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# Electromagnetic tensors.

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## Résumé

The electromagnetic potential  $A^i$  is a quadrivector in Minkowski space-time  $x^k$  and its gradient  $a_k^i$  is a tensor of rank two whose elements are the sixteen partial derivatives  $\partial A^i / \partial x^k$ . We study in this article the properties of a family of tensors resulting from  $[a_k^i]$ . We first introduce the covariant tensor  $[a_{ki}]$ . Four initial tensors are obtained by separating  $[a_k^i]$  on the one hand, and  $[a_{ki}]$  on the other hand into their symmetric and antisymmetric parts. These are  $(s_k^i, [f_k^i], [S_{ki}], [F_{ki}])$ . As the lowering-raising index operations and symmetrization-antisymmetrization operations do not commute, these four tensors are different. We associate a Lagrangian density  $\mathcal{L}$  to the determinant of  $[a_k^i]$  which is invariant in an operation of symmetry of the Poincaré group. In the first part of the article, we show that there is a particular coordinate system where the scalar potential obeys the Helmholtz equation. The solutions allow to describe the "electromagnetic particles", characterized by three quantum numbers  $n, \ell$  and  $m$ . We give the tensors corresponding to the first five solutions. They describe energy and electric charge distributions. The condition of existence of these particles is related to a property of the electron described in the Wheeler-Feynman's absorber theory. In a second part, we first check that  $[F_{ki}]$  is the usual electromagnetic tensor whose components are the electric and magnetic fields. We prove that Maxwell's equations are obtained by applying the principle of least action to the 4-potential endowed with  $\mathcal{L}$ . The source terms ( $\rho$  and  $\vec{j}$ ) are expressed in terms of the components of  $[S_{ki}]$ . The results obtained are covariant. The formulation of these tensors being independent of scale, they unify the human and the electron scales, giving a new way to understand elementary particles.

**Keywords :** Electromagnetism. Elementary particles.

## 1 Introduction.

Classical textbooks in electromagnetism[1, 2, 3] describe the electromagnetic antisymmetric rank-2 tensor  $[F_{ki}]$  as an object which neatly groups the components of the electric and magnetic fields. An essential application of this tensor is to show that a pure electric field in a standing system of coordinates gives a magnetic field in a moving system through a Lorentz transformation.

However it can be readily seen that  $[F_{ki}]$  is only one member of a family of tensors which are obtained from the gradient  $[a_k^i]$  of the electromagnetic 4-potential  $A^i$  at each event  $M$  in Minkowski spacetime. The members of this family are generated through the use of two operations which do not commute :

- 1) The raising-lowering (r.l.) index operation which is done with the metric tensor and
- 2) the antisymmetric-symmetric (a.s.) splitting which makes use of the transpose of the tensors.

Four different tensors are thus obtained. One of them is  $[F_{ki}]$ , it results from the lowering index operation acting on  $[a_k^i]$  followed by the a.s. operation.

This preliminary remark has led us to study the properties of each of these tensors and to develop a general theory where the sole assumption is that of a 4-potential  $A^i$  in a flat spacetime. A characteristic of these tensors is that their formulation is scale-independent. They should thus be perfect tools to unify phenomena at ordinary and microscopic scales.

Indeed, many unsolved problems in modern physics are due to the difficulties associated with scaling. One of the most important is the structure of a "point-like" particle. In string theories, such an idealized particle is represented by a one dimensional object called a string. However, despite the abundance of developments in these theories the solution remains questionable[4, 5]. We show in this article that this family of tensors

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offers a new way to solve it and especially a method to unify macroscopic and microscopic scales.

The first authors to use tensor techniques to study the properties of an elementary particle, the electron, were Max Born and Leopold Infeld in the year 1934[6]. However, their attempt was not successful because their tensor was phenomenological : the antisymmetric part was indeed  $[F_{ki}]$  but the symmetric part was chosen to be the metric tensor and both parts of the tensor were of a different nature. The tensors we use in this article were obtained from the total derivative of the electromagnetic potential and both parts are combinations of partial derivatives.

Due to the non-commutativity mentioned above, the theory is divided into two parts :

The first part is based on the study of the symmetric ( $[s_k^i]$ ) and antisymmetric ( $[f_k^i]$ ) parts of  $[a_k^i]$ . We show that a proper time exists in which the scalar potential obeys a Helmholtz equation whose solutions describe "electromagnetic particles". Splitting  $[a_k^i]$  into its symmetric and antisymmetric parts leads to associate the former to the description of the mass and the latter to the field which could explain the particle-field duality. The second part is based on the study of the symmetric ( $[S_{ki}]$ ) and antisymmetric ( $[F_{ki}]$ ) parts of  $[a_{ki}]$ . There is absolutely no reason to reduce  $[a_{ki}]$  to its antisymmetric part  $[F_{ki}]$  : we will show that both parts contribute to Maxwell equations. If  $[S_{ki}]$  is forgotten, which is the case in classical electromagnetism, it becomes necessary to replace it with phenomenological quantities. These are charges and currents : we demonstrate that they can be expressed in terms of the elements of  $[S_{ki}]$ .

A key point of the theory is the association of a Lagrangian density with  $[a_k^i]$ . This density is a local scalar invariant of the tensor and plays a central role : it allows the description of the structure, or the geometrical distribution of energy in the particle where it is used to compute the canonical momenta. In the second part, it is used to deduce Maxwell's equations from  $[a_{ki}]$  through the least action principle.

The second section of this paper describes the few basic assumptions and the notations.

The third section develops the properties of the electromagnetic particles. We show that the scalar potential obeys a Helmholtz equation in the proper time of  $[a_k^i]$ . Solutions of this equation describe the electromagnetic particles which are characterized by three quantum numbers  $n, \ell, m$ . We studied the spatial distribution of energy and the electric charge of these solutions. Here an interesting result occurs : while the distribution of field and mass energies are different, their sum, when integrated over spacetime, vanishes. A second important result concerns the electric charge : it is found that it can be different from zero for the even and odd solutions  $n = 1, \ell = m = 0$  only, which confers the status of electron and positron to these solutions. Finally, we obtained the tensors of a spinning particle.

In the fourth section we computed the electromagnetic inductions which are the derivatives of the Lagrangian with respect to the fields. The second set of Maxwell's equations is obtained from Euler-Lagrange equations. A new result is the expression of the sources (charge  $\rho$  and current  $\vec{j}$  densities) in terms of the elements of  $[S_{ki}]$ .

Associated with this subject is the question of the preeminence of potentials and fields which arises in standard textbooks on classical electromagnetism : In ref.[1] , electric and magnetic fields are deduced from a 4-potential. In refs.[2, 3] , the 4-potential is deduced from the fields. In classical theory, the fields are more fundamental than the potential because they are observable quantities and the 4-potential is a *Deus ex machina* which can be seen only through its gradient  $[a_k^i]$ . In quantum theory, it is the potential that is more fundamental as illustrated by Aharonov-Bohm effect[7, 8]. Recently, two articles[9, 10] have been published where the authors develop Richard Feynman's idea of introducing potentials before fields[11]. The theory which is presented here brings the proof that the potential is also more fundamental than the fields in classical electromagnetism.

## 2 Foundations and notations.

The theory that we are developing is nothing but the study of the properties of the electromagnetic potential and its derivatives in different frames of observation. It is based on few ingredients :

1- One consider a point  $M$  (an "event") in spacetime where the observer is located. Given a coordinates frame with an origin  $O$ ,  $M$  is defined by the vector  $\overrightarrow{OM}$ . We will make use of the direct and the inverse spaces where  $\overrightarrow{OM}$  is defined respectively by its contravariant and covariant components. Convention  $\eta = \text{diag}(1, -1, -1, -1)$  is taken for the metric tensor  $\eta$ .

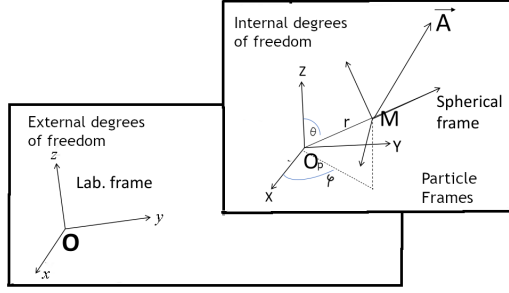


FIGURE 1 – Schematics of the three frames of coordinates in the geometrical space.  $M$  is the point of observation. Tensors leading to Maxwell's equations are expressed in the laboratory frame. Electromagnetic particles tensors are first described at  $M$  in spherical coordinates  $r, \theta, \varphi$  with  $M$  fixed with respect to the origin  $O_p$ . A particle spinning around the  $\mathbf{Z}$  axis is described by the tensors obtained from a local Lorentz transformation at  $M$ . Note that  $M$  close to  $O_p$  allows the description of the structure of the particle. The long range behavior of the particle is obtained when  $M$  is far from  $O_p$ .

The natural coordinates[14] will be used to describe the electromagnetic particles in section 3. They happen to be the proper time and the geometrical spherical system centered at  $M$ . On the other hand, Maxwell's equations are covariant and can be computed in any coordinate system.

We will thus have two independent cartesian coordinate systems with two different origins, the first is the laboratory frame with origin  $\mathbf{O}$ . Let us name the second the particle frame with  $O_p$  as an origin. The external and internal coordinates of  $M$  are noted respectively as  $x^k = (ct, x, y, z)$  and  $X^k = (cT, X, Y, Z)$  in Cartesian coordinates. Here,  $t$  or  $T$  stands for the time,  $(x, y, z)$  or  $(X, Y, Z)$  for the geometrical coordinates and  $c$  is the speed of light. We will also use a spherical system of coordinates at  $M$ . We will see that this system is the *natural* system[14] of electromagnetic particles. Figure (1) shows the three coordinate frames which will be used. The two systems of coordinates  $(x, y, z)$  or  $(X, Y, Z)$  are the analog of those of a moving body, where the internal degrees of freedom allow the description of its shape, its rotations (Euler's angles) and its deformations and where the independent external coordinates are used to describe the motion of the center of gravity.

2- A four-potential  $A^i = (\phi/c, A^x, A^y, A^z)$  is associated with each point  $M$ .  $\phi/c$  is the scalar potential and  $(A^x, A^y, A^z)$  are the three contravariant components of the vector potential in the direct space. This is the standard notation in classical electromagnetism. Covariant components of the potential in the reciprocal space are :  $A_i = (\phi/c, A_x, A_y, A_z) = (\phi/c, -A^x, -A^y, -A^z)$ .

3- The 16 partial derivatives  $\partial A^i / \partial x^k$  of the potential are the components of the gradient tensor  $[a_k^i]$ . They are obtained from the total derivative of each component  $A^i$  at  $M$ . The extended form of  $[a_k^i]$  in the cartesian frame is :

$$[a_k^i] = \begin{bmatrix} (\phi/c)_{,t} & A^x_{,t} & A^y_{,t} & A^z_{,t} \\ (\phi/c)_{,x} & A^x_{,x} & A^y_{,x} & A^z_{,x} \\ (\phi/c)_{,y} & A^x_{,y} & A^y_{,y} & A^z_{,y} \\ (\phi/c)_{,z} & A^x_{,z} & A^y_{,z} & A^z_{,z} \end{bmatrix} \quad (1)$$

We have adopted the conventional notation :

$$\begin{aligned} (\phi/c)_{,t} &\equiv \frac{\partial(\phi/c)}{c\partial t} & (\phi/c)_{,x} &\equiv \frac{\partial(\phi/c)}{\partial x} \quad \dots \\ A^x_{,t} &\equiv \frac{\partial A^x}{c\partial t} & A^x_{,x} &\equiv \frac{\partial A^x}{\partial x} \quad \dots \end{aligned}$$

$k$  and  $i$  respectively indicate the line and the column index for a reason which will appear in eq.(73). The above derivatives become more complicated in curvilinear coordinates where they include Christoffel's coefficients([13]).

The theory is essentially local : a particle will be described by vector and tensor fields. Global properties are obtained with integration over spacetime.

