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# SHARP INEQUALITIES RELATED TO WILKER RESULTS

Abd Raouf Chouikha \*

## Abstract

In this paper, we interested in Wilker inequalities. We provide finer bounds than known previous. Moreover, bounds are obtained for the following trigonometric function

$$g_n(x) = \left(\frac{\sin(x)}{x}\right)^2 \left(1 - \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) + \frac{\tan(x)}{x}, \quad n \geq 0$$

*Key Words and phrases:* Trigonometric functions; Sinc function; Inequalities.<sup>1</sup>

## 1 Introduction

For  $0 < x < \pi/2$  we know

$$\frac{\sin x}{x} < 1 < \frac{\tan x}{x},$$

or equivalently

$$\frac{x}{\tan x} < 1 < \frac{x}{\sin x}.$$

Exploiting these inequalities to the aim to provide other similar Wilker proposed open problems [21]. One of them is

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

Moreover, Wilker asked about the largest constant  $c$  such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

These problems was solved by Sumner and al. [20] who proved in addition

$$2 + \frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x.$$

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As we may remark these constants  $\frac{16}{\pi^4}$  and  $\frac{8}{45}$  are the limits at 0 and  $\pi/2$  of the function

$$x \rightarrow \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}.$$

Chen and Cheung [5] proved that this function decreases and provided

$$2 + \frac{8x^3}{45} + \frac{16x^5}{315} \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8x^3}{45} + \left(\frac{2}{\pi}\right)^6 x^5 \tan x,$$

$$2 + \frac{8x^3}{45} + \frac{16x^5}{315} + \frac{104x^7}{4725} \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8x^3}{45} + \frac{16x^5}{315} + \left(\frac{12}{\pi}\right)^8 x^7 \tan x.$$

Moreover, all these constants are the best possible constants.

Later in using a natural approach Mortici [17] gave improvements of that inequalities and produced the following for  $0 < x < \pi/2$

$$2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} \right) < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} \right).$$

Malesevic and al. [12] further refined the above inequality for  $0 < x < \pi/2$  and  $n \in \mathbb{N}$  :

$$2 + \frac{1}{\cos x} \sum_{k=2}^{2n+1} (-1)^k D_k x^{2k} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x} \sum_{k=2}^{2n} (-1)^k D_k x^{2k},$$

$$\text{where } D_k = \frac{(-9+3^{2k+2}-40k-32k^2)}{4(2k+2)!}.$$

On the other hand, Wu and Srivastava [22, Lemma 3] proved the following dual inequality for  $0 < x < \pi/2$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2, \quad 0 < x < \frac{\pi}{2}$$

Mortici [17] improved that result

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \frac{2x^4}{45}.$$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > \frac{2 \sin(x)}{x} + \frac{\tan(x)}{x}.$$

Malesevic and al. [12] further improved the above result for  $0 < x < \pi/2$  and  $n \in \mathbb{N}$  :

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \sum_{k=2}^n \frac{|B_{2k}| (2k-2)2^{2k}}{(2k)!} x^{2k}.$$

**Statement 1.** [12, Theorem 4 p.9] For  $0 < x < \frac{\pi}{2}$

$$(i) \quad 2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650} \right) <$$

$$\left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} \right).$$

$$(ii) \quad 2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650} + \frac{283x^{12}}{20638800} - \frac{3503x^{14}}{6810604000} \right) <$$

$$\left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} + \frac{19x^8}{4725} - \frac{37x^{10}}{133650} + \frac{283x^{12}}{20638800} \right).$$

**Statement 2.** [12, Theorem 5 p.11] For  $0 < x < \frac{\pi}{2}$

$$(i) \quad 2 + \frac{2x^4}{45} + \frac{8x^6}{945} < \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < 2 + \frac{2x^4}{45} + \left( \frac{2}{\pi} \right)^6 \left( -2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} \right) x^6.$$

$$(ii) \quad 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} < \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} <$$

$$2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \left( \frac{2}{\pi} \right)^8 \left( -2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560} \right) x^8.$$

$$(iii) \quad 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + \frac{16x^{10}}{93555} < \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} <$$

$$2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + \left( \frac{2}{\pi} \right)^{10} \left( -2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560} - \frac{\pi^8}{201600} \right) x^{10}.$$

In this paper, we aim to refine the inequalities mentioned above.

## 2 The first Wilker inequality

One proved [..]

