that ensures synchronization of the overall network while optimizing some cost functions. We provide an approach that significantly reduces the computational effort required to obtain the controller by exploiting the clustered network structure. The dense interconnections result in a fast convergence inside the cluster toward a local agreement and then slowly towards the global consensus. We use this property to divide the control design problem into computationally tractable sub-problems using Time-Scale Separation (TSS). The fast variables represent the synchronization error inside the clusters, whereas the slow variables represent the aggregate behavior of the agent states within each cluster.

The design of our controller is based on the two time-scale behavior of the clustered network. First, we perform the *Time-Scale Modeling (TSM)* to represent the network dynamics in *Standard Singular Perturbation From (SSPF)*. As a result of TSM, internal control is associated with fast dynamics, while external control is associated with slow dynamics. Then, using time-scale separation, we decouple the dynamics into slow and fast subsystems, and this decoupling allows us to independently design the internal and the external controller. The internal controller, associated with the fast dynamics, is designed to achieve the consensus inside

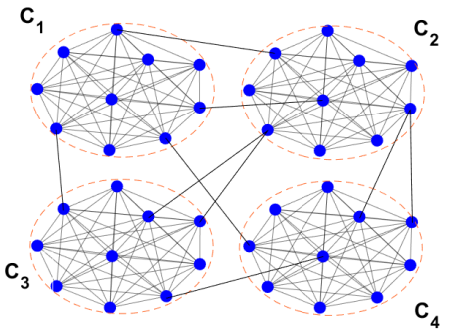

Fig. 1. A network partitioned into 4 clusters.

the cluster while minimizing an internal cost. The convergence of agents inside the cluster towards the consensus is fast; the cluster roughly merges into a single node after the fast transient and external behavior is defined by the slow dynamics. The long-term behavior of the network depends on this slow dynamics. Finally, the external control is designed to synchronize all the clusters based on the satisfaction equilibrium approach [29], i.e., external control is designed such that the external cost associated with each cluster is bounded under a given threshold. In addition, we also provide an approximation of the cluster cost as a sum of the internal and external costs associated with internal and external control, respectively.

The main contributions of this paper can be outlined as follows, (1) we formulate a singular perturbation model of the clustered network using time-scale modeling in the synchronization framework, where each agent has its individual dynamics, (2) based on the obtained model in (1), we propose a computationally efficient sub-optimal control design scheme that synchronizes the network by splitting the controllers into two parts corresponding to the slow and fast dynamics, and (3) finally, we provide an approximation of the cluster cost bound induced by the time-scale separation.

The remainder of the paper is organized as follows. The model and the control objectives are stated in Section 2. The time-scale modeling and decoupling into slow and fast dynamics using time-scale separation is described in detail in Section 3. Then, the internal and external controller design procedures are developed in Section 4. In Section 5, we provide the global system analysis and an approximation of the cluster cost. Finally, numerical results are presented in Section 6 before concluding in Section 7. The proofs are presented in the Appendix to make the paper easily readable.

1.1 Notation and Preliminaries

The symbol $\otimes$ represents the Kronecker product. Let $(x, y) \in \mathbb{R}^{n+m}$ stand for $[x^\top y^\top]^\top$. The identity matrix of size n is denoted by I_n and by $\mathbb{1}_n \in \mathbb{R}^n$, the column vector whose components are all 1. For a matrix

$A \in \mathbb{R}^{m\times n}$, $A^\top$ denotes the transpose of A. For a vector $x \in \mathbb{R}^n$, we denote by $\|x\|_2 := \sqrt{x^\top x}$ its Euclidean norm and, for a matrix A, $\|A\|_2 := \sqrt{\lambda_{max}(A^\top A)}$. For a square matrix $M \in \mathbb{R}^{n\times n}$, let $\lambda_{min}(M)$ and $\lambda_{max}(M)$ be the minimum and the maximum eigenvalue, respectively. The *measure of the square matrix* M is defined as $\nu(M) = \frac{1}{2}\lambda_{max}(M + M^\top)$. We said that a matrix $M \in \mathbb{R}^{n\times n}$ is orthonormal if $M^\top M = MM^\top = I_n$. We denote by $M_{-k} \in \mathbb{R}^{(n-1)\times(n-1)}$ the matrix M with its k-th row and column removed. By $B = \text{diag}(B_1, ..., B_N)$, we denote a block-diagonal matrix with the entries B_1,...,B_N on the diagonal and $B_{-k} := \text{diag}(B_1, ..., B_{k-1}, B_{k+1}, ..., B_N)$ the block-matrix with the k-th block removed. A function $f(t) : (0, \infty) \to \mathbb{R}^n$ is $\mathcal{O}(\epsilon)$ iff there exists constant $c > 0$ such that $\|f(t)\| \leq c\epsilon$. A connected, undirected graph is represented as $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, n\}$ is the agent set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. The adjacency matrix $\mathcal{A} = (a_{ij})_{n\times n}$ is defined as: $a_{ij} \neq 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian of the graph $\mathcal{G}$ is defined as $\mathcal{L}$, has $-a_{ij}$ off-diagonal elements and $\sum_{j=1}^n a_{ij}$ diagonal ones. Let $\mathbf{G} = (\mathcal{V}, \{\mathcal{K}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$, be a strategic form game, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of players (agents), $\mathcal{K}_i$ is the set of strategies of the player i , and u_i is a utility function of the player i and $\{f_1, \ldots, f_n\}$ be n set-valued satisfaction functions. Then the strategy profile $\mathcal{K}^* = (\mathcal{K}_1^*, \ldots, \mathcal{K}_n^*)$ is a *Satisfaction Equilibrium (SE)* if and only if, for all $i \in \mathcal{V}$, we have, $\mathcal{K}_i^* \in f_i(\mathcal{K}_{-i}^*)$, where $\mathcal{K}_{-i}^* := (\mathcal{K}_1^*, \ldots, \mathcal{K}_{i-1}^*, \mathcal{K}_{i+1}^*, \ldots, \mathcal{K}_n^*)$ denotes the reduced profile with the component $\mathcal{K}_i^*$ removed.

2 Problem Statement

2.1 Model Description

Consider a network of n agents partitioned into m non-empty clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m \subset \mathcal{V}$. Clustered network refers to a network that is divided into distinct groups of agents having dense connection structure, whereas the connections between the clusters are sparse. Let us denote by $\mathcal{M} := \{1, 2, \ldots, m\}$, the set of clusters while n_k represents the cardinality of the cluster $\mathcal{C}_k$ and $n = \sum_{k=1}^m n_k$. Each agent in the network is identified by a couple $(k, i) \in \mathcal{C}_k$, where, k refers to the cluster $\mathcal{C}_k$ and i the index of the agent in the cluster $\mathcal{C}_k$. The notation $(k, j) \in \mathcal{N}_{k,i}$ represents the neighbors of the agent (k, i) in the same cluster $\mathcal{C}_k$. To each agent $(k, i) \in \mathcal{C}_k$, $k \in \mathcal{M}$, one assigns a state $x_{k,i} \in \mathbb{R}^{n_x}$ whose dynamics is

$$\dot{x}_{k,i} = Ax_{k,i} + Bu_{k,i}, \tag{1}$$

where $u_{k,i} \in \mathbb{R}^{n_u}$, $A \in \mathbb{R}^{n_x\times n_x}$ and $B \in \mathbb{R}^{n_x\times n_u}$.

For each cluster $\mathcal{C}_k$, let $x_k := (x_{k,1}, \ldots, x_{k,n_k}) \in \mathbb{R}^{n_k\cdot n_x}$ be the cluster state and $u_k := (u_{k,1}, \ldots, u_{k,n_k}) \in \mathbb{R}^{n_k\cdot n_u}$ the cluster control. Thus, the cluster dynamics takes the following form

$$\dot{x}_k = (I_{n_k} \otimes A)x_k + (I_{n_k} \otimes B)u_k, \quad \forall k \in \mathcal{M}. \tag{2}$$

Moreover, in the presence of clusters, the Laplacian of the network can be written as $\mathcal{L} := \mathcal{L}^{int} + \mathcal{L}^{ext}$. The internal Laplacian of the network $\mathcal{L}^{int} := \text{diag}(\mathcal{L}_1^{int}, ..., \mathcal{L}_m^{int})$ is a block-diagonal matrix, with each block $\mathcal{L}_k^{int}$ referring to the Laplacian of the cluster $\mathcal{C}_k$ excluding the external connections. The external Laplacian $\mathcal{L}^{ext}$ represents the connections between agents from different clusters.

The next two assumptions are the necessary condition to ensure the synchronization of the network and the presence of the cluster in the network, respectively.

Assumption 1 *The graph of clusters is connected.*

Due to the dense communications between the agents inside the clusters, we make the following assumption on the communications inside the cluster.

Assumption 2 *The internal graphs are complete for all clusters.*

Remark 1 *Under Assumption 2, the zero eigenvalue of the internal Laplacian $\mathcal{L}_k^{int}$ is simple and all the non-zero eigenvalues are n_k for all $k \in \mathcal{M}$. This assumption is imposed only for the control design purpose, and the obtained controller can be implemented even if Assumption 2 does not hold. The main advantage of this assumption is that it greatly reduces the computational effort required to obtain the control.*

2.2 Control Design Outline

We consider the problem of network synchronization, and the network is said to be asymptotically synchronized for all $(k, i) \in \mathcal{C}_k$, $(l, j) \in \mathcal{C}_l$ and $k, l \in \mathcal{M}$, when $\lim_{t\to+\infty} \|x_{k,i}(t) - x_{l,j}(t)\| = 0$. Our first objective is to design a distributed control synchronizing the network while minimizing some cost function.

If the network dimension is large, the control design problem under certain cost constraints becomes difficult as the computational complexity increases with an increase in network dimension. To simplify the calculations and minimize the computational efforts, for each cluster $\mathcal{C}_k, k \in \mathcal{M}$, we propose a composite control of the form:

$$u_k := u_k^{int} + u_k^{ext}, \quad \forall k \in \mathcal{M}, \tag{3}$$

where $u_k^{int} := (u_{k,1}^{int}, ..., u_{k,n_k}^{int})$, $u_k^{ext} := (u_{k,1}^{ext}, ..., u_{k,n_k}^{ext})$ and

$$\begin{cases} u_{k,i}^{int} := -K_k^{int} \sum_{(k,j)\in\mathcal{N}_{k,i}} (x_{k,i} - x_{k,j}), \\ u_{k,i}^{ext} := -K_k^{ext} \sum_{(l,p)\in\mathcal{N}_{k,i}} (x_{k,i} - x_{l,p}), \end{cases} \tag{4}$$

where $K_k^{int}, K_k^{ext} \in \mathbb{R}^{n_u\times n_x}$.

The notation $(l, p) \in \mathcal{N}_{k,i}$ indicates the neighbors belonging to a different cluster, that is $l \neq k$. The internal control u_k^{int}

is the effort required to achieve local agreement, whereas the external control u_k^{ext} is the energy necessary to synchronize the agents between the clusters.

