



**HAL**  
open science

## Quasi-Transfer Continuity and Nash Equilibrium \*

Rabia Nessah, Tarik Tazdait

► **To cite this version:**

Rabia Nessah, Tarik Tazdait. Quasi-Transfer Continuity and Nash Equilibrium \*. International Game Theory Review, 2019, 10.1142/S021919891950004X . hal-03592758

**HAL Id: hal-03592758**

**<https://hal.science/hal-03592758>**

Submitted on 1 Mar 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Quasi-Transfer Continuity and Nash Equilibrium\*

Rabia Nessah<sup>†</sup>

IESEG School of Management, CNRS-LEM (UMR 8179)  
3 rue de la Digue 59000 Lille, France

Tarik Tazdait<sup>‡</sup>

CNRS - EHESS - CIRED  
Campus du Jardin Tropical, 45 bis  
Av. de la Belle Gabrielle  
94736 Nogent sur Marne Cedex, France

January 14, 2019

## Abstract

We introduce a new notion of continuity, called *quasi-transfer continuity*, and show that it is enough to guarantee the existence of Nash equilibria in compact, quasiconcave normal form games. This holds true in a large class of discontinuous games. We show that our result strictly generalizes the pure strategy existence theorem of Carmona [G. Carmona, An existence result for discontinuous games, *Journal of Economic Theory* 144 (2009) 1333-1340]. We also show that our result is neither implied by nor does it imply the existence theorems of Reny [J.P. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica* 67 (1999) 1029-1056] and Baye, Tian, and Zhou [M.R. Baye, G. Tian, J. Zhou, Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs, *The Review of Economic Studies* 60 (1993) 935-948].

*Keywords:* Nash equilibrium, discontinuity, quasi-transfer continuity.

*JEL Classifications:* C70 - General, C72 - Noncooperative Games

## 1 Introduction

Nash's concept of equilibrium is probably the most important solution in non-cooperative game theory. It is immune to unilateral deviations; given that other players do not deviate from their

---

\*We thank Louis Eeckhoudt and an anonymous referee for helpful comments and suggestions that significantly improved the exposition of the paper.

<sup>†</sup>E-mail address: r.nessah@ieseg.fr

<sup>‡</sup>E-mail address: tazdait@centre-cired.fr

strategy, each player has no incentive to deviate from his/hers. Baye *et al.* [1993] established necessary and sufficient conditions for the existence of Nash equilibria in non-cooperative games that may have discontinuous and/or non-quasiconcave payoffs, but satisfy conditions called *diagonal transfer quasiconcavity* and *diagonal transfer continuity*. Reny [1999] established the existence of Nash equilibria in compact and quasiconcave games where the game is *better-reply secure*. Reny [1999] showed that better-reply security can be imposed separately as *reciprocal upper semicontinuity* and *payoff security*. Bagh and Jofre [2006] introduced the notion of *weak reciprocal upper semicontinuity* (wrusc) and showed that it is a strict weakening of reciprocal upper semicontinuity. They also prove that a game that is wrusc, together with payoff security, implies better-reply security. More recently, Carmona [2009] established the existence of Nash equilibria in compact and quasiconcave games where the game is *weakly upper semicontinuous* and *weakly payoff secure*. This paper investigates the existence of Nash equilibria in discontinuous and convex games. We introduce the notion of *quasi-transfer continuity*, *quasicontinuity* and *individual upper semicontinuity*. We show that the quasi-transfer continuity condition is a strict weakening of weakly upper semicontinuous and weakly payoff secure, and prove that a game is compact, quasiconcave and quasi-transfer continuous has a pure strategy Nash equilibrium. We also show that our result is neither implied by nor does it imply the existence theorems of Baye *et al.* [1993], Reny [1999] and Reny [2009].