The first Wilker inequality [21]

$$\left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} > 2, \quad x \in \left( 0, \frac{\pi}{2} \right)$$

was intensively studied by many authors, e.g, [3],[7],[13],[18],[23] Mortici [17] and Malesevic [12] proved

$$2 + \left( \frac{8}{45} x^4 - \frac{8}{105} x^6 \right) \left( \frac{1}{\cos x} \right) < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \left( \frac{8x^4}{45 \cos(x)} \right).$$

The following result gives bounds which permit to refine some previous ones

**Theorem 2-1** For  $0 < x < \pi/2$  the following inequalities holds for  $1 \leq m \leq n$  and  $p \leq n$

$$\begin{aligned} & \left( 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^\beta)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^n a^k x^{2k-2} \right) \times \\ & \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)} x^{2k} \right) < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < \\ & \left( 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} + \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2} \right) \times \\ & \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \left( \frac{2x}{\pi} \right)^{2p+2} \frac{1}{1 - \left( \frac{2x}{\pi} \right)^2} \right), \end{aligned}$$

where  $B_{2k}$  are the Bernoulli numbers and  $\beta = 2 + \frac{\ln(1-\frac{6}{\pi^2})}{\ln 2} = 0.6491\dots$  is the Alzer constant.

**Proof of Theorem 2-1** Remark at first we may write obviously

$$\left( \frac{\sin(x)}{x} \right)^2 \left( 1 + \frac{2x}{\sin(2x)} \right) = \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

**Lemma 2-2** For  $0 < x < \pi/2$  the following inequalities holds for any integer  $m \geq 5$

$$\begin{aligned} & \left( \frac{\sin(x)}{x} \right)^2 < 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} + \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2}, \\ & 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^n a^k x^{2k-2} < \left( \frac{\sin(x)}{x} \right)^2. \end{aligned}$$

**Lemma 2-3** For  $0 < x < \pi/2$  consider the function

$$f(x) = (\sin x)^2 - x^3 \cot x - \frac{x^6}{15} + \frac{x^8}{945}$$

then  $f(x)$  can be expressed as power series  $f(x) = \sum_{k \geq 5} a_k x^{2k}$ ,

$$a_k = \frac{2^{2k-2} |B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right),$$

where  $B_{2k}$  are the Bernoulli numbers. Moreover, the coefficients  $a_k, k \geq 5$  are all positive:

$$a_5 = \frac{1}{255150}, \quad a_6 = \frac{2}{15436575}, \quad a_7 = \frac{103}{8300667375}, \dots$$

*Proof of Lemma 2-3*

The following series expansions can be found in [1],[10],[11]

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, \quad x \in (0, \pi)$$

and

$$\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k}, \quad x \in (0, \frac{\pi}{2})$$

$$\begin{aligned} (\sin x)^2 - x^3 \cot x = & \left( \frac{1}{15} x^6 - \frac{1}{945} x^8 + \frac{1}{2835} x^{10} + \frac{8}{467775} x^{12} + \frac{206}{91216125} x^{14} + \right. \\ & \left. \frac{139}{638512875} x^{16} + \frac{10861}{488462349375} x^{18} + \frac{438628}{194896477400625} x^{20} + O(x^{22}) \right). \end{aligned}$$

To prove the positivity of the coefficients  $a_k$ ,  $k$  even we will use the following inequality for Bernoulli numbers established by D'Aniello [9]:

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}| < \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}.$$

For any odd value of  $k$ , we have

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right) = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{1}{(2k-1)k} \right) > 0.$$

Consider the even case, then

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{-1}{(2k-1)k} \right).$$

By definition of Bernoulli numbers

$$S_n(p) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k} B_k n^{p+1-k}.$$

Then for  $k$  even

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| - \frac{1}{(2k-1)k} \right) > \frac{2^{2k-2}}{(2k-2)!} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)} - \frac{1}{(2k-1)k} \right)$$

and

$$a_k < \frac{2^{2k-2}}{(2k-2)!} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right).$$

Therefore

$$\begin{aligned} \frac{2^{2k-2} \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)}}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} &< a_k = \frac{2^{2k-2} |B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \\ &< \frac{2^{2k-2} \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)}}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!}. \end{aligned}$$

and

$$\sum_{k \geq 5} \left( \frac{2^{2k-1}}{2\pi^{2k-2}(2^{2k-2}-1)} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \right) x^{2k} < f(x).$$

Thus it implies the left hand of (4) since for any integer  $n \geq 5$  we have

$$\begin{aligned} \cos x &\leq \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} \right) + \sin x \sum_{5 \leq k \leq n} a_k x^{2k-3} \\ &\leq \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} \right) + \sin x \sum_{5 \leq k \leq \infty} a_k x^{2k-3} = \left( \frac{\sin x}{x} \right)^3. \end{aligned}$$

$$(\sin x)^2 - x^3 \cot x = (\sin x)^2 - x^3 \left( \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \right) =$$

$$(\sin x)^2 - x^2 + \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k+2} < (\sin x)^2 - x^2 + \sum_{k=1}^{\infty} \frac{2^{2k+1} x^{2k+2}}{\pi^{2k} (2^{2k}-2)}.$$