Such a decomposition of the control allows us to decouple the overall optimization in the following way. We split the cost functions corresponding to the cluster into the sum of internal, external, and cross term. First, we define the cluster cost, J_k associated with each cluster $\mathcal{C}_k, k \in \mathcal{M}$, is defined as a

$$J_k = \int_0^{+\infty} x_k^\top(t)(\mathcal{L}_k^{int} \otimes I_{n_x})x_k(t) + x^\top(t)(\mathcal{L}_k^{ext} \otimes I_{n_x})x(t) + u_k^\top(t)(I_{n_k} \otimes R_k)u_k(t)\,\mathrm{d}t, \tag{5}$$

where the internal Laplacian $\mathcal{L}_k^{int} \in \mathbb{R}^{n_k \times n_k}$ captures the connections inside the cluster $\mathcal{C}_k$, and the external Laplacian $\mathcal{L}_k^{ext} \in \mathbb{R}^{n \times n}$ expresses the external connections between $\mathcal{C}_k$ and the neighboring clusters. Then, substituting the composite control (3), we recast the cost function (5) as the sum of internal, external and a cross term as

$$J_k = J_k^{int} + J_k^{ext} + J_k^{cross}, \tag{6}$$

where,
$$\begin{aligned} J_k^{int} &= \int_0^{+\infty} x_k^\top(t)(\mathcal{L}_k^{int} \otimes I_{n_x})x_k(t) + u_k^{int\top}(t)(I_{n_k} \otimes R_k)u_k^{int}(t)\,\mathrm{d}t, \\ J_k^{ext} &= \int_0^{+\infty} x^\top(t)(\mathcal{L}_k^{ext} \otimes I_{n_x})x(t) + u_k^{ext\top}(t)(I_{n_k} \otimes R_k)u_k^{ext}(t)\,\mathrm{d}t, \\ J_k^{cross} &= 2\int_0^{+\infty} u_k^{ext\top}(t)(I_{n_k} \otimes R_k)u_k^{int}(t)\,\mathrm{d}t\,. \end{aligned} \tag{7}$$

In this way, we replace the original problem of optimizing the cost function for the overall network given (5) with the problem of optimization of the internal and the external cost. This way, the initial optimization problem is recast as a problem of finding the internal (K_k^{int}) and external (K_k^{ext}) control gains. The internal and external gains are designed independently, and the obtained internal gain is optimal while the external gain is sub-optimal because the internal cost is minimized and the external cost is capped below a certain threshold, respectively.

The next objective is to bound the total cluster cost with the sum of internal (J_k^{int}) and external (J_k^{ext}) cost and a constant term. The cross term in the equation (7) can be bounded by a constant term multiplied by the norm of the initial conditions (See Theorem 2).

To solve this problem, we propose an approach based on time-scale separation that we describe in the following section.

3 Time-scale Separation

This section provides a procedure to decouple the closed-loop network dynamics into two subsystems, evolving on different time-scales. First, we perform a coordinate transformation to exhibit the collective dynamics of the network: the average and the synchronization error dynamics. Then, we apply the TSS to decouple the collective dynamics into slow and fast subsystems. In two time-scale, the slow variable corresponds to the average of the agent's state while the fast variable corresponds to the synchronization error.

3.1 Coordinate Transformation

Following from [24], we introduce the coordinate transformation for the cluster $\mathcal{C}_k$ based on the internal Laplacian $\mathcal{L}_k^{int}$. For a connected graph, the Jordan decomposition of the symmetric Laplacian matrix is,

$$\mathcal{L}_k^{int} = T_k \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_k^{int} \end{bmatrix} T_k^\top, \quad \forall k \in \mathcal{M}, \tag{8}$$

where $T_k \in \mathbb{R}^{n_k \times n_k}$ is an orthonormal matrix and $\Lambda_k^{int} = \mathrm{diag}(\lambda_{k,2}^{int}, \ldots, \lambda_{k,n_k}^{int}) \in \mathbb{R}^{(n_k-1)\times(n_k-1)}$ collects the $n_k - 1$ positive eigenvalues of $\mathcal{L}_k^{int}$. Moreover, the matrix T_k can be expressed as

$$T_k = \begin{bmatrix} v_{k,1} & V_k \end{bmatrix}, \quad \forall k \in \mathcal{M}, \tag{9}$$

where $v_{k,1}^\top = \frac{1}{\sqrt{n_k}}\mathbb{1}_{n_k}^\top$ is the eigenvector associated with the 0 eigenvalue and the matrix $V_k \in \mathbb{R}^{n_k \times (n_k-1)}$ contains the eigenvectors corresponding to the nonzero eigenvalues of $\mathcal{L}_k^{int}$. Furthermore, it can be verified that, $v_{k,1}^\top V_k = 0$ and $V_k^\top V_k = I_{n_k-1}$.

Now, we define the coordinate transformation as

$$\overline{x}_k := \begin{bmatrix} y_k \\ \xi_k \end{bmatrix} = (\frac{1}{\sqrt{n_k}} T_k^\top \otimes I_{n_x})x_k, \quad \forall k \in \mathcal{M}. \tag{10}$$

Then, from (9) and (10), the change of variables yields, for all $k \in \mathcal{M}$,

$$y_k := \left(\frac{\mathbb{1}_{n_k}^\top}{n_k} \otimes I_{n_x}\right) x_k =: H_k x_k \in \mathbb{R}^{n_x} \tag{11}$$

$$\xi_k := \left(\frac{V_k^\top}{\sqrt{n_k}} \otimes I_{n_x}\right) x_k =: Z_k x_k \in \mathbb{R}^{(n_k-1).n_x}. \tag{12}$$

The first component y_k corresponds to the average of the respective agents' states in the cluster $\mathcal{C}_k$. The second component, ξ_k corresponds to the synchronization error. Since the matrix T_k is orthonormal i.e., $T_k^\top = T_k^{-1}$, the inverse of the transformation (10), for all $k \in \mathcal{M}$ yields,

$$x_k = (\mathbb{1}_{n_k} \otimes I_{n_x})y_k + (\sqrt{n_k}V_k \otimes I_{n_x})\xi_k =: \widetilde{H}_k y_k + \widetilde{Z}_k \xi_k. \tag{13}$$

In vector form, let $x := (x_1, \ldots, x_m) \in \mathbb{R}^{n.n_x}$, $y := (y_1, \ldots, y_m) \in \mathbb{R}^{m.n_x}$ and $\xi := (\xi_1, \ldots, \xi_m) \in \mathbb{R}^{(n-m).n_x}$, respectively. Then for overall network, we obtain

$$y = Hx, \quad \xi = Zx \quad \text{and} \quad x = \widetilde{H}y + \widetilde{Z}\xi, \tag{14}$$

where $H = \text{diag}(H_1, ..., H_m)$ and $Z = \text{diag}(Z_1, ..., Z_m)$ and $\widetilde{H} = \text{diag}(\widetilde{H}_1, ..., \widetilde{H}_m)$ and $\widetilde{Z} = \text{diag}(\widetilde{Z}_1, ..., \widetilde{Z}_m)$. Now, we recast the overall network dynamics in terms of the new coordinate variables. The overall network dynamics in the presence of the control (3) is

$$\begin{aligned}\dot{x} = ((I_n \otimes A) - (I_n \otimes B)K^{int}(\mathcal{L}^{int} \otimes I_{n_x}) \\ - (I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x}))x,\end{aligned} \tag{15}$$

where $K^{int} = \text{diag}((I_{n_1} \otimes K_1^{int}), \ldots, (I_{n_m} \otimes K_m^{int}))$ and $K^{ext} = \text{diag}((I_{n_1} \otimes K_1^{ext}), \ldots, (I_{n_m} \otimes K_m^{ext}))$. Then, using (14), the overall dynamics (15) is recast in new coordinates as follows,

$$\begin{cases} \dot{y} = \bar{A}_{11}y + \bar{A}_{12}\xi, \\ \dot{\xi} = \bar{A}_{21}y + (\bar{A}_{22}^1 + \bar{A}_{22}^2)\xi, \end{cases} \tag{16}$$

where

$$\begin{cases} \bar{A}_{11} = ((I_m \otimes A) - H(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{H}), \\ \bar{A}_{12} = -H(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{Z}, \\ \bar{A}_{21} = -Z(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{H}, \\ \bar{A}_{22}^1 = -Z(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{Z}, \\ \bar{A}_{22}^2 = ((I_{n-m} \otimes A) - (I_n \otimes B)K_{n-m}^{int}(\Lambda^{int} \otimes I_{n_x})), \end{cases} \tag{17}$$

and $K_{n-m}^{int} = \text{diag}((I_{n_1-1} \otimes K_1^{int}), \ldots, (I_{n_m-1} \otimes K_m^{int}))$ and $\Lambda^{int} = \text{diag}(\Lambda_1^{int}, \ldots, \Lambda_m^{int})$. We recall that y and ξ corresponds to the average of the agents' states and the synchronization error, respectively.

3.2 Network Dynamics in Two time-scale

In the absence of agents' individual dynamics in [10], [22], the consensus dynamics is expressed in *Standard Singular Perturbation From (SSPF)* based only on the density of connections inside and between the clusters. This was sufficient because the convergence required in [10], [22] is dictated only by the density of the connections in the consensus framework. However, in our case, the convergence towards synchronization manifold depends also on the control gains K^{int} and K^{ext} and we need to take them into account for the time-scale analysis i.e., while defining the network parameter (ϵ).

Thus, to study the time-scale behavior and analyze the synchronizing behavior, we define the network parameters as follows,

$$\begin{cases} \mu^{ext} := \|(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\|, \\ \mu^{int} := \min\limits_{k \in \mathcal{M}} \|(\Lambda_k^{int} \otimes BK_k^{int})\|, \\ \epsilon := \frac{\mu^{ext}}{\mu^{int}}. \end{cases} \tag{18}$$

The network parameter ϵ is the ratio of the strength of the controls between and within the clusters. It's worth noting, in our case, the network parameter ϵ can be tuned by the choice of the control gains.

Remark 2 *For the rest of this section, we assume that ϵ is small enough such that time-scale separation occurs, and the control design presented in the following section will serve, among others to ensure this property.*

Upon analyzing the orders of the state matrices of the collective dynamics (16), knowledge of the order of matrix A is crucial for transformation in SSPF. In the following, we state the following assumption on the order of the matrix A which is necessary for representing the dynamics (16) in SSPF.

Assumption 3 *The state matrix A satisfies the following*

$$\|A\| \leq \mathcal{O}(\mu^{ext}).$$

We here note that since μ^{ext} depends on K^{ext}, we can always choose K^{ext} sufficiently large such that the assumption 3 is satisfied. In the following lemma, we analyze the order of the matrices in equation (17) under Assumption 3.

Lemma 1 *Under Assumption 3, the matrices in* (17) *satisfy the following conditions,*

- $\|\bar{A}_{11}\|$, $\|\bar{A}_{12}\|$, $\|\bar{A}_{21}\|$, $\|\bar{A}_{22}^1\|$ *are of order* $\mathcal{O}(\epsilon\mu^{int})$,
- $\|\bar{A}_{22}^2\|$ *is of order* $\mathcal{O}(\mu^{int})$.

PROOF. See Appendix. ■

As a consequence of the Assumption 3, all the matrices in (17) are of order $\mathcal{O}(\epsilon\mu^{int})$ except $\bar{A}_{22}^2$, which is of order $\mathcal{O}(\mu^{int})$. Since the dynamics of the variables y and ξ are dominated by the matrix $\bar{A}_{11}$ and $\bar{A}_{22}^2$, the variables y and ξ behaves as a slow and fast variables, respectively.

Now, to reveal the TSS, following the idea of [10], we rescale the time with μ^{int} to obtain a fast time-scale as $t_f = \mu^{int}t$, and a slow time-scale $t_s = \epsilon t_f$. This allows us to represent the overall dynamics (16) in SSPF as follows,

$$\frac{dy}{dt_s} = A_{11}y + A_{12}\xi, \tag{19a}$$

$$\epsilon\frac{d\xi}{dt_s} = \epsilon A_{21}y + (\epsilon A_{22}^1 + A_{22}^2)\xi. \tag{19b}$$

where,

$$\begin{aligned} A_{11} &= \frac{\bar{A}_{11}}{\epsilon\mu^{int}}, \qquad A_{12} = \frac{\bar{A}_{12}}{\epsilon\mu^{int}}, \qquad A_{21} = \frac{\bar{A}_{21}}{\epsilon\mu^{int}}, \\ A_{22}^1 &= \frac{\bar{A}_{22}^1}{\epsilon\mu^{int}}, \qquad A_{22}^2 = \frac{\bar{A}_{22}^2}{\mu^{int}}. \end{aligned} \tag{20}$$

Next, we analyze the slow and fast dynamics of the singularly perturbed system (19).

3.3 Slow Dynamics

To define the slow dynamics of the system (19), we follow the standard approach of singular perturbation analysis [17]. Setting $\epsilon = 0$ in (19), we obtain that equation (19b) degenerates into equation $\xi_s(t_s) = 0$. Substituting this into equation (19a), we obtain the slow dynamics as

$$\frac{dy_s}{dt_s} = A_{11} y_s. \tag{21}$$

where y_s and ξ_s are the slow parts of the variables y and ξ, respectively. Equivalently, since $t_s = \epsilon t_f = \epsilon \mu^{int} t$, it yields,

$$\dot{y}_s(t) = (I_m \otimes A) y_s(t) + (I_m \otimes B) u_s(t), \tag{22}$$

where $u_s(t) = -HK^{ext}(\mathcal{L}^{ext} \otimes I_{n_x}) \widetilde{H} y_s(t)$. Notice that in our setting the slow dynamics (22) represents the collective behavior of the cluster and it may or may not be stable.