## 2 Preliminaries

Consider the following non-cooperative game in a normal form:

$$G = (X_i, u_i)_{i \in I}$$

where  $I = \{1, \dots, n\}$  is a finite set of players,  $X_i$  is player  $i$ 's strategy space that is a nonempty subset of a locally convex space  $E_i$ , and  $u_i : X \rightarrow \mathbb{R}$  is the payoff function of player  $i$ . Denote by  $X = \prod_{i \in I} X_i$  the set of strategy profiles of the game. For each player  $i \in I$ , denote by  $-i$  all players except  $i$  and by  $X_{-i} = \prod_{j \neq i} X_j$  the set of strategies of the players set  $-i$ .

We say that a game  $G = (X_i, u_i)_{i \in I}$  is compact if, for all  $i \in I$ ,  $X_i$  is compact and  $u_i$  is bounded. We say that a game  $G = (X_i, u_i)_{i \in I}$  is quasiconcave if, for every  $i \in I$ ,  $X_i$  is convex and the function  $u_i$  is quasiconcave in  $x_i$ .

Carmona [2009] studied the existence of pure strategy Nash equilibria in discontinuous games by introducing the concepts of weakly upper semicontinuous and weakly payoff secure.

**DEFINITION 2.1** (Carmona [2009]) A game  $G = (X_i, u_i)_{i \in I}$  is weakly payoff secure if for all  $i \in I$ ,  $x \in X$  and  $\epsilon > 0$ , there exists a neighborhood  $\mathcal{N}(x_{-i})$  of  $x_{-i}$  so that for each  $x'_{-i} \in \mathcal{N}(x_{-i})$ , there exists  $\bar{x}_i \in X_i$  such as  $u_i(\bar{x}_i, x'_{-i}) \geq u_i(x_i, x_{-i}) - \epsilon$ .

Definition 2.1 is equivalent to the following function  $v_i : X_{-i} \rightarrow \mathbb{R}$  defined by  $v_i(x_{-i}) = \sup_{z_i \in X_i} u_i(z_i, x_{-i})$  is lower semicontinuous on  $X_{-i}$ , for all  $i \in I$ .

Let us consider the following function  $\bar{u} : X \times X \rightarrow \mathbb{R}^n$  defined by  $\bar{u}(x, y) = (u_i(x_i, y_{-i}), i \in I)$  and let  $\Gamma = \text{cl}(\text{Graph}(\bar{u})) \setminus \text{Graph}(\bar{u})$  the frontier of the graph of function  $\bar{u}$ .

**DEFINITION 2.2** (Carmona [2009]) A game  $G = (X_i, u_i)_{i \in I}$  is weakly upper semicontinuous if for all  $(x, y, \alpha) \in \Gamma$ , there exists a player  $i$  and a strategy  $\hat{x}_i \in X_i$  such that  $u_i(\hat{x}_i, y_{-i}) > \alpha_i$ .

Corollary 2 in Carmona [2009] shows that a  $G = (X_i, u_i)_{i \in I}$  possesses a Nash equilibrium if it is compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure.

### 3 Existence Result

In this section we investigate the existence of pure strategy Nash equilibria in games that may be discontinuous. We provide the main result on the existence of pure strategy Nash equilibria in discontinuous games, which strictly generalizes the existence result of Carmona [2009]. In Section 3.2, we show that our existence result is unrelated to those of Baye *et al.* [1993], Reny [1999] and Reny [2009].

#### 3.1 Quasi-Transfer Continuity and Nash Equilibrium

We start by introducing the following notions. Let us consider the best reply correspondence of game  $G$ ,  $F : X \rightrightarrows X$  defined by

$$F(x) = \{y \in X \text{ such that } u_i(z_i, x_{-i}) \leq u_i(y_i, x_{-i}), \forall z \in X, \forall i \in I\}.$$

**DEFINITION 3.1** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *quasicontinuous* if,  $y \notin F(x)$  for some  $x, y \in X$ , then there exist a neighborhood  $\mathcal{N}(x)$  of  $x$  and a neighborhood  $\mathcal{N}(y)$  of  $y$  such that: for all  $(x', y') \in \mathcal{N}(x) \times \mathcal{N}(y)$ , there exist a player  $j$  and a strategy  $z'_j \in X_j$  so as  $u_j(z'_j, x'_{-j}) > u_j(y'_j, x'_{-j})$ .