In the other hand we know that for  $k > 1$

$$(2k)! > \sqrt{4\pi k} \left( \frac{2k}{e} \right)^{2k} e^{\frac{1}{24k+1}}.$$

It implies

$$\begin{aligned} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) &> \left( \frac{2\sqrt{4\pi(k-1)}\left(\frac{2k-2}{e}\right)^{2k-2}e^{\frac{1}{48k-23}}}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) > \\ \left( \frac{2\sqrt{4\pi(k-1)}\left(\frac{2k-2}{\pi e}\right)^{2k-2}}{(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) &> \left( \frac{2\sqrt{4\pi(k-1)}\left(\frac{2k-2}{\pi e}\right)^{2k-2}}{2^{2k-2}} - \frac{1}{(2k-1)k} \right) > \\ \left( 2\sqrt{4\pi(k-1)}\left(\frac{k-1}{\pi e}\right)^{2k-2} - \frac{1}{(2k-1)k} \right). \end{aligned}$$

Thanks to *Maple* we may easily verify that the last expression is non negative as soon as  $k > 6$ . This means that  $a_k$  are non negative.

By this Lemma we deduce

$$\frac{f(x)}{x^2} = \left( \frac{\sin(x)}{x} \right)^2 - x \cot x - \frac{x^4}{15} + \frac{x^6}{945} = \sum_{k \geq 5} a^k x^{2k-2},$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right).$$

Thus for any integer  $n \geq 5$

$$\left( \frac{\sin(x)}{x} \right)^2 = x \cot x + \frac{x^4}{15} - \frac{x^6}{945} + \sum_{k \geq 5} a^k x^{2k-2} \geq x \cot x + \frac{x^4}{15} - \frac{x^6}{945} + \sum_{k=5}^n a^k x^{2k-2}$$

since coefficients  $a_k > 0$ .

Therefore since by [6, p.] for  $x \in (0, \pi)$

$$\begin{aligned} \cot x &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \\ &= \frac{1}{x} - \sum_{k=1}^{n-1} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} - \sum_{k=n}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, \end{aligned}$$

and by [9]

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}| < \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}. \quad (1)$$

Then we deduce the inequalities

$$\sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k-1} < \cot x - \frac{1}{x} + \sum_{k=1}^n \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} < \sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k-1}.$$

Notice that Alzer [2] provides a further result

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}| < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2^\beta)} < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2)} \quad (2)$$

where  $\beta = 2 + \frac{\ln(1-\frac{6}{\pi^2})}{\ln 2} = 0.6491\dots$  is the best possible constant in the sense that it can not be replaced respectively by any bigger and smaller constant in the double inequality.

We then obtain better inequalities for  $\cot x$

$$\sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-2^\beta)} x^{2k-1} < \cot x - \frac{1}{x} + \sum_{k=1}^n \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}.$$

On the other hand, it follows by the same way for any integer  $m \geq 5$

$$\left(\frac{\sin(x)}{x}\right)^2 < 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2},$$

$$1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^\beta)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^n a^k x^{2k-2} < \left(\frac{\sin(x)}{x}\right)^2.$$

Let us consider now expansions trigonometric functions with power series. We will use the Taylor expansions of  $\sin(x)$ ,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where  $0 < \theta < 1$ . It is easy to remark that

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

for  $0 < x < \frac{\pi}{2}$ . We then deduce bounds for  $\left(\frac{\sin x}{x}\right)^2$

$$\sum_{k=1}^{2p} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2} < \left(\frac{\sin x}{x}\right)^2 < \sum_{k=1}^{2p+1} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2}.$$

On the other hand, we know that [11 p.145]

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \dots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k-1}.$$

We then derive the following inequalities for any  $n \geq 1$

**Lemma 2-4** For  $0 < x < \pi/2$  the following inequalities holds for any integer  $p \geq 1$

$$2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)} x^{2k} < 1 + \frac{2x}{\sin 2x} <$$

$$2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \left(\frac{2x}{\pi}\right)^{2p+2} \frac{1}{1 - \left(\frac{2x}{\pi}\right)^2}.$$

Indeed, by [11 p.146] one has

$$1 + \frac{2x}{\sin 2x} = 2 + \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} > 2 + \sum_{k=1}^n \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k}.$$

Then it follows by (1)

$$\begin{aligned} 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)} x^{2k} &< 1 + \frac{2x}{\sin 2x} < \\ &2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \sum_{k=p+1}^{\infty} \frac{2^{2k}}{\pi^{2k}} x^{2k} = \\ &2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \left(\frac{2x}{\pi}\right)^{2p+2} \frac{1}{1 - \left(\frac{2x}{\pi}\right)^2}. \end{aligned}$$

