The mathematical background of proving processes in discrete optimization - Exemplification with Research Situations for the Classrooms

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Abstract. Discrete mathematics brings interesting problems to teach and learn proof with accessible objects such as integers (arithmetic), graphs (modeling, order) or polyominoes (geometry). A lot of still open problems can be explained to a large public. The objects can be manipulated by simple dynamic operations (removing, adding, "gluing", contracting, splitting, decomposing, etc.). All these operations can be seen as tools for proving. This article particularly explores the field of "discrete optimization". A theoretical background is defined by taking two main axes into account: the epistemological analysis of discrete problems studied by contemporary researchers in discrete optimization and the design of adidactical situations for classrooms in the frame of the Theory of Didactical Situations. Two problems coming from ongoing research in discrete optimization (the Pentamino Exclusion and the Eight Queens problems) are developed. They underscore the learning potentialities of discrete mathematics and epistemological obstacles about proving processes. They emphasize the understanding of a necessary condition and a sufficient condition and problematize the difference between optimal and optimum. They provide proofs involving partitioning strategies, greedy algorithms but also primal-dual methods leading to the concept of duality. The way such problems can be implemented in the classrooms is described in a collaborative work between mathematicians and mathematics education researchers (*Maths à Modeler* Research Federation) through the Research Situations for the Classrooms.
Keywords: discrete mathematics, optimization, proving processes, research situations for the classrooms, polyominoes, eight queens problem.

Introduction

Since 1999, the French *Maths à Modeler* Research Federation has promoted a collaborative work between mathematicians in discrete mathematics and mathematics education researchers. We design innovative Research Situations for the Classroom (RSCs) from real problems coming from the ongoing mathematical research (Grenier & Payan 2003; Ouvrier-Buffet 2009). The aims of RSCs are to put students in the role of a researcher and to engage them in a proving process or more generally in an inquiry process including several transversal skills such as making conjecture, modeling, defining etc. Several RSCs have been designed and implemented in different contexts by *Maths à Modeler* with the help of the Theory of Didactical Situations (TDS, Brousseau 1997). All these experiments are based on a strong epistemological analysis of discrete problems taking into account the needs of proof.

The literature in mathematics education about proof has produced fundamental findings regarding the epistemological, philosophical and didactical aspects of proof (e.g. Hanna & De Villiers 2012; Hanna, Jahnke & Pulte 2010; Mariotti et al. 2018; Stylianides 2016). New perspectives for education also emerge from the research based on interviews with active mathematicians (e.g. Ouvrier-Buffet 2015; Weber 2011). In all the research about proof, the fields of arithmetic, number theory and combinatorics are often used. Such discrete mathematics (DM) clearly favors an "authentic mathematical activity" (Lampert 1990) with a work on heuristics in non-routine problem solving (e.g. Mousoulides & Sriraman 2020). It also brings a new beginning for students and teachers (Goldin 2010). The concepts involved (mainly integers) are not an obstacle to the solving process, the students "come to the problem fresh" (Maher et al., 2010, p.10), the proof is therefore central.

All the research dealing specifically with the teaching of DM at different school levels (e.g. Rosenstein et al. 1997; Heinze et al. 2004; Hart & Sandefur 2018) has pointed out the advantages of DM to teach and learn proof, with the use of problems whose resolution mobilizes notions and skills such as implication, induction, etc. (e.g. Grenier & Payan 1998, 1999; Ouvrier-Buffet 2020). Nevertheless, few of the research works have characterized precisely proving processes which are specific to DM from the epistemological point of view. Thus, the didactical transposition of such proving processes in the classrooms must be questioned. That is why we will focus on the following research questions: what kinds of proving processes are

specific to DM from the epistemological point of view? Are these proving processes accessible for students from the didactical transposition's perspective?

We chose the field of discrete optimization to explore these questions.

Discrete optimization or combinatorial optimization is a part of mathematical programming which is "everything of direct or indirect use concerning the problem of optimizing a function of many variables, often subject to a set of constraints"[1].

We will explore its epistemological features through the following topics:

- Concepts of bound, framing optimal values, in particular: lower and upper bounds *versus* necessary condition and sufficient condition
- Minimal, minimum: local *versus* global optimality
- Dual problems
- Algorithmic approach and heuristics.

Note that the word *heuristic* is often used in mathematics education as Polyá (1954) defined it. In problem-solving, it includes, for instance, analogy, generalization, induction, decomposing and recombining, working backward. We will keep in mind the definition of *heuristic* coming from operations research, which is non-contradictory with that of Polyá: "In general, for a given problem, a heuristic procedure is a collection of rules or steps that guide one to a solution that may or may not be the best (optimal) solution" (Laguna & Marti 2013).

In this article, we will present our background designed collaboratively by discrete mathematicians and didacticians: the *Maths à Modeler* Research Federation and the didactical features of RSCs (Section 1). Then, we will develop two problems in discrete optimization (the *Pentamino Exclusion* and the *Eight Queens* problems) (Section 2) and their transpositions for the classrooms (Section 3). From an epistemological point of view, it will allow us to highlight and formalize proving processes which are transversal to DM and specific to discrete optimization such as partitioning strategies, greedy algorithms, primal-dual methods (Sections 4 and 5). The aim is to provide a refined theoretical mathematical background for these proving processes and the epistemological obstacles they imply for an educational perspective. Indeed, discrete optimization is less studied for itself in mathematics education than other fields of DM (see for instance the main view of the teaching and learning of DM worldwide in Hart & Sandefur 2018 or Heinze et al. 2004). It is also an ongoing contemporary field with proving processes and open problems

[1] Aims and scope of the journal *Mathematical Programming*, Springer.

in constant evolution. During the mathematical developments, we will mention students' works coming from several experiments designed with a common methodology (RSCs) to illustrate the above-mentioned epistemological features of discrete optimization. The conclusion will open perspectives regarding the teaching and learning of proof through DM and RSCs.

1. Features of discrete mathematics and the frame of *Maths à Modeler*

1.1. Discrete objects and proving processes in DM

The etymology of the word *discrete* leads to the word *countable* but also *separable*: DM deals with countable objects which have the property of splitting such as integers, graphs, sets, relations, orders, words, automata, language etc. In this way, DM contains finite mathematics.

In mathematics education, DM arouses unanimous interest because it offers a new field with non-routine problem solving for the learning and the teaching of different kinds of proof (e.g. Grenier & Payan 1999; Heinze et al. 2004; Hart & Sandefur 2018). Proofs by exhaustion, proofs by induction, algorithmic proofs, generic proofs but also probabilistic methods, decomposition techniques, and structuration of objects (such as coloration in graph theory) are applied to solve various kinds of problems in DM: existence, characterization, recognition, optimization and extremal problems. The accessibility of discrete objects by their representations, and the importance of the defining process linked to proof (e.g. Lakatos 1976; Ouvrier-Buffet 2015) complete this variety of proof techniques. The representation of a discrete object can spread a generic characteristic which facilitates the production of proofs. Moreover, discrete objects generally admit several equivalent definitions involving different representations, some of them being 'natural' and close to the formal definition. However, this seemingly accessibility of discrete concepts does not mean that solving discrete problems and searching proofs is easy. Besides, in contemporary mathematics, basic definitions are not stabilized yet for some discrete objects. For instance, there is no standard definition of either *orientation* or *cycle* for the *hypergraphs* (collection of finite sets of a given vertex set) (e.g. Berge 1973), or of objects in discrete geometry leading to complex questionings related to axiomatic theory (e.g. Reveillès 1991). This fact is particularly interesting to engage students in a wider mathematical activity.