So, a game is quasicontinuous if for every  $x, y \in X$  with  $\sup_{z_i \in X_i} u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$  for some player  $i$ , there exist a neighborhood  $\mathcal{N}(y)$  of  $y$  and a neighborhood  $\mathcal{N}(x)$  of  $x$  such that for every strategy profile  $(x', y')$  in the neighborhood  $\mathcal{N}(x) \times \mathcal{N}(y)$ , some players  $j$  have a securing strategy profile, will be strictly better above  $u_j(y'_j, x'_{-j})$ .

**DEFINITION 3.2** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *individual upper semicontinuous* (IUSC) if, for all  $i \in I$ ,  $x \in X$  and for each  $\epsilon > 0$ , there exists a neighborhood  $\mathcal{N}(x_i)$  of  $x_i$  such that for all  $x'_i \in \mathcal{N}(x_i)$ , we have  $u_i(x) \geq u_i(x'_i, x_{-i}) - \epsilon$ .

**DEFINITION 3.3** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *quasi-transfer continuous* if,  $y \notin F(x)$  for some  $x, y \in X$ , then there exist a neighborhood  $\mathcal{N}(x)$  of  $x$  and a neighborhood  $\mathcal{N}(y)$  of  $y$  such that for all  $x' \in \mathcal{N}(x)$ , there exist a player  $i$  and a strategy  $\bar{y}_i \in X_i$  so as  $u_i(\bar{y}_i, x'_{-i}) > u_i(y'_i, x'_{-i})$ , for each  $y'_i \in \mathcal{N}(y_i)$ .

So, a game is quasi-transfer continuous if for every  $x, y \in X$  with  $y \notin F(x)$ , then there exist a neighborhood  $\mathcal{N}(x)$  of  $x$  and a neighborhood  $\mathcal{N}(y)$  of  $y$  such that for every strategy profile  $x'$  in the neighborhood  $\mathcal{N}(x)$ , then for some players  $i$  have a strategy  $\bar{y}_i \in X_i$  so as  $u_i(\bar{y}_i, x'_{-i})$  will be strictly better above  $u_i(y'_i, x'_{-i})$ , for each  $y'_i \in \mathcal{N}(y_i)$ . It is clear that  $G$  is quasi-transfer continuous if it is both quasicontinuous and individually upper semicontinuous.

The following proposition shows that a game  $G$  compact, weakly upper semicontinuous and weakly payoff secure is sufficient to make the game quasi-transfer continuous.

**PROPOSITION 3.1** *If  $G = (X_i, u_i)_{i \in I}$  is compact, weakly upper semicontinuous and weakly payoff secure, then it is quasi-transfer continuous.*

**PROOF.** Let  $G = (X_i, u_i)_{i \in I}$  be a compact, weakly upper semicontinuous and weakly payoff secure. Suppose, by way of contradiction, that the game is not quasi-transfer continuous. Then, there exists  $x, y \in X$ , with  $y \notin F(x)$ , Then for some  $i \in I$ , we have

$$\sup_{z_i \in X_i} u_i(z_i, x_{-i}) > u_i(y_i, x_{-i}), \quad (3.1)$$

such that for every neighborhood  $\mathcal{N}(x) \times \mathcal{N}(y)$  of  $(x, y)$ , there exists  $x' \in \mathcal{N}(x)$  so as for each player  $j \in I$  and for each  $\bar{z}_j \in X_j$ , there exists  $y'_j \in \mathcal{N}(y_j)$  so that