We may also prove the following frame

$$\frac{2 \left(\frac{2x}{\pi}\right)^{2n+2}}{1 - \left(\frac{2x}{\pi}\right)^2} - 2 \sum_{k=1}^n \frac{1}{2^k - 1} \left(\frac{2x}{\pi}\right)^{2k} < 1 + \frac{2x}{\sin 2x} < \frac{2}{1 - \left(\frac{2x}{\pi}\right)^2}.$$

Finally, we get a lower bound for the product

$$\begin{aligned} \left(\frac{\sin(x)}{x}\right)^2 \left(1 + \frac{2x}{\sin 2x}\right) &> \left(1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2^\beta)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^n a^k x^{2k-2}\right) \times \\ &\left(2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)} x^{2k}\right) = \phi_{m,n,p}(x). \end{aligned}$$

The upper bound is

$$\begin{aligned} \left(\frac{\sin(x)}{x}\right)^2 \left(1 + \frac{2x}{\sin 2x}\right) &< \left(1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} - \sum_{k=1}^m \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2}\right) \times \\ &\left(2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \left(\frac{2x}{\pi}\right)^{2p+2} \frac{1}{1 - \left(\frac{2x}{\pi}\right)^2}\right) = \psi_{m,p}(x). \end{aligned}$$

Thus the following inequalities hold for  $0 < x < \pi/2$  and for integers  $1 \leq m; 3 \leq n; 1 \leq p \leq n$

$$\phi_{m,n,p}(x) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < \psi_{m,p}(x).$$

Theorem 2-1 is then proved.

**Examples 2-5** Let  $0 < x < \pi/2$ . By Theorem 4-1 we are able to precise the lower bound of the Wilker inequality in putting different values of  $n, p$ .

- Taking  $n = 3$  and  $p = m = 2$  we have

$$\phi_{2,4,2} = 2 + \frac{8x^4}{45} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

Then we find again a result of [3] since

$$\frac{1}{\cos x} \left( \frac{8x^4}{45} - \frac{4x^6}{105} \right) > \frac{\frac{8x^4}{45} - \frac{4x^6}{105}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} > \frac{8x^4}{45}.$$

- Taking  $n = 4$  and  $p = m = 3$  we find

$$\phi_{3,4,3} = 2 + \frac{8x^4}{45} + \frac{16x^6}{315} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

- Taking  $n = 5$  and  $p = m = 3$  we have

$$\phi_{3,5,3} = 2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

- Taking  $n = 6$  and  $p = m = 5$  we have

$$\phi_{5,6,5} = 2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592x^{10}}{66825} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

This permits to find again statement 1 (i) [4, p.9]. indeed, since (see lemma 2-6 below) for  $0 < x < \pi/2$

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}$$

we then deduce

$$(\cos x) \left( \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592x^{10}}{66825} \right) > \frac{8}{45} x^4 - \frac{4}{105} x^6 + \frac{19}{4725} x^8 - \frac{37}{133650} x^{10}.$$

- Taking  $n = 7$  and  $p = m = 4$  we obtain

$$\phi_{4,7,4} = 2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592x^{10}}{66825} + \frac{152912x^{12}}{42567525} < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.$$

By the same way, using again the lower bound of  $\cos x$  we find a result of [4, p.10]

$$(\cos x) \left( 2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592x^{10}}{66825} + \frac{152912x^{12}}{42567525} \right) >$$

$$\frac{8}{45}x^4 - \frac{4}{105}x^6 + \frac{19}{4725}x^8 - \frac{37}{133650}x^{10} + \frac{283}{20638800}x^{12} - \frac{3503}{6810804000}x^{14}.$$

Etc...

In the sequel we will find upper and lower bounds of Wilker inequalities which appear to be finer than known previous. Consider at first

**Lemma 2-6** For  $0 < x < \pi/2$  the following inequalities holds for any integer  $p \geq 1$

$$\sum_{k=1}^{2p} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2} < \left( \frac{\sin x}{x} \right)^2 < \sum_{k=1}^{2p+1} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2},$$

$$\sum_{k=1}^{2p+1} (-1)^k \frac{1}{(2k)!} x^{2k} < \cos x < \sum_{k=1}^{2p} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

Let us consider expansions trigonometric functions with power series. We will use the Taylor expansions of  $\sin x$ ,  $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{k+1} \frac{\cos \theta x}{(2k+2)!} x^{2k+2}$$

where  $0 < \theta < 1$ . It is easy to remark that

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

for  $0 < x < \frac{\pi}{2}$ . We then deduce bounds for  $\left( \frac{\sin x}{x} \right)^2 \cos x$

$$\sum_{k=1}^{2p} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2} < \left( \frac{\sin x}{x} \right)^2 < \sum_{k=1}^{2p+1} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2}.$$

$$\sum_{k=1}^{2p+1} (-1)^k \frac{1}{(2k)!} x^{2k} < \cos x < \sum_{k=1}^{2p} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