3.4 Fast Dynamics

Now, representing the dynamics (19) in fast time scale t_f and setting $\epsilon = 0$, we have $dy_f/dt_f = 0$ and we obtain the fast dynamics as follows,

$$\frac{d\xi_f}{dt_f} = A_{22}^2 \xi_f. \tag{23}$$

where y_f and ξ_f are fast parts of the corresponding variable in (19). The fast dynamics (23) in original time-scale t is

$$\dot{\xi}_f(t) = (I_{n-m} \otimes A) \xi_f(t) + (I_{n-m} \otimes B) u_f(t), \tag{24}$$

where $u_f(t) = -K_{n-m}^{int} (\Lambda^{int} \otimes I_{n_x}) \xi_f(t)$.

The fast dynamics (24) corresponds to the intra-cluster dynamics, and hence the dynamics is dominated by the internal gain and the eigenvalues of the intra-cluster Laplacian. With the suitable choice of the internal gain K_k^{int}, the system (24) is exponentially stable.

Remark 3 *We note that the stabilization of the synchronization error dynamics, i.e., the stabilization of dynamics ξ implies the synchronization inside the cluster.*

Note that we use slow (t_s) and fast (t_f) time-scales for the analysis while the original time-scale (t) for the control design. This is possible because the transformations are invertible, and it can be verified by the definition of t_f and t_s.

3.5 Singular Perturbation Approximation

Now, we provide an approximation of the original system by the reduced-order subsystems in the following theorem. The proof follows from Theorem 5.1, [17]. But before stating the result, we make the following assumption on the existence of the control gains.

Assumption 4 *There exists an internal gain K^{int} and an external gain K^{ext} such that the slow dynamics* (22) *is synchronized and the fast dynamics* (24) *is stabilized.*

Remark 4 *Although we assume the existence of the synchronizing internal and external gain, it will be ensured by design in the next section that such gains exist.*

Theorem 1 *Under the Assumption 3 and 4, if Re $\lambda(A_{22}^2) < 0$, there exists a $\epsilon^* > 0$ such that, for all $\epsilon \in (0, \epsilon^*]$, the original system* (19) *starting from any bounded initial conditions y_0 and ξ_0, is approximated for all finite time $t \geq t_0$ by*

$$\begin{cases} y = y_s(t_s) + \mathcal{O}(\epsilon) \\ \xi = \xi_f(t_f) + \mathcal{O}(\epsilon), \end{cases} \tag{25}$$

where $y_s \in \mathbb{R}^{m.n_x}$ and $\xi_f \in \mathbb{R}^{(n-m).n_x}$ are the respective slow and the fast variables.

PROOF. See Appendix. ■

In the next section, we present the control design strategy i.e., the design of gains K^{int} and K^{ext}, to stabilize the fast subsystems and synchronize the slow subsystems, respectively.

4 Control Design Scheme

In this section, we present a controller design strategy for the system (16). Using the idea of timescale separation, we split the design procedure into two parts corresponding to the internal and the external control.

First, based on the fast dynamics (24) we design an internal control using the local information that ensures the synchronization inside the cluster. Then, we use the slow dynamics (22) for the design of external control to achieve the synchronization between the clusters. While the internal controller is optimal, the external control is designed to ensure the cost is below a given threshold. Finally, Theorem 1 is used to justify such a separation of the system analysis in two steps and to approximate overall network behavior in terms of fast and slow dynamics.

In what follows, we first address the internal control design and give an analytical gain expression for the case of complete graph inside clusters. The fast dynamics obtained after the time-scale separation represent the synchronization dynamics of an isolated cluster. Under assumption 2, i.e., the graph is complete, this dynamics can be further decoupled.

4.1 Internal (Fast) Control Design

As the fast variable ξ_f is an approximation of the synchronization error ξ inside the clusters, it is still relevant to consider the fast subsystems (24) for the internal control design. We denote by $\xi_{f,k} \in \mathbb{R}^{(n_k-1).n_x}$ the component of

$\xi_f := (\xi_{f,1}, ..., \xi_{f,m})$ corresponding to the k-th cluster. For each cluster $\mathcal{C}_k$, for $k \in \mathcal{M}$, we have the following dynamics

$$\begin{cases} \dot{\xi}_{f,k}(t) &= (I_{n_k-1} \otimes A)\xi_{f,k}(t) + (I_{n_k-1} \otimes B)u_{f,k}(t), \\ u_{f,k}(t) &= -(\Lambda_k^{int} \otimes K_k^{int})\xi_{f,k}(t). \end{cases} \tag{26}$$

The cluster cost associated with the cluster $\mathcal{C}_k$ takes the form

$$J_{f,k} = \int_0^{+\infty} \xi_{f,k}^\top (\Lambda_k^{int} \otimes I_{n_x})\xi_{f,k} + u_{f,k}^\top (I_{n_k-1} \otimes R_k)u_{f,k} \, \mathrm{dt}\,. \tag{27}$$

Instead of considering the internal cluster cost (7), we approximate the internal cost by the cost function (27) and the validity of the approximation is justified in the Proposition 3. Since, the matrices in equations (26) and (27) have block-diagonal form, they can decoupled into $n_k - 1$ independent subsystems. For each cluster $\mathcal{C}_k$, similarly to ξ_k defined in equation (12), let us denote the fast subsystems and the associated control by $\xi_{f,k} := (\xi_{f,k,1}, ..., \xi_{f,k,n_k-1})$ and $u_{f,k} := (u_{f,k,1}, ..., u_{f,k,n_k-1})$, respectively. Then, for $i = 1, ..., n_k - 1$ and for all $k \in \mathcal{M}$, the dynamics are

$$\begin{cases} \dot{\xi}_{f,k,i}(t) &= A\xi_{f,k,i}(t) + n_k B u_{f,k,i}(t), \\ u_{f,k,i}(t) &= -K_k^{int}\xi_{f,k,i}(t), \end{cases} \tag{28}$$

and the associated individual cost is

$$J_{f,k,i} = \int_0^{+\infty} n_k \xi_{f,k,i}^\top \xi_{f,k,i} + n_k^2 u_{f,k,i}^\top R_k u_{f,k,i} \, \mathrm{dt}\,. \tag{29}$$

Thus, the cost (27) can be expressed as the sum of individual costs (29) as follows, $J_{f,k} = \sum_{i=1}^{n_k-1} J_{f,k,i}, \quad \forall k \in \mathcal{M}$.

Remark 5 *The decoupling of dynamics* (26) *into* $n_k - 1$ *subsystems* (28) *is not only limited to all-to-all connections. In the case, where we know the eigenvalues of the Laplacian or the Laplacian eigenvalues can be characterized in terms of* n_k *(for example, star graph), similar decoupling can be achieved.*

Remark 6 *It is noteworthy that the gain* K_k^{int} *is the same for all the agents belonging to the same cluster* $\mathcal{C}_k$*. As a result, the rewriting of* (27) *as a sum of individual cost* (29) *reduces the computational effort for the control design. Indeed, one can solve only one optimization problem* (28)-(29) *for each cluster and it is equivalent to optimizing the cluster cost* (27)*.*

Next, we show that the system (28) is stabilizable with a simple linear controller, while we recall that the system (28) corresponds to fast dynamics of our original system. Finally, we apply the LQ-control [16] to stabilize (28) while minimizing the cost (29).

Lemma 2 *Consider the system* (28)*, under assumption 1, if the pair* (A, B) *is stabilizable and* $(A, (R_k)^{1/2})$ *is detectable, then for every* $k \in \mathcal{M}$*, the system* (28) *is stabilizable while minimizing the cost* (29) *by a controller* $u_{f,k,i}(t) = -K_k^{int}\xi_{f,k,i}(t)$ *with the gain*

$$K_k^{int} = \frac{R_k^{-1}}{n_k} B^\top P_k^{int}, \quad k \in \mathcal{M}, \tag{30}$$

where P_k^{int} *is the solution of the Algebraic Riccati Equation (ARE)*

$$P_k^{int} A + A^\top P_k^{int} - P_k^{int} B R_k^{-1} B^\top P_k^{int} + n_k I_{n_x} = 0. \tag{31}$$

From Lemma 2, we observe that the fast dynamics (28) is exponentially stable i.e., $\xi_f(t) \to 0$ as $t \to \infty$ and we pass to the design of the external controller.

4.2 External (Slow) Control Design

In this sub-section, we present the external controller design based on the slow dynamics (22). To achieve the synchronization between the clusters, we propose a method based on [29]. First, the synchronization problem is transformed into a stabilization problem using a change of variable. Then, we design the control to stabilize the system while upper bounding the associated cost.

Recall that if the clusters are synchronized, each cluster behave like a single node, and the number of nodes representing the external network equals the number of clusters. Thus, the external graph of agents between clusters is only connected, and hence the standard optimization or the optimal control approaches cannot be applied directly. In this context, inspired by the notion in game theory, we use the satisfaction equilibrium approach, and satisfaction games [27]. A set of actions are said to be in satisfaction equilibrium when the individual cost for each agent is upper-bounded by a given threshold.

4.2.1 Average Dynamics

The slow dynamics obtained after time-scale separation in equation (22) defines the dynamics of the average of each cluster. Following from equation (22), the average dynamics can be written as

$$\dot{y}_s(t) = ((I_m \otimes A) - (I_m \otimes B)\overline{K}^{ext}(\overline{\mathcal{L}}^{ext} \otimes I_{n_x}))y_s(t), \tag{32}$$

where $\overline{K}^{ext} = \mathrm{diag}\,(K_1^{ext}, ..., K_m^{ext})$ is the external gain and $(\overline{\mathcal{L}}^{ext} \otimes I_{n_x}) = H(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{H}$ with the following form

$$\overline{\mathcal{L}}^{ext} = \begin{pmatrix} \sum_{l=2}^{m} \frac{a_{1l}^{ext}}{n_1} & -\frac{a_{12}^{ext}}{n_1} & \cdots & -\frac{a_{1m}^{ext}}{n_1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{m1}^{ext}}{n_m} & -\frac{a_{m2}^{ext}}{n_m} & \cdots & \sum_{l=1}^{m-1} \frac{a_{ml}^{ext}}{n_m} \end{pmatrix} \in \mathbb{R}^{m \times m},$$

is the average Laplacian matrix related to (32). In average Laplacian, $\overline{\mathcal{L}}^{ext}$ the diagonal elements represent the total number of external connections from a cluster $k \in \mathcal{M}$ to the rest of the network and the non-diagonal entries a^{ext}_{kl} represents the total number of connections between cluster $\mathcal{C}_k$ and $\mathcal{C}_l$.

Let us denote by $y_{s,k} \in \mathbb{R}^{n_x}$ the k-th component of the variable y_s. Then, the average dynamics of each cluster $\mathcal{C}_k$, for $k \in \mathcal{M}$, based on equation (32) is

$$\begin{cases} \dot{y}_{s,k} &= Ay_{s,k} + Bu^{ext}_{s,k}, \\ u^{ext}_{s,k} &= -K^{ext}_k \sum\limits_{l \in \mathcal{N}_{\mathcal{C}_k}} \frac{a^{ext}_{kl}}{n_k}(y_{s,k} - y_{s,l}) \end{cases} \tag{33}$$

where, $u^{ext}_{s,k}$ can be viewed as the control on the cluster level, since it represents the sum of the individual controllers. For system (33) we define the average cost for each cluster $\mathcal{C}_k, k \in \mathcal{M}$, as

$$\overline{J}^{ext}_k = \int_0^{+\infty} \sum_{l \in \mathcal{N}_{\mathcal{C}_k}} \frac{a^{ext}_{kl}}{n_k}(y_{s,k} - y_{s,l})^2 + n_k \sum_{i=1}^{n_k} \widehat{u}^{ext\top}_{k,i} R_k \widehat{u}^{ext}_{k,i} \, \mathrm{dt} \tag{34}$$

where

$$\widehat{u}^{ext}_{k,i} := -K^{ext}_k \sum_{l \in \mathcal{N}_{\mathcal{C}_k}} \frac{a^{ext}_{(k,i)\leftrightarrow C_l}}{n_k}(y_{s,k} - y_{s,l}) \quad \forall i \in C_k, \tag{35}$$

and $a^{ext}_{(k,i)\leftrightarrow \mathcal{C}_l}$ is the total number of connections between the i-th agent belonging to $\mathcal{C}_k$ and the cluster $\mathcal{C}_l$ and clearly $a^{ext}_{(k,i)\leftrightarrow \mathcal{C}_l} \leq n_k$. The control $\widehat{u}^{ext}_{k,i}$ is the external control (4) expressed in the average variable y_s. In addition, we have the relation $u^{ext}_{s,k} = \sum_{i=1}^{n_k} \widehat{u}^{ext}_{k,i}$ and $a^{ext}_{kl} = \sum_{i=1}^{n_k} a^{ext}_{(k,i)\leftrightarrow \mathcal{C}_l}$.