4- The fourth ingredient is the set of constraints that are imposed by nature :

a) Coordinate transformations form the Poincaré group : they include translations with respect to time and

TABLE 1 – Synopsis of the different electromagnetic tensors. Tensors in the upper part are obtained from the gradient  $a_k^i$  of the 4-potential  $A^i$  at point  $M$ .  $a_k^i$  is split into its symmetric and antisymmetric parts  $s_k^i$  and  $f_k^i$  in the coordinate frame where  $M$  is at rest with respect to the origin (a.s. splitting). Then these tensors are transformed into their covariant form  $s_{ki}$  and  $f_{ki}$  by lowering the index  $i$  with the metric tensor (r.l. operations). A Lorentz transform  $L$  is performed on these tensors to find those of the spinning particle.  $A_i$  and  $a_{ki}$  in the lower part are obtained by the lowering index operation acting on  $A^i$  or  $a_k^i$ . Then  $a_{ki}$  is split into its symmetric part  $S_{ki}$  and its antisymmetric part  $F_{ki}$  which is the usual electromagnetic tensor. The non-commutativity of the r.l. and a.s. operations is fundamental.

$A^i$ , $[a_k^i]$	a.s. splitting : $[a_k^i] = [s_k^i] + [f_k^i]$ lowering index $i$ : $f_{ki} = [f_k^i] \cdot \eta$ $s_{ki} = [s_k^i] \cdot \eta$ Lorentz transformation : $[\bar{s}_{ki}] = L \cdot [s_{ki}] \cdot L$ $[\bar{f}_{ki}] = L \cdot [f_{ki}] \cdot L$
lowering index $i$	
$A_i$ , $[a_{ki}]$	a.s. splitting : $[a_{ki}] = [S_{ki}] + [F_{ki}]$

space, rotations, and Lorentz transformations (or boosts). Tensors are characterized by **invariants** in such transformations. Our attention should thus be fixed on these invariants.

b) The evolution of a system is subject to the principle of least action which is expressed by the Euler-Lagrange equations.

c) The principle of relativity couples time and the geometrical coordinates of  $M$  in two frames in relative motion.

5- Finally, transformations of vectors or tensors have to obey mathematical rules. A fundamental concept is the splitting of a tensor into its antisymmetric (or skew-symmetric) and symmetric parts (a.s. operation). This operation does not commute with other manipulations such as the raising or lowering index operations (r.l. operations) performed with the metric tensor. This non-commutativity is illustrated in Appendix A. This simple law is at the origin of the splitting of electromagnetism into two parts where the first give the description of electromagnetic particles (section 3) and the second describes how Maxwell's equations can be deduced from the gradient tensor (section 4).

Table (1) shows the different tensors  $a_{ki}, S_{ki}, F_{ki}, a_k^i, s_k^i, f_k^i, \bar{s}_{ki}, \bar{f}_{ki}$  which will be described as we progress. One has to distinguish between those tensors which are defined in the pure direct space ( $a_k^i, s_k^i, f_k^i$ ) where contravariant components  $A^i$  and  $x^k$  only are involved and ( $a_{ki}, S_{ki}, F_{ki}, \bar{s}_{ki}, \bar{f}_{ki}$ ) where the geometrical coordinates can depend on time and where we need covariant components  $A_i$ . One should note that covariant or contravariant tensors keep their symmetry or antisymmetry properties in a coordinate transformation which is not the case of a mixed tensor[12].

We summarize below the key points of the study.

1-Assumptions :

-Minkowski's spacetime :  $x^k = (ct, x, y, z)$ .

-4 dimensional potential  $A^i$  : scalar potentiel  $\phi/c$  and vector potentiel  $A^x, A^y, A^z$ .

2- Study of the gradient of the potential :  $[a_k^i] = [\partial A^i / \partial x^k]$  (rank 2 tensor, 16 components).

Main invariants of  $[a_k^i]$  in symmetry operations of the Poincaré group : determinant  $||a_k^i||$  and trace  $\sum_i a_i^i$ . The determinant  $||a_k^i||$  is demonstrated to be associated with the local Lagrangian density  $\mathcal{L}$ .

3- Splitting of  $[a_k^i]$  into its symmetric and antisymmetric parts :  $[a_k^i] = [s_k^i] + [f_k^i]$ .

(a) Being symmetric,  $[s_k^i]$  can be diagonalized  $\Rightarrow \exists$  a **proper time** where  $s_1^2 = s_1^3 = s_1^4 = 0$ .

(b) The trace of  $[s_k^i]$  is invariant in a time translation

(a) and (b)  $\Rightarrow$  Helmholtz equation for  $\phi/c \Rightarrow$  Solutions describe "Electromagnetic Particles".

4- Study of the covariant tensor  $[a_{ki}]$  (lowering index operation). Splitting of  $[a_{ki}]$  into its symmetric and antisymmetric parts :  $[a_{ki}] = [S_{ki}] + [F_{ki}]$ .

$[F_{ki}]$  is the usual electromagnetic tensor.

Applying the principle of least action, Maxwell equations are obtained from  $[a_{ki}]$ .

### 3 Tensors of Electromagnetic Particles.

The main part of this section deals with tensors written in the pure direct space where the geometrical coordinates do not depend on time (the observation point  $M$  is fixed with respect to the origin of coordinates). We first show that the natural coordinates of the tensor  $[a_k^i]$  are introduced by a Helmholtz equation whose solutions describe the electromagnetic particles. We first derive this equation, then illustrate some of its solutions and the associated tensors. Some local and global properties are given. The case of a spinning particle is finally studied.

The point  $M$  in Fig.(1) is **motionless** with respect to the origin  $O_p$ . We will first write the derivatives in the local cartesian frame  $(T, X, Y, Z)$ .

We use the transpose  $[a_i^k]$  of  $[a_k^i]$  to split  $[a_k^i]$  into its symmetric ( $[s_k^i]$ ) and antisymmetric ( $[f_k^i]$ ) parts :

$$[a_k^i] = [s_k^i] + [f_k^i] \quad , \quad (2)$$

with :

$$[s_k^i] = \frac{1}{2} \left( [a_k^i] + [a_i^k] \right) = \frac{1}{2} \begin{bmatrix} 2(\phi/c)_{,t} & A_{,t}^X + (\phi/c)_{,X} & A_{,t}^Y + (\phi/c)_{,Y} & A_{,t}^Z + (\phi/c)_{,Z} \\ (\phi/c)_{,X} + A_{,t}^X & 2A_{,X}^X & A_{,X}^Y + A_{,Y}^X & A_{,X}^Z + A_{,Z}^X \\ (\phi/c)_{,Y} + A_{,t}^Y & A_{,Y}^X + A_{,X}^Y & 2A_{,Y}^Y & A_{,Y}^Z + A_{,Z}^Y \\ (\phi/c)_{,Z} + A_{,t}^Z & A_{,Z}^X + A_{,X}^Z & A_{,Z}^Y + A_{,Y}^Z & 2A_{,Z}^Z \end{bmatrix} \quad (3)$$

and :

$$[f_k^i] = \frac{1}{2} \left( [a_k^i] - [a_i^k] \right) = \frac{1}{2} \begin{bmatrix} 0 & A_{,t}^X - (\phi/c)_{,X} & A_{,t}^Y - (\phi/c)_{,Y} & A_{,t}^Z - (\phi/c)_{,Z} \\ (\phi/c)_{,X} - A_{,t}^X & 0 & A_{,X}^Y - A_{,Y}^X & A_{,X}^Z - A_{,Z}^X \\ (\phi/c)_{,Y} - A_{,t}^Y & A_{,Y}^X - A_{,X}^Y & 0 & A_{,Y}^Z - A_{,Z}^Y \\ (\phi/c)_{,Z} - A_{,t}^Z & A_{,Z}^X - A_{,X}^Z & A_{,Z}^Y - A_{,Y}^Z & 0 \end{bmatrix} \quad (4)$$

This pair of tensors will be a good candidate to explain the matter-field duality :  $[f_k^i]$  describes the field properties and  $[s_k^i]$  contains matter waves.  $[s_k^i]$  will be referred to as the matter tensor and  $[f_k^i]$  as the field tensor. Both tensors will display fundamental properties such as energies or electric charge.

#### 3.1 The Helmholtz equation.

An Helmholtz equation for the scalar potential is demonstrated in the following way :

1- Being symmetric,  $[s_k^i]$  can be diagonalized provided its determinant does not vanish. The consequence is that a time coordinate  $\bar{t}$  exists for which terms  $\bar{s}_1^2 = \bar{s}_1^3 = \bar{s}_1^4 = \bar{s}_2^1 = \bar{s}_2^3 = \bar{s}_2^4 = 0$ . We referred to  $\bar{t}$  the proper time of the tensor. In the proper time system, we use the barred symbols  $c\bar{t}, \bar{X}, \bar{Y}, \bar{Z}$  and :  $\bar{A} = (\bar{\phi}/c, \bar{A}^X, \bar{A}^Y, \bar{A}^Z)$ . The ordinary derivatives of a Cartesian system are replaced in a general system by absolute derivatives including Christoffel coefficients[13]. One obtains the equations :

$$\overline{A^X_{,t} + (\phi/c)_{,X}} = \overline{A^Y_{,t} + (\phi/c)_{,Y}} = \overline{A^Z_{,t} + (\phi/c)_{,Z}} = 0 \quad (5a)$$

$$\text{or} \quad \overrightarrow{\text{grad}(\phi/c)} + \frac{\partial \bar{A}}{c \partial \bar{t}} = 0. \quad (5b)$$

Here a symbol like  $\overline{(\phi/c)_{,X}}$  generally stands for the absolute derivative of  $\overline{(\phi/c)}$  with respect to  $\bar{X}$ . The second formulation is tensorial. Eq.(5b) means that in the proper time, the temporal derivative  $\overline{A^i_{,t}}$  is compensated by the spatial derivative of  $\overline{(\phi/c)}$  in the direction  $\bar{i}$ .

$[s_k^i]$  and  $[f_k^i]$  are written in the proper time frame :

$$[s_k^i] = \frac{1}{2} \left( [a_k^i] + [a_i^k] \right) = \frac{1}{2} \begin{bmatrix} 2\overline{(\phi/c)_{,\bar{i}}} & 0 & 0 & 0 \\ 0 & 2\overline{A^X_{,\bar{X}}} & \overline{A^Y_{,\bar{X}} + A^X_{,\bar{Y}}} & \overline{A^Z_{,\bar{X}} + A^X_{,\bar{Z}}} \\ 0 & \overline{A^X_{,\bar{Y}} + A^Y_{,\bar{X}}} & 2\overline{A^Y_{,\bar{Y}}} & \overline{A^Z_{,\bar{Y}} + A^Y_{,\bar{Z}}} \\ 0 & \overline{A^X_{,\bar{Z}} + A^Z_{,\bar{X}}} & \overline{A^Y_{,\bar{Z}} + A^Z_{,\bar{Y}}} & 2\overline{A^Z_{,\bar{Z}}} \end{bmatrix} \quad (6)$$

and :

$$[f_k^i] = \frac{1}{2} \left( [a_k^i] - [a_i^k] \right) = \frac{1}{2} \begin{bmatrix} 0 & 2\overline{A^X_{,\bar{i}}} & 2\overline{A^Y_{,\bar{i}}} & 2\overline{A^Z_{,\bar{i}}} \\ 2\overline{(\phi/c)_{,\bar{X}}} & 0 & \overline{A^Y_{,\bar{X}} - A^X_{,\bar{Y}}} & \overline{A^Z_{,\bar{X}} - A^X_{,\bar{Z}}} \\ 2\overline{(\phi/c)_{,\bar{Y}}} & \overline{A^X_{,\bar{Y}} - A^Y_{,\bar{X}}} & 0 & \overline{A^Z_{,\bar{Y}} - A^Y_{,\bar{Z}}} \\ 2\overline{(\phi/c)_{,\bar{Z}}} & \overline{A^X_{,\bar{Z}} - A^Z_{,\bar{X}}} & \overline{A^Y_{,\bar{Z}} - A^Z_{,\bar{Y}}} & 0 \end{bmatrix} \quad (7)$$

One should note that, in the proper time, when geometrical coordinates are independent of time (the point  $M$  is at rest with respect to the origin  $O$ ), one has :  $\partial \bar{x}^j / \partial(ct) = 0$  for  $j = 2, 3, 4$  and  $\partial \bar{ct}^j / \partial(ct) = 1$  for  $j = 1$ . It follows that the potential  $A^i = (\phi/c, A^1, A^2, A^3)$  becomes transformed into  $A^i = (\phi/c, \bar{A}^1, \bar{A}^2, \bar{A}^3)$  in a geometrical transformation in the proper time. In the same way, one finds the elements of the first line of the transformed tensor remain  $A_1^i = ((\phi/c), 0, 0, 0)$ .