By Lemma 2-4 we then derive the following which improves Theorem 2-1

**Theorem 2-7** For  $0 < x < \pi/2$  the following inequalities holds for any  $q \geq 1, 1 \leq p \leq n$

$$\left( \sum_{k=1}^{2q} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k-2} \right) \times \left( 2 + \sum_{k=1}^p \frac{2^{2k} (2^{2k} - 2) |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{2^{2k+1} (2^{2k} - 2)}{\pi^{2k} (2^{2k} - 1)} x^{2k} \right)$$

$$\begin{aligned}
&< \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < \\
&\left( \sum_{k=1}^{2q+1} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k-2} \right) \times \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + 2 \left( \frac{2x}{\pi} \right)^{2p+2} \frac{1}{1 - \left( \frac{2x}{\pi} \right)^2} \right),
\end{aligned}$$

where  $B_{2k}$  are the Bernoulli numbers.

**Corollary 2-8** For  $0 < x < \pi/2$  the following inequalities holds for any  $n, p, 1 \leq p \leq n$

$$\begin{aligned}
&\left( \frac{\sin(x)}{x} \right)^2 \times \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \frac{2 \left( \frac{2x}{\pi} \right)^{2n+2}}{1 - \left( \frac{2x}{\pi} \right)^2} - 2 \sum_{k=1}^n \frac{1}{2^k - 1} \left( \frac{2x}{\pi} \right)^{2k} \right) < \\
&\left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < \\
&\left( \frac{\sin(x)}{x} \right)^2 \times \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \frac{2 \left( \frac{2x}{\pi} \right)^{2n+2}}{1 - \left( \frac{2x}{\pi} \right)^2} \right).
\end{aligned}$$

where  $B_{2k}$  are the Bernoulli numbers.

Corollary 2-8 means that the following inequalities hold

$$\begin{aligned}
&\left( \frac{\sin(x)}{x} \right)^2 \times \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} - 2 \sum_{k=1}^n \frac{1}{2^k - 1} \left( \frac{2x}{\pi} \right)^{2k} \right) < \\
&g_n(x) = \left( \frac{\sin(x)}{x} \right)^2 \left( 1 - \frac{2 \left( \frac{2x}{\pi} \right)^{2n+2}}{1 - \left( \frac{2x}{\pi} \right)^2} \right) + \frac{\tan(x)}{x} < \\
&\left( \frac{\sin(x)}{x} \right)^2 \times \left( 2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} \right) < \left( \frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x}.
\end{aligned}$$

The function  $g_n(x)$  is growing as  $n$  increasing. We have

$$\begin{aligned}
g_2(x) &= \left( \frac{\sin(x)}{x} \right)^2 \left( 1 - \frac{2 \left( \frac{2x}{\pi} \right)^2}{1 - \left( \frac{2x}{\pi} \right)^2} \right) + \frac{\tan(x)}{x} < g_n(x) < \\
\lim_{n \rightarrow \infty} g_n(x) &= \left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x}.
\end{aligned}$$

Moreover, we may compute the limit when  $x$  tends to  $(\frac{\pi}{2})^-$

$$\lim_{x \rightarrow \frac{\pi}{2}} g_n(x) = \frac{2(3+4n)}{\pi^2},$$

and

$$g_n(x) - \frac{2(3+4n)}{\pi^2} = \frac{-24n^2 + 24n + 33 + 2\pi^2}{3\pi^3}(\pi - 2x) + \dots$$

**Examples 2-9** Let  $0 < x < \pi/2$ .

- Taking  $n = 3$  and  $p = 2$  we find

$$2 + \frac{8x^4}{45} < g_3(x) = \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - \left(\frac{\sin x}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^8}{1 - \left(\frac{2x}{\pi}\right)^2}\right) <$$

$$\frac{2306x^{10}}{467775} - \frac{2x^8}{63} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2 < \frac{16x^6}{315} + \frac{8x^4}{45} + 2.$$

- Taking  $n = 4$  and  $p = 3$  we find

$$2 + \frac{8x^4}{45} + \frac{16x^6}{315} < g_4(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^{10}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) <$$

$$\frac{61232x^{12}}{30405375} - \frac{868x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2 < \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2.$$

- Taking  $n = 5$  and  $p = 3$  we obtain

$$2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} < g_5(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^{12}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) <$$

$$\frac{1566172x^{14}}{1915538625} - \frac{480604x^{12}}{91216125} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2 <$$

$$\frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2.$$

We then improve by another way statement 1 (i) ([4, p.9]) since by Lemma 2-4

$$\left(\frac{8}{45}x^4 + \frac{16}{315}x^6 + \frac{104}{4725}x^8 + \frac{592}{66825}x^{10}\right) \cos(x) < \frac{8}{45}x^4 - \frac{4}{105}x^6 + \frac{19}{4725}x^8$$

$$\left(\frac{8}{45}x^4 + \frac{16}{315}x^6 + \frac{104}{4725}x^8 + \frac{592}{66825}x^{10} + \frac{152912}{42567525}x^{12}\right) \cos(x) >$$

$$\frac{8}{45}x^4 - \frac{4}{105}x^6 + \frac{19}{4725}x^8 - \frac{37}{133650}x^{10} + \frac{283}{20638800}x^{12}.$$