$$u_j(\bar{z}_j, x'_{-j}) \leq u_j(y'_j, x'_{-j}).$$

Then, we can find a directed system of neighborhoods  $\{\mathcal{V}^\alpha(x, y)\}_{\alpha \in \Lambda}$  and a net  $(x^\alpha, y^\alpha)^\alpha \in \mathcal{V}^\alpha(x, y)$  with  $(x^\alpha, y^\alpha)^\alpha \rightarrow_\alpha (x, y)$  and a net  $\{\epsilon(\alpha) > 0\}^\alpha$  with  $\epsilon(\alpha) \rightarrow_\alpha 0$  such that for each  $\alpha$ , we have

$$\sup_{z_j \in X_j} u_j(z_j, x_{-j}^\alpha) \leq u_j(y_j^\alpha, x_{-j}^\alpha) + \epsilon(\alpha), \text{ for each } j \in I.$$

By Theorem 1 in Carmona [2009], the function  $v_i(x_{-i}) = \sup_{z_i \in X_i} u_i(z_i, x_{-i})$  is continuous over  $X_{-i}$ , for each  $i \in I$  if  $G = (X_i, u_i)_{i \in I}$  is compact, weakly upper semicontinuous and weakly payoff secure. Since  $G$  is compact and  $u_i$  is bounded for each  $i$ , we may assume that  $\{u_j(y_j^\alpha, x_{-j}^\alpha)\}_\alpha$  converges to  $u_j$ , for each  $j \in I$ . Thus,  $(y, x, u) \in \text{cl}(\text{Graph}(\bar{u}))$  and

$$v_j(x_{-j}) \leq u_j \text{ for each } j \in I.$$

If  $(y, x, u) \in \Gamma = \text{cl}(\text{Graph}(\bar{u}) \setminus \text{Graph}(\bar{u}))$ , then by weak upper semicontinuity of  $G$ , there exist  $i \in I$ , a strategy  $\hat{y}_i \in X_i$  so that  $u_i(\hat{y}_i, x_{-i}) > u_i$ . Thus,  $v_i(x_{-i}) = \sup_{z_i \in X_i} u_i(z_i, x_{-i}) \leq u_i <$

$u_i(\hat{y}_i, x_{-i})$  which is impossible. Therefore  $(y, x, u) \in \text{Graph}(\bar{u})$ , i.e.  $u_j = u_j(y_j, x_{-j})$  for each  $j \in I$  and by (3.1), we obtain  $u_i(y_i, x_{-i}) < \sup_{z_i \in X_i} u_i(z_i, x_{-i}) = v_i(x_{-i}) \leq u_i = u_i(y_i, x_{-i})$ , which is impossible. Hence, the game must be quasi-transfer continuous. ■

Now we are ready to state our main result, which strictly generalizes Corollary 2 of Carmona [2009].

**THEOREM 3.1** *Suppose  $G = (X_i, u_i)_{i \in I}$  is convex, compact, bounded, quasiconcave, and quasi-transfer continuous. Then  $G$  possesses a pure strategy Nash equilibrium.*

**PROOF.** For each  $x \in X$ , let us consider the best-reply correspondence defined by

$$F(x) = \{y \in X, \text{ such that } u_i(z_i, x_{-i}) \leq u_i(y_i, x_{-i}), \text{ for all } i \in I, z_i \in X_i\}.$$

By compactness and quasi-transfer continuity of game  $G$ , we can easily show that  $F$  has nonempty and compact values and a closed graph. Furthermore, it also has convex values whenever the game is quasiconcave. Then by Fan-Glicksberg's fixed point theorem, the best-reply correspondence  $F$  has a fixed point, and so the game has a pure Nash equilibrium. ■

The quasi-transfer continuity is strictly weaker than the weak upper semicontinuity and weak payoff security (Carmona [2009]). To see this, consider the following example.

**EXAMPLE 3.1** Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$  studied by Carmona [2009].

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}, \text{ and } u_2(x_1, x_2) = 1 - u_1(x_1, x_2)$$

Carmona [2009] shows that this game is not weakly upper semicontinuous. Indeed, let  $x = y = (\frac{1}{2}, 0)$  and  $u = (1, 1)$ . It is clear that  $(x, y, u) \in \Gamma$ , but we have for all  $i \in I$ , and for every  $\hat{x}_i \in X_i$ ,  $u_i(\hat{x}_i, y_{-i}) \leq u_i = 1$ . Thus, this game is not weakly upper semicontinuous, so Corollary 2 of Carmona [2009] cannot be applied.