1.2. A collaboration between researchers in DM and in mathematics education

In France, at the beginning of the 90s, a desire to promote 'doing maths' within school led to the creation of two main actions in education. A group working on *open problems* in Lyon initiated by G.Arsac (Arsac & Mante 2007) has developed *didactic engineerings* (Artigue 2014) for primary and secondary schools. Simultaneously, the *Math.en.Jeans* project[2] was initiated by a discrete mathematician, P.Duchet, in order to set up research groups in mathematics for middle and high school.

The research team *Maths à Modeler* supported by the French Ministry of Research was structured in 1999 by the discrete mathematicians S.Gravier and C.Payan in Grenoble. The research developed in *Maths à Modele*r is based on a close collaboration between didacticians and mathematicians, generating co-supervision of 11 PhD theses in didactic of mathematics and several implementations of research activities in the classrooms and in teacher training. It extends the research in mathematics education about proof and proving in the specific field of DM. *Maths à Modeler* has become a Research Federation led by the University of Grenoble Alps with antennas in France, Belgium, Brazil and Algeria, furthering the research on the teaching and popularization of mathematics through RSCs.

1.3. The definition of Research Situations for the Classrooms (RSCs)

In *Maths à Modeler*, innovative RSCs from real problems coming from the ongoing mathematical research are designed (e.g. Grenier & Payan 1999; Ouvrier-Buffet 2009) and implemented mainly in secondary and university levels, and also in popularization events.

The aim of RSCs is to put students in the role of a mathematical researcher. From a didactical point of view, RSCs are designed in the frame of the TDS (Brousseau 1997) and mainly mobilize the concepts of *didactical variables*, the *devolution* and the dialectics of the situations of *action*, *formulation* and *validation*. In this theoretical framework, notice that some research about proof have pointed out the difficult shift in the validation process (Balacheff 2010), the crucial role of the

[2] https://www.mathenjeans.fr

teacher's guidance (Mariotti et al. 2018; Stylianides 2016), and the difficulties in *institutionalizing* (Stylianides, 2007).

RSCs must fulfill the six following criteria (derived from Grenier & Payan 1999, 2003; Ouvrier-Buffet 2009). Some of them are close to problem solving.

- Firstly, the research field should be 'huge', have roots in mathematical ongoing research: a large part of it should be accessible to the students.
- Secondly, the mathematical problem should be easily understood.
- Thirdly, there should be few notional traps to start the research. Initial strategies without pre-requisites exist. These criteria foster the *devolution* of the mathematical problem. Therefore, there are only criteria of local resolution (local ending criteria) and possibly no final ending: an answered question often leads to a new question.
- Fourthly, the students should manage their research themselves: among the *didactical variables*, at least one of them, called a *research variable*, is left to the students, while the others can be set by the teacher. These variables are chosen depending on the didactical interest of the questions they can generate for the mathematical processes.
- Fifthly, many strategies may put the research forward and several developments are possible to enable the research process and the emergence of mathematical skills (such as proof, modeling etc.) and knowledge.
- Sixthly, at the end of the research process, students are invited to share their results, e.g. through posters in their school or a presentation at university (in front of other school students, who are also there to present their results, and researchers from university).

We underline that research is an activity that requires a big amount of time, so if one expects students to do research, one must grant them enough time to search: a long-term situation is essential.

1.4. Experiments with RSCs in the frame of the TDS

The TDS (Brousseau 1997) is used in order to design, analyze and implement RSCs. In Brousseau's engineering, the situation is divided into three adidactical phases (*action*, *formulation*, *validation*) to "foster the discovery and demonstration, by the children, of a sequence of theorems" (ibid. p.4). The interactions between the student(s) and the environment (*milieu*) constitutes the *dialectic of action*. This production of a set of experiences and results should be rich enough to nourish the

collective discussion in the next phase. The *dialectic of formulation* consists in "progressively establishing a shared language", making "possible the explanation of actions and modes of action". During the *validation* phase, students "have to put forward propositions and to prove to an opponent that they are either true or false" (ibid. p.4).

In the case of RSCs, a specific *research contract* is implemented to avoid the bias of the usual *didactical contract*. The main idea is to encourage students' research processes and to foster a mathematical community, close to the sociomathematical norms where the criteria for acceptability of arguments are negotiated in the classroom (e.g. Lampert 1990; Yackel & Cobb 1996). In RSCs, the initial problem comes from mathematical research, and nobody (neither the students nor the teacher) knows the solution. The teacher's reactions should not be an implicit indicator for the students of the value of their processes. Besides, the social dimension of exploratory proving remains fundamental (e.g. Balacheff 2010; Stylianides 2016). After the *devolution* of a RSC, the action phase engages students in the exploration of the problem, often with manipulatives. The formulation phase has an explicit social dimension: it allows to make the actions explicit. Validation takes place when interactions with peers include explanations and parts of proofs. The implementation of RSCs should preserve the dialectics of action, formulation and validation (Grenier & Payan 2003). The closure of a RSC with posters and/or oral presentation by students of their results ends the research process of the students and contributes to the *institutionalization* process (even if students' resolutions lead to new research questions).

Most of the PhD theses defended in *Maths à Modeler* offer RSCs with didactic engineerings in the frame of TDS with an epistemological background in DM: Colipan (combinatorial games) (2014), Deloustal-Jorrand (implication and necessary and sufficient conditions) (2004), Dissa (arithmetic and geometry) (2020), Giroud (modeling) (2011), Godot (open-problems and RSCs) (2005), Modeste (algorithm) (2012), Ouvrier-Buffet (defining) (2003). Several RSCs have been designed and implemented in different contexts: elementary and secondary levels, university level, pre-service and in-service teacher training, popularization events, and also with autistic children in a medical environment (e.g. Coffin et al. 2006). In this article, we will use students' works from all these experiments

implemented in various contexts to illustrate specific proving processes and skills in discrete optimization.

2. Two problems and their theoretical mathematical framework

2.1. *Hunting the Beast!* based on the *Pentamino Exclusion* problem

The *Hunting the beast!* problem involves polyominoes popularized by the game Tetris and the Martin Gardner's puzzles. A polyomino is a shape "made by connecting certain numbers of equal-sized squares, each joined together with at least 1 other square along an edge" (Golomb, 1966, p.19).

Here is the usual way to present the problem to students (Figure 1) in *Maths à Modeler*:

> Your garden is a collection of adjacent squares and a beast is a collection of squares. Your objective is to prevent a beast from entering your garden. To do this, you can buy traps. A trap is represented by a single square that can be placed on any square of the garden. The question is: what is the minimum number of traps you need to place so that no beast can land on your garden?

These rules allow the beast to be rotated (90,180 or 270 degrees) or reflected (flipped over) at will.