Notice that the average cost (34) is different from the external cost function that appears in equation (7) in several ways:

- the average variable $y_{s,k}$ is used instead of the original state variables x_k for each cluster, and
- although the clusters have merged into a single node, the agents still apply the individual control (4) rather than the average control (33). Thus, we express the individual external control (4) in average variables y_s in equation (35) and define the average cost (34) in terms of the original control. It is possible to define the cost function as a function of average control $u^{ext}_{s,k}$ as follows,

$$\overline{J}^{ext}_k = \int_0^{+\infty} \sum_{l \in \mathcal{N}_{\mathcal{C}_k}} \frac{a^{ext}_{kl}}{n_k}(y_{s,k} - y_{s,l})^2 + u^{ext\top}_{s,k} R_k u^{ext}_{s,k} \, \mathrm{dt}, \tag{36}$$

however, we remark that optimization of the average cost does not necessarily imply optimization of individual cost.

In the following, we perform the change of variables to design an external gain synchronizing the network of clusters.

4.2.2 *Change of Variables*

To study the consensus between the clusters, we define the external error variable for each cluster $\mathcal{C}_k, k \in \mathcal{M}$ as follows,

$$Y_k := \begin{pmatrix} y_{s,1} - y_{s,k} \\ \dots \\ y_{s,k-1} - y_{s,k} \\ y_{s,k+1} - y_{s,k} \\ \dots \\ y_{s,m} - y_{s,k} \end{pmatrix} \in \mathbb{R}^{(m-1).n_x}. \tag{37}$$

Then, based on equation (33), the corresponding external error dynamics is

$$\begin{aligned} &\dot{Y}_k = \mathbf{A}_k Y_k + \mathbf{B}_k u^{ext}_{s,k}, \quad \forall k \in \mathcal{M}, \\ &\mathbf{A}_k = (I_{m-1} \otimes A) - (I_{m-1} \otimes B)\overline{K}^{ext}_{-k}(\overline{\mathcal{L}}^{ext}_{-k} \otimes I_{n_x}), \\ &\mathbf{B}_k = -(\mathbb{1}_{m-1} \otimes B). \end{aligned} \tag{38}$$

Here, $\overline{K}^{ext}_{-k} = \mathrm{diag}(K^{ext}_1, \dots, K_{k-1}, K_{k+1}, \dots, K_m)$ is not a control action, but it represents the behavior of the network.

To recast the average cost function (34) in terms of new variables Y_k, we introduce the following notations. First we look into the structure of the external Laplacian which have the block form as follows,

$$\mathcal{L}^{ext} = \begin{pmatrix} \mathcal{L}^{ext}_{1,1} & \mathcal{L}^{ext}_{1,2} & \dots & \mathcal{L}^{ext}_{1,m} \\ \mathcal{L}^{ext}_{2,1} & \mathcal{L}^{ext}_{2,2} & \dots & \mathcal{L}^{ext}_{2,m} \\ \vdots & \vdots & & \vdots \\ \mathcal{L}^{ext}_{m,1} & \mathcal{L}^{ext}_{m,2} & \dots & \mathcal{L}^{ext}_{m,m} \end{pmatrix} \in \mathbb{R}^{n \times n}, \tag{39}$$

where $\mathcal{L}^{ext}_{p,q} \in \mathbb{R}^{n_p \times n_q}$ for $p, q \in \mathcal{M}$. We denote by $\mathcal{L}^{ext}_{k,row} \in \mathbb{R}^{n_k \times n}$ the k-th row of the block-matrix (39) for all $k \in \mathcal{M}$. It describes the connections of the cluster $\mathcal{C}_k$ with the rest of the agents in the network. The matrix $\mathcal{L}^{ext}_{k,red} \in \mathbb{R}^{n_k \times (n-n_k)}$ is obtained by removing the $\mathcal{L}^{ext}_{k,k}$ block from the $\mathcal{L}^{ext}_{k,row}$. For example, $\mathcal{L}^{ext}_{2,row} = [\mathcal{L}^{ext}_{2,1} \ \mathcal{L}^{ext}_{2,2} \ \dots \ \mathcal{L}^{ext}_{2,m}]$ and $\mathcal{L}^{ext}_{2,red} = [\mathcal{L}^{ext}_{2,1} \ \mathcal{L}^{ext}_{2,3} \ \dots \ \mathcal{L}^{ext}_{2,m}]$. Then, we rewrite the external cost (34) in terms of new variables as

$$\overline{J}^{ext}_k = \int_0^{+\infty} Y_k^\top Q^{ext}_{k,1} Y_k + Y_k^\top \frac{Q^{ext}_{k,2}}{n_k} Y_k \, \mathrm{dt} \tag{40}$$

where

$$\begin{aligned} &Q^{ext}_{k,1} = \left(\mathrm{diag}\left(\frac{a^{ext}_{k,1}}{n_k}, ..., \frac{a^{ext}_{k,k-1}}{n_k}, \frac{a^{ext}_{k,k+1}}{n_k}, ..., \frac{a^{ext}_{k,m}}{n_k} \right) \otimes I_{n_x} \right), \\ &Q^{ext}_{k,2} = U^\top_{-k}(\mathcal{L}^{ext\top}_{k,red}\mathcal{L}^{ext}_{k,red} \otimes K^{ext\top}_k R_k K^{ext}_k) U_{-k}, \\ &U = (\mathrm{diag}(\mathbb{1}_{n_1}, \dots, \mathbb{1}_{n_m}) \otimes I_{n_x}), \\ &R_k > 0. \end{aligned} \tag{41}$$

The matrices $Q_{k,1}^{ext}$ and $Q_{k,2}^{ext}$ simplify the expressions in (34) such that $Y_k^\top Q_{k,1}^{ext} Y_k = \sum_{l \in \mathcal{N}_{C_k}} \frac{a_{kl}^{ext}}{n_k}(y_{s,k} - y_{s,l})^2$ and $Y_k^\top \frac{Q_{k,2}^{ext}}{n_k} Y_k = n_k \sum_{i=1}^{n_k} \widehat{u}_{k,i}^{ext\top} R_k \widehat{u}_{k,i}^{ext}$.

4.2.3 Control Design

We will use the error dynamics (38) to design the external gain profile using satisfaction equilibrium approach. It characterizes the external gain profile synchronizing the network in such a way that each cost (34) is bounded, i.e.,

$$\overline{J}_k^{ext} \leq \gamma^{ext} \|Y_k(0)\|^2, \quad \text{for} \quad k \in \mathcal{M}. \tag{42}$$

The term $\|Y_k(0)\|$ represents the initial condition of the cluster $\mathcal{C}_k$ while γ^{ext} is a given threshold. In particular, the following proposition is valid.

Proposition 1 (Prop 1, [29]) *Let a gain profile $\overline{K}^{ext} = \operatorname{diag}\left(K_1^{ext}, ..., K_m^{ext}\right)$ be given. The following statements are equivalent,*

(1) The gain profile $\overline{K}^{ext}$ is an SE of the satisfaction game (38) for all $k \in \mathcal{M}$.
(2) For all $k \in \mathcal{M}$, there exists a positive-definite matrix $P_k^{ext} > 0$ such that

$$\begin{cases} P_k^{ext} \mathbf{A}_{k,cl}(K_k^{ext}) + \mathbf{A}_{k,cl}^\top(K_k^{ext}) P_k^{ext} + \mathbf{Q}_k^{ext}(K_k^{ext}) < 0, \\ P_k^{ext} - \gamma^{ext} I_{(m-1).n_x} < 0, \end{cases} \tag{43}$$

where

$$\begin{cases} \mathbf{A}_{k,cl}(K_k^{ext}) = \mathbf{A}_k + \mathbf{B}_k K_k^{ext}(F_k \otimes I_{n_x}), \\ F_k = \left(\frac{a_{k,1}^{ext}}{n_k}, ..., \frac{a_{k,k-1}^{ext}}{n_k}, \frac{a_{k,k+1}^{ext}}{n_k}, ..., \frac{a_{k,m}^{ext}}{n_k}\right), \\ \mathbf{Q}_k^{ext} = \left(Q_{k,1}^{ext} + \frac{Q_{k,2}^{ext}}{n_k}\right). \end{cases} \tag{44}$$

Next, we present the algorithm that allows us to obtain the gain (K^{ext}) in satisfaction equilibrium. This algorithm greatly reduces the computational effort of obtaining the synchronizing gain for large-scale networks.

4.3 Algorithm

Consider a network of m clusters (the number of clusters in our case) with their respective dynamics. We aim to design a synchronizing gain profile $K^{ext} = (K_1^{ext}, ..., K_m^{ext})$ satisfying the cost constraints.

In the following algorithm, we first calculate the internal gain by solving the algebraic Riccati equation (31). To design the external gain (K^{ext}), we start with the initial gain profile that satisfies the LMI (43). Then we multiply the gain from the previous iteration with a scalar $\alpha^{ext} \in \mathbb{R}_+ \setminus \{0\}$ and check if it satisfies the LMI (43), to obtain the sub-optimal gain. One approach could be to start with a high gain and decrease α^{ext} until the condition (43) is not satisfied and use the smallest gain that satisfied the condition.

Furthermore, we should also make sure the network parameter ϵ is small so that control design using time-scale separation holds. Thus, to ensure this, we multiply the internal gain K_k^{int} with ϵ/ϵ^* to obtain the new internal gain such that $\epsilon \leq \epsilon^*$.

Algorithm 1 Sequential Satisfaction Algorithm

Data: A, B and $n_k, k \in \mathcal{M}$;
Set: iterations $itr = 1$, maximum number of iterations itr_{max}, $0 < \epsilon^* \ll 1$ and $K^{ext}(0) = (K_1^{ext}(0), ..., K_m^{ext}(0))$ initial gain profile synchronizing the system ;
Calculate: P_k^{int} and K_k^{int} using equation (31) and (30) for all $k \in \mathcal{M}$, respectively;
while LMIs (43) not satisfied **OR** $itr \leq itr_{max}$ **do**
  $K^{ext}(itr+1) \leftarrow \alpha^{ext} K^{ext}(itr), \quad \alpha^{ext} \in \mathbb{R}_+ \setminus \{0\}$;
  Calculate: ϵ;
  if $\epsilon > \epsilon^*$ **then**
    $K_k^{int}(itr+1) \leftarrow \frac{\epsilon}{\epsilon^*} K_k^{int}(itr)$;
  else
    $K_k^{int}(itr+1) \leftarrow K_k^{int}(itr)$;
  end if
end while

Remark 7 *Notice that with such an approach, we only scale the whole matrix K_k^{int} and K_k^{ext} on each step while keeping the structure of the matrix intact.*

In the algorithm 1, to obtain the initial gain profile $K^{ext}(0)$ we use the algorithm in [3] which has the computational complexity of $\mathcal{O}(m)$ for m clusters. Then, the computational complexity to obtain the internal gain is of order $\mathcal{O}(m)$. Notice that the dimension of the matrix P_k^{int} in equation (31) does not depend on the number of agent (n_k) in the cluster, thus the problem of finding the internal control K_k^{int} is independent of the number of agents in the cluster. To obtain the external gain, K^{ext} we use the SeDuMi [18]. The computational complexity of verifying, if the gain profile satisfies the LMI condition (50) using SeDuMi is $\mathcal{O}(m^{5.5})$. Thus, the overall computational complexity of the Algorithm 1 is $\mathcal{O}(m) + \mathcal{O}(m) + \mathcal{O}(m^{5.5})$. Moreover, from Lemma 2 we obtain the stabilizing internal gain K^{int} and if the algorithm successfully converges to synchronizing external gain (K^{ext}) that satisfies LMI conditions (43), then they will satisfy the Assumption 4.