However, it is quasi-transfer continuous. Indeed, let  $x, y \in X$  with  $\sup_{z_i \in X_i} u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$ , which is equivalent to  $i = 1$  and  $y_1 > \frac{1}{2}$  (Because if  $i = 2$ , then  $u_2(x_1, y_2) = 0$  if and only if  $x_1 \leq \frac{1}{2}$ , but in this case  $\sup_{z_2 \in X_2} u_2(x_1, z_2) = 0$ ). Thus, there exist a neighborhood  $\mathcal{N}(x) \subset [0, 1] \times [0, 1]$  of  $x$  and a neighborhood  $\mathcal{N}(y) \subset (\frac{1}{2}, 1] \times [0, 1]$  of  $y$  such that for all  $x' \in \mathcal{N}(x)$ , there exist a player  $j = 1$  and a strategy  $\bar{z}_j = \frac{1}{2}$  so as  $1 = u_1(\bar{z}_1, x'_2) > 0 = u_1(y'_1, x'_2)$ , for each  $y'_1 \in \mathcal{N}(x) \times \mathcal{N}(y_1)$ . Since the game is also compact and quasiconcave, so by Theorem 3.1, the game considered possesses a Nash equilibrium.

The notion of quasi-transfer continuity can be easily extended to the generalized games, quasi-(symmetric) games and to games with mixed strategies. Moreover, every game that is quasicontinuous in the extended sense has a mixed strategy Nash equilibrium, even if the payoff functions are not quasiconcave.

### 3.2 Examples and Related Literature

The following examples show that the quasi-transfer continuity introduced in the present paper neither implies nor is implied by diagonal transfer continuity in Baye *et al.* [1993] or better-reply security in Reny [1999] and/or single deviation property of Reny [2009].

**EXAMPLE 3.2** Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$ .

$$u_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 + x_2 \geq \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}, \text{ and } u_2(x_1, x_2) = x_1 + x_2$$

The game considered in this example is compact and quasiconcave. It is also both diagonally transfer continuous and diagonally transfer quasiconcave. Indeed, let  $x, y \in X$  such that  $U(x, y) > U(x, x)$  with  $U(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2)$ .

- 1) If  $x_1 + x_2 \geq \frac{3}{2}$ . Then,  $U(x, x) = 2x_1 + x_2$  and  $x_1 + x_2 < y_1 + y_2$ . Therefore, there exist  $\bar{y} = (1, 1)$  and a neighborhood  $\mathcal{N}(x)$  of  $x$  such that for all  $x' \in \mathcal{N}(x)$ ,  $x'_1 + x'_2 < y_1 + y_2$  and  $U(x', \bar{y}) > U(x', x')$ .
- 2) If  $x_1 + x_2 < \frac{3}{2}$ . Then,  $U(x, x) = x_1 + x_2$ . Therefore, there exist  $\bar{y} = (1, 1)$  and a neighborhood  $\mathcal{N}(x)$  of  $x$  such that for all  $x' \in \mathcal{N}(x)$ ,  $x'_1 + x'_2 < \frac{3}{2}$  and  $U(x', \bar{y}) > U(x', x')$ .

Let  $Y^m = \{y^1, \dots, y^m\}$  a subset of  $X$  and define  $x^k = (1, 1)$ , for each  $k = 1, \dots, m$ . Then, for any subset  $\{x^{k_1}, \dots, x^{k_s}\}$  of  $Y^m$  with  $1 \leq s \leq m$ , if  $x^0 \in \text{co}(\{x^{k_1}, \dots, x^{k_s}\}) = \{(1, 1)\}$  and so for each  $j = 1, \dots, s$ ,  $U(x^0, y^{k_j}) \leq U(x^0, x^0)$ . So, by Theorem 1 of Baye *et al.* [1993], the game considered possesses a Nash equilibrium.

