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# Mathematical analysis of an interspecific density-dependent model with mutual inhibition and distinct removal rates

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## Abstract

This paper deals with a complete analysis of a density-dependent model in a chemostat describing the competition of two species for a single nutrient with a mutual-inhibitory relationship. In the presence of species mortality and under general growth functions, we give a quite comprehensive analysis of the existence and local stability of all steady states of the three-dimensional system. The nullcline method permits us to show that if a positive steady state exists, then it is unique and unstable. In this case, the system exhibits a bi-stability where the behavior of the process depends on the initial condition. Our mathematical analysis proves that at most one species can survive which confirms the competitive exclusion principle. Comparing with the study of the inter and intraspecific model, we conclude that adding only intraspecific competition in the classical chemostat model is not sufficient to show the coexistence of two species even considering mortality in the dynamics of two species.

## Keywords

Chemostat; Coexistence; Competitive Exclusion Principle; Mortality; Mutual-inhibitory relationship

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## I INTRODUCTION

The Competitive Exclusion Principle (CEP) states that, in continuous culture and under specific assumptions, when two or more microbial species compete for the same limiting nutrient, only the species with the lowest "break-even concentration" survives while all other species are extinct, see for instance [2, 8]. Although this prediction has been corroborated by the experiences of Hansen and Hubbell [1], the biodiversity that is observed in microbial ecosystems as well as in wastewater treatment processes and bioreactors seems to contradict the CEP. Thus, the classical chemostat model is unable to reproduce reality and new assumptions should be considered in order to reconcile the theory and the experimental results.

In this vein, several mechanisms of coexistence in the chemostat were considered in the existing literature, such as the density-dependence of the growth functions [6, 8], the predator-prey relationship between species [9, 12, 13, 15], the obligate mutualistic relationship [10]. Additionally, the intra and interspecific interference (see for instance [7]) can explain the coexistence of species in the chemostat. In [7], it was considered a model of two species in competition for a single nutrient with a mutual-inhibitory relationship, that is, each species inhibits the growth

of the other. The authors show, without or with death rates, that there can be one or more positive locally exponentially stable steady states. In the case of more than one, bi-stability can occur, for certain values of the operating parameters. A particular case of the model proposed in [7] was studied by Lobry et al. [4] where only the intraspecific interferences are taken into account. It was proved how the intraspecific interferences are a mechanism of coexistence for competing species in the chemostat.

We give a particular attention on the special case where only the interspecific interferences are present. In this case, in [5, 10] it was proved that a system with the same removal rates predicts the CEP and when a coexistence steady state exists (always unstable) the system exhibits a bi-stability. Recently, the study of Alshafi et al. [14] provides an extension of the results in [5, 10] to the case where a virus associated on the first species is present. They predict that the CEP is still fulfilled and the coexistence of species is impossible.

In our knowledge, in the existing literature it has not yet been studied the effect of mortality (that is, one has distinct removal rates for each species) on the behavior of the model proposed in [5, 9], which is the matter of the present paper. In particular, with distinct removal rates, we give a quite comprehensive analysis of the model. Using the nullcline method [7, 12, 13, 15], we present a geometric characterization that describes all steady states of the model and their stability. We show that the system can have at most four types of steady states: the washout of species, two boundary steady states and a positive steady state. These steady states are studied analytically by providing their necessary and sufficient conditions of existence and local stability, and showing their uniqueness for each type of steady state. Mathematical analysis shows that a positive steady state is unstable if it exists, that is, the coexistence of two species is impossible. This result was obtained in [5, 9] where the mortality of species are neglected and it was shown that at least one species goes extinct. Thus, even if we add the species mortality terms in interspecific density-dependent model of two species with a mutual inhibitory relationship, the CEP is still fulfilled.

The rest of this paper is organized in the following way. In the next Section II, we develop our model and we present general assumptions on the growth rates. Section III is devoted to the analysis of the model which describes the existence and stability conditions of steady states. Section IV deals with numerical simulations. The final Section V contains a brief conclusion of our results. Most of the proofs are reported in Appendix A. Finally, all the parameter values used in simulations are provided in Appendix B.