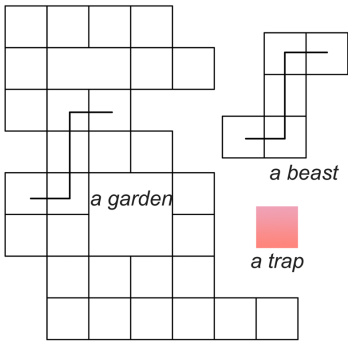

Fig.1. A garden, a beast and a trap

In this problem, placing a trap on each square is clearly not optimal, except if the beast is reduced to a single square (monomino). The difficulty thus lies in finding a configuration with the *smallest* number of traps. In the literature, this problem can be seen as a variation of the *Pentomino Exclusion* problem introduced by Golomb (1966). In the latter problem, the garden is a $k \times n$ rectangular board and the aim is to minimize the number of monominoes so that no pentomino can be placed on it. For both problems, it turns out that the computation of the minimum number of

traps is NP-complete[3] in terms of algorithmic complexity. However, for some special cases of the board and the beast, results exist about the minimum number of traps (e.g. Dorbec 2007; Gravier & Payan 2001). Such problem leads to new research questionings in graph theory (e.g. Gravier, Moncel & Payan 2007) so as to new integer programming models still in progress (e.g. Kartak & Fabarisova 2019).

2.2. *Fireworks* (based on the *Eight Queens* problem)

Here is the usual way to present the problem to students in *Maths à Modeler*:

> In a warehouse, we try to store the maximum number of cases of fireworks under the following constraint: if one case explodes, it does not damage any other case.
>
> For our purpose, the warehouse will be a subset of the grid. A case of fireworks consists of a single cell of the grid for which we attached a security zone depending on the explosion mode. A placed case of fireworks in the warehouse can therefore prohibit a certain set of cells (Figure 2).

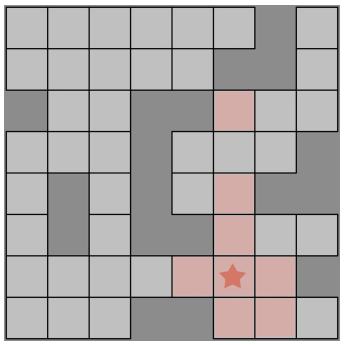

Fig.2. Example of a warehouse and a security zone

If the warehouse is the 8×8 chessboard and the security zone of a case of fireworks placed on a cell is the set of cells on the same row, column and the two diagonals, then we get the so-called well-known *Eight Queens* problem. One attributes this problem to the German chessman M. Bezzel (1848). It was generalized for *n* queens by the mathematician F. Nauck (see for historical references Campbell 1977). Lucas (1882) published the first complete solutions of this problem. Since then, several extensions of this problem have still been studied such as dominating and stable sets in grid-like graphs, generalized Latin squares, etc.

2.3. General mathematical problem

[3] i.e. there is no known algorithm that computes an optimal solution with a polynomial number of elementary operations.

A lot of problems of discrete optimization consists of finding a subset of a given *ground set* satisfying some given *constraints* with maximum or minimum cardinality (called *OPT*).

A *feasible solution* is a set satisfying the required constraints. Finding a feasible solution of cardinality k is an *existence* problem **(P1)** which gives an *upper bound* $OPT \leq k$ whenever *OPT* is a minimum. Analogously, it gives a lower bound $OPT \geq k$ whenever OPT is a maximum.

> For instance, a feasible solution of *Hunting the beast!* is a set of cells of the garden (the ground set) where one places traps with the following constraint: each beast in the garden contains at least one cell with a trap.
>
> Similarly, for *Fireworks*, a feasible solution consists of storing a set of cells of the warehouse (the ground set) where one places cases of fireworks with the following constraint: if one case explodes, it does not damage any other case.

A *solution* of the optimization problem is a feasible solution with optimal cardinality. Proving that a feasible solution of cardinality k is optimal is a difficult *non-existence* problem **(P2)** which involves an universal quantifier ($\forall$). Indeed, if *OPT* is a minimum, we have to prove that any set of size n, with $n < k$, is not a feasible solution. This leads to the *lower bound* $OPT \geq k$.

A *weak* version of the problem could be to determine *OPT*.

This would be interesting if, for instance, we had a proof[4] of the optimality value without an explicit description of a solution. Since we are interested in proposing this kind of problem for a wide public, it is easier to ask to exhibit an optimal solution.

We will highlight which kinds of proofs and proof techniques might be involved in finding solutions. In discrete optimization, (P1) and (P2) are usually explicitly stated and their proofs are distinct, whereas in continuous optimization problems, the resolutions of (P1) and (P2) are often put together in an asymptotic argument. It is particularly interesting to clearly distinguish these two problems since the notions of upper and lower bounds are related to sufficient and necessary conditions. Proposing discrete optimization problems allows us to work on sufficient and necessary conditions in a ‘concrete’ way.

[4] For instance, probabilistic methods bring such proofs.

Indeed, for minimizing optimization problems, the lower bound $OPT \geq k$ shows that k elements are necessary to satisfy all *constraints*. Moreover, exhibiting a feasible solution of cardinality k shows that k elements are sufficient to satisfy all constraints.

For a lot of combinatorial optimization problems, the constraints can be expressed as an intersection's condition between a feasible solution X and a given family $\mathcal{F}$ of subsets of the ground set. We will focus on the condition on cardinality of this intersection. Let us now define two problems (P1) as follows:

(T) Find a subset X of the ground set of cardinality k such that $|X \cap C| \geq 1$ for all $C \in \mathcal{F}$.

(M) Find a subset X of the ground set of cardinality k such that $|X \cap C| \leq 1$ for all $C \in \mathcal{F}$.

A feasible solution of (T) is called a *transversal* of $\mathcal{F}$ and a feasible solution of (M) is called a *matching*.

Observe that the ground set is a feasible solution of (T) and the empty-set ones of (M). Therefore, it is interesting to minimize the set of feasible solutions of (T) and to maximize those of (M). Therefore, for the rest of the paper, one considers that a (T) (resp. (M)) problem is a minimizing (resp. maximizing) problem.

> *Hunting the beast!* is a (T) problem with $\mathcal{F}$ be the set of all beasts in the ground set. *Fireworks* is a (M) problem with $\mathcal{F}$ be the set of all security zones.

3. Towards Research Situations for the Classrooms

3.1. The RSC *Hunting the beast!*

Presenting the problem in a general context leads to discussions to reduce the space of research. One may simplify the problem by setting the following didactical variables.

- The number of squares used to define the beast: naturally, hunting small beasts appears to be more accessible. A beast is connected but not necessarily convex. To hunt convex beasts is a simplification of the problem.
- The shape of the beast plays an important part: for example, hunting a rectangle (whatever its size) is a problem that is completely solvable through elementary considerations. On the contrary, the problems may become

harder if the beast is a non-connected set of squares. Several beasts are given considering the topology of the beasts below.

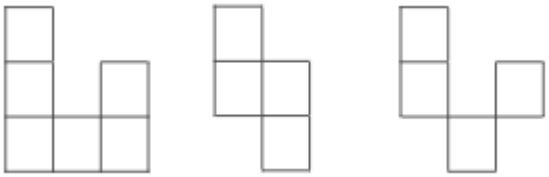

Fig.3. A connected but not convex beast, a convex beast, a non-connected beast

- The number of types of beasts that are simultaneously excluded complexifies the research: in the initial case, we consider only one type of beast. In the *Pentomino Exclusion Problem*, all the polyominoes of size 5 are hunted, which is very complex. After having considered each type of polyomino separately, excluding all of them simultaneously is also interesting.
- The size and the topology of the board are also important: it may be relevant to let the students work on larger sizes, or also play on rectangular boards.