5 Global System Analysis

In this section, we analyze the overall networked system with the controller gains K^{int} and K^{ext} defined by the Algorithm 1 and designed for reduced slow and fast subsystems.

First, we present the proposition which ensures that the slow and fast controllers, designed independently of each other, synchronize the overall network. And finally, we prove that the cluster cost $J_k(T, +\infty)$ is approximated only by the external cost $J_k^{ext}(T, +\infty)$, where $T > 0$ is a finite time at which each cluster has reached internal synchronization.

5.1 Overall Network Behavior

Based on the controller design procedure presented in section 4, we ensure that Assumption 4 is satisfied, i.e., the internal gain stabilizing the fast dynamics and the external gain synchronizing the slow dynamics exists. Note that the presented design strategy optimizes the cost function (27) associated with the internal controller and upper bound the cost function (34) corresponding to the external controller. Hence, the obtained internal control gain is optimal while the external control gain is sub-optimal. We apply these gains to achieve synchronization in the overall network, and the following proposition ensures synchronization.

Proposition 2 *Consider the closed-loop network dynamics* (15)*, and equivalently, the dynamics in new coordinates* (16)*. Let the internal and external control gains are chosen based on Lemma 2 and Proposition 1, then the overall network synchronizes and satisfies the following bounds,*

$$\begin{aligned} y(t) &= y_s(t) + \mathcal{O}(\epsilon) \\ \xi(t) &= \xi_f(\mu^{int}t) + \mathcal{O}(\epsilon). \end{aligned} \tag{45}$$

PROOF. The proof follows from Theorem 2. ■

5.2 Cost Approximation

In this subsection, we prove that the cluster cost can be approximated by the average cost after finite time T. The motivation is derived from the fact that the internal dynamics converge rapidly to the consensus, and external dynamics exhibit the dominating network behavior. We prove that for the time $t \in [T, +\infty)$, the cluster cost J_k is approximated by n_k times the average external cost, i.e., $n_k\overline{J}_k^{ext}$.

To provide this approximation result, we first define the internal error bound, which helps us characterize the time $T > 0$. And secondly, we ensure that the exponential stability of the fast dynamics (24) implies the exponential stability of the error dynamics (16).

The necessity of the internal error bound arises in the approximation of the cluster cost. During the control design, we recall that the internal consensus is considered to be achieved before designing the external control. Thus, we need to characterize an error bound for the internal cost in finite time T, at which the cluster is very close to the internal consensus. More precisely, the bound at the time $T > 0$ such that $|\xi_{f,k}(T)| \leq \epsilon$ for all $k \in \mathcal{M}$.

The closed-loop fast dynamics is

$$\dot{\xi}_{f,k}(t) = \big((I_{n_k-1} \otimes A) - (\Lambda_k^{int} \otimes BK_k^{int})\big)\xi_{f,k}(t),$$

and

$$\xi_{f,k}(t) = e^{Cl_{f,k}t}\xi_{f,k}(0),$$

where $Cl_{f,k} := \big((I_{n_k-1} \otimes A) - (\Lambda_k^{int} \otimes BK_k^{int})\big)$ and $Cl_{f,k} < 0$ due to Lemma 2. Now, taking norm on both sides and from the definition of the measure of the matrix (see notations and preliminaries), we obtain,

$$\|\xi_{f,k}(t)\| = e^{\nu(Cl_{f,k})t}\|\xi_{f,k}(0)\| \leq e^{\nu(Cl_f)t}\|\xi_{f,k}(0)\|$$

where $\nu(Cl_f) = \max_{k\in\mathcal{M}} \nu(Cl_{f,k})$. Then, as an internal error bound, we choose smallest $T \geq 0$ such that

$$\|\xi_{f,k}(T)\| \leq e^{\nu(Cl_f)T} \max_{k\in\mathcal{M}} \|\xi_{f,k}(0)\| \leq \epsilon.$$

This bound characterizes the local consensus inside each cluster in the finite time T. And hence, it yields

$$\|\xi_{f,k}(t)\| \leq \epsilon e^{\nu(Cl_f)(t-T)} \qquad \forall k \in \mathcal{M},$$

and

$$\|\xi_f(t)\| \leq \epsilon\sqrt{n-m}.e^{\nu(Cl_f)(t-T)}. \tag{46}$$

Next, in equation (25), we notice that the approximation of ξ defined in equation (16) depends on the fast variable ξ_f and the slow variable y_s, but the slow variable may or may not be stable. For the network to achieve synchronization, ξ should be stable. Thus, we prove the following lemma, which ensures the exponential stability of ξ provided that ξ_f is exponentially stable.

Lemma 3 *The exponential stability of the fast dynamics* (24) *and the external error dynamics* (38) *implies the exponential stability of the error dynamics in* (16)*.*

PROOF. See Appendix. ■

Next, with error bound for a finite time, T we present the cluster cost approximation for $t \in [T, +\infty)$. The proposition is stated as follows:

Proposition 3 *During the time interval* $[T, +\infty)$*, the following approximation holds,*

$$J_k(T, +\infty) = n_k\overline{J}_k^{ext}(T, +\infty) + \mathcal{O}(\epsilon), \quad \forall k \in \mathcal{M}. \tag{47}$$

PROOF. See Appendix. ■

Finally, we present the following theorem that bounds the total cluster cost with the sum of internal, external and the constant term.

Theorem 2 *The total cluster cost for all clusters $\mathcal{C}_k$, $k \in \mathcal{M}$ satisfy the following bound:*

$$J_k \leq (\|P_k^{int}\| + n_k\gamma_k + \boldsymbol{C}_k)\|x(0)\|^2 + \mathcal{O}(\epsilon) \tag{48}$$

where P_k^{int} is the solution of the Riccati equation (31) *and $\boldsymbol{C}_k$ is a constant.*

PROOF. See Appendix. ■

6 Simulation

This section provides numerical results to illustrate the effectiveness of the control procedure and the cost approximation using three scenarios. The agent's dynamics are given by (1), where

$$A = \begin{pmatrix} 0.15 & 0.98 \\ -0.98 & 0.15 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{49}$$

The external graph between the agents in different clusters is generated using Erdos-Renyi [14] random graph generator. Then the internal graph with all-to-all connections for each cluster is generated and added to the external graph to obtain the network graph. For the numerical illustration, we consider the multiple scenarios.

- **Scenario 1:** Graph $\mathcal{G}_1$ with four clusters $m = 4$ with 630 agents in total. Each cluster has **all-to-all** internal connections and 299 external connections between the clusters in total. The threshold for the external cost is $\gamma^{ext} = 0.8$.
- **Scenario 2:** Same as Scenario 1 with **dense** internal connections instead of all-to-all internal connections.
- **Scenario 3:** Comparison of control design presented in this paper with the satisfactory control approach in [29] and guaranteed cost approach proposed in [3].

The details of the simulations are present in Tables 2 - 5. In the tables, n_k represent the number of agents in cluster $\mathcal{C}_k$, $error(k) = \frac{|J_k - n_k\bar{J}_k^{ext}|}{J_k} \times 100$, is the error percentage between the total cost and the external cost after time T, and K^{ext} and K^{int} are the respective external and internal gains.

6.1 Scenario 1: All-to-all connections in Clusters

The Figure 2 represents the synchronization of the agents in a network with graph $\mathcal{G}_1$. For the graph $\mathcal{G}_1$, the network parameter is $\epsilon_1 = 0.06$. In the figure, we can observe the four branches appearing and merging into one. Each branch represents the local agreement within the clusters. Next, Figure 3 illustrates the cost approximation for the cluster $\mathcal{C}_4$ by comparing the total cluster cost J_4 and the external cost $n_4\bar{J}_4^{ext}$, after finite time $T = 2s$. More details of the simulations are presented in the table.

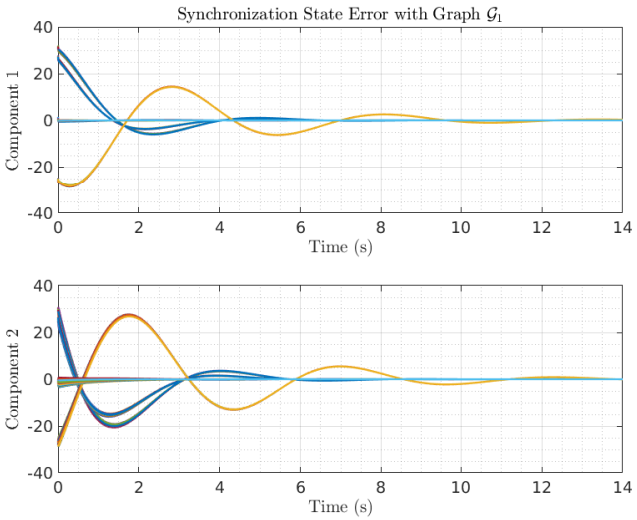

Fig. 2. Evolution of the error between the agents' state in graph $\mathcal{G}_1$ with all-to-all connections inside clusters.

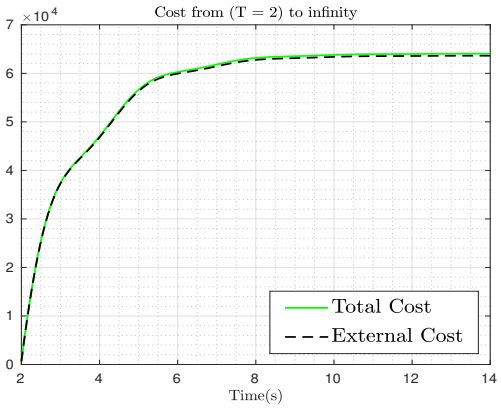

Fig. 3. Evolution of the costs J_4 and $n_4\bar{J}_4^{ext}$ with all-to-all connections inside clusters.

Clusters	K^{int}	K^{ext}
$\mathcal{C}_1$	$[1.5352, -0.1102]$	$[0.85, 0.16]$
$\mathcal{C}_2$	$[1.5349, -0.1114]$	$[1.17, 0.22]$
$\mathcal{C}_3$	$[1.5346, -0.1128]$	$[0.59, 0.11]$
$\mathcal{C}_4$	$[1.5344, -0.1137]$	$[1.05, 0.2]$

Table 1
Internal and External gains

6.2 Scenario 2: Connected Clusters

In this scenario, we consider the graph where the clusters have dense interconnections instead of the all-to-all connections compared to Scenario 1. However, the number of agents and number of external connections remain the same as in the graph $\mathcal{G}_1$. Let us denote this graph as $\mathcal{G}_2$. The same gains from Scenario 1 (Table 1) are applied to the network system with the graph $\mathcal{G}_2$. The details of the simulation are

$\epsilon = 0.06$, $\gamma = 0.8$			
	n_k	$J_k(\times 10^5)$	$error(k)$
$\mathcal{C}_1$	120	0.8966	0.45%
$\mathcal{C}_2$	140	0.5768	0.86%
$\mathcal{C}_3$	170	1.8950	0.24%
$\mathcal{C}_4$	200	0.6405	0.65%

Table 2
Network with 630 agents and 299 external connections with dense connections in clusters.

presented in table 3.

$\epsilon = 0.06$, $\gamma = 0.8$			
	n_k	$J_k(\times 10^5)$	$error(k)$
$\mathcal{C}_1$	120	0.8983	0.64%
$\mathcal{C}_2$	140	0.5780	1.07%
$\mathcal{C}_3$	170	1.8975	0.37%
$\mathcal{C}_4$	200	0.6415	0.81%

Table 3
Network with 630 agents and 299 external connections, with all-to-all connections inside every cluster.