- Taking  $n = 6$  and  $p = 3$  we obtain

$$2 + \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^8}{4725} + \frac{592}{66825}x^{10} < g_6(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - \left(\frac{\sin(x)}{x}\right)^2 \left(\frac{2\left(\frac{2x}{\pi}\right)^{14}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) <$$

$$\frac{161934166x^{16}}{488462349375} - \frac{123992x^{14}}{58046625} + \frac{152912x^{12}}{42567525} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2 <$$

$$\frac{152912x^{12}}{42567525} + \frac{592x^{10}}{66825} + \frac{104x^8}{4725} + \frac{16x^6}{315} + \frac{8x^4}{45} + 2$$

The last estimate improves statement 1 (ii) ([4, p.10]).

Etc...

### 3 Second Wilker inequality

Wu and Srivastava [23, Lemma 3] proved the following dual inequality for  $0 < x < \pi/2$  also called second Wilker inequality

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2, \quad 0 < x < \frac{\pi}{2}$$

Mortici [17] improved it and gave

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \frac{2x^4}{45}.$$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > \frac{2\sin(x)}{x} + \frac{\tan(x)}{x}.$$

Malesevic and al. [12, Theorem 5] gave an improvement of that and provided the following bounds for  $0 < x < \pi/2$  and any integer  $m \geq 2$

$$2 + \sum_{k=2}^m \frac{(2k-2)2^{2k} |B_{2k}|}{(2k)!} x^{2k} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} <$$

$$2 + \sum_{k=2}^{m-1} \frac{(2k-2)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \left(\frac{2x}{\pi}\right)^{2n} \left(\frac{\pi^2}{4} - 2 - 2 + \sum_{k=2}^{m-1} \frac{(2k-2)2^{2k} |B_{2k}|}{(2k)!} \left(\frac{\pi}{2}\right)^{2k}\right).$$

It seems that Theorem 3-1 below improves that result.

**Theorem 3-1** For  $0 < x < \pi/2$  and any  $p \geq 1$  the following inequalities hold

$$\left(1 + \sum_{k=1}^{2p} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) \left(1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k}\right) <$$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} <$$

$$\left(1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) \left(1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2^\beta)} x^{2k}\right),$$

where  $B_{2k}$  are the Bernoulli numbers and  $\beta = 2 + \frac{\ln(1-\frac{6}{\pi^2})}{\ln 2} = 0.6491\dots$  is the Alzer constant.

Proof of Theorem 3-1

Recall the following series expansions which can be found in [11] or [10]

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, \quad \frac{\sin x}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

Then we derive

$$\left(\frac{x}{\sin x}\right)^2 = -x^2 \frac{d \cot x}{dx} = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k}.$$

Notice that

$$\begin{aligned} \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} &= \left(\frac{x}{\sin x}\right)^2 \left(1 + \frac{\sin 2x}{2x}\right) = \\ &= \left(1 + \sum_{k=1}^{\infty} \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k}\right) \left(1 + \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k+1)!}\right). \end{aligned}$$

By Taylor expansions we may deduce

$$\sum_{k=0}^{2p} (-1)^k \frac{x^{2k}}{(2k+1)!} < \frac{\sin x}{x} < \sum_{k=0}^{2p+1} (-1)^k \frac{x^{2k}}{(2k+1)!}.$$

On the other hand, writing

$$\sum_{k=1}^{\infty} \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} = \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k}.$$

Combining with inequalities (2)

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}| < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2^\beta)} < \frac{2^{2k+1}(2k)!}{\pi^{2k}(2^{2k}-2)}$$

it follows for  $0 < x < \pi/2$  and any  $p \geq 1$

$$\begin{aligned} \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} &> \left(1 + \sum_{k=1}^{2p} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) \times \\ &= \left(1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k}\right), \end{aligned}$$

as well as

$$\begin{aligned} & \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < \left( 1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!} \right) \times \\ & \left( 1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} \right) < \\ & \left( 1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!} \right) \left( 1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} \right). \end{aligned}$$