2- Now we will use the invariants of  $[\bar{s}_k^i]$  in a time translation. There are four scalar invariants that are the coefficients of the characteristic polynomial. The most well-known are the trace and the determinant. We use the property of the trace of  $[\bar{s}_k^i]$  to be invariant in a time translation to obtain :

$$\frac{\partial}{c\partial \bar{t}} \left( \frac{\partial(\overline{\phi/c})}{c\partial \bar{t}} + \frac{\partial \bar{A}^X}{\partial \bar{X}} + \frac{\partial \bar{A}^Y}{\partial \bar{Y}} + \frac{\partial \bar{A}^Z}{\partial \bar{Z}} \right) = 0 \quad (8a)$$

$$\text{or} \quad \frac{\partial^2(\overline{\phi/c})}{c^2 \partial \bar{t}^2} + \text{div} \frac{\vec{\partial \bar{A}}}{c\partial \bar{t}} = 0. \quad (8b)$$

Note that the trace of  $[\bar{s}_k^i]$  is the term that appears in Lorenz's gauge. Equations (5b) and (8b) are combined to give :

$$\frac{\partial^2(\overline{\phi/c})}{c^2 \partial \bar{t}^2} - \Delta(\overline{\phi/c}) = 0 \quad , \quad (9)$$

where the symbol  $\Delta$  stands for the Laplacian.

In the following, we will be interested in permanent oscillatory potentials which are proportional to  $\cos \omega t$  or  $\sin \omega t$ . These potentials obey a Helmholtz-type equation :

$$\frac{\omega^2}{c^2} (\overline{\phi/c})_{\text{spatial}} + \Delta(\overline{\phi/c})_{\text{spatial}} = 0, \quad (10)$$

where  $(\overline{\phi/c})_{\text{spatial}}$  represents the spatial part of  $(\overline{\phi/c})$ . This equation is a tensor equation that remains the same in any system of spatial coordinates.

### 3.2 Electromagnetic particles.

Solutions of eq.(10) will describe spatial distributions of the scalar potential  $\overline{\phi/c}$  in geometrical space. The components of the vector potential can be determined from  $\overline{\phi/c}$  thanks to eq.(5b). The knowledge of the components of the four-potential for each solution will lead to the corresponding tensor of derivatives at each point to be solved. These solutions are studied in this section.

Helmholtz equation can be written in the spherical reference frame attached to  $M$  with the proper time  $\bar{t}$  and the geometrical coordinates  $(r, \theta, \varphi)$  such that :

$$\bar{X} = r \sin \theta \cos \varphi \quad , \quad \bar{Y} = r \sin \theta \sin \varphi \quad , \quad \bar{Z} = r \cos \theta \quad , \quad (11)$$

The advantage of the  $(r, \theta, \varphi)$  system is that it makes use of the spherical or cylindrical symmetry of the solutions that we are going to use. The set  $\bar{ct}, r, \theta, \varphi$  is the natural set of coordinates of the particles[14].

We introduce the *normalized* distance to the origin  $O_p$  of the coordinates :  $x = \omega r/c$ . (Note the typography which is different from that of the coordinate  $x$ ). This distance will thus be measured in units of the reference length  $c/\omega$ .

Eq.(10) has been studied extensively in the context of the hydrogen atom where some of its solutions describe the electronic orbitals[15].

Solutions of eq.(10) can be split into normal and coupled modes :

1- Coupled angular-radial modes describe simultaneous vibrations on the three coordinates. They are obtained from the *ansatz* :  $(\overline{\phi/c})_{\text{spatial}} = R(r) \Theta(\theta) \Phi(\varphi)$  where  $R(r)$ ,  $\Theta(\theta)$ , and  $\Phi(\varphi)$  are functions of  $r, \theta$ , and  $\varphi$  respectively. One thus obtains the coupled angular-radial modes in terms of spherical Bessel functions of order  $\ell$   $J_\ell(x)$  and spherical harmonics  $Y_\ell^m(\theta, \varphi)$  :

$$\phi_{\ell,m}(x, \theta, \phi) = \mathcal{A} J_\ell(x) Y_\ell^m(\theta, \varphi) \begin{cases} \cos \omega t \\ \sin \omega t \end{cases} \quad (12)$$

There are two solutions which are even or odd in time. Quantities  $\mathcal{A}$  and  $\omega$  are the constants of integration and are not determined at this stage. Amplitude  $\mathcal{A}$  has the dimension  $[\mathcal{A}] = ML^2T^{-2}Q^{-1}$  in the standard nomenclature.

$J_\ell(x)$  is a solution to the radial equation :

$$x^2 + \frac{1}{R} \frac{\partial}{\partial x} \left( x^2 \frac{\partial R}{\partial x} \right) = \ell(\ell + 1), \quad (13)$$

The  $Y_\ell^m(\theta, \varphi)$  describe the solutions of the angular part of eq.(10).

2- Normal modes describe independent vibrations on one of the three coordinates. They are obtained from the *ansatz* :  $(\phi/c)_{spatial} = R(r) + \Theta(\theta) + \Phi(\varphi)$ . There is an important difference between the equations which describe normal and coupled modes of an oscillator. Typically an oscillator receives a sustaining energy from the outside and loses energy from, for instance, mechanical friction or electromagnetic radiation. Equilibrium is attained when both energies compensate each other. Normal and coupled modes need a source term (a seed) to develop. This source is generally a noise (again mechanical vibrations or electromagnetic waves), it starts the oscillation whose amplitude increases until the equilibrium state. In the case of coupled modes, the source is the coupling term and the noise does not need to be explicitly written : the source term for the radial oscillation is  $\ell(\ell + 1)$  in eq.(13). This is in contrast with the equation of a normal mode where the source term must appear. In our case, this term originates from the noise in which the particle is embedded. We consider that it is an homogeneous background (independent of the point of observation  $M$ ). We thus introduce the total potential  $\mathbb{A}^i = (\Phi/c, \mathbb{A}^r, \mathbb{A}^\theta, \mathbb{A}^\varphi)$  at  $M$  as the sum of the oscillating potential  $A^i = (\phi/c, A^r, 0, 0)$  and the noise potential  $\mathbf{N}^i = (N, N^r, N^\theta, N^\varphi)$ . Equations (5b) and (8b) become :

$$\overrightarrow{\text{grad}}(\phi/c + N) + \frac{\partial \overrightarrow{\mathbb{A}}}{c \partial t} = 0. \quad (14)$$

and :

$$\frac{\partial^2(\phi/c + N)}{c^2 \partial t^2} + \text{div} \frac{\partial \overrightarrow{\mathbb{A}}}{c \partial t} = 0. \quad (15)$$

Eqs. (14) and (15) give :

$$\Delta(\phi/c + N) - \frac{\partial^2(\phi/c + N)}{c^2 \partial t^2} = 0. \quad (16)$$

$N$  being homogeneous, its spatial derivatives nullify and  $\Delta N = 0$ . The second term is :

$$-\frac{\partial^2(\phi/c + N)}{c^2 \partial t^2} = \omega^2 (\phi/c + N) \quad (17)$$

which give an Helmholtz equation for  $\phi/c$  :

$$\Delta(\phi/c) + \omega^2 (\phi/c) + \omega^2 N = 0. \quad (18)$$

Physically acceptable spatial solutions are the spherical Bessel functions  $J_n$  which are obtained when the spatial part of  $N$  is  $N_{spatial} = n(n + 1)$ .

Grouping coupled and normal modes together shows that the potential which describes a solution finally depends on three quantum numbers  $n, \ell, m$  :  $\phi/c = \phi/c(n, \ell, m)$ . Its general expression is :

$$\phi/c(x, \theta, \phi) = \mathcal{A} J_n(x) Y_\ell^m(\theta, \varphi) \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}. \quad (19)$$

Explicit values of the first spherical harmonics are :

$$\begin{aligned} Y_0^0 &= \sqrt{\frac{1}{4\pi}} & Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_1^1 &= -\sqrt{\frac{3}{4\pi}} \sin \theta \cos \varphi, & Y_1^{-1} &= \sqrt{\frac{3}{4\pi}} \sin \theta \sin \varphi \end{aligned} \quad (20)$$

The first spherical Bessel functions are :

$$J_0 = J_0(x) = \frac{\sin x}{x} \quad (21a)$$

$$J_1 = J_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad (21b)$$

Equation (5b) allows the computation of the components of the vector potential in the proper time frame from the scalar potential.

These solutions obey the physical boundary conditions : they become asymptotically null far from the origin and they are either finite or null at the origin. Each solution describes an "electromagnetic particle" (e.m. particle).

The general expression of the gradient tensor in the spherical system of coordinates associated with point  $M$  is for the resonant solutions :

$$[a_k^i] = \begin{bmatrix} \phi_{,t} & A_{,t}^r & A_{,t}^\theta & A_{,t}^\varphi \\ \phi_{,r} & A_{,r}^r & A_{,r}^\theta & A_{,r}^\varphi \\ \phi_{,\theta}/r & \frac{1}{r} (A_{,\theta}^r - A^\theta) & \frac{1}{r} (A_{,\theta}^\theta + A^r) & \frac{1}{r} A_{,\theta}^\varphi \\ \frac{\phi_{,\varphi}}{r \sin \theta} & \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} & \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) & \frac{1}{r \sin \theta} A_{,\varphi}^\varphi + \frac{1}{r} A^r + \frac{1}{r} \frac{\cos \theta}{\sin \theta} A^\theta \end{bmatrix} \quad (22)$$

This tensor does not include the noise and will characterize the particle only. Its symmetric part (the mass part) is :

$$[s_k^i] = \frac{1}{2} \begin{bmatrix} 2\phi_{,t} & A_{,t}^r + \phi_{,r} & A_{,t}^\theta + \phi_{,\theta}/r & A_{,t}^\varphi + \frac{\phi_{,\varphi}}{r \sin \theta} \\ \phi_{,r} + A_{,t}^r & 2A_{,r}^r & A_{,r}^\theta + \frac{1}{r} (A_{,\theta}^r - A^\theta) & A_{,r}^\varphi + \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} \\ \phi_{,\theta}/r + A_{,t}^\theta & \frac{1}{r} (A_{,\theta}^r - A^\theta) + A_{,r}^\theta & 2\frac{1}{r} (A_{,\theta}^\theta + A^r) & \frac{1}{r} A_{,\theta}^\varphi + \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) \\ \frac{\phi_{,\varphi}}{r \sin \theta} + A_{,t}^\varphi & \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} + A_{,r}^\varphi & \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) + \frac{1}{r} A_{,\theta}^\varphi & 2 \left( \frac{1}{r \sin \theta} A_{,\varphi}^\varphi + \frac{1}{r} A^r + \frac{1}{r} \frac{\cos \theta}{\sin \theta} A^\theta \right) \end{bmatrix} \quad (23)$$

The antisymmetric part (the field part) is :

$$[f_k^i] = \frac{1}{2} \begin{bmatrix} 0 & A_{,t}^r - \phi_{,r} & A_{,t}^\theta - \phi_{,\theta}/r & A_{,t}^\varphi - \frac{\phi_{,\varphi}}{r \sin \theta} \\ \phi_{,r} - A_{,t}^r & 0 & A_{,r}^\theta - \frac{1}{r} (A_{,\theta}^r - A^\theta) & A_{,r}^\varphi - \frac{A_{,\varphi}^r}{r \sin \theta} + \frac{A^\varphi}{r} \\ \phi_{,\theta}/r - A_{,t}^\theta & \frac{1}{r} (A_{,\theta}^r - A^\theta) - A_{,r}^\theta & 0 & \frac{1}{r} A_{,\theta}^\varphi - \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) \\ \frac{\phi_{,\varphi}}{r \sin \theta} - A_{,t}^\varphi & \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} - A_{,r}^\varphi & \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) - \frac{1}{r} A_{,\theta}^\varphi & 0 \end{bmatrix} \quad (24)$$