6.3 Scenario 3

In the last scenario, we consider a network of $m = 4$ clusters with $n_k = 10$ agents in each. We recall that $\gamma^{ext} = 1$ is chosen for both controls. A comparison is made between the composite control proposed in this paper and the satisfactory control approach proposed in [29]. The design procedure in [29] needs 13752 seconds (3.8 hours) to compute the gains for $n = 40$ agents, while the composite design in this paper requires 13 seconds. However, we can observe an incontestable difference in performance on the cluster costs due to satisfactory control, as shown in table 4. This emphasizes the trade-off between the computing time/resources to obtain the required controller. Despite being less effective, we must keep in mind that the composite control suits better for large-scale networks and presents an essential benefit in computation loads and time.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
n_k	10	10	10	10
J_k	17204	5452	6943	16949
J_k^*	10164	3303	3080	9714

Table 4
Comparison of cost.

Next, we compare the strategy in [3] with the composite control. In [3], each of the agents applies the same gain independently of their neighborhoods and aims to bound a global cost. Applying the control [3] on the graph $\mathcal{G}_1$, it results in a cluster cost, which we label by $J_k^\dagger$. From Table 5, we observe that our strategy significantly outperforms the approach in [3], the first cluster cost obtained via the composite control is 20 times smaller. One may observe the same for the other clusters.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
n_k	120	140	170	200
$J_k(\times 10^6)$	0.385	0.269	0.689	0.262
$J_k^\dagger(\times 10^6)$	6.7	8.1	16.7	20.5

Table 5
Comparison of the cost.

7 Conclusion

In this paper, we propose a distributed composite control design strategy for the clustered network. Using a coordinate transformation, the network dynamics is transformed into standard singular perturbation form and decoupled into slow and fast dynamics using time-scale separation. This decoupling of the network dynamics also decouple the control into fast (internal) and slow (external). The internal control is responsible for intra-cluster synchronization, while the external synchronize the network while satisfying the imposed cost criterion. This independent design greatly reduces the computational effort required to obtain the control. Finally, we show that the cluster cost is approximated only by the external cost after a short time period.

A Proofs

Proof of Lemma 1
We know from [19] $\|(A \otimes B)\| = \|A\|\|B\|$ for any matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$. Let us define $\bar{n} = \max_k n_k$ and $\underline{n} = \min_k n_k$. In addition, $\|H\| = \frac{1}{\sqrt{\underline{n}}}$, $\|\widetilde{H}\| = \sqrt{\bar{n}}$ and $\|Z\| = \frac{1}{\sqrt{\underline{n}}}$, $\|\widetilde{Z}\| = \sqrt{\bar{n}}$. From the Assumption 3, there exists a strictly positive constant $c_1 \in \mathbb{R}$ such that $\|A\| = c_1\mu^{ext}$. It follows that,

$$\begin{aligned}\|\bar{A}_{11}\| &= \|(I_m \otimes A) - H(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{H}\| \\ &\leq \|A\| + \|H\|.\|(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\|.\|\widetilde{H}\| \\ &= (c_1 + \sqrt{\tfrac{\bar{n}}{\underline{n}}})\mu^{ext} = (c_1 + \sqrt{\tfrac{\bar{n}}{\underline{n}}})\epsilon\mu^{int}.\end{aligned} \tag{A.1}$$

The bounds of $\bar{A}_{12}$, $\bar{A}_{21}$ and $\bar{A}_{22}^1$ are derived similarly, that's why we only prove for $\bar{A}_{12}$,

$$\begin{aligned}\|\bar{A}_{12}\| &= \|H(I_n \otimes B)K^{ext}(\mathcal{L}^{ext} \otimes I_{n_x})\widetilde{Z}\| \\ &\leq \sqrt{\tfrac{\bar{n}}{\underline{n}}}\mu^{ext} = \epsilon\sqrt{\tfrac{\bar{n}}{\underline{n}}}\mu^{int}.\end{aligned} \tag{A.2}$$

Then, we lower-bound the matrix $\bar{A}_{22}^2$ such that

$$\begin{aligned}&\|\bar{A}_{22}^2\| = \|(I_{n-m} \otimes A) - (I_{n-m} \otimes B)K_{n-m}^{int}(\Lambda^{int} \otimes I_{n_x})\| \\ &\geq |\|A\| - \|(I_{n-m} \otimes B)K_{n-m}^{int}(\Lambda^{int} \otimes I_{n_x})\||.\end{aligned} \tag{A.3}$$

From (18), we understand that the second term in (A.3) is much larger than the first one. Thus, by taking the difference between the largest value of the first term and the smallest value of the second term, it yields a lower-bound as

$$\|\bar{A}_{22}^2\| \geq |c_1\epsilon\mu^{int} - \mu^{int}| = |1 - c_1\epsilon|\mu^{int}, \tag{A.4}$$

where $\mu^{int} = \min_{k\in\mathcal{M}} \|(\Lambda_k^{int} \otimes BK_k^{int})\|$.

Proof of Theorem 1

The proof of Theorem 1 follows the block-diagonalization technique provided in [17]. The singularly perturbed system dynamics (19) is slightly different from the one in the [17] thus we adapt the result to our system model to obtain the approximation results. Using the following transformation

$$\begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} I_{m.n_x} & \epsilon\Psi(\epsilon) \\ -\Omega(\epsilon) & I_{n_x(n-m)} - \epsilon\Omega(\epsilon)\Psi(\epsilon) \end{bmatrix} \begin{bmatrix} y_s \\ \xi_f \end{bmatrix}$$
$$\begin{bmatrix} y_s \\ \xi_f \end{bmatrix} = \begin{bmatrix} I_{m.n_x} - \epsilon\Psi(\epsilon)\Omega(\epsilon) & -\epsilon\Psi(\epsilon) \\ \Omega(\epsilon) & I_{n_x(n-m)} \end{bmatrix} \begin{bmatrix} y \\ \xi \end{bmatrix}, \tag{A.5}$$

where the functions Ω and Ψ should satisfy the following,

$$\begin{aligned} R(\Omega(\epsilon),\epsilon) &= \epsilon A_{21} - \epsilon A_{22}^1\Omega(\epsilon) - A_{22}^2\Omega(\epsilon) \\ &\quad +\epsilon\Omega(\epsilon)A_{11} - \epsilon\Omega(\epsilon)A_{12}\Omega(\epsilon) = 0, \\ S(\Psi(\epsilon),\epsilon) &= \epsilon A_{11}\Psi(\epsilon) + A_{12} - \epsilon A_{12}\Omega(\epsilon)\Psi(\epsilon) \\ &\quad -\epsilon\Psi(\epsilon)A_{22}^1 - \Psi(\epsilon)A_{22}^2 - \epsilon\Psi(\epsilon)\Omega(\epsilon)A_{12} = 0 \end{aligned}$$

the dynamics (19) can be decoupled into two independent two time-scale slow and fast subsystems.

The approximation of Ω and Ψ, obtained with the Taylor development w.r.t. ϵ, are

$$\begin{aligned} \Omega(\epsilon) &= \epsilon(A_{22}^2)^{-1}A_{21} + \mathcal{O}(\epsilon^2), \\ \Psi(\epsilon) &= A_{12}(A_{22}^2)^{-1} + \epsilon((A_{22}^2)^{-1}A_{11}A_{12}(A_{22}^2)^{-1} - A_{12}) \\ &\quad +\mathcal{O}(\epsilon^2). \end{aligned} \tag{A.6}$$

From Lemma (3), we know that $\xi(t)$ and $\xi_f(t_f)$ converge to zero exponentially as t and t_f tend to $+\infty$, respectively. Thus, we can claim that $\Omega(\epsilon)y_s(t)$ has an exponential decrease to zero w.r.t. t. Finally, from the above transformation (A.5), we have,

$$y = y_s(t_s) + \epsilon\Psi(\epsilon)\xi_f \tag{A.7}$$
$$\xi = \xi_f(t_f) - \Omega(\epsilon)y_s(t_s) - \epsilon\Omega(\epsilon)\Psi(\epsilon)\xi_f. \tag{A.8}$$

Then from (A.6), we have that $\Omega(\epsilon) = \mathcal{O}(\epsilon)$ and we obtain the approximations (25).

Proof of Lemma 3

Integrating the error dynamics in (16), we obtain

$$\begin{aligned} \xi(t) &= e^{\bar{A}_{22}t}\xi(0) + \textstyle\int_0^t e^{\bar{A}_{22}(t-\tau)}\bar{A}_{21}y(\tau)\,\mathrm{d}\tau \\ &= e^{\bar{A}_{22}t}\xi(0) + \textstyle\int_0^t e^{\bar{A}_{22}(t-\tau)}\bar{A}_{21}(y_s(\tau) + \epsilon\Psi(\epsilon)\xi_f(\tau))\,\mathrm{d}\tau \\ &= e^{\bar{A}_{22}t}\xi(0) + \textstyle\int_0^t e^{\bar{A}_{22}(t-\tau)}Z^TMY(\tau)\,\mathrm{d}\tau \\ &\quad +\epsilon\textstyle\int_0^t e^{\bar{A}_{22}(t-\tau)}\bar{A}_{21}\Psi(\epsilon)\xi_f(\tau)\,\mathrm{d}\tau \end{aligned}$$

where $M = diag(M_1, ..., M_m)$ and $M_k = (\mathcal{L}_{k,red}^{ext} \otimes BK_k^{ext})U_{-k}$. By taking norm on both sides, we have

$$\begin{aligned} \|\xi(t)\| &\leq \|e^{\bar{A}_{22}t}\|\|\xi(0)\| + \|Z^TM\|\textstyle\int_0^t \|e^{\bar{A}_{22}(t-\tau)}\|\|Y(\tau)\|\,\mathrm{d}\tau \\ &\quad +\epsilon\|\bar{A}_{21}\Psi(\epsilon)\|\textstyle\int_0^t \|e^{\bar{A}_{22}(t-\tau)}\|\|\xi_f(\tau)\|\,\mathrm{d}\tau \end{aligned} \tag{A.9}$$

Also, from the design of internal and external control, we know that, for all $t \geq 0$,

$$\begin{cases} Y(t) = e^{\mathbf{A}_{cl}t}Y(0) \\ \xi_f(t) = e^{\bar{A}_{22}^2t}\xi_f(0) \end{cases} \Rightarrow \begin{cases} \|Y(t)\| \leq e^{\nu(\mathbf{A}_{cl})t}\|Y(0)\| \\ \|\xi_f(t)\| \leq e^{\nu(\bar{A}_{22}^2)t}\|\xi_f(0)\| \end{cases} \tag{A.10}$$

where $\mathbf{A}_{cl} = diag(\mathbf{A}_{1,cl}, ..., \mathbf{A}_{m,cl})$ is the closed-loop dynamics of the external error (38). Then, it follows that

$$\begin{aligned} \|\xi(t)\| &\leq e^{\nu(\bar{A}_{22})t}\|\xi(0)\| \\ &\quad +\|Z^TM\|\|Y(0)\|\textstyle\int_0^t e^{\nu(\bar{A}_{22})(t-\tau)}e^{\nu(\mathbf{A}_{cl})\tau}\,\mathrm{d}\tau \\ &\quad +\epsilon\|\bar{A}_{21}\Psi(\epsilon)\|\|\xi_f(0)\|\textstyle\int_0^t e^{\nu(\bar{A}_{22})(t-\tau)}e^{\nu(\bar{A}_{22}^2)\tau}\,\mathrm{d}\tau\,. \end{aligned}$$

By integrating the second term in (A.9), we have

$$\begin{aligned} &\|Z^TM\|\|Y(0)\|\textstyle\int_0^t e^{\nu(\bar{A}_{22})(t-\tau)}e^{\nu(\mathbf{A}_{cl})\tau}\,\mathrm{d}\tau \\ &\quad = \|Z^TM\|\|Y(0)\|e^{\nu(\bar{A}_{22})t}\textstyle\int_0^t e^{(\nu(\mathbf{A}_{cl})-\nu(\bar{A}_{22}))\tau}\,\mathrm{d}\tau \\ &= \tfrac{\|Z^TM\|\|Y(0)\|}{\nu(\mathbf{A}_{cl})-\nu(\bar{A}_{22})}\left[e^{\nu(\mathbf{A}_{cl})t} - e^{\nu(\bar{A}_{22})t}\right]. \end{aligned}$$