During experiments (from elementary to university levels), we set the didactical variables in order to imply the devolution of the problem and an effective search of optimal configurations and proofs. It appears that playing on a rectangular board is reasonable. A first possible step may consist in letting the students choose a reasonable sized beast (consisting of less than 6 squares) and a 8×8 or a 7×7 board. The shape of the beast is then a research variable. After this first step, three kinds of polyominoes are chosen for the experiments in the classrooms to make the proving process evolve. We usually choose a 5×5 garden, a monomino for a trap and three kinds of beast (Fig. 4). We will use this configuration in the following analysis of proving processes at stake. Considering a 5×5 board is already enough to do nontrivial effective search of optimal configurations and to formulate arguments and proofs. It is also possible to open the problem with other values of the didactical variables and then to question the generalization of the results. For instance, for rectangular gardens n×m with the straight tromino, the whole part of nm/3 is the optimal value (Dorbec 2007).

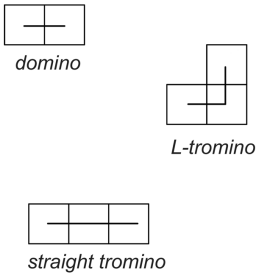

Fig.4. The beasts

3.2. The RSC *Fireworks*

We propose simpler formulations of the problem so that it is accessible at elementary school. Even in the simplified formulations, interesting proving processes arise. In a similar manner as for *Hunting the beast!*, the didactical variables of *Fireworks* concern the shape and the size of the warehouse and the characterization of the security zone. The topology of a case of fireworks can also be a didactical variable, but here we only consider a single cell.

From now on, the warehouse will be a $n \times n$ grid. To build a RSC based on this problem, we leave n as a research variable available to the students.

This RSC is usually scheduled with a first step which enables the devolution. We ask students to set a (reasonable) security zone and solve the problem on a set warehouse (generally a 8×8 or 7×7 grid). During this step, symmetric properties on the security zone are discussed: for example, if the cell $(i,j+t)$ is in the security zone of cell (i,j), then one may assume that cell $(i,j-t)$ also belongs to its security zone. Indeed, in a feasible solution, cells $(i,j-t)$ and (i,j) can occur simultaneously. This first step mainly allows us to identify the questions which are problems (P1) and (P2). As it appears that finding answers for these problems seems difficult with original security zones, we propose to study special security zones such as those of chess moves.

We refer to Gandit et al. (2014) for a detailed resolution of this problem when the security zone is defined as the king's move in chess. Later, we will use the king's move to illustrate a less effective general method for the other moves.

From now on, we consider the security zone defined by the fool's move in chess, that is the set of cells belonging to the same diagonals of the place where the case of fireworks (fool) is. Formally, if we place a case of fireworks on the cell of coordinates (i, j), then its security zone is the set $\{(a, b) \mid a+b=i+j \text{ or } a-b=i-j\}$. We

note the diagonals $D_{i+j} = \{(a, b) \mid a+b=i+j\}$ and $D_{i-j} = \{(a, b) \mid a-b=i-j\}$. Students often used a graphical register to represent the security zone or the diagonals instead of a formal one (Figure 5).

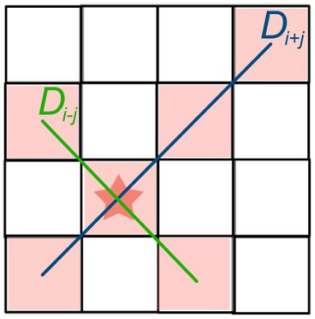

Fig.5. The diagonals

In the following sections, we will describe the main phases which occur in proofs when solving combinatorial optimization problems. We will analyze the construction of feasible solutions (P1) (§4) and the proof of the optimality of our best feasible solution or the proof of the bound of the optimal value (P2) (§5). The described strategies will be illustrated with students' works.

4. Construction of feasible solutions – Existence problem (P1)

We will develop a mathematical analysis of both problems (§4.1 & 4.2) and illustrate it with students' works. The aim is twofold: to reach an overview of the general mathematical framework (§4.3) and to point out the accessibility of such problems and proving processes in the classrooms.

4.1. *Hunting the beast!* Local arguments and difference between minimal and minimum

We remind the reader that to solve the problem, it is necessary to exhibit a way of placing a certain number of traps on the garden so as to exclude the given polyomino (P1).

One can develop local strategies of optimality when searching for the use of a minimum of traps. Additionally, the symmetry of the ground set suggests finding a symmetrical feasible solution as well. We will give an example of such a strategy (which often occurs in the classrooms) with the straight tromino.

To exclude a beast of a single row off a 5×5 garden, we need at least one trap. This trap must be on the middle square. The same is true for columns. Consider the first

row and the last row by symmetry, on the one hand, and the first and the last columns on the other hand: we obtain the partial selection of the traps in Figure 6a. Still by symmetric argument, we choose the midpoint of the 5×5 garden (Fig. 6b). The cell (2;2) is one of the remaining cells of the garden which contains the greatest number of straight trominoes (Fig. 6c), then we select this cell by a local optimality argument. Finally, by symmetry, we obtain the solution of Figure 6d.

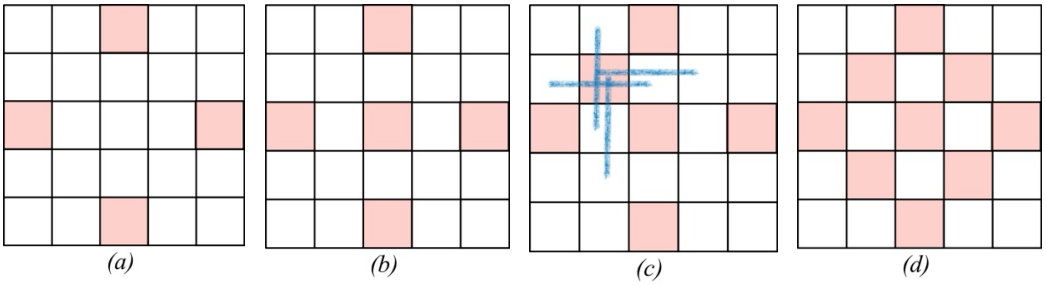

Fig.6. Strategy based on symmetrical and local arguments

Such a strategy is usable for any shape of the beast. It generates a feasible solution which can have the following property of *minimality*: if one removes any trap from such a feasible solution, then one gets a set of traps which is not a feasible solution (Figures 7).

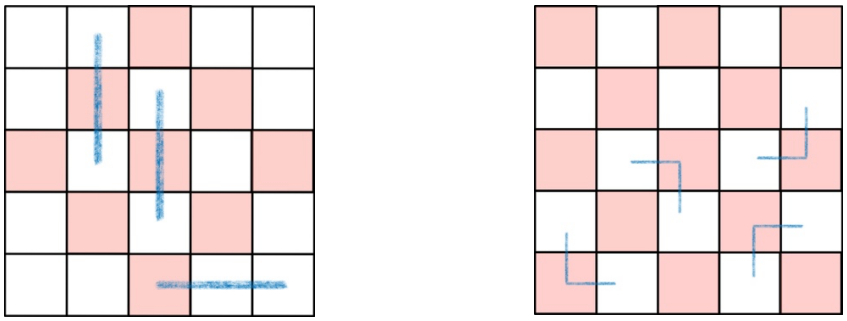

Fig.7. Minimality of feasible solutions for the straight tromino and the L-tromino

This kind of argument is used by students to claim the optimality of a solution: "If I remove any trap, a beast can land in the garden. Therefore, my value is optimal." But it only proves that the feasible solution cannot be improved by removing certain traps: it means that this feasible solution is *minimal* (not necessarily *minimum*). Students may be convinced that this argument fails because they get a better solution after a few trials (Figure 8).