Expressions of these tensors are obtained in the proper time where the antisymmetric part is :

$$[f_k^i] = \begin{bmatrix} 0 & E^r/c & E^\theta/c & E^\varphi/c \\ -E^r/c & 0 & B^\varphi & -B^\theta \\ -E^\theta/c & -B^\varphi & 0 & B^r \\ -E^\varphi/c & B^\theta & -B^r & 0 \end{bmatrix} \quad (25)$$

The components of the field are *defined* by the relations :

$$\begin{aligned} E^r/c &:= \frac{1}{2} (A_{,t}^r - \phi_{,r}) = A_{,t}^r = -\phi_{,r} \\ E^\theta/c &:= \frac{1}{2} (A_{,t}^\theta - \phi_{,\theta}/r) = A_{,t}^\theta = -\phi_{,\theta}/r \\ E^\varphi/c &:= \frac{1}{2} (A_{,t}^\varphi - \frac{\phi_{,\varphi}}{r \sin \theta}) = A_{,t}^\varphi = -\frac{\phi_{,\varphi}}{r \sin \theta} \end{aligned} \quad (26)$$

$$\begin{aligned} B^r &:= \frac{1}{2} \left( \frac{1}{r} A_{,\theta}^\varphi - \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) \right) \\ B^\theta &:= \frac{1}{2} \left( \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} - A_{,r}^\varphi \right) \\ B^\varphi &:= \frac{1}{2} \left( A_{,r}^\theta - \frac{1}{r} (A_{,\theta}^r - A^\theta) \right) \end{aligned} \quad (27)$$

Note that these components are *different* from those obtained in eq.(60) in the next section from the tensor  $a_{ik}$  in the real space (direct+dual spaces). The difference arises from the fact that the electric field is defined here from the components of an antisymmetric *mixed* tensor while they are defined from an antisymmetric *covariant* tensor in the general case.

Explicit formulas for the potential components, their derivatives and the corresponding tensors are given in Appendix B for the first 5 even and odd solutions corresponding to  $n = \ell = m = 0$  (solutions  $g$  and  $g^*$ ),  $n = 1, \ell = m = 0$  (solutions  $e$  and  $e^*$ ) and  $n = 1, \ell = 1, m = 0, \pm 1$  (solutions  $q_0, q_1, q_{-1}$  and  $q_0^*, q_1^*, q_{-1}^*$ ).

### 3.3 Particles properties.

Some properties resulting from the tensorial description of the particles are listed in this section, without giving illustrations, some of them can be found in previous articles[16, 17]. These properties are illustrated here with the use of the even "e" solution ( $n = 1, \ell = m = 0$ ) which leads to the tensor :

$$[a_k^i]_e = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \times \begin{bmatrix} -\sin \omega t J_1 & -\cos \omega t J_1' & 0 & 0 \\ \cos \omega t J_1' & -\sin \omega t J_1'' & 0 & 0 \\ 0 & 0 & -\sin \omega t \frac{J_1'}{x} & 0 \\ 0 & 0 & 0 & -\sin \omega t \frac{J_1'}{x} \end{bmatrix} \quad (28)$$

The symmetric part is diagonal, the antisymmetric part contains the electric field and there is no magnetic field.

#### 3.3.1 Waves.

Each element of the tensor (28) is a system of stationary waves. Their amplitudes vary radially and are maximum in the vicinity of the center. For the moment, one can recognize several kinds of waves :

1 - The electric field is defined in eq.(26) and is expressed by the formula :

$$E^r/c = -\phi_{,r} = A^r_{,t} = -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \cos \omega t J_1' \quad (29)$$

where  $J_1' = \sin x/x + 2 \cos x/x^2 - 2 \sin x/x^3$ .

$E^r$  is *longitudinally polarized* and corresponds to the usual electrostatic field. It is a *standing wave* that extends to infinity. When  $x$  becomes large, the long range field is proportional to :

$$E^r \sim \cos \omega t \sin x/x = 1/2x (\sin \omega(t + r/c) - \sin \omega(t - r/c)) \quad (30)$$

This expression displays a travelling advanced wave  $\sin \omega(t + r/c)$  and a retarded wave  $\sin \omega(t - r/c)$  which correspond exactly to those described in Wheeler-Feynman's absorber theory[18]. Thus, we have adopted their interpretation based on causality : the outgoing wave is emitted by the particle and is absorbed by the surrounding medium which also acts as the emitter of a wave that is absorbed by the particle. Equilibrium is obtained when both incoming and outgoing waves have the same energy. This is the condition for the stability of the particle and the existence of permanent solutions. It follows that the amplitude  $\mathcal{A}$  is fixed by this condition which is certainly scale-dependent. Moreover,  $E^r$  is the classical electrostatic field. Its nature is completely different from the electromagnetic field of eq.(60) which corresponds to transverse waves emitted by a moving charge. We will see below that a different field characterizes a spinning charge

2 - The second kind of waves appears in  $s_k^i$  : these are  $s_1^1$  and  $s_2^2$ . They can be interpreted as "matter waves" and should correspond to de Broglie's pilot wave[19]. Moreover they should be the origin of gravitation. They intervene in self-interference effects in particle diffraction experiments. The physical interpretations of  $s_1^1$  and  $s_2^2$  are different :

- $s_1^1$  originates from the time derivative of the scalar potential  $\phi/c$ . The standing wave is scalar.

- $s_2^2$  originates from the radial derivative of the radial potential  $A^r$ . The standing wave is radially polarized. In the following, we will find other kinds of waves (e.g. induction waves) emitted by the particle.

#### 3.3.2 Energy.

The Hamiltonian density describes the local density of energy at point  $M$ . We have to find the Lagrangian which is linked to the Hamiltonian through a Legendre transform. An integration over spacetime will

give the total energy of the particle.

We have two hints to find the Lagrangian associated with an electromagnetic particle :

- The invariance of a Lagrangian in a coordinate change. Among the four invariants of  $[a_k^i]$ , one is proportional to the Lagrangian density  $\mathcal{L}$ .

- The second hint is that the global Lagrangian of a particle should be finite. In other words, the Hamiltonian density  $\mathcal{H}$ , when integrated over the whole volume, should converge, giving the total energy of the particle.

This integral is :

$$\int \equiv \int_0^{2\pi} d(\omega t)/2\pi \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \mathcal{H}$$

in the spherical system of coordinates. The first summation will give the mean value over a period of time.

These conditions are met by the determinant of  $[a_k^i]$  only and we are led to the equation :

$$\mathcal{L} = \mathcal{C} \parallel a_k^i \parallel \quad (31)$$

where the double bar is the symbol for the determinant. The proportionality constant  $\mathcal{C}$  is a physical quantity that has the dimensions  $[\mathcal{C}] = M^{-3}L^{-2}T^2Q^4$ .

The hamiltonian is given by the Legendre transform :

$$\mathcal{H} = \sum_{ik} a_k^i \frac{\partial \mathcal{L}}{\partial a_k^i} - \mathcal{L} \quad (32)$$

This equation uses the *canonical momentum*  $\mathcal{L}'_i^k = \partial \mathcal{L} / \partial a_k^i$  associated with  $a_k^i$ . This term appears in the Euler–Lagrange equations(41).

Since  $\mathcal{L}$  is the determinant of  $[a_k^i]$ , one sees that  $\mathcal{L}'_i^k$  is the minor relative to the element  $a_k^i$  (with the sign  $(-1)^{i+k}$ ) and that  $\sum_k \partial \mathcal{L} / \partial a_k^i a_k^i = \mathcal{L}$  (development of the determinant with respect to the elements of line  $k$ ). The simple equation follows :

$$\mathcal{H} = 3\mathcal{L} \quad (33)$$

Integration over spacetime with the 4-volume element  $dv$  gives a quantity which is proportional to the total energy of the particle :

$$W_{total} \propto \int \mathcal{L} dv = \int \parallel a_k^i \parallel dv \quad (34)$$

Each term of this integral can be split into a product of 4 integrals of the form  $\int_a^b \partial A^i / \partial x^m dx^m = A^i|_{-\infty}^{+\infty}$  or  $A^i|_0^{2\pi}$ . Assuming that the potential is periodic and vanishes at infinity, one finds that the total energy of an electromagnetic particle is identically zero.

This fundamental result is illustrated now in the case of solution  $e$ .

The Lagrangian density which corresponds to this solution is proportional to the determinant of (28) :

$$\mathcal{L} = \mathcal{C} \frac{1}{(4\pi)^2} \left( \frac{\mathcal{A}\omega}{c^2} \right)^4 \left( J_1 J''_1 \frac{J_1'^2}{x^2} \sin^4 \omega t + \frac{J_1'^4}{x^2} \sin^2 \omega t \cos^2 \omega t \sin^2 \omega t \right) \quad (35)$$

The first term in the sum is the mass term :

$$\mathcal{L}_s = \mathcal{C} \frac{1}{(4\pi)^2} \left( \frac{\mathcal{A}\omega}{c^2} \right)^4 J_1 J''_1 \frac{J_1'^2}{x^2} \sin^4 \omega t \quad (36)$$

It corresponds to the determinant of the symmetric part of  $[a_k^i]$  : The second term is the field term :

$$\mathcal{L}_f = \mathcal{C} \frac{1}{(4\pi)^2} \left( \frac{\mathcal{A}\omega}{c^2} \right)^4 \frac{J_1'^4}{x^2} \sin^2 \omega t \cos^2 \omega t \sin^2 \omega t \quad (37)$$

It is the difference between  $\mathcal{L}$  and  $\mathcal{L}_s$ .

The mass energy density is :

$$W_s = \mathcal{C} \sum_{ik} s_k^i \frac{\partial \mathcal{L}}{\partial s_k^i} - \mathcal{L}_s = 3\mathcal{L}_s$$

The mean value of  $\mathcal{L}$  over a period  $2\pi/\omega$  is :

$$\int_1^{2\pi} d(\omega t)/2\pi \mathcal{L} = C \frac{1}{(4\pi)^2} \left( \frac{\mathcal{A}\omega}{c^2} \right)^4 \times \left( \frac{3}{8} J_1 J''_1 \frac{J_1'^2}{x^2} + \frac{1}{8} \frac{J_1'^4}{x^2} \right) \quad (38)$$

The total energy  $W$  is proportional to the integral :

$$W \propto \int_1^\pi \sin \theta \, d\theta \int_1^{2\pi} d\varphi \times \int_1^\infty r^2 \, dr \left( 3 J_1 J''_1 \frac{J_1'^2}{x^2} + \frac{J_1'^4}{x^2} \right) \quad (39)$$

Integration over angles  $\theta$  and  $\varphi$  give  $4\pi$ .  
Integrating over  $r = x \, \omega/c$  gives :

$$\begin{aligned} (\omega/c)^3 \int_1^\infty x^2 \, dx \left( 3 J_1 J''_1 \frac{J_1'^2}{x^2} + \frac{J_1'^4}{x^2} \right) (\omega/c)^3 \int_1^\infty dx \left( \frac{\partial(J_1'^3)}{\partial x} \right) = \\ = (\omega/c)^3 \left. J_1'^3 \right|_1^\infty = 0 \quad (40) \end{aligned}$$

The integral vanishes because  $J_1'$  vanishes when  $x \rightarrow 0$  and  $x \rightarrow \infty$ .

This result means that the **total** field and mass energies which are respectively associated with the antisymmetric and symmetric parts of the tensor, are equal and opposite. One can verify that this property occurs for all solutions. It is a wonder that mathematics can build a world from zero total energy, the negative part of which being the mass, and the positive part the field energy. Graphical illustrations of the radial distributions of energy have been published elsewhere([16]).