This game is better-reply secure. Indeed, since the unique Nash equilibrium is given by  $x_1 = x_2 = 1$ , any nonequilibrium strategy profile  $(x_1, x_2)$  contains a component that is not equal to one and let  $(x, u)$  be in the closure of the graph of its vector function (In this case we have  $u = (u_1, u_2) \in \{(x_1, x_1 + x_2), (0, x_1 + x_2)\}$ ).

- 1) If  $x_1 < 1$  and  $x_2 > \frac{1}{2}$ . Then, there exist  $\bar{y}_1 = 1$  and a neighborhood  $\mathcal{N}(x_2) \subset (\frac{1}{2}, 1]$  of  $x_2$  such that for all  $x'_2 \in \mathcal{N}(x_2)$ ,  $u_1(\bar{y}_1, x'_2) = 1 > u_1 \in \{x_1, 0\}$ .
- 2) If  $x_1 < 1$  and  $x_2 \leq \frac{1}{2}$ . Then, there exist  $\bar{y}_2 = 1$  and a neighborhood  $\mathcal{N}(x_1) \subset [x_1 - \delta, 1)$  of  $x_1$  (with  $\delta < \frac{1}{2}$ ) such that for all  $x'_1 \in \mathcal{N}(x_1)$ ,  $u_2(x'_1, \bar{y}_2) = 1 + x'_1 > u_2 = x_1 + x_2$ .
- 3) If  $x_1 = 1$ , then  $x_2 < 1$ . Choose  $\epsilon > 0$  with  $x_2 + \epsilon < 1$ . Then, there exist  $\bar{y}_2 = 1$  and a neighborhood  $\mathcal{N}(x_1) \subset (1 - \epsilon, 1]$  of  $x_1$  such that for all  $x'_1 \in \mathcal{N}(x_1)$ ,  $u_2(x'_1, \bar{y}_2) = 1 + x'_1 > u_2 = x_1 + x_2$ .

So, by Theorem 3.1 of Reny [1999], the game considered possesses a Nash equilibrium.

However, it is not quasi-transfer continuous. Indeed, let  $x = (\frac{1}{2}, \frac{1}{2})$  and  $y = (\frac{1}{2}, 1)$ , therefore  $y \notin F(x)$  ( $\sup_{z_1 \in X_1} u_1(z_1, x_2) = 1 > 0 = u_1(y_1, x_2)$ ). Then, for each neighborhood  $\mathcal{N}(x) \subset [0, 1] \times [0, 1]$  of  $x$  and any neighborhood  $\mathcal{N}(y) \subset [0, 1] \times [0, 1]$  of  $y$  there exists  $x' \in \mathcal{N}(y)$  with  $x'_2 < \frac{1}{2}$  such that for each  $z \in X$ , there exists  $y' \in \mathcal{N}(y)$  with  $y'_2 = 1$  so as  $u_1(z_1, x'_2) = 0 \leq u_1(y'_1, x'_2)$ , and  $u_2(x'_1, z_2) = x'_1 + z_2 \leq x'_1 + 1 = u_2(x'_1, y'_2)$ . So Theorem 3.1 cannot be applied.

Reny [2009] introduced the following definition weaker than better-reply security and diagonally transfer continuity.

**DEFINITION 3.4** Reny [2009] A game  $G = (X_i, u_i)_{i \in I}$  has the *single deviation property* if whenever  $x \in X$  is not an equilibrium, there exist a strategy profile  $y \in X$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for every  $x' \in \mathcal{V}(x)$ , there exists a player  $i$  such as  $u_i(y_i, x'_{-i}) > u_i(x')$ .

When a game  $G$  has the single deviation property, it means that if  $x$  is not an equilibrium, some of its neighborhood  $\mathcal{V}(x)$  of  $x$  does not contain any equilibrium.