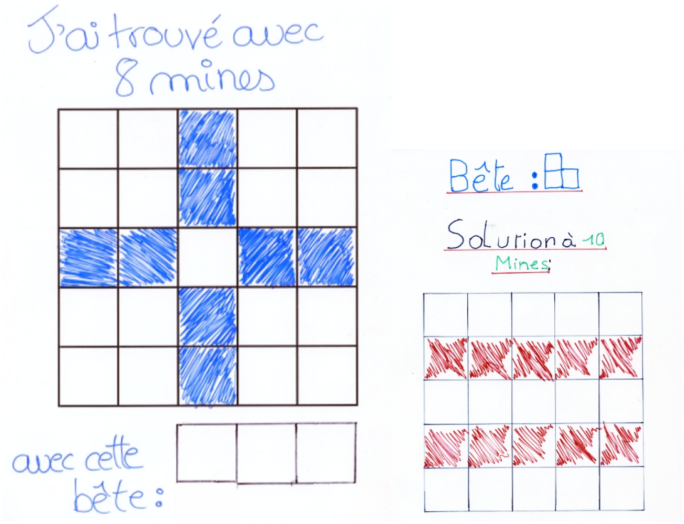

Fig.8. Students' works of feasible solutions [straight tromino (left) and L-tromino (right)]

4.2. *Fireworks* - Partitioning strategy and difference between maximal and maximum

To generate a feasible solution for *Fireworks*, we can use a *heuristic* (*local strategy*) consisting in placing one case of fireworks, removing its security zone, choosing another one and iterating this process until there are no more cells. This strategy is often used by students even if it is not explicitly stated. The choice of the first element is not necessarily justified. Some geometrical arguments may be proposed (corner or center) without a clear relationship with the problem.

An example of such a strategy for the security zone defined by the fool's move whenever n=8 is given in Figure 9. Then, with this heuristic, we obtain a *maximal* solution with 9 cases of fireworks. Here, *maximal* means that if we add another case of fireworks, then we get a non-feasible set i.e. this new case of fireworks will damage another one if it explodes. So, we get the first bound OPT≥9.

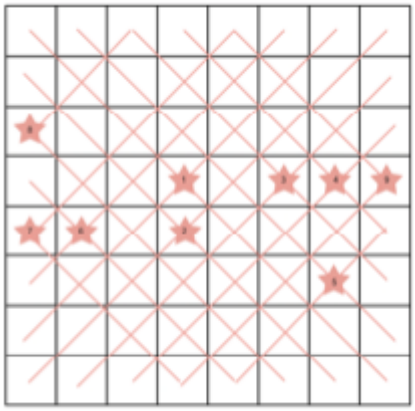

Fig.9. Local strategy for fool's move on a 8×8 chessboard

A new investigation gives a solution with 14 fools (*2n-2* in the general case) in Figure 10. This allows us to discuss the difference between *maximal* and *maximum*.

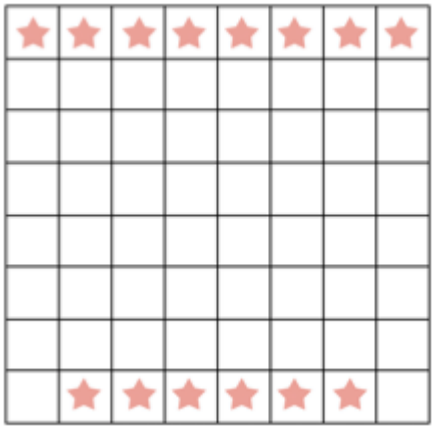

Fig.10. Solution with 14 fools

After more attempts, we fail to improve this solution, so the conjecture that we have reached the optimal value can be formulated.

When the security zone is bounded (not depending on *n*) such as for the king's move (left, right, up, bottom), then it may induce a partitioning strategy: it consists in tiling the ground set (by translation) with tiles with one marked cell corresponding to the place of the king. We have to make sure that this tiling provides a feasible solution. This strategy is illustrated for n=8 in Figure 11.

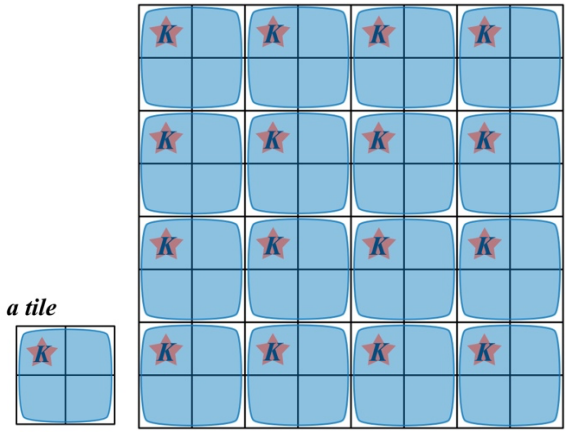

Fig.11. Partitioning strategy to get a feasible solution with 14 kings

4.3. Proving processes: the greedy algorithm and the primal-dual method

4.3.1. A heuristic: the greedy algorithm

For both problems, we have described local strategies to build feasible solutions. These strategies are particular cases of the *heuristic* so-called *greedy algorithm*. It consists in building a feasible solution, element by element, following local choice

at each step to try to reach the optimality of the solution obtained i.e. always choosing the next element that offers an *immediate benefit*. This approach never reconsiders the choices taken previously.

It means:

- a trap which excludes a beast not excluded by the other selected traps in *Hunting the beast!*
- a new firework which does not damage another one if it explodes in *Fireworks*.

When the algorithm stops, we get a feasible solution.

To choose the next element, some heuristics may be used. For instance, in the case of a minimization problem: if, at some step, one has chosen a subset X of elements, then one chooses a new element e such that the set $X+e$ satisfies a maximum number of constraints. One can still get a minimal feasible solution which is not necessarily optimal even if at each step the heuristic chooses an optimal element. The solution can then be locally optimal[5] but not necessarily globally optimal. It is not natural at all and appears as an epistemological obstacle: "Doing the best at each step does not guarantee the best at the end".

It is worth pointing out that greedy algorithms are intensively studied because they are simple and practical. A first application of such algorithms was used by Dijkstra for a graph theoretical problem (Dijkstra 1959). Ongoing research of mathematicians is to understand for which problems there is an effective greedy algorithm. Matroids theory has attempted to provide an axiomatic framework describing the problems for which a greedy algorithm applied with any order of selection of the elements guarantees the optimality of the result (Welsh 2010).

4.3.2. The primal-dual method - towards duality

For *Fireworks*, we have explained a partitioning strategy based on the description of a tiling into *parts containing exactly one element* of the feasible solution.