**Corollary 3-2** For  $0 < x < \pi/2$  and any  $n \geq p \geq 1$  the following inequalities hold

$$\begin{aligned} a_{p,n}(x) &= \left( 1 + \sum_{k=1}^{2p} (-1)^k \frac{(2x)^{2k}}{(2k+1)!} \right) \left( 1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \sum_{k=p+1}^n \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} \right) \\ &< \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < b_p(x) = \left( 1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!} \right) \left( 1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \right. \\ & \left. \frac{2 \left( \frac{x}{\pi} \right)^{2p+2}}{\left( 1 - \left( \frac{x}{\pi} \right)^2 \right)^2} \left[ (1-2p) \left( \frac{x}{\pi} \right)^2 + 2p+1 \right] + \frac{2 \left( \frac{x}{2\pi} \right)^{2p+2}}{\left( 1 - \left( \frac{x}{2\pi} \right)^2 \right)^2} \left[ (1-2p) \left( \frac{x}{2\pi} \right)^2 + 2p+1 \right] \right). \end{aligned}$$

**Proof**

Let us consider the function

$$\frac{1}{1-2^{1-2x}}, \quad x \geq 1.$$

For  $1 \leq x$  this function can be majored

**Lemma 3-3** For  $1 \leq x$  the following inequality holds

$$\frac{1}{1-2^{1-2x}} \leq (1+2^{2-2x}).$$

Indeed, consider the difference

$$\alpha(x) = \frac{1}{1-2^{1-2x}} < (1+2^{2-2x}).$$

Its derivative is

$$\frac{d\alpha(x)}{dx} = -2 \frac{2^{1-2x} \ln(2)}{(1-2^{1-2x})^2} + 4 \cdot 2^{1-2x} \ln(2) = 2 \frac{2^{1-2x} \ln(2) (1-4 \cdot 2^{1-2x} + 2 (2^{1-2x})^2)}{(-1+2^{1-2x})^2}.$$

The study suggest us that this function is non positive for  $x \neq 1$ . It is such that  $\alpha(1) = 0$ ,  $\lim_{x \rightarrow \infty} \alpha(x) = 0$  and increases between 1 and  $\infty$ . Thus, we get for  $1 \leq x$ ,  $\frac{1}{1-2^{1-2x}} \leq (1 + 4.2^{-2x})$ .

By Lemma 3-3 we have for any  $k \geq 1$

$$\frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} = 2(2k-1) \frac{1}{1-2^{1-2x}} \left(\frac{x}{\pi}\right)^{2k} \leq$$

$$2(2k-1)(1+4.2^{-2k}) \left(\frac{x}{\pi}\right)^{2k} = 2(2k-1) \left(\frac{x}{\pi}\right)^{2k} + 8(2k-1) \left(\frac{x}{2\pi}\right)^{2k}.$$

We then deduce the sum since  $0 < x < \pi/2$

$$\sum_{k=p+1}^{\infty} \frac{(2k-1)2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} \leq \sum_{k=p+1}^{\infty} 2(2k-1) \left(\frac{x}{\pi}\right)^{2k} + 8(2k-1) \left(\frac{x}{2\pi}\right)^{2k} =$$

$$\frac{2 \left(\frac{x}{\pi}\right)^{2p+2}}{\left(1 - \left(\frac{x}{\pi}\right)^2\right)^2} \left[ (1-2p) \left(\frac{x}{\pi}\right)^2 + 2p+1 \right] + \frac{2 \left(\frac{x}{2\pi}\right)^{2p+2}}{\left(1 - \left(\frac{x}{2\pi}\right)^2\right)^2} \left[ (1-2p) \left(\frac{x}{2\pi}\right)^2 + 2p+1 \right].$$

Replacing in Theorem 3-1 we get Corollary 3-2.

**Examples 3-4** Let  $0 < x < \pi/2$  we will use *Maple*

. - Taking  $n=3$ ,  $p=2$  one obtains

$$a_{2,3}(x) = 2 + \frac{2x^4}{45} + \frac{8x^6}{945} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} <$$

$$b_2(x) = \left(2 - \frac{2x^2}{3} + \frac{2x^4}{15} - \frac{4x^6}{315}\right) \left(1 + \frac{x^2}{3} + \frac{x^4}{15} + \frac{85}{8} \frac{x^6}{\pi^6}\right) < 2 + \frac{2x^4}{45} + \left(\frac{85}{4\pi^6} - \frac{4}{315}\right)x^6$$

The last expression is lower than

$$2 + \frac{2x^4}{45} + \frac{64}{\pi^6} \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360}\right) x^6$$

since  $\left(\frac{85}{4\pi^6} - \frac{4}{315}\right) - \frac{64}{\pi^6} \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360}\right) \approx -0.003697$ .

Then, this case is finer than statement 2(i).