### 3.3.3 Electric charge.

This section briefly describes the method to find the electric charge  $Q$  associated with an e.m. particle. The main steps are as follows :

- 1- The second couple of Maxwell's equation (63) relates the electric displacement  $\vec{D}$  to the charge density  $\rho$ . We use this equation here to *define*  $\rho$ . For the moment  $\rho$  is an intermediate quantity that will be integrated to obtain the total charge of the particle and used later in the case of a spinning particle.
- 2- There are few fundamental principles that fix the dynamics of a physical system. The first of them is the principle of least action which is expressed by the Euler–Lagrange equations :

$$\sum_k \frac{\partial}{\partial x^k} \left( \frac{\partial \mathcal{L}}{\partial A_{,k}^i} \right) - \frac{\partial \mathcal{L}}{\partial A^i} = 0 \quad (41)$$

We apply this equation to the component  $\phi/c$ . It can be written in the extended form as :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\phi/c)} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{\partial t} \right)} \right) - \frac{\partial}{\partial r} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{\partial r} \right)} \right) \\ - \frac{\partial}{\partial \theta} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{\partial \theta} \right)} \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{\partial \theta} \right)} \right) = 0 \quad . \quad (42) \end{aligned}$$

The first term is nullified because  $\mathcal{L}$  does not depend on  $\phi/c$  explicitly. The other terms represent the 4-divergence of the electric displacement (which is the induction divided by  $c$ ). If we use the expression for  $E^r/c$  in equations (26) we see that the induction reduces to the radial component for solutions  $g$  and  $e$  :

$$D_r = \frac{\partial \mathcal{L}}{\partial E^r} = - \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{\partial r} \right)} \quad (43)$$

In the general case, where  $E_\theta$  and  $E_\varphi$  do not vanish, one has :

$$\frac{\partial}{c\partial t} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial(\phi/c)}{c\partial t} \right)} \right) = -div \vec{D} \quad (44)$$

One sees that the term  $\partial \mathcal{L} / \partial \left( \frac{\partial(\phi/c)}{c\partial t} \right)$  is simply the minor relative to  $\frac{\partial(\phi/c)}{c\partial t}$  in  $[a_k^i]$ . This minor is the determinant which groups the spatial derivatives in  $a_k^i$ . One gets :

$$div \vec{D} = -\frac{\partial}{c\partial t} \left\| \begin{array}{ccc} A_{,r}^r & A_{,r}^\theta & A_{,r}^\varphi \\ \frac{1}{r} (A_{,\theta}^r - A^\theta) & \frac{1}{r} (A_{,\theta}^\theta + A^r) & \frac{1}{r} A_{,\theta}^\varphi \\ \frac{A_{,\varphi}^r}{r \sin \theta} - \frac{A^\varphi}{r} & \frac{1}{r \sin \theta} (A_{,\varphi}^\theta - \cos \theta A^\varphi) & \frac{1}{r \sin \theta} A_{,\varphi}^\varphi + \frac{1}{r} A^r + \frac{1}{r} \frac{\cos \theta}{\sin \theta} A^\theta \end{array} \right\| \quad (45)$$

When applied to solution  $e$ , one obtains :

$$\frac{\partial D_r}{\partial r} = 3 \left( \frac{1}{\sqrt{4\pi}} \frac{A_e \omega_e}{c^2} \right)^3 J''_1 \frac{J_1'^2}{x_1^2} \sin^2 \omega_e t \cos \omega_e t \quad (46)$$

This expression shows that  $\vec{D}$  is also a longitudinally polarized standing wave, with a modulus such that :

$$\frac{\partial}{\partial r} (mod D_r) = 3 \left( \frac{1}{\sqrt{4\pi}} \frac{A_e \omega_e}{c^2} \right)^3 J''_1 \frac{J_1'^2}{x_1^2} \sin^2 \omega_e t, \quad (47)$$

whose mean value over a period is :

$$|\rho| = \frac{3}{2} \left( \frac{1}{\sqrt{4\pi}} \frac{A_e \omega_e}{c^2} \right)^3 J''_1 \frac{J_1'^2}{x_1^2}, \quad (48)$$

and the total charge is :

$$\begin{aligned} Q &= \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi |\rho| \\ &= \frac{3}{2} \left( \frac{1}{\sqrt{4\pi}} \frac{A_e \omega_e}{c^2} \right)^3 4\pi \frac{c^3}{\omega_e^3} \int_0^\infty x_1^2 dx_1 J''_1 \frac{J_1'^2}{x_1^2} \\ &= \frac{1}{2\sqrt{4\pi}} \frac{A_e^3}{c^3} \frac{J_1'^3}{3} \Big|_0^\infty. \end{aligned} \quad (49)$$

In the vicinity of the origin, one has the series :  $J_1' \sim 1/3 - x^2/10 + x^4/168 - x^6/6480 + x^8/443520 + \dots$  and the limit value of  $J'$  when  $x \rightarrow 0$  is  $1/3$ . Moreover  $J_1' \rightarrow 0$  when  $x \rightarrow \infty$ . Finally :

$$Q = \frac{1}{2} \frac{1}{3^4} \frac{A_e^3}{\sqrt{4\pi}} \frac{1}{c^3} \quad (50)$$

When applied to the odd solution  $e^*$ , one obtains :

$$div \vec{D} = -3 \left( \frac{1}{\sqrt{4\pi}} \frac{A_e \omega_e}{c^2} \right)^3 J''_1 \frac{J_1'^2}{x_1^2} \cos^2 \omega_e t \sin \omega_e t \quad (51)$$

which shows that the charge of the odd solution  $e^*$  is the same but with the opposite sign to that of the even solution  $e$ .

The same calculation done with solution  $g$  brings the result that the total charge is zero because  $J'_0 = 0$ . The same result happens for all other solutions. The *only charged electromagnetic particles are described by solutions  $e$  and  $e^*$* . This result leads us to name  $e$  and  $e^*$  the electron and the positron. We note that the total charge (50) is not invariant in a coordinate change. It is the spinning electron that will give the *global* invariant.

### 3.4 rotating tensors.

Up to now, the electromagnetic tensors have been written for points  $M$  at rest with respect to the local (or internal) center of coordinates  $O_p$ , *i.e.*, in the inertial system where the geometrical coordinates are time-independent. We next studied the situation where  $M$  is subject to a rotation around the  $z$  axis. We have computed the expressions for the components of  $[a_k^i]$  in the frame where  $M$  is at rest. We consider now the local infinitesimal length elements  $c dt, dr, r d\theta, r \sin\theta d\varphi$  which define the volume element  $dv$  around  $M$  and which are measured by a rotating observer attached to  $M$ . These coordinates become  $c dt', dr', r' d\theta', r' \sin\theta' d\varphi'$  for a fixed observer at  $M$ . Both sets of coordinates are linked by a local, tangential, Lorentz transformation. The elementary motion is a translation along the  $\varphi'$  axis and the two other local axes  $r'$  and  $\theta'$  are perpendicular to it. It follows that the coordinates  $dr$  and  $r d\theta$  are not affected by the rotation. Only the length element  $r \sin\theta d\varphi$  and the time element  $c dt$  at event  $M$  are subject to the Lorentz transformation :

$$\begin{aligned} c dt' &= \gamma c dt + \gamma \beta r \sin\theta d\varphi, \quad r' = r, \quad \theta' = \theta, \\ r \sin\theta d\varphi' &= \gamma r \sin\theta d\varphi + \gamma \beta c dt. \end{aligned} \quad (52)$$

We use the standard notation  $\beta$  for the relative tangential velocity, and  $\gamma$  for the Lorentz factor :

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (53)$$

where  $\beta$  and  $\gamma$  depend upon the coordinates.

Note that the temporal phase  $\omega t$  (which is a true scalar) is invariant in a Lorentz transformation :  $\omega t = \bar{\omega} \bar{t}$ . Note also that the factor  $c/\omega$  which appears in the equation of the potential or its derivatives is the normalization parameter that transforms the invariant radial coordinate  $r$  into the non-dimensioned quantity  $x$ . This parameter is not modified here.

The Jacobian of the transformation is the Lorentz matrix :

$$\mathbb{J}_L = \begin{bmatrix} \gamma & 0 & 0 & \gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta & 0 & 0 & \gamma \end{bmatrix} \quad (54)$$

The expression of the potential, as seen by a non-rotating observer at  $M$ , is :

$$\begin{bmatrix} \phi'/c \\ A'^r \\ A'^\theta \\ A'^\varphi \end{bmatrix} = \mathbb{J}_L \begin{bmatrix} \phi/c \\ A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \begin{bmatrix} \gamma \phi/c + \gamma \beta A^\varphi \\ A^r \\ A^\theta \\ \gamma \beta \phi/c + \gamma A^\varphi \end{bmatrix}$$

Let us apply the transformation to solution  $e$ . We split the tensor(28) into its symmetric and antisymmetric parts which we transform into their covariant forms to use both direct and reciprocal spaces. The Lorentz transform gives the rotating tensors  $[\bar{f}_{ki}]$  and  $[\bar{s}_{ki}]$ . The field tensor is :

$$[\bar{f}_{ki}] = \mathbb{J}_L \cdot [f_{ki}] \cdot \mathbb{J}_L = \begin{bmatrix} 0 & -\gamma E_r/c & 0 & 0 \\ -\gamma E_r/c & 0 & 0 & -\beta \gamma E_r/c \\ 0 & 0 & 0 & 0 \\ 0 & -\beta \gamma E_r/c & 0 & 0 \end{bmatrix} \quad (55)$$

As expected one notes the appearance of the magnetic field  $B_\theta = \beta \gamma E_r/c$ . It is oriented along the local axis  $\vec{\theta}$ . The behavior of  $B_\theta$  is governed by  $\beta$ .

We have studied the properties of the fields in a separate article[17]. Applying the principle of energy conservation, one finds that  $\gamma$  is very close to 2 which confers to every point  $M$  the same tangential speed but a variable rotation frequency. Another interesting conclusion arises from the fact that the charge  $Q$  which has been computed for the non-rotating particle is modified by the rotation. The observable electron charge  $\bar{Q}$  is different from  $Q$ .

A final remark is in order. While the electromagnetic field tensor  $F_{ik}$  is antisymmetric, the electrostatic field tensor  $\bar{f}_{ik}$  is symmetric. The contracted product of these two tensors vanishes, which illustrates the different nature of both fields.

## 4 Tensors in the laboratory frame.

This section shows that the standard electromagnetic tensor  $F_{ki}$  is the antisymmetric part of  $a_{ki}$  expressed in the general frame of the laboratory. The Lagrangian density which is associated with  $a_k^i$  allows the calculation of inductions. Then Euler–Lagrange equations are applied to find Maxwell’s equations.

### 4.1 Splitting the covariant derivative.

One starts with the contravariant components of a general electromagnetic four-potential vector  $A^i$  ( $i = 0, 1, 2, 3$ ). This potential can be related to those studied in the preceding section in each particular problem. The scalar potential is noted again  $A^0 = \phi/c$  and the set  $\vec{A} = (A^x, A^y, A^z)$  represents also the vector potential. An event  $M$  in real Minkowski’s spacetime is defined by its coordinates  $x^k = (ct, x, y, z)$  and a four-potential corresponds to each event :  $A^i = A^i(M)$ . The coordinates are defined in the Cartesian frame spanned by the normalized basis vectors  $(\vec{e}_t, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ . All the theory described here is *local* : the point  $M$  is surrounded by an arbitrarily small volume. There are quantities, like fields, which are defined at  $M$  and densities which are defined around  $M$ .

To obtain the corresponding covariant components  $A_i$  in the dual space, we use again the  $(+, -, -, -)$  convention for the metric tensor  $[\eta_{mn}]$  and, therefore, one has the relation :  $A_i = \eta_{im} A^m$  written with Einstein’s summation convention.

The 16 partial derivatives  $a_k^i = \partial A^i / \partial x^k$  at  $M$  are the components of the tensor  $[a_k^i]$  which are given in the expression(1).