We remind that a *partition* is a collection of subsets (parts) which have two properties:

- They are pairwise disjoint. Such a collection is called a *Packing*.
- Their union contains the ground set. Such a collection is called a *Covering*.

[5] Sometimes the heuristic gives a minimal or a maximal solution (as described in *Fireworks*).

To obtain a generalization of the partitioning strategy, we first need to determine, for a given (P1) problem, denoted (Q), a collection of subsets $\mathcal{B}$ (the parts) of the ground set satisfying the following property (*):

> If (Q) is a problem of maximization, then *any feasible solution of (Q)* contains *at most one* element in each subset of $\mathcal{B}$.
>
> Else *any feasible solution of* (Q) contains *at least one* element in each subset of $\mathcal{B}$.
>
> > If (Q) is the *Fireworks* problem with king's move, then let $\mathcal{B}$ be the set of 2×2 squares of the warehouse. Since at most one king can be placed on a 2×2 square in a feasible solution of (Q), then $\mathcal{B}$ satisfies (*).
> >
> > If (Q) is the *Hunting the beast!* problem, then let $\mathcal{B}$ be the set of all beasts of the garden. Since at least one trap must be placed on each beast in a feasible solution of (Q), then $\mathcal{B}$ satisfies (*).

Secondly, we define the problem (D) of $\mathcal{B}$, or (D) if there is no ambiguity, as follows:

> If (Q) is a problem of maximization, then (D) is "Find a *covering* of the ground set using only sets of $\mathcal{B}$".
>
> Else (D) is "Find a *packing* of the ground set using only sets of $\mathcal{B}$".

Suppose now that (Q) is a maximization problem (M) or a minimization problem (T) of a given collection $\mathcal{F}$, i.e:

> (M) Find a subset *X* of ground set of cardinality *k* such that $|X \cap C| \leq 1$ for all $C \in \mathcal{F}$, or
>
> *(T)* Find a subset *X* of ground set of cardinality *k* such that $|X \cap C| \geq 1$ for all $C \in \mathcal{F}$.

In both cases, the collection $\mathcal{F}$ satisfies (*). The problem (D) of $\mathcal{F}$ is called the *dual* of (Q) and denoted by $(Q)^*$:

> If (Q) is a (M) problem, then $(Q)^*$ is a *Covering* problem (restricted to subsets in $\mathcal{F}$).
>
> If (Q) is a (T) problem, then $(Q)^*$ is a *Packing* problem (restricted to subsets in $\mathcal{F}$).

In that case, we have that $(Q) = ((Q)^*)^*$.

Furthermore, a feasible solution of such a *Packing* problem is a sub-collection $\mathcal{C}$ of $\mathcal{F}$ of pairwise disjoint sets which can be rewritten as: $|\mathcal{C} \cap e| \leq 1$ for all elements e

of the ground set. Therefore, if you (re)define the element e of the ground set as the "set of all subsets in $\mathcal{F}$ which contains e", then the *Packing* problem can be seen as a *Matching* problem.
Similarly, a *Covering* problem can be seen as a *Transversal* problem.
We obtain that (M) and (T) are dual problems as well as *covering* and *packing*.

We can prove that the word *duality* has the same meaning in geometry, algebra and combinatorics: it is linked to the notions of respectively sphere covering, generating set and transversal for (Q) and sphere packing, independent set and matching for (Q)*.
From this terminology, the partitioning strategy can be seen as a *primal-dual method* by solving at the same time the *primal* and the *dual* problems. Such methods consist in using a dual solution to build a feasible solution of the primal problem which satisfies *optimally* each (or a maximum of) constraint(s) of the dual solution. The primal-dual method, proposed by Egerváry in the early 1930s in combinatorial optimization and then by Dantzig, Ford, and Fulkerson in 1956 in linear programming, is nowadays modified to deal with NP-hard problems (Goemans & Williamson 1996).
From a didactical point of view, this approach should be carefully checked since it could reveal a confusion between necessary and sufficient conditions.

5. Proof of optimality (P2)

5.1. *Hunting the beast!*

In the previous section, we have exhibited strategies to construct feasible solutions of (P1). We now prove that no fewer traps could have been used for the same purpose "by whatever combinatorial reasoning or tricks suggest themselves" (Golomb, 1966, p.42).

Proof by exhaustion (also called proof by cases, or by case analysis)
Examining all possible cases is always possible, mainly for small cases, but remains laborious and leads to combinatorial explosion phenomenon in the general cases. It motivates the use of another kind of reasoning.

Necessity and sufficient proof
The manipulation of beasts brings the necessity proof, accessible in primary schools, and questions the sufficient proof. Observe that any beast in the garden

forces to place a trap on at least one of the cells occupied by it. This observation may suggest considering the *dual packing* problem i.e. to fill all the garden with disjoint beasts.

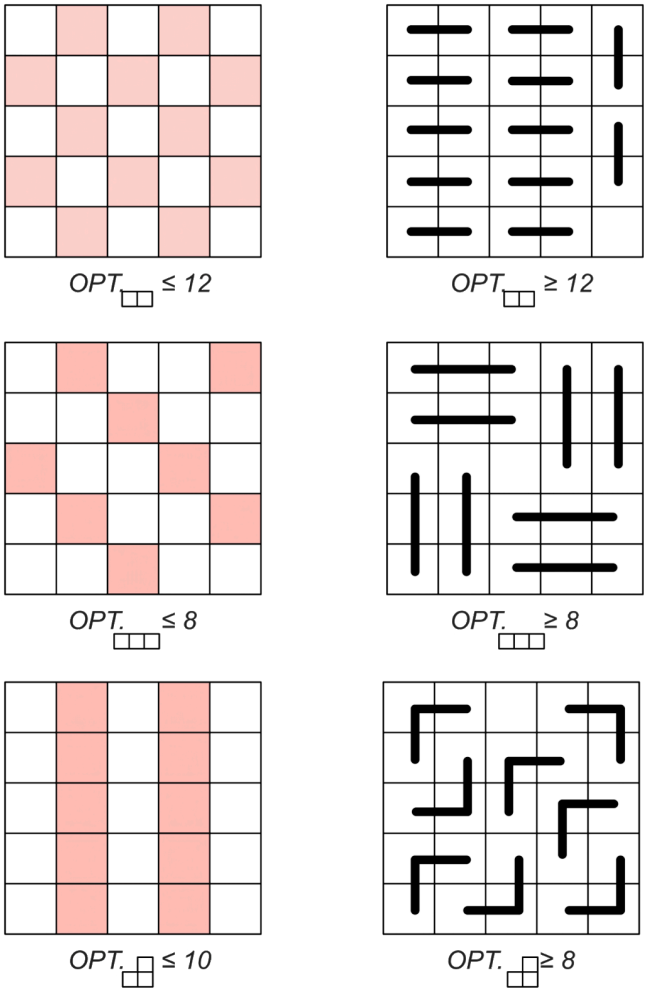

Fig.12. Proof illustrations for *Hunting the beast!*

For the domino (resp. straight tromino), the packing technique proves that 12 nonoverlapping dominoes (resp. 8 straight trominoes) are necessary to cover the garden. Then, a minimum of 12 traps (resp. 8) is necessary to exclude the domino (resp. straight tromino): it is a *lower bound*. To exhibit a placement of 12 traps (resp. 8) is sufficient to end the proof. Domino and tromino can be seen as generic examples and can lead to a generic argument linked to the following proposition (which can be proved): “If one covers the garden with n non-overlapping beasts, then at least n traps are needed to exclude the beast.” For the L-tromino, this kind of proof is not enough. Indeed, only 8 L-trominoes cover the 5×5 garden, but 10 traps are needed to exclude the L-tromino. Then, the optimal value is framed between 8 and 10.