In the same manner, the third term is

$$\begin{aligned} &\epsilon\|\bar{A}_{21}\Psi(\epsilon)\|\|\xi_f(0)\|\int_0^t e^{\nu(\bar{A}_{22})(t-\tau)}e^{\nu(\bar{A}_{22}^2)\tau}\,\mathrm{d}\tau \\ &\quad = \frac{\epsilon\|\bar{A}_{21}\Psi(\epsilon)\|\|\xi_f(0)\|}{\nu(\bar{A}_{22}^2)-\nu(\bar{A}_{22})}\left[e^{\nu(\bar{A}_{22}^2)t} - e^{\nu(\bar{A}_{22})t}\right]. \end{aligned} \tag{A.11}$$

Finally, we have

$$\begin{aligned} \|\xi(t)\| &\leq \mathbf{C}_1e^{\nu(\mathbf{A}_{cl})t} + \epsilon\mathbf{C}_2e^{\nu(\bar{A}_{22}^2)t} \\ &\quad +\left(\|\xi(0)\| - \mathbf{C}_1 - \epsilon\mathbf{C}_2\right)e^{\nu(\bar{A}_{22})t}, \end{aligned} \tag{A.12}$$

where $\mathbf{C}_1 = \frac{\|Z^TM\|\|Y(0)\|}{\nu(\mathbf{A}_{cl})-\nu(\bar{A}_{22})}$ and $\mathbf{C}_2 = \frac{\|\bar{A}_{21}\Psi(\epsilon)\|\|\xi_f(0)\|}{\nu(\bar{A}_{22}^2)-\nu(\bar{A}_{22})}$. Moreover, we know that $\nu(\bar{A}_{22}^2) < \nu(\bar{A}_{22}) < \nu(\mathbf{A}_{cl}) < 0$. Thus, we conclude that ξ converges exponentially to zero and the rate of convergence can be bounded as

$$\|\xi(t)\| \leq \|\xi(0)\|e^{\nu(\mathbf{A}_{cl})t}. \tag{A.13}$$

Proof of Proposition 3

The cost J_k is split into the sum of the internal and external costs and composite term, as shown in equation (7). Then, we bound the internal and external costs from time T to infinity. We proceed similarly with the composite term.

Internal Cost: Substituting $x_k = \widetilde{H}_k y_k + \widetilde{Z}_k \xi_k$ from equation (13) into J_k^{int} in equation (7) and with $\widetilde{H}_k^\top(\mathcal{L}_k^{int} \otimes I_{n_x}) = 0$, it yields

$$\begin{aligned}
&J_k^{int}(T,+\infty) = \textstyle\int_T^{+\infty} \xi_k^\top \widetilde{Z}_k((\mathcal{L}_k^{int} \otimes I_{n_x}) \\
&\quad +(\mathcal{L}_k^{int\top}\mathcal{L}_k^{int} \otimes K_k^{int\top} R_k K_k^{int}))\widetilde{Z}_k\xi_k \,\mathrm{dt}, \\
&= \textstyle\int_T^{+\infty} n_k \xi_k^\top \Bigg((\Lambda_k^{int} \otimes I_{n_x}) \\
&\quad + \Big(\Lambda_k^{int2} \otimes P_k^{int\top} B \tfrac{R_k^{-1}}{n_k^2} B^\top P_k^{int} \Big) \Bigg) \xi_k \,\mathrm{dt}, \\
&= \int_T^{+\infty} n_k \xi_k^\top \big((\Lambda_k^{int} \otimes I_{n_x}) \\
&+ \big(I_{n_k-1} \otimes P_k^{int\top} B R_k^{-1} B^\top P_k^{int} \big) \big) \xi_k \,\mathrm{dt}, \\
&\leq \mathbf{C}_{3,k} \textstyle\int_T^{+\infty} \|\xi_k\|^2 \,\mathrm{dt} \leq \mathbf{C}_{3,k} \int_T^{+\infty} \|\xi(t)\|^2 \,\mathrm{dt}\,.
\end{aligned}$$

where,
$\mathbf{C}_{3,k} = \|n_k \big((\Lambda_k^{int} \otimes I_{n_x}) + \big(I_{n_k-1} \otimes P_k^{int\top} B R_k^{-1} B^\top P_k^{int} \big) \big) \|$.
From Lemma 3 and equation (A.13), we have $\|\xi(t)\| \leq \|\xi(T)\| e^{\nu(\mathbf{A}_{cl})(t-T)}$, for all $t \in [T, +\infty)$. Thus, with $\nu(\mathbf{A}_{cl}) < 0$, we have,

$$\int_T^{+\infty} \|\xi(t)\|^2 \,\mathrm{dt} \leq -\frac{\|\xi(T)\|^2}{2\nu(\mathbf{A}_{cl})} = \mathbf{C}_4 \|\xi(T)\|^2 \tag{A.14}$$

where $\mathbf{C}_4 := \big(-\frac{1}{2\nu(\mathbf{A}_{cl})} \big)$. Thus, from (A.14)-(A.14) and the approximation of ξ in equation (25),

$$\begin{aligned}
&J_k^{int}(T,+\infty) \leq \mathbf{C}_{3,k}\mathbf{C}_4 \|\xi_f(T) + \mathcal{O}(\epsilon)\|^2 \\
&\leq \mathbf{C}_{3,k}\mathbf{C}_4 \left(\|\xi_f(T)\|^2 + 2\mathcal{O}(\epsilon)\|\xi_f(T)\| + \mathcal{O}(\epsilon^2) \right).
\end{aligned}$$

Finally, replacing $\|\xi_f(T)\| \leq \epsilon\sqrt{n-m}$ from (46) we have

$$J_k^{int}(T,+\infty) \leq \mathcal{O}(\epsilon^2). \tag{A.15}$$

External cost: First, we recast the collective external control (4) in the external error variable Y_k, as follows

$$\begin{aligned}
&u_k^{ext}(t) = -(I_{n_k} \otimes K_k^{ext})(\mathcal{L}_{k,row}^{ext} \otimes I_{n_x})x(t) \\
&= -(\mathcal{L}_{k,row}^{ext} \otimes K_k^{ext})(\widetilde{H} y(t) + \widetilde{Z}\xi(t)) \\
&= -(\mathcal{L}_{k,row}^{ext} \otimes K_k^{ext})(\widetilde{H} y_s(t) + \epsilon\widetilde{H}\Psi(\epsilon)\xi_f(t_f) + \widetilde{Z}\xi(t)) \\
&= (\mathcal{L}_{k,red}^{ext} \otimes K_k^{ext}) U_{-k} Y_k(t) \\
&\quad - (\mathcal{L}_{k,row}^{ext} \otimes K_k^{ext})(\epsilon\widetilde{H}\Psi(\epsilon)\xi_f(t_f) + \widetilde{Z}\xi(t)),
\end{aligned} \tag{A.16}$$

where $\mathcal{L}_{k,row}^{ext}$ is the k-th block-row of $\mathcal{L}^{ext}$ and $\mathcal{L}_{k,red}^{ext}$ is obtained by removing the $\mathcal{L}_{k,k}^{ext}$ block from $\mathcal{L}_{k,row}^{ext}$. Then, it yields

$$\begin{aligned}
&u_k^{ext\top}(t)(I_{n_k} \otimes R_k) u_k^{ext}(t) \\
&= Y_k^\top(t) Q_{k,2}^{ext} Y_k(t) + \epsilon^2 \xi_f^\top(t_f) D_{1,k} \xi_f(t_f) + \xi^\top(t) D_{2,k} \xi(t) \\
&- \epsilon Y_k^\top(t) D_{3,k} \xi_f(t_f) - Y_k^\top(t) D_{4,k} \xi(t) + \epsilon \xi^\top(t) D_{5,k} \xi_f(t_f),
\end{aligned} \tag{A.17}$$

where

$$\begin{cases}
Q_{k,2}^{ext} = U_{-k}^\top (\mathcal{L}_{k,red}^{ext\top} \mathcal{L}_{k,red}^{ext} \otimes K_k^{ext\top} R_k K_k^{ext}) U_{-k}, \\
D_{1,k} = \Psi(\epsilon)^\top \widetilde{H}^\top (\mathcal{L}_{k,row}^{ext^\top} \mathcal{L}_{k,row}^{ext} \otimes K_k^{ext^\top} R_k K_k^{ext}) \widetilde{H} \Psi(\epsilon), \\
D_{2,k} = \widetilde{Z}^\top (\mathcal{L}_{k,row}^{ext^\top} \mathcal{L}_{k,row}^{ext} \otimes K_k^{ext^\top} R_k K_k^{ext}) \widetilde{Z}, \\
D_{3,k} = 2U_{-k}^\top (\mathcal{L}_{k,red}^{ext^\top} \mathcal{L}_{k,row}^{ext} \otimes K_k^{ext^\top} R_k K_k^{ext}) \widetilde{H} \Psi(\epsilon), \\
D_{4,k} = 2U_{-k}^\top (\mathcal{L}_{k,red}^{ext\top} \mathcal{L}_{k,row}^{ext} \otimes K_k^{ext^\top} R_k K_k^{ext}) \widetilde{Z}, \\
D_{5,k} = 2\widetilde{Z}^\top (\mathcal{L}_{k,row}^{ext\top} \mathcal{L}_{k,row}^{ext} \otimes K_k^{ext^\top} R_k K_k^{ext}) \widetilde{H} \Psi(\epsilon).
\end{cases}$$

Secondly, let consider the state part in the external cost. To simplify the expression, we use $(\mathcal{L}_k^{ext} \otimes I_{n_x})\widetilde{H} y_s(t) = -(\mathcal{L}_{k,col}^{ext} \otimes I_{n_x}) U_{-k} Y_k(t)$ where $\mathcal{L}_{k,col}^{ext}$ is the matrix $\mathcal{L}_k^{ext}$ with its k-th block-column removed. Then, we obtain

$$\begin{aligned}
&x^\top(t)(\mathcal{L}_k^{ext} \otimes I_{n_x}) x(t) \\
&= \star^\top (\mathcal{L}_k^{ext} \otimes I_{n_x})(\widetilde{H} y_s(t) + \epsilon\widetilde{H}\Psi(\epsilon)\xi_f(t_f) + \widetilde{Z}\xi(t)) \\
&= n_k Y_k^\top(t) Q_{k,1}^{ext} Y_k(t) + \epsilon^2 \xi_f(t_f)^\top M_{1,k} \xi_f(t_f) \\
&+ \xi^\top(t) M_{2,k} \xi(t) - \epsilon Y_k^\top(t) M_{3,k} \xi_f(t_f) - Y_k^\top(t) M_{4,k} \xi(t) \\
&+ \epsilon \xi^\top(t) M_{5,k} \xi_f(t_f)
\end{aligned} \tag{A.18}$$

where

$$\begin{cases}
M_{1,k} = \Psi(\epsilon)^\top \widetilde{H}^\top (\mathcal{L}_k^{ext} \otimes I_{n_x}) \widetilde{H} \Psi(\epsilon) \\
M_{2,k} = \widetilde{Z}^\top (\mathcal{L}_k^{ext} \otimes I_{n_x}) \widetilde{Z} \\
M_{3,k} = 2U_{-k}^\top (\mathcal{L}_{k,col}^{ext\top} \otimes I_{n_x}) \widetilde{H} \Psi(\epsilon) \\
M_{4,k} = 2U_{-k}^\top (\mathcal{L}_{k,col}^{ext\top} \otimes I_{n_x}) \widetilde{Z} \\
M_{5,k} = 2\widetilde{Z}^\top (\mathcal{L}_k^{ext} \otimes I_{n_x}) \widetilde{H} \Psi(\epsilon).
\end{cases}$$