The covariant form  $[a_{ki}]$  is written explicitly as :

$$\begin{aligned} [a_{ki}] &= \left[ \sum_m a_k^m \eta_{mi} \right] \\ &= \begin{bmatrix} (\phi/c)_{,t} & -A_{,t}^x & -A_{,t}^y & -A_{,t}^z \\ (\phi/c)_{,x} & -A_{,x}^x & -A_{,x}^y & -A_{,x}^z \\ (\phi/c)_{,y} & -A_{,y}^x & -A_{,y}^y & -A_{,y}^z \\ (\phi/c)_{,z} & -A_{,z}^x & -A_{,z}^y & -A_{,z}^z \end{bmatrix} \end{aligned} \quad (56)$$

This tensor is divided into its symmetric and antisymmetric parts :

$$[S_{ki}] = \frac{1}{2} ([a_{ki}] + [a_{ik}]) \quad , \quad (57)$$

$$[F_{ki}] = \frac{1}{2} ([a_{ki}] - [a_{ik}]) \quad . \quad (58)$$

The antisymmetric part of  $[a_{ki}]$  is :

$$[F_{ki}] = 1/2 \begin{bmatrix} 0 & -A_{,t}^x - (\phi/c)_{,x} & -A_{,t}^y - (\phi/c)_{,y} & -A_{,t}^z - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A_{,t}^x & 0 & A_{,y}^x - A_{,x}^y & A_{,z}^x - A_{,x}^z \\ (\phi/c)_{,y} + A_{,t}^y & A_{,x}^y - A_{,y}^x & 0 & A_{,z}^y - A_{,y}^z \\ (\phi/c)_{,z} + A_{,t}^z & A_{,x}^z - A_{,z}^x & A_{,y}^z - A_{,z}^y & 0 \end{bmatrix} \quad (59)$$

The electromagnetic field is defined from the components of  $[F_{ki}]$ . Below are the usual equations which condense these definitions :

$$\vec{E} := -\frac{\partial \vec{A}}{\partial t} - \overrightarrow{grad} \phi \quad \vec{B} := \overrightarrow{curl} \vec{A} \quad (60)$$

The fields which are used in Maxwell’s equations are pseudovectors. We will use the special notation  $E^x, E^y, E^z, B^x, B^y, B^z$  to distinguish them from usual vectors.

The electromagnetic tensor is written :

$$[F_{ki}] = \frac{1}{2} \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & -B^z & B^y \\ -E^y/c & B^z & 0 & -B^x \\ -E^z/c & -B^y & B^x & 0 \end{bmatrix} \quad (61)$$

The preceding formulas are not new : they belong to the basic knowledge of electromagnetism. This is not the case for the symmetric part of  $[a_{ki}]$  :

$$[S_{ki}] = \frac{1}{2} \begin{bmatrix} 2(\phi/c)_{,t} & -A_{,t}^x + (\phi/c)_{,x} & -A_{,t}^y + (\phi/c)_{,y} & -A_{,t}^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_{,t}^x & -2A_{,x}^x & -A_{,y}^x - A_{,x}^y & -A_{,z}^x - A_{,x}^z \\ (\phi/c)_{,y} - A_{,t}^y & -A_{,x}^y - A_{,y}^x & -2A_{,y}^y & -A_{,z}^y - A_{,y}^z \\ (\phi/c)_{,z} - A_{,t}^z & -A_{,x}^z - A_{,z}^x & -A_{,y}^z - A_{,z}^y & -2A_{,z}^z \end{bmatrix} \quad (62)$$

$[S_{ki}]$  has been ignored in textbooks[1, 2] and in the specialized literature. This neglect leads to its replacement by charge and current densities, which are phenomenological quantities. We name this tensor the source part because it is responsible for the source terms which, as seen below, will appear in Maxwell's equations.

## 4.2 The Lagrangian.

Maxwell's equations are written as :

$$\begin{aligned} \text{div} \vec{B} &= 0 & \overrightarrow{\text{curl}} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{div} \vec{D} &= \rho & \overrightarrow{\text{curl}} \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{j} \end{aligned} \quad (63)$$

The first pair shows the relation between electric  $\vec{E}$  and magnetic  $\vec{B}$  fields. The second couple links the inductions  $\vec{D}$  and  $\vec{H}$  to the sources  $\rho$  (charge density) and  $\vec{j}$  (current density). Inductions are defined from the Lagrangian  $\mathcal{L}$  which is proportional to the determinant :

$$\mathcal{L} \propto \|a_k^i\| = \begin{vmatrix} (\phi/c)_{,t} & A_{,t}^x & A_{,t}^y & A_{,t}^z \\ (\phi/c)_{,x} & A_{,x}^x & A_{,x}^y & A_{,x}^z \\ (\phi/c)_{,y} & A_{,y}^x & A_{,y}^y & A_{,y}^z \\ (\phi/c)_{,z} & A_{,z}^x & A_{,z}^y & A_{,z}^z \end{vmatrix}$$

This expression will not be explicitly used in the following. However, it shows that the induction term which is the derivative of  $\mathcal{L}$  with respect to the term  $A_k^i$  is the determinant of the minor relative to  $A_k^i$  (accompanied by the proper sign). For instance :

$$\frac{\partial \mathcal{L}}{\partial((\phi/c)_{,t})} \propto \begin{vmatrix} A_{,x}^x & A_{,x}^y & A_{,x}^z \\ A_{,y}^x & A_{,y}^y & A_{,y}^z \\ A_{,z}^x & A_{,z}^y & A_{,z}^z \end{vmatrix}$$

The dimension of  $\mathcal{L}$  is that of a density of energy in 4-dimensional space :  $[\mathcal{L}] = \text{M L}^{-2}\text{T}^{-2}$ . The dimension of the induction is that of  $\partial \mathcal{L} / \partial A_k^i$  which is  $\text{Q L}^{-2}\text{T}^{-1}$ .

## 4.3 Maxwell's equations.

**The induction tensors.** The first pair of Maxwell's equations are identities which are nicely expressed by the equation[6] :

$$\frac{\partial F_{k\ell}}{\partial x^m} + \frac{\partial F_{\ell m}}{\partial x^k} + \frac{\partial F_{mk}}{\partial x^\ell} = 0 \quad (64)$$

Let us now show that the source terms  $\rho$  and  $\vec{j}$  in the second pair are related to the induction tensor  $[\mathcal{L}'^k_i]$ .

An element  $\mathcal{L}'^k_i$  of  $[\mathcal{L}'^k_i]$  is obtained from the derivative of the Lagrangian with respect to the element  $a_k^i$ . The developed form is :

$$\begin{aligned} [\mathcal{L}'^k_i] &= \left[ \frac{\partial \mathcal{L}}{\partial a_k^i} \right] = \\ &= \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,t}} & \frac{\partial \mathcal{L}}{\partial A_{,t}^x} & \frac{\partial \mathcal{L}}{\partial A_{,t}^y} & \frac{\partial \mathcal{L}}{\partial A_{,t}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial \mathcal{L}}{\partial A_{,x}^x} & \frac{\partial \mathcal{L}}{\partial A_{,x}^y} & \frac{\partial \mathcal{L}}{\partial A_{,x}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A_{,y}^x} & \frac{\partial \mathcal{L}}{\partial A_{,y}^y} & \frac{\partial \mathcal{L}}{\partial A_{,y}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial \mathcal{L}}{\partial A_{,z}^x} & \frac{\partial \mathcal{L}}{\partial A_{,z}^y} & \frac{\partial \mathcal{L}}{\partial A_{,z}^z} \end{bmatrix} \end{aligned} \quad (65)$$

Note that the elements of this tensor are defined in the reciprocal space. An element  $\mathcal{L}'^k_i$  is the *canonical momentum* corresponding to  $A_k^i$ .

Now the corresponding covariant tensor  $[\mathcal{L}'_{ki}]$  is split into its symmetric and antisymmetric parts which are then transformed back into mixed tensors. One obtains the separation of  $[\mathcal{L}'^k_i]$  into two parts :  $[\mathcal{L}'^k_i] = [\mathbb{D}^k_i] + [\mathbb{S}^k_i]$ .

The first part is directly linked to the usual induction tensor, it corresponds to the antisymmetric part of  $[\mathcal{L}'_{ki}]$ . The second part corresponds to the symmetric part of  $[\mathcal{L}'_{ki}]$  and will be referred to as the source tensor. The expressions of these tensors are :

$$[\mathbb{D}^k_i] = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial \mathcal{L}}{\partial A_t^x} + \frac{\partial \mathcal{L}}{\partial(\phi/c),x} & \frac{\partial \mathcal{L}}{\partial A_t^y} + \frac{\partial \mathcal{L}}{\partial(\phi/c),y} & \frac{\partial \mathcal{L}}{\partial A_t^z} + \frac{\partial \mathcal{L}}{\partial(\phi/c),z} \\ \frac{\partial \mathcal{L}}{\partial A_t^x} + \frac{\partial \mathcal{L}}{\partial(\phi/c),x} & 0 & \frac{\partial \mathcal{L}}{\partial A_x^y} - \frac{\partial \mathcal{L}}{\partial A_y^x} & \frac{\partial \mathcal{L}}{\partial A_x^z} - \frac{\partial \mathcal{L}}{\partial A_z^x} \\ \frac{\partial \mathcal{L}}{\partial A_t^y} + \frac{\partial \mathcal{L}}{\partial(\phi/c),y} & \frac{\partial \mathcal{L}}{\partial A_x^y} - \frac{\partial \mathcal{L}}{\partial A_y^x} & 0 & \frac{\partial \mathcal{L}}{\partial A_y^z} - \frac{\partial \mathcal{L}}{\partial A_z^y} \\ \frac{\partial \mathcal{L}}{\partial A_t^z} + \frac{\partial \mathcal{L}}{\partial(\phi/c),z} & \frac{\partial \mathcal{L}}{\partial A_x^z} - \frac{\partial \mathcal{L}}{\partial A_z^x} & \frac{\partial \mathcal{L}}{\partial A_y^z} - \frac{\partial \mathcal{L}}{\partial A_z^y} & 0 \end{bmatrix} \quad (66)$$

and ;

$$[\mathbb{S}^k_i] = \frac{1}{2} \begin{bmatrix} 2 \frac{\partial \mathcal{L}}{\partial(\phi/c),t} & \frac{\partial \mathcal{L}}{\partial A_t^x} - \frac{\partial \mathcal{L}}{\partial(\phi/c),x} & \frac{\partial \mathcal{L}}{\partial A_t^y} - \frac{\partial \mathcal{L}}{\partial(\phi/c),y} & \frac{\partial \mathcal{L}}{\partial A_t^z} - \frac{\partial \mathcal{L}}{\partial(\phi/c),z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c),x} - \frac{\partial \mathcal{L}}{\partial A_t^x} & 2 \frac{\partial \mathcal{L}}{\partial A_x^x} & \frac{\partial \mathcal{L}}{\partial A_x^y} + \frac{\partial \mathcal{L}}{\partial A_y^x} & \frac{\partial \mathcal{L}}{\partial A_x^z} + \frac{\partial \mathcal{L}}{\partial A_z^x} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c),y} - \frac{\partial \mathcal{L}}{\partial A_t^y} & \frac{\partial \mathcal{L}}{\partial A_x^y} + \frac{\partial \mathcal{L}}{\partial A_y^x} & 2 \frac{\partial \mathcal{L}}{\partial A_y^y} & \frac{\partial \mathcal{L}}{\partial A_y^z} + \frac{\partial \mathcal{L}}{\partial A_z^y} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c),z} - \frac{\partial \mathcal{L}}{\partial A_t^z} & \frac{\partial \mathcal{L}}{\partial A_x^z} + \frac{\partial \mathcal{L}}{\partial A_z^x} & \frac{\partial \mathcal{L}}{\partial A_y^z} + \frac{\partial \mathcal{L}}{\partial A_z^y} & 2 \frac{\partial \mathcal{L}}{\partial A_z^z} \end{bmatrix} \quad (67)$$

Electric and magnetic inductions are given by the derivatives of the Lagrangian with respect to the components of the fields  $\vec{E}/c$  and  $\vec{B}$  ; One applies the chain rule and equations (60) to obtain :

$$\mathbb{D}_x = \frac{\partial \mathcal{L}}{\partial(E^x/c)} = -\frac{\partial \mathcal{L}}{\partial(\phi/c),x} - \frac{\partial \mathcal{L}}{\partial A_t^x} \quad (68)$$

$$H_x = \frac{\partial \mathcal{L}}{\partial B^x} = \left( \frac{\partial \mathcal{L}}{\partial A_y^z} - \frac{\partial \mathcal{L}}{\partial A_z^y} \right) \quad (69)$$

Other components  $\mathbb{D}_y, \mathbb{D}_z, H_y$  and  $H_z$  are obtained from circular permutations of  $x, y, z$ .