In some experiments, it appears that after solving the problem with straight tromino, the students started to solve the problem with L-tromino by placing beasts in the garden and they claimed "the minimum of traps is 8 since there are 8 beasts" (Figure 13). Clearly, there is a confusion here between *necessary* and *sufficient* conditions. Then, one may ask students to give a solution with 8 beasts.

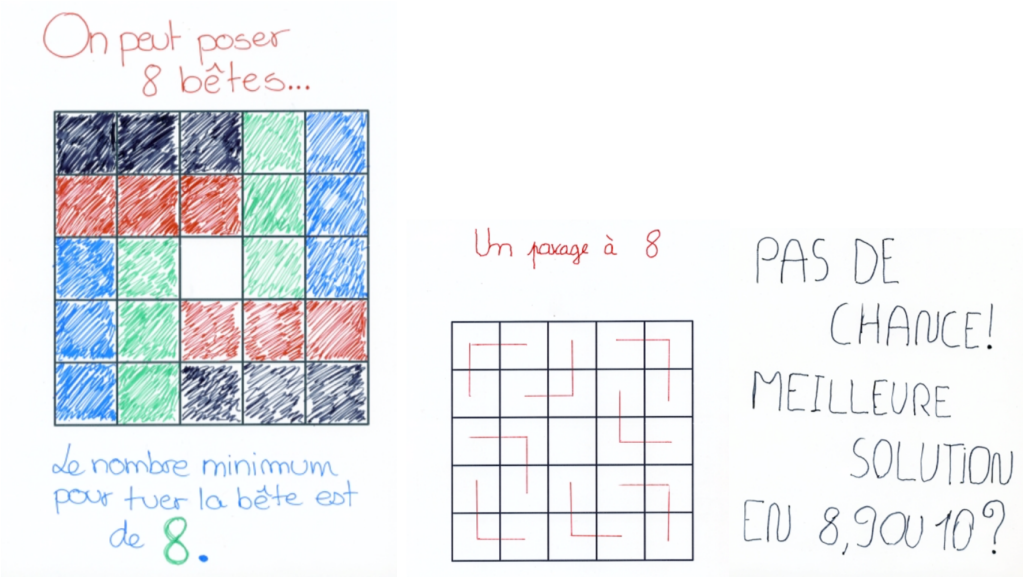

Fig.13. Students' slides for an oral presentation at the end of an experiment

So, studying L-tromino enables work on the differences between necessary and sufficient conditions. In any case, whenever the beast is the L-tromino, one may study smaller gardens. This may allow us to remark that 2 traps are needed to keep the L-tromino off a 2×2 garden (Figure 14). Then, one trap is needed for the beast landed on this garden and one more trap on the cell not covered by a beast in order to keep another neighboring beast off the 2×2 garden.

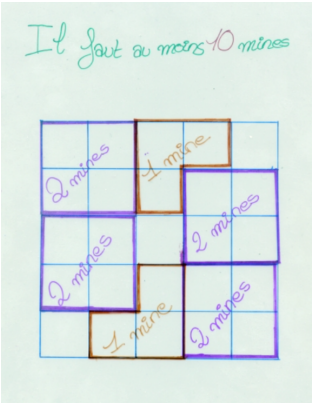

Fig.14. Students' representation of the lower bound to exclude the L-tromino

So, for each 2×2 garden, 2 traps are needed. When one structures the 5×5 garden with 2×2 squares (Figure 15), four 2×2 squares 'at worst' appear and two other beasts can come: then, 4 times 2 traps and 2 traps are required, so 10 traps in order

to keep the beast off the 5×5 garden. We have solved *Hunting the beast!* on the 5×5 garden.

Induction

If we generalize the problem to rectangular $k \times n$ gardens (with a fixed small k), this problem leads to reasoning related to *induction*.

Other very interesting mathematical proofs exist, exploring rows and columns arguments (Ouvrier-Buffet et al. 2017) or covering numbers, but are too long to be included in this article. Another perspective is to exclude both straight and L-trominoes.

5.2. *Fireworks*

We remind that for the fool's move, we have built a solution with 14 fools on the 8×8 board. After several attempts, we failed to improve this solution, so the conjecture that we reached the optimal value can be formulated. At this stage, the students tried a *case analysis*, but there were too many cases to consider, and they failed to find a full proof. One suggested studying the smallest warehouses and therefore a *partitioning strategy* could appear and then a work on the *dual covering* problem: to place the smallest number of diagonals whose union covers the entire 8×8 board.

As for *Hunting the beast!* the students used *partitioning strategy* as a proof argument, but they did not see it as a new problem.

Quickly, the students placed $2n$ such diagonals: $D_{1\text{-}1}$, …, $D_{1\text{-}n}$, $D_{2\text{-}1}$, …, $D_{n\text{-}1}$

(*(1, 1)* corresponds to the cell on the top left on the $n \times n$ board).

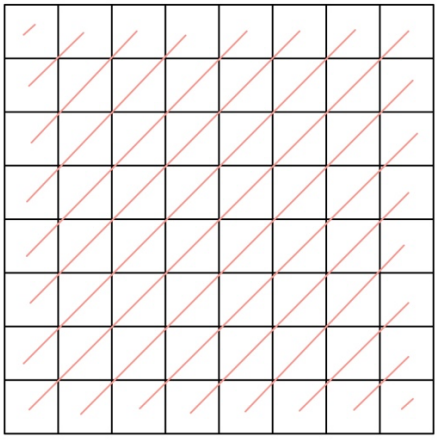

Fig.15. $2n$ diagonals on a $n \times n$ board

So, we got the closest frame on optimal value: $2n\text{-}2 \leq OPT \leq 2n\text{-}1$.

To conclude, it is enough to observe that diagonals D_{n-1} and D_{1-n} are reduced to one cell. Moreover, the security zone of D_{n-1} contains D_{1-n} and vice-versa. So any solution of (P2) has at most one of the cells D_{n-1} and D_{1-n}. This property is sometimes expressed in terms of *contradiction*: "If there is a case of fireworks in D_{n-1} and D_{1-n} then D_{1+1} contains 2 fools". Finally, we get the desired upper bound $OPT \leq 2n-2$, which corresponds to a necessary condition on the value of OPT.

To continue the mathematical problem and the RSC, we can change the value of some didactical variables. For instance, we could propose other chess moves to investigate and propose forthcoming works with a $n \times m$ board ($m \neq n$).

5.3. General proving process for (P2)

To solve (P2) for a given problem (Q), the exhaustive methods are quickly given up on because of the combinatorial explosion phenomenon.

Another strategy consists in determining a problem (D) as described in §4.3.2. The goal is to find constraints defined by subsets $\mathcal{B}$ for which any feasible solution of (P1) has a bounded intersection such that:

> If (Q) is a problem of maximization, then *any feasible solution of (Q)* contains *at most one* element in each subset of $\mathcal{B}$.
>
> Else *any feasible solution of* (Q) contains *at least one* element in each subset of $\mathcal{B}$.