The following examples show that the quasi-transfer continuity (Definition 3.1) neither implies nor is implied by single deviation property (Definition 3.4).

**EXAMPLE 3.3** Recall Example 3.2. Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$ .

$$u_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 + x_2 \geq \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}, \text{ and } u_2(x_1, x_2) = x_1 + x_2$$

This game is not quasi-transfer continuous (see the proof in Example 3.2) but it does have the single deviation property. Indeed, the unique Nash equilibrium is  $x_1 = x_2 = 1$ , and any nonequilibrium strategy profile  $(x_1, x_2)$  contains a component that is not equal to one. Then, there exist a strategy  $y = (1, 1)$  and a neighborhood  $\mathcal{N}(x)(1, 1)$  of  $x$  such that for each  $x' \in \mathcal{N}(x)$ , we have: If  $x'_2 < 1$ , then there exists  $i = 2$ , so that  $u_2(x'_1, y_2) = 1 + x'_1 > u_1(x') = x'_2 + x'_1$ . If  $x'_2 = 1$ , thus  $x'_1 < 1$ , then there exists  $i = 1$ , so that  $u_1(y_1, x'_2) = 1 > u_1(x') = \begin{cases} x'_1 & \text{if } x'_1 \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$

**EXAMPLE 3.4** Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$ .

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise.} \end{cases}$$

This game is quasi-transfer continuous. Indeed, let  $x, y$  with  $\sup_{z_i \in X_i} u_i(z_i, x_{-i}) = 1 > 0 = u_i(y_i, x_{-i})$ . Thus,  $y_i \neq x_{-i}$ . Therefore, there exists a neighborhood  $\mathcal{N}(y_i) \times \mathcal{N}(x_{-i}) \subset [0, 1] \times$



$[0, 1]$  of  $(y_i, x_{-i})$  with  $y'_i \neq x'_{-i}$  such that for each  $x'_{-i} \in \mathcal{N}(x_{-i})$ , there exists  $\bar{y}_i = x'_{-i}$  so as  $u_i(\bar{y}_i, x'_{-i}) = 1 > 0 = u_i(y'_i, x'_{-i})$ , for each  $y'_i \in \mathcal{N}(y_i)$ .

However, it does not have the single deviation property. Indeed, let  $x = (1, 0)$ , clearly  $x$  is not a Nash equilibrium. Notice nevertheless that, for all  $y' \in [0, 1] \times [0, 1]$  and any neighborhood  $\mathcal{V}(x) \subset X$  of  $x$ , choosing  $x' \in \mathcal{V}(x)$  with  $x'_1 \neq y'_2$  and  $x'_2 \neq y'_1$ , we obtain that  $u_i(y'_i, x'_{-i}) = 0 \leq u_i(x')$ . Therefore, this game does not have the single deviation property.

With a counterexample (Example 3.1 in Reny [2009]), Reny [2009] shows that a game  $G = (X_i, u_i)_{i \in I}$  is quasiconcave, compact, bounded and has the single-deviation property, it may not possess a pure strategy Nash equilibrium. However, he proved that if the mixed extension of  $G$  has the single-deviation property, then it possesses a mixed strategy Nash equilibrium.

**REMARK 3.1** By Examples 3.3-3.4, we conclude that in the extended mixed strategy games, our result is neither implied by nor does it imply the existence theorem of Reny [2009].

## References

- A. Bagh, A. Jofre, Reciprocal upper semicontinuity and better-reply secure Games: A comment, *Econometrica* 74 (2006) 1715-1721.
- M.R. Baye, G. Tian, J. Zhou, Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs, *The Review of Economic Studies* 60 (1993) 935-948.
- G. Carmona, An existence result for discontinuous games, *Journal of Economic Theory* 144 (2009) 1333-1340.
- J.P. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica* 67 (1999) 1029-1056.
- J.P. Reny, Further results on the existence of Nash equilibria in discontinuous games, *mimeo* (2009).