We use the special notation  $(\mathbb{D}_x, \mathbb{D}_y, \mathbb{D}_z)$  to make a distinction from the usual displacement vector  $(D_x, D_y, D_z)$  which appears in Maxwell's equations. These components are defined from the derivatives of  $\mathcal{L}$  with respect to the electric field. These quantities are proportional  $(\mathbb{D}_x = cD_x, \dots)$ . The dimension of  $D_x$  is a density of dipoles in 4-space : Q L/(L<sup>4</sup>) while that of  $\mathbb{D}_x$  is Q L<sup>-2</sup> T<sup>-1</sup>.

We have also used a lower index notation  $\mathbb{D}_x, \mathbb{D}_y, \dots$  to stress the fact that the components of the induction pseudovectors are those of a type  $[\mathbb{D}^k_i]$  tensor :

$$[\mathbb{D}^k_i] = \frac{1}{2} \begin{bmatrix} 0 & -\mathbb{D}_x & -\mathbb{D}_y & -\mathbb{D}_z \\ -\mathbb{D}_x & 0 & H_z & -H_y \\ -\mathbb{D}_y & -H_z & 0 & H_x \\ -\mathbb{D}_z & H_y & -H_x & 0 \end{bmatrix} \quad (70)$$

The relations between the pseudovectors  $\mathbb{D}^x, \dots, H^x \dots$  and  $\mathbb{D}_x, \dots, H_x \dots$  are :

$$\mathbb{D}^x = -\mathbb{D}_x \quad , \quad \mathbb{D}^y = -\mathbb{D}_y \quad , \quad \mathbb{D}^z = -\mathbb{D}_z \quad (71a)$$

$$H^x = H_x \quad , \quad H^y = H_y \quad , \quad H^z = H_z \quad (71b)$$

In the following we will use this notation to write Maxwell's equations in both direct and inverse spaces. In passing, one should note that the splitting of  $[a_{ki}]$  into its symmetric and antisymmetric parts allows the study of special cases where one of the tensor is nullified in some regions of space while the other still exists. An illustration is the Aharonov-Bohm effect[7] which shows that a potential can exist in a region of space even in the absence of any field ( $F_{ki} = 0$ ). One sees that in such a situation it is the source tensor  $[S_{ki}]$  which can change the phase of the electron when it crosses this region.

**Euler–Lagrange equations.** Equation (41) expresses the principle of least action and introduces the conjugate momenta  $\partial\mathcal{L}/\partial A_k^i$  which are the elements of the induction tensor (65). We will show that Maxwell's equations are a consequence of this principle.

The Lagrangian density  $\mathcal{L}$  does not depend explicitly on the potentials (only on its derivatives) and equation (41) reduces to the first term. It introduces the tensor  $[\mathcal{L}'^k_i]$  whose elements have been written before :

$$\mathcal{L}'^k_i = \frac{\partial\mathcal{L}}{\partial a_k^i} \quad (72)$$

This tensor is defined in the reciprocal space and the summation-derivation operation  $\sum_k \partial/\partial x^k$  in eq.(41), corresponds to a contraction over the  $k$  index that conforms to the tensorial dimensionality of (41) which is that of a covariant vector.

When  $\partial\mathcal{L}/\partial A^i = 0$ , equation (41) can be written in matrix form :

$$\left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) [\mathcal{L}'^k_i] = (0, 0, 0, 0) \quad (73)$$

This expression groups four equations and we show now that it leads to the second pair of Maxwell's equations in reciprocal space.

We use the separation of  $[\mathcal{L}'^k_i]$  into its two parts  $[\mathbb{D}^k_i]$  and  $[\mathbb{S}^k_i]$  and write eq.(73) in compressed notation :

$$(\partial) [\mathcal{L}'^k_i] = (0) \quad \text{or} \quad (\partial) ([\mathbb{D}^k_i]) = -(\partial) ([\mathbb{S}^k_i]) \quad (74)$$

Expressions for  $\mathbb{D}^k_i$  and  $\mathbb{S}^k_i$  are given by eqs. (66) and (67). In the following we skip the factor 1/2 before  $\mathbb{D}^k_i$  and  $\mathbb{S}^k_i$  which simplifies eq.(74).

The first term  $(\partial) (\mathbb{D}^k_i)$  is computed first :

$$\begin{aligned} (\partial)([\mathbb{D}^k_i]) &= \\ &= \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 0 & -cD_x & -cD_y & -cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{bmatrix} \\ &= \left( -c \operatorname{div} \vec{D}_I, \left( -\operatorname{curl} \vec{H} - \frac{\partial \vec{D}_I}{\partial t} \right) \right) \end{aligned} \quad (75)$$

$\vec{D}_I = (D_x, D_y, D_z)$  is the symbol for the induction in the reciprocal space.

The resulting 4-vector in this equation has a time component  $-c \operatorname{div} \vec{D}_I$  et 3 space components  $(-\operatorname{curl} \vec{H} - \frac{\partial \vec{D}_I}{c\partial t})$ . These are the induction components in Maxwell's equations. The right hand side term  $(\partial) ([\mathbb{S}^k_i])$  in eq.(74) is computed now :

$$\begin{aligned} (\partial) [\mathbb{S}^k_i] &= \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 2 \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,t}} & \frac{\partial\mathcal{L}}{\partial A_{,t}^x} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial\mathcal{L}}{\partial A_{,t}^y} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial\mathcal{L}}{\partial A_{,t}^z} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} - \frac{\partial\mathcal{L}}{\partial A_{,t}^x} & 2 \frac{\partial\mathcal{L}}{\partial A_{,x}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^y} + \frac{\partial\mathcal{L}}{\partial A_{,y}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^z} + \frac{\partial\mathcal{L}}{\partial A_{,z}^x} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} - \frac{\partial\mathcal{L}}{\partial A_{,t}^y} & \frac{\partial\mathcal{L}}{\partial A_{,y}^x} + \frac{\partial\mathcal{L}}{\partial A_{,x}^y} & 2 \frac{\partial\mathcal{L}}{\partial A_{,y}^y} & \frac{\partial\mathcal{L}}{\partial A_{,y}^z} + \frac{\partial\mathcal{L}}{\partial A_{,z}^y} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} - \frac{\partial\mathcal{L}}{\partial A_{,t}^z} & \frac{\partial\mathcal{L}}{\partial A_{,z}^x} + \frac{\partial\mathcal{L}}{\partial A_{,x}^z} & \frac{\partial\mathcal{L}}{\partial A_{,z}^y} + \frac{\partial\mathcal{L}}{\partial A_{,y}^z} & 2 \frac{\partial\mathcal{L}}{\partial A_{,z}^z} \end{bmatrix} \\ &= \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,t}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ -\frac{\partial\mathcal{L}}{\partial A_{,t}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^y} & \frac{\partial\mathcal{L}}{\partial A_{,x}^z} \\ -\frac{\partial\mathcal{L}}{\partial A_{,t}^y} & \frac{\partial\mathcal{L}}{\partial A_{,y}^x} & \frac{\partial\mathcal{L}}{\partial A_{,y}^y} & \frac{\partial\mathcal{L}}{\partial A_{,y}^z} \\ -\frac{\partial\mathcal{L}}{\partial A_{,t}^z} & \frac{\partial\mathcal{L}}{\partial A_{,z}^x} & \frac{\partial\mathcal{L}}{\partial A_{,z}^y} & \frac{\partial\mathcal{L}}{\partial A_{,z}^z} \end{bmatrix} \end{aligned} \quad (76)$$

The second expression is obtained after simplification by eq.(73). Equating each component of the 4-vector of eq.(74) gives :

$$\operatorname{div} \vec{D}_I = \frac{\partial}{c\partial t} \frac{\partial\mathcal{L}}{(\phi/c)_{,t}} - \frac{\partial}{\partial x} \frac{\partial\mathcal{L}}{A_{,t}^x} - \frac{\partial}{\partial y} \frac{\partial\mathcal{L}}{A_{,t}^y} - \frac{\partial}{\partial z} \frac{\partial\mathcal{L}}{A_{,t}^z} \quad (77a)$$

$$\begin{aligned} \frac{\partial D_x}{\partial t} + \left[ \operatorname{curl} \vec{H} \right]_x &= \\ -\frac{\partial}{c\partial t} \frac{\partial\mathcal{L}}{(\phi/c)_{,x}} + \frac{\partial}{\partial x} \frac{\partial\mathcal{L}}{A_{,x}^x} + \frac{\partial}{\partial y} \frac{\partial\mathcal{L}}{A_{,x}^y} + \frac{\partial}{\partial z} \frac{\partial\mathcal{L}}{A_{,x}^z} \end{aligned} \quad (77b)$$

The two remaining equations along the  $y$  and  $z$  axis are obtained from circular permutations of  $x, y, z$  and  $X, Y, Z$ .

We use these equations to introduce the following new 4-vectors in spacetime :

$$\vec{\mathcal{L}}_k = \left( \frac{\partial \mathcal{L}}{(\phi/c)_{,k}}, -\frac{\partial \mathcal{L}}{A_{,k}^x}, -\frac{\partial \mathcal{L}}{A_{,k}^y}, -\frac{\partial \mathcal{L}}{A_{,k}^z} \right) \quad (78)$$

where  $k$  stands for  $t, x, y,$  or  $z$ .

One sees that the r.h.s. of eqs(77a,77b) are all 4-divergences of these vectors :

$$\begin{aligned} c \operatorname{div} \vec{D}_t &= \operatorname{div} \vec{\mathcal{L}}_t \\ \frac{\partial D_x}{\partial t} + [\overrightarrow{\operatorname{curl}} \vec{H}]_x &= -\operatorname{div} \vec{\mathcal{L}}_x \end{aligned}$$

These divergences define the source terms :

$$\begin{aligned} \rho &:= \frac{1}{c} \operatorname{div} \vec{\mathcal{L}}_t & j_x &:= -\operatorname{div} \vec{\mathcal{L}}_x \\ j_y &:= -\operatorname{div} \vec{\mathcal{L}}_y & j_z &:= -\operatorname{div} \vec{\mathcal{L}}_z \end{aligned} \quad (79)$$

The lower indices  $x, y, z$  label the components of the covector  $\vec{j}_i = (j_x, j_y, j_z)$ . One thus obtain Maxwell's equations in matrix form in the reciprocal space :

$$\begin{aligned} (c\rho, j_x, j_y, j_z) &= \\ \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) &\begin{bmatrix} 0 & -cD_x & -cD_y & -cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{bmatrix} \end{aligned} \quad (80)$$

Finally, Maxwell's equations in the direct space are obtained after transforming the covariant quadrivector  $(c\rho, j_x, j_y, j_z)$  into its contravariant counterpart  $(c\rho, -j^x, -j^y, -j^z)$  and the pseudovector  $\vec{D}_i$  into  $\vec{D}$ . These operations give the desired result :

$$\begin{aligned} \operatorname{div} \vec{D} &= \rho \\ [\overrightarrow{\operatorname{curl}} \vec{H}]^x &= \frac{\partial D^x}{\partial t} + j^{\vec{x}} \quad (x \text{ component}) \end{aligned}$$

or in matrix form :

$$\begin{aligned} (c\rho, -j^x, -j^y, -j^z) &= \\ \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) &\begin{bmatrix} 0 & cD^x & cD^y & cD^z \\ cD^x & 0 & H^z & -H^y \\ cD^y & -H^z & 0 & H^x \\ cD^z & H^y & -H^x & 0 \end{bmatrix} \end{aligned} \quad (81)$$

One can use the above formulae to verify the continuity equation :

$$\operatorname{div} \vec{j} = -\frac{\partial \rho}{\partial t} \quad (82)$$

We have demonstrated in this section that Maxwell's equations can be deduced from a few basic operations :

- 1 - The covariant tensor of derivatives  $[a_{ki}]$  has been split into its symmetric and antisymmetric parts in our real space, *i.e.*, the space in which the geometrical coordinates are allowed to depend on time.
- 2 - The Lagrangian density has been associated with the determinant of  $[a_k^i]$ .
- 3 - Induction tensors have been computed.

4 - The principle of least action has been applied.

The study which is presented in this section is very general and does not need any particular form of potential. It connects the well-known electromagnetic tensor to the antisymmetric part of  $[a_{ki}]$  and the source terms to the symmetric part. It shows that the fundamental quantity is the potential and that Maxwell's equations are a consequence of the least action principle. The potential is very general and can be related to the potential which describes a particle in particular situations.