Remember that the problem (D) of $\mathcal{B}$, or (D), is defined by:

> If (Q) is a problem of maximization, then (D) is "Find a *covering* of the ground set using only sets of $\mathcal{B}$".
>
> Else (D) is "Find a *packing* of the ground set using only sets of $\mathcal{B}$".

This will help to solve (P2) since (D) has the following property (**):

> If (Q) is a maximization problem, then the cardinality of a feasible solution of (D) is *greater* than that of a feasible solution of (Q).
>
> Else the cardinality of a feasible solution of (D) is *smaller* than that of a feasible solution of (Q).

In particular, if (Q) is a maximization problem, then the minimum of (D) is *greater* than the maximum of (Q). Else, the maximum of (D) is *smaller* than the minimum of (Q). This relation between optimal values of (Q) and (D) is called *weak duality* whenever $(D) = (Q)^*$.

Besides, as shown in §4.3.2, *packing* and *covering* problems are existence problems (P1) for which we know effective procedures to get feasible solutions. So, the challenge to solve (P2) now is to look for a suitable collection $\mathcal{B}$.

One way is to study small ground sets and exhaustively prove the required intersection constraint for these subsets.

> For instance, diagonals in *Fireworks* with fool allowed to define a *covering* problem.

When (D) is defined, solving it is now an *existence* problem like (P1). Thus, the universal quantifier of (P2) is only needed to determine the constraints defining (D).

> For *Hunting the beast!* problem (T), the problem (P2) is directly transformed, by duality, into an existence $(T)^*$ = find a *packing* of the ground set with beast.

Now, to solve (P2), it is enough to use (**) relation between (D) and (Q).

> For *Hunting the beast!* with domino and straight tromino, the inequality is an equality.

In that case, we say that the primal and dual problems have a *strong duality* relationship.

> Now, to solve *Hunting the beast!* with L-tromino, we considered more restrictive constraints (2 traps are necessary to exclude all the beasts off a 2×2 square).

In general, suppose that we find a subset *A* of *X* not in $\mathcal{F}$ such that any solution of *(P1)* contains at least *a* (≥ 1) elements of *A*. So, one may add this constraint in (D) as follows:

> Find a sub-collection $\mathcal{C}$ of pairwise disjoint sub-subsets of $\mathcal{F} \cup A$ which maximizes $|\mathcal{C}| + |\mathcal{C} \cap A|.(a-1)$.

A solution of this new problem is one of (D) and has a greater or equal cardinality. This technique may improve the upper bound on the size of a solution of (P1).

> Similarly, for *Fireworks*, the subsets $\mathcal{F} = \{D_{1-1}, \ldots, D_{1-n}\} \cup \{D_{2-1}, \ldots, D_{n-1}\}$ were not sufficient to solve (P2). So, it was proved that adding subset $A = D_{1-1} \cup D_{n-1}$ (with $a = 1$) allowed to get the optimal value.

Of course, this process can be repeated. We obtain a weighted version of *(D)*.

For instance, for *Hunting the beast!* the sub-collection $\mathcal{C}$ is composed of disjoint subsets of $\mathcal{F} \cup A$ where $\mathcal{F}$ are the beasts and A are the 2×2 squares in the ground set. Figure 14 illustrates such a feasible solution.

This technique of *adding new constraints* is classical in combinatorial optimization and can be seen in the case of linear programming as the *cutting plane method* introduced by Gomory (1960). The idea is to attain a new problem *(D)*' for which the size of an optimal solution is equal to one of (P1).

Conclusion and perspectives

Discrete mathematics has a huge potential to design learning situations of proof and proving processes and, more generally, mathematical inquiry. Even if a lot of discrete mathematics problems can be easily explained, most of them are really difficult to solve. A non-trivial selection of "good" mathematical problems for classrooms has to be validated by epistemological and didactical *a priori* analysis, but also by several experiments in classrooms in a long-time process.

Exploiting two contemporary problems, we have emphasized several proving processes used in discrete optimization. From the mathematical point of view, we have underscored specific proving processes and heuristics, mainly: partitioning strategies, greedy algorithms, primal-dual methods, weak and strong duality between packing and covering problems, cutting planes method. In such discrete problems, more classical proving processes such as proof by exhaustion or proof by induction can also be used. In all these proving processes, we have also pointed

out epistemological obstacles which lie in confusions or misunderstandings of the following topics:

- Necessary and sufficient conditions can make sense through the notions of lower and upper bounds.
- Definition of dual problems gives a general framework to obtain these bounds.
- The distinction between optimal and optimum appears during a problem of optimizing the cardinality of a set when the students are used to optimizing numerical functions.
- Again, the distinction between local and global optimality arises when the students produce feasible solutions with simple algorithms like greedy ones.

Using the TDS to transpose mathematical research activities developed in *Maths à Modeler*, we have designed two RSCs (*Hunting the beast!* and *Fireworks*) with the *Pentamino Exclusion* and the *Eight Queens* problems. Several experiments, mainly realized in action-research contexts but carefully planned and implemented with the help of TDS, have revealed regularities in students' processes at all school levels (from primary school to university) (e.g. Gandit et al. 2014; Ouvrier-Buffet et al. 2017). Most of these regularities were predicted in the *a priori* analysis whose mathematical components are developed in Sections 3, 4 and 5. In particular, the two RSCs are particularly efficient to work on the distinction between necessary and sufficient conditions and on optimality (local versus global, optimal versus optimum). In the experiments, the proof of the sufficient condition which corresponds to solving an existence problem involved strategies, sometimes even algorithms, to produce a 'good' solution i.e. a solution that students cannot improve. Greedy algorithms emerged in students' works. They are also used by mathematicians. Students generally adopted a partitioning strategy to prove the necessary condition: this technique allowed them to prove that the previously obtained solution could not be improved. There are two clearly distinct ways to prove the sufficient condition and the necessary condition respectively. Nevertheless, the students did not see that their partitioning strategy could be seen as a new problem i.e. the dual and/or detected this new problem as the solution to the previous one. This points out the limits of adidactical situations for proving processes and transversal concepts such as primal-dual methods and duality. Indeed, in an educational context, and nowadays in the frame of Inquiry Based

Mathematics Education (Artigue & Blomhøj 2013), researchers in mathematics education must question the relevant skills at stake in a mathematical inquiry. In the examples developed in this paper, it leads to the following new questions: How to choose proving processes to teach for curricula in mathematics? How to transpose such processes which are relevant in a mathematical inquiry to the classrooms? How to institutionalize such processes which *a priori* required a long-term process in the classrooms? Finally how to redesign teacher training? Such questions are crucial at the international level both for DM (Rosenstein 2018) and mathematics in general.

It is obvious that a collaborative work with research-active mathematicians can generate frameworks and tools for education at the interplay between mathematics education and mathematics (e.g. Alcock & Simpson 2009). The research dealing with proof and ongoing mathematics using interviews with mathematicians (e.g. Lockwood et al. 2016; Ouvrier-Buffet 2015; Weber 2011) and the RSCs clearly open new perspectives for the teaching and learning of proving processes in DM: the mathematical background developed in this paper allows the design of new didactic engineerings in discrete optimization, including teacher training.

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