Then, replacing (A.17) and (A.18) into the external cost (J_k^{ext}) in equation (7), we get

$$\begin{aligned}
&J_k^{ext}(T,+\infty) \\
&= n_k \int_T^{+\infty} Y_k^\top(t) Q_{k,1}^{ext} Y_k(t) + Y_k^\top(t) \frac{Q_{k,2}^{ext}}{n_k} Y_k(t) \,\mathrm{dt} + \Delta_1 \\
&= n_k \overline{J}_k^{ext}(T,+\infty) + \Delta_1,
\end{aligned} \tag{A.19}$$

where $\Delta_1 = \Delta_1^1 + \Delta_1^2 + \Delta_1^3 + \Delta_1^4 + \Delta_1^5$ and

$$\begin{cases}
\Delta_1^1 = \epsilon^2 \int_T^{+\infty} \xi_f(t_f)^\top (M_{1,k} + D_{1,k}) \xi_f(t_f) \,\mathrm{dt}, \\
\Delta_1^2 = \int_T^{+\infty} \xi^\top(t) (M_{2,k} + D_{2,k}) \xi(t) \,\mathrm{dt}, \\
\Delta_1^3 = -\epsilon \int_T^{+\infty} Y_k^\top(t) (M_{3,k} + D_{3,k}) \xi_f(t_f) \,\mathrm{dt}, \\
\Delta_1^4 = -\int_T^{+\infty} Y_k^\top(t) (M_{4,k} + D_{4,k}) \xi(t) \,\mathrm{dt}, \\
\Delta_1^5 = \epsilon \int_T^{+\infty} \xi^\top(t) (M_{5,k} + D_{5,k}) \xi_f(t_f) \,\mathrm{dt}\,.
\end{cases} \tag{A.20}$$

$$\Delta_1^1 \leq \epsilon^2 \|M_{1,k}+D_{1,k}\| \int_T^{+\infty} \|\xi_f(t_f)\|^2 \,\mathrm{dt} \leq -\epsilon^2 \frac{\|M_{1,k}+D_{1,k}\|\|\xi_f(0)\|^2}{2\nu(\bar{A}_{22}^2)} e^{2\nu(\bar{A}_{22}^2)T} = \mathcal{O}(\epsilon^2).$$

$$\Delta_1^2 \leq \mathbf{C}_{3,k}\|M_{2,k}+D_{2,k}\| \int_T^{+\infty} \|\xi(t)\|^2 \|\xi(T)\|^2 \,\mathrm{dt} \leq \mathbf{C}_{3,k}\|M_{2,k}+D_{2,k}\| \left(\|\xi_f(T)\|^2 + 2\mathcal{O}(\epsilon)\|\xi_f(T)\| + \mathcal{O}(\epsilon^2)\right) \leq \mathcal{O}(\epsilon^2).$$

$$\Delta_1^3 \leq \epsilon \|M_{3,k}+D_{3,k}\|\|Y(0)\|\|\xi_f(0)\| \int_T^{+\infty} e^{\nu(\mathbf{A}_{cl})t} e^{\nu(\bar{A}_{22}^2)t} \,\mathrm{dt} = -\epsilon \frac{\|M_{3,k}+D_{3,k}\|\|Y(0)\|\|\xi_f(0)\|}{\nu(\mathbf{A}_{cl})+\nu(\bar{A}_{22}^2)} e^{\left(\nu(\mathbf{A}_{cl})+\nu(\bar{A}_{22}^2)\right)T} = \mathcal{O}(\epsilon).$$

Similarly, Δ_1^4 and Δ_1^5 are of order $\mathcal{O}(\epsilon)$. Finally, from (A.19) and bounds in (A.20) for Δ_1, we obtain

$$J_k^{ext}(T,+\infty) = n_k \bar{J}_k^{ext}(T,+\infty) + \mathcal{O}(\epsilon). \tag{A.21}$$

Composite term: We rewrite the external control (A.16) and the internal control (4) as

$$\begin{aligned} u_k^{ext}(t) &= -\mathbf{C}_{5,k}Y_k(t) - \epsilon \mathbf{C}_{6,k}\xi_f(t_f) - \mathbf{C}_{7,k}\xi(t) \\ u_k^{int}(t) &= (\mathcal{L}_k^{int} \otimes K_k^{int})\widetilde{Z}_k \xi_k(t) =: \mathbf{C}_{8,k}\xi_k(t). \end{aligned} \tag{A.22}$$

where $\mathbf{C}_{5,k} = (\mathcal{L}_{k,red}^{ext} \otimes K_k^{ext})U_{-k}$, $\mathbf{C}_{6,k} = (\mathcal{L}_{k,row}^{ext} \otimes K_k^{ext})\widetilde{H}\Psi(\epsilon)$ and $\mathbf{C}_{7,k} = (\mathcal{L}_{k,row}^{ext} \otimes K_k^{ext})\widetilde{Z}$. Then, taking the norm and substituting from equations (A.22) into the J_k^{cross} term in equation (7), we get,

$$\begin{aligned} J_k^{cross}(T,+\infty) &\leq 2\|R_k\| \int_T^{+\infty} \|{u_k^{ext}}^\top(t)\|\|u_k^{int}(t)\| \,\mathrm{dt} \\ &\leq 2\|R_k\| \int_T^{+\infty} \|\mathbf{C}_{5,k}Y_k(t) + \epsilon \mathbf{C}_{6,k}\xi_f(t_f) + \mathbf{C}_{7,k}\xi(t)\| \cdot \\ &\qquad \|\mathbf{C}_{8,k}\xi_k(t)\| \,\mathrm{dt} \end{aligned} \tag{A.23}$$

With simple calculation it can be shown that the first integral in the above equation is of order $\mathcal{O}(\epsilon)$ and the second and the third integrals are of order $\mathcal{O}(\epsilon^2)$. Thus, we have,

$$J_k^{cross}(T,+\infty) \leq \mathcal{O}(\epsilon). \tag{A.24}$$

Finally, from (7), (A.15), (A.21) and (A.24), we conclude the proof.

Proof of Theorem 2

Internal Cost: Following the similar approximation as the approximation of the internal cost in Proposition 3, we obtain the following approximation for the internal cost for

$$J_k^{int} = n_k J_{f,k} + \mathcal{O}(\epsilon) \tag{A.25}$$

Moreover, due to LQ-control design, the optimal fast cost $J_{f,k} = \xi_{f,k}(0)^\top (I_{n_{k-1}} \otimes P_k^{int})\xi_{f,k}(0)$. The substituting the approximation $\xi_k = \xi_{f,k} + \mathcal{O}(\epsilon)$, we get, $J_{f,k} = \xi_k(0)^\top (I_{n_{k-1}} \otimes P_k^{int})\xi_k(0) + \mathcal{O}(\epsilon)$. Then, from the transformation (12), it yields,

$$\begin{aligned} J_k^{int} &= n_k . x_k(0)^\top Z_k^\top (I_{n_{k-1}} \otimes P_k^{int}) Z_k x_k(0) + \mathcal{O}(\epsilon) \\ &= x_k(0)^\top (I_{n_k} \otimes P_k^{int}) x_k(0) + \mathcal{O}(\epsilon) \\ &\leq \|P_k^{int}\| \|x_k(0)\|^2 + \mathcal{O}(\epsilon). \end{aligned} \tag{A.26}$$

External Cost: Substituting $x_k = \widetilde{H}y + \widetilde{Z}\xi$ in the external cost J_k^{ext} in equation (7), and performing the similar operation as in the approximation of the external cost in Proposition 3, we obtain,

$$J_k^{ext} \leq n_k \overline{J}_k^{ext} + \Pi_1 + \mathcal{O}(\epsilon) \tag{A.27}$$

with

$$\begin{aligned} \Pi_1 = 2 \int_0^{+\infty} & \xi^\top \widetilde{Z}^\top \big((\mathcal{L}_k^{ext} \otimes I_{n_x}) + \\ & ({\mathcal{L}_{k,row}^{ext}}^\top \mathcal{L}_{k,row}^{ext} \otimes {K_k^{ext}}^\top R_k K_k^{ext})\big) \widetilde{H} y \,\mathrm{dt} \\ + \int_0^{+\infty} & \xi^\top \big(\widetilde{Z}^\top (\mathcal{L}_k^{ext} \otimes I_{n_x}) \\ & + ({\mathcal{L}_{k,row}^{ext}}^\top \mathcal{L}_{k,row}^{ext} \otimes {K_k^{ext}}^\top R_k K_k^{ext})\widetilde{Z}\big)\xi \,\mathrm{dt}\,. \end{aligned} \tag{A.28}$$

Furthermore, substituting $(\mathcal{L}_k^{ext} \otimes I_{n_x})\widetilde{H}y_s = -(\mathcal{L}_{k,col}^{ext} \otimes I_{n_x})U_{-k}Y_k$ and $(\mathcal{L}_{k,row}^{ext} \otimes I_{n_x})\widetilde{H}y_s = -(\mathcal{L}_{k,red}^{ext} \otimes I_{n_x})U_{-k}Y_k$ in equation (A.28) and taking the norm we have,

$$\Pi_1 \leq \mathbf{C}_{9,k}\|x(0)\|^2 + \mathcal{O}(\epsilon) \tag{A.29}$$

where $\mathbf{C}_{9,k} := \mathbf{C}_4\|Z\|\big(\|Y_k(0)\|\|2\widetilde{Z}^\top \big((\mathcal{L}_{k,col}^{ext} \otimes I_{n_x}) + ({\mathcal{L}_{k,row}^{ext}}^\top \mathcal{L}_{k,red}^{ext} \otimes {K_k^{ext}}^\top R_k K_k^{ext})\big)U_{-k}\| + \|\big(\widetilde{Z}^\top (\mathcal{L}_k^{ext} \otimes I_{n_x}) + ({\mathcal{L}_{k,row}^{ext}}^\top \mathcal{L}_{k,row}^{ext} \otimes {K_k^{ext}}^\top R_k K_k^{ext})\widetilde{Z}\big)\mathbf{C}_4\big)$.

Cross Term: Substituting from equation (A.22) and from Theorem 1 into the cross term in equation (7) and after further calculation, we get,

$$\begin{aligned} J_k^{cross} \leq\ & 2\|R_k\|\|\mathbf{C}_{5,k}\|\|\mathbf{C}_{8,k}\|\|\mathbf{C}_4\|\|Y_k(0)\|\|Z\|\|x(0)\| \\ & + 2\|R_k\|\|\mathbf{C}_{7,k}\|\|\mathbf{C}_{8,k}\|\|\mathbf{C}_4\|\|Z\|^2\|x(0)\|^2 + \mathcal{O}(\epsilon) \end{aligned}$$

By definition of the variable Y_k in equation (37), it satisfies $\|Y_k\| \leq \sqrt{n_k}\|H\|\|x(0)\| + \mathcal{O}(\epsilon)$ and substituting it in the above equation leads to

$$J_k^{cross} \leq \boldsymbol{C}_{10,k}\|x(0)\|^2 + \mathcal{O}(\epsilon), \tag{A.30}$$

where $\boldsymbol{C}_{10,k} := 2\|R_k\|\|\mathbf{C}_{8,k}\|\|\mathbf{C}_4\|\big(\sqrt{n_k}\|H\|\|\mathbf{C}_{5,k}\| + \|\mathbf{C}_{7,k}\|\|Z\|\big)\|Z\|$. Then from equation (7), (A.26), (A.27), (A.29) and (A.30), we have,

$$J_k \leq \|P_k^{int}\|\|x_k(0)\|^2 + n_k \overline{J}_k^{ext} + \boldsymbol{C}_k\|x(0)\|^2 + \mathcal{O}(\epsilon)$$

where $\boldsymbol{C}_k := (\boldsymbol{C}_{9,k} + \boldsymbol{C}_{10,k})$. Moreover, we have $\|Y_k(0)\| \leq \|x(0)\|^2 + \mathcal{O}(\epsilon)$ and substituting from equation (42),

$$\begin{aligned} J_k &\leq \|P_k^{int}\|\|x_k(0)\|^2 + n_k\gamma_k\|x(0)\|^2 + \boldsymbol{C}_k\|x(0)\|^2 + \mathcal{O}(\epsilon) \\ &\leq (\|P_k^{int}\| + n_k\gamma_k + \boldsymbol{C}_k)\|x(0)\|^2 + \mathcal{O}(\epsilon) \end{aligned} \tag{A.31}$$

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