- Taking  $n=4$ ,  $p=3$  one obtains

$$a_{3,4}(x) = 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < b_3(x) =$$

$$\left(2 - \frac{2x^2}{3} + \frac{2x^4}{15} - \frac{4x^6}{315}\right) \left(1 + \frac{x^2}{3} + \frac{x^4}{15} + \frac{2}{189} x^6 + \frac{16x^8\pi^2 - 12x^{10}}{\pi^6(x^2 - \pi^2)^2} +$$

$$\frac{1}{4} \frac{16x^8\pi^2 - 3x^{10}}{\pi^6(x^2 - 4\pi^2)^2} - 2 \frac{x^8}{\pi^6(-x^2 + \pi^2)} + \frac{x^8}{-32\pi^8 + 8x^2\pi^6}\right) <$$

$$2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \left(-\frac{34}{14175} + \frac{455}{16}\pi^{-8}\right)x^8.$$

The last expression is lower than

$$2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \frac{256}{\pi^8} \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560}\right)x^8$$

since  $-\frac{34}{14175} + \frac{455}{16}\pi^{-8} - \frac{256}{\pi^8} \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560}\right) \approx -0.0012804$ .

This case is also finer than statement 2(ii).

- Taking  $n=5$ ,  $p=4$  one obtains (details omitted)

$$a_{3,4}(x) = 2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + \frac{16x^{10}}{93555} < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < b_3(x) < \\ 2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \frac{2}{1575}x^8 + \left(-\frac{4}{18711} + \frac{2313}{64}\pi^{-10}\right)x^{10}.$$

The last expression is lower than

$$2 + \frac{2x^4}{45} + \frac{8x^6}{945} + \frac{2x^8}{1575} + \left(\frac{2}{\pi}\right)^{10} \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560} - \frac{\pi^8}{201600}\right)x^{10}$$

since  $\left(-\frac{4}{18711} + \frac{2313}{64}\right) - \left(-2 + \frac{\pi^2}{4} - \frac{\pi^4}{360} - \frac{\pi^6}{7560} - \frac{\pi^8}{201600}\right) \approx -0.00007469$ .

This case is also finer than statement 2(iii).

## 4 C

oncluding remarks

1 - Examples 2-5 and Examples 2-9 above suggest us that Corollary 2-7 is finer than Theorem 4 of [12]. Therefore, we may ask if the following is valid for  $0 < x < \pi/2$  and  $2 \leq n$ :

$$(i) \quad 2 + \frac{1}{\cos x} \sum_{k=2}^{2n+1} (-1)^k D_k x^{2k} < g_n(x) = \left(\frac{\sin(x)}{x}\right)^2 \left(1 - \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) + \frac{\tan(x)}{x}.$$

$$(ii) \quad 2 + \frac{1}{\cos x} \sum_{k=2}^{2n+1} (-1)^k D_k x^{2k} < \left(\frac{\sin(x)}{x}\right)^2 \left(2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \right. \\ \left. \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1 - \left(\frac{2x}{\pi}\right)^2} - 2 \sum_{k=1}^n \frac{1}{2^k - 1} \left(\frac{2x}{\pi}\right)^{2k}\right)$$

and

$$(iii) \quad \left(\frac{\sin(x)}{x}\right)^2 \left(2 + \sum_{k=1}^p \frac{2^{2k}(2^{2k}-2) |B_{2k}|}{(2k)!} x^{2k} + \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1 - \left(\frac{2x}{\pi}\right)^2}\right) < 2 + \frac{1}{\cos x} \sum_{k=2}^{2n} (-1)^k D_k x^{2k},$$

where  $D_k = \frac{(-9+3^{2k+2}-40k-32k^2)}{4(2k+2)!}$ .

**2** - Examples 3-4 suggest us that Corollary 3-2 is finer than Theorem 5 of [12] and we may naturally ask if more generally the following inequality

$$b_m(x) < 2 + \sum_{k=2}^{m-1} \frac{|B_{2k}| (2k-2)4^k}{(2k)!} x^{2k} + \left(\frac{2x}{\pi}\right)^{2n} \left( \frac{\pi^2}{4} - 2 - \sum_{k=2}^{m-1} \frac{|B_{2k}| (2k-2)4^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \right),$$

holds for any  $m \geq 2$ , where

$$b_p(x) = \left(1 + \sum_{k=1}^{2p+1} (-1)^k \frac{(2x)^{2k}}{(2k+1)!}\right) \left(1 + \sum_{k=1}^p \frac{(2k-1)2^{2k} |B_{2k}|}{(2k)!} x^{2k} + \frac{2 \left(\frac{x}{\pi}\right)^{2p+2}}{\left(1 - \left(\frac{x}{\pi}\right)^2\right)^2} \left[ (1-2p) \left(\frac{x}{\pi}\right)^2 + 2p+1 \right] + \frac{2 \left(\frac{x}{2\pi}\right)^{2p+2}}{\left(1 - \left(\frac{x}{2\pi}\right)^2\right)^2} \left[ (1-2p) \left(\frac{x}{2\pi}\right)^2 + 2p+1 \right] \right).$$

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