## 5 Conclusion.

We have described in this article the properties of a family of tensors whose elements are obtained from partial derivatives of a potential  $A^i$  at each point M in Minkowski's spacetime. The primary tensor is the gradient  $[a_k^i]$  at M.

In the first main part of the study,  $[a_k^i]$  is divided into its two symmetric ( $[s_k^i]$ ) and antisymmetric ( $[f_k^i]$ ) parts. We use the properties of these tensors to show that there is a proper time where the scalar potential obeys the Helmholtz equation. The solutions of this equation describe electromagnetic particles. These are characterized by an accumulation of energy around the origin of the coordinates and by far fields in  $1/x$ . The condition of existence of these particles corresponds to Wheeler-Feynman's theory where an equilibrium must exist between the incoming and outgoing waves of the electron. Each solution can be even or odd and is characterized by three quantum numbers. We have given the tensors corresponding to the first five solutions. The essential result of this part is the union in a single expression of field and matter properties. In the second main part of the study,  $[a_k^i]$  is first transformed into the covariant tensor  $[a_{ki}]$  which is then divided into its symmetric ( $[S_{ki}]$ ) and antisymmetric ( $[F_{ki}]$ ) parts. These two tensors are different from ( $[s_{ki}]$ ) and ( $[f_{ki}]$ ) because the lowering-index operation does not commute with the symmetric-antisymmetric splitting. We find that  $F_{ki}$  is the well-known tensor of classical electromagnetism. By applying the Euler-Lagrange equations, we find Maxwell's equations. We prove that the source terms are expressed as functions of the derivatives of the potential.

The theory that has been described is simple, synthetic and powerful.

It is **simple** because it is based (1) on a single assumption, that of a 4-potential in Minkowski's spacetime, (2) on two non-commuting mathematical manipulations of tensors i.e. raising-lowering operation and antisymmetric-symmetric splitting, (3) on fundamental physical principles which are the principle of least action, the principle of symmetry and the principle of relativity. It is **synthetic** because it groups in a family of tensors classical electromagnetism and a description of new fundamental particles. It is **powerful** because it contains a wealth of developments : The mathematical existence of the electromagnetic particles is thought-provoking : if they are physically proved, they will change our perception of the microscopic and cosmological universe.

## 6 Appendix A.

The non-commutativity of the antisymmetrization-symmetrization operation (a-s operation) and the lowering-raising operation (l-r operation) is illustrated in this appendix with the use of  $2 \times 2$  matrices to shorten the notation (or in a one dimensional geometrical space).

The starting tensor is written as a mixed tensor in the real space :

$$[\mathbf{a}_k^i] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (83)$$

The metric tensor is :

$$[\mathbf{g}_m^n] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (84)$$

Let us consider first the lowering index operation (1) acting on  $[\mathbf{a}_k^i]$  :

$$[\mathbf{a}_{ki}] = [\mathbf{a}_k^i] \cdot [\mathbf{g}_m^n] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$$

and followed by the a-s operation (2) :

$$[f_{ki}] = \frac{1}{2} ([\mathbf{a}_{ki}] - [\mathbf{a}_{ik}]) = \frac{1}{2} \begin{bmatrix} 0 & -b-c \\ c+b & 0 \end{bmatrix}$$

$$[s_{ki}] = \frac{1}{2} ([\mathbf{a}_{ki}] + [\mathbf{a}_{ik}]) = \frac{1}{2} \begin{bmatrix} 2a & -b+c \\ c-b & -2d \end{bmatrix}$$

Consider now the a-s operation (2) acting on  $[\mathbf{a}_k^i]$  :

$$[\bar{s}_k^i] = \frac{1}{2} ([\mathbf{a}_k^i + [\mathbf{a}_i^k]) = \frac{1}{2} \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} \quad (85)$$

$$[\bar{f}_k^i] = \frac{1}{2} ([\mathbf{a}_k^i - [\mathbf{a}_i^k]) = \frac{1}{2} \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} \quad (86)$$

It is followed by the lowering index operation (1) :

$$[\bar{s}_{ki}] = \frac{1}{2} \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2a & -(b+c) \\ c+b & -2d \end{bmatrix}$$

$$[\bar{f}_{ki}] = \frac{1}{2} \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & c-b \\ c-b & 0 \end{bmatrix}$$

One sees that  $[s_{ki}] \neq [\bar{s}_{ki}]$  and  $[f_{ki}] \neq [\bar{f}_{ki}]$ . It is this non-commutativity which leads to the two branches of electromagnetism :

The first branch is based on the tensors(1) (expressed in the cartesian frame of coordinates) :

$$[a_k^i] = \begin{bmatrix} (\phi/c)_{,t} & A_{,t}^x & A_{,t}^y & A_{,t}^z \\ (\phi/c)_{,x} & A_{,x}^x & A_{,x}^y & A_{,x}^z \\ (\phi/c)_{,y} & A_{,y}^x & A_{,y}^y & A_{,y}^z \\ (\phi/c)_{,z} & A_{,z}^x & A_{,z}^y & A_{,z}^z \end{bmatrix} \quad (87)$$

$$[s_k^i] = \frac{1}{2} \begin{bmatrix} (2\phi/c)_{,t} & A_{,t}^x + (\phi/c)_{,x} & A_{,t}^y + (\phi/c)_{,y} & A_{,t}^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} + A_{,t}^x & 2A_{,x}^x & A_{,x}^y + A_{,y}^x & A_{,x}^z + A_{,z}^x \\ (\phi/c)_{,y} + A_{,t}^y & A_{,y}^x + A_{,x}^y & 2A_{,y}^y & A_{,y}^z + A_{,z}^y \\ (\phi/c)_{,z} + A_{,t}^z & A_{,z}^x + A_{,x}^z & A_{,z}^y + A_{,y}^z & 2A_{,z}^z \end{bmatrix} \quad (88)$$

$$[f_k^i] = \frac{1}{2} \begin{bmatrix} 0 & A_{,t}^x - (\phi/c)_{,x} & A_{,t}^y - (\phi/c)_{,y} & A_{,t}^z - (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_{,t}^x & 0 & A_{,x}^y - A_{,y}^x & A_{,x}^z - A_{,z}^x \\ (\phi/c)_{,y} - A_{,t}^y & A_{,y}^x - A_{,x}^y & 0 & A_{,y}^z - A_{,z}^y \\ (\phi/c)_{,z} - A_{,t}^z & A_{,z}^x - A_{,x}^z & A_{,z}^y - A_{,y}^z & 0 \end{bmatrix} \quad (89)$$

$$[s_{ki}] = \frac{1}{2} \begin{bmatrix} (2\phi/c)_{,t} & -A_{,t}^x - (\phi/c)_{,x} & -A_{,t}^y - (\phi/c)_{,y} & -A_{,t}^z - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A_{,t}^x & -2A_{,x}^x & -A_{,x}^y - A_{,y}^x & -A_{,x}^z - A_{,z}^x \\ (\phi/c)_{,y} + A_{,t}^y & -A_{,y}^x - A_{,x}^y & -2A_{,y}^y & -A_{,y}^z - A_{,z}^y \\ (\phi/c)_{,z} + A_{,t}^z & -A_{,z}^x - A_{,x}^z & -A_{,z}^y - A_{,y}^z & -2A_{,z}^z \end{bmatrix} \quad (90)$$

$$[f_{ki}] = \frac{1}{2} \begin{bmatrix} 0 & -A_{,t}^x + (\phi/c)_{,x} & -A_{,t}^y + (\phi/c)_{,y} & -A_{,t}^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_{,t}^x & 0 & -A_{,x}^y + A_{,y}^x & -A_{,x}^z + A_{,z}^x \\ (\phi/c)_{,y} - A_{,t}^y & -A_{,y}^x + A_{,x}^y & 0 & -A_{,y}^z + A_{,z}^y \\ (\phi/c)_{,z} - A_{,t}^z & -A_{,z}^x + A_{,x}^z & -A_{,z}^y + A_{,y}^z & 0 \end{bmatrix} \quad (91)$$

The second branch is based on the tensors starting from (56) :

$$[a_{ki}] = \begin{bmatrix} (\phi/c)_{,t} & -A_{,t}^x & -A_{,t}^y & -A_{,t}^z \\ (\phi/c)_{,x} & -A_{,x}^x & -A_{,x}^y & -A_{,x}^z \\ (\phi/c)_{,y} & -A_{,y}^x & -A_{,y}^y & -A_{,y}^z \\ (\phi/c)_{,z} & -A_{,z}^x & -A_{,z}^y & -A_{,z}^z \end{bmatrix} \quad (92)$$

$$[S_{ki}] = \frac{1}{2} \begin{bmatrix} 2(\phi/c)_{,t} & -A_{,t}^x + (\phi/c)_{,x} & -A_{,t}^y + (\phi/c)_{,y} & -A_{,t}^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_{,t}^x & -2A_{,x}^x & -A_{,x}^y - A_{,y}^x & -A_{,x}^z - A_{,z}^x \\ (\phi/c)_{,y} - A_{,t}^y & -A_{,y}^x - A_{,x}^y & -2A_{,y}^y & -A_{,y}^z - A_{,z}^y \\ (\phi/c)_{,z} - A_{,t}^z & -A_{,z}^x - A_{,x}^z & -A_{,z}^y - A_{,y}^z & -2A_{,z}^z \end{bmatrix} \quad (93)$$

$$[F_{ki}] = \frac{1}{2} \begin{bmatrix} 0 & -A_{,t}^x - (\phi/c)_{,x} & -A_{,t}^y - (\phi/c)_{,y} & -A_{,t}^z - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A_{,t}^x & 0 & -A_{,x}^y + A_{,y}^x & -A_{,x}^z + A_{,z}^x \\ (\phi/c)_{,y} + A_{,t}^y & -A_{,y}^x + A_{,x}^y & 0 & -A_{,y}^z + A_{,z}^y \\ (\phi/c)_{,z} + A_{,t}^z & -A_{,z}^x + A_{,x}^z & -A_{,z}^y + A_{,y}^z & 0 \end{bmatrix} \quad (94)$$

$[F_{ki}]$  is the standard electromagnetic tensor. Fields are defined from the usual equations (60).

## 7 Appendix B.

This appendix gives explicit formulas for the potentials and the corresponding tensors for the first 5 even electromagnetic particles. Odd solutions are obtained by changing  $\omega t$  into  $\omega t - \pi/2$ .

**g solution** ( $n = \ell = m = 0$ ) :

Potential :

$$\begin{aligned} \phi/c &= \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}}{c} J_0 \cos \omega t \\ A^r &= -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}}{c} J'_0 \sin \omega t \quad , \quad A_r = -A^r \\ A^\theta &= 0 \quad , \quad A^\varphi = 0 \end{aligned} \quad (95)$$

Mixed tensor :

$$[a_k^i] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \begin{bmatrix} -\sin \omega t J_0 & -\cos \omega t J'_0 & 0 & 0 \\ \cos \omega t J'_0 & -\sin \omega t J_0 & 0 & 0 \\ 0 & 0 & -\sin \omega t \frac{J'_0}{x} & 0 \\ 0 & 0 & 0 & -\sin \omega t \frac{J'_0}{x} \end{bmatrix} \quad (96)$$

**e solution** ( $n = 1, \ell = m = 0$ ).

The same formulas are obtained by replacing  $J_0$  by  $J_1$ .

**q<sub>0</sub> solution** ( $n = 1, \ell = 1, m = 0$ ).

Potential :

$$\begin{aligned} \phi/c &= \sqrt{\frac{3}{4\pi}} \frac{\mathcal{A}}{c} J_1 \cos \theta \cos \omega t \\ A^r &= -\sqrt{\frac{3}{4\pi}} \frac{\mathcal{A}}{c} J'_1 \cos \theta \sin \omega t \quad , \quad A_r = -A^r \\ A^\theta &= \sqrt{\frac{3}{4\pi}} \frac{\mathcal{A}}{c} \frac{J_1}{x} \sin \theta \sin \omega t \quad , \quad A_\theta = -A^\theta \\ A^\varphi &= 0 \end{aligned} \quad (97)$$



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