## II MODEL DESCRIPTION AND ASSUMPTIONS

The system we focus on consists of two microbial species ( $x_1$  and  $x_2$ ), which are competitors for a single nutrient  $S$ . Additionally, each species inhibits the growth of the other species. This system can be modeled by the following set of differential equations

$$\dot{S} = D(S_{\text{in}} - S) - f_1(S, x_2)x_1 - f_2(S, x_1)x_2, \quad (1)$$

$$\dot{x}_1 = (f_1(S, x_2) - D_1)x_1, \quad (2)$$

$$\dot{x}_2 = (f_2(S, x_1) - D_2)x_2, \quad (3)$$

where  $D$  and  $S_{\text{in}}$  are, respectively, the dilution rate and the input substrate concentration in the chemostat. At time  $t$ , let  $S(t)$  denote the substrate concentration and  $x_i(t)$ ,  $i = 1, 2$ , the concentrations of species  $x_i$ . For  $i = 1, 2$ ,  $f_i$  is the density-dependent growth rate of species  $x_i$ . It is assumed to be increasing in  $S$  and decreasing in  $x_j$ ,  $j \neq i$ . For  $i = 1, 2$ ,  $D_i$  is the removal rate of species  $x_i$  and can be modeled as in [11, 15] by

$$D_i = \alpha_i D + a_i, \quad (4)$$

where the coefficient  $\alpha_i$  belongs to  $[0, 1]$  and represents the fraction of species leaving the reactor to model a biomass reactor attached to the support or to decouple the residence time of solids and the hydraulic residence time ( $1/D$ );  $a_i$  is the nonnegative death rate of the species  $x_i$ .

We assume the following general assumptions on the given growth functions  $f_i$ , for  $i = 1, 2$ ,  $j = 1, 2$ ,  $i \neq j$ .

(H0)  $f_i$  belong to  $C^1(\mathbb{R}_+, \mathbb{R}_+)$ :

The functions must be sufficiently smooth so that existence, uniqueness and continuity of the solution for all nonnegative time and nonnegative initial condition are satisfied.

(H1)  $f_i(0, x_j) = 0$  for all  $x_j \geq 0$ :

Growth rate of species  $x_i$  is take place if and only if the substrate  $S$  is present.

(H2)  $\frac{\partial f_i}{\partial S}(S, x_j) > 0$  for all  $S \geq 0$  and  $x_j > 0$ :

Growth rate of species  $x_i$  is favored by the substrate  $S$ .

(H3)  $\frac{\partial f_i}{\partial x_j}(S, x_j) < 0$  for all  $S > 0$  and  $x_j \geq 0$ :

Each species inhibits the growth of the other species, namely, by decreasing the growth of each by the other.

It is straightforward to show that all solutions of (1)-(3) are nonnegative and bounded. In fact, similar to [15], one can prove the following proposition. Let  $D_{\min} = \min(D, D_1, D_2)$  and  $\Omega = \{(S, x_1, x_2) \in \mathbb{R}_+^3 : S + x_1 + x_2 \leq DS_{\text{in}}/D_{\min}\}$ . Then, for any non-negative initial condition,

1. the solution of system (1)-(3) exists for all  $t \geq 0$ , remains nonnegative and is bounded.
2.  $\Omega$  is positively invariant and is a global attractor for the dynamics (1)-(3).

### III STEADY STATES: EXISTENCE AND LOCAL STABILITY

In this section, we discuss the existence and local stability conditions of all steady states of system (1)-(3). For convenience, we use the abbreviation LES for Locally Exponentially Stable steady state. In all figures, the straight line  $\delta$  is defined by the equation  $D_1x_1/D + D_2x_2/D = S_{\text{in}}$  and the curves  $\gamma_1$  and  $\gamma_2$  are in blue and red, respectively. In what follows, we need to define the set

$$M := \{(x_1, x_2) \in \mathbb{R}_+^2 : D_1x_1/D + D_2x_2/D \leq S_{\text{in}}\} \quad (5)$$

and the following notations:

$$E = \frac{\partial f_1}{\partial S}, \quad F = \frac{\partial f_2}{\partial S}, \quad G = -\frac{\partial f_1}{\partial x_2}, \quad H = -\frac{\partial f_2}{\partial x_1}. \quad (6)$$

We have used the opposite sign of the partial derivatives  $G = -\frac{\partial f_1}{\partial x_2}$  and  $H = -\frac{\partial f_2}{\partial x_1}$ , such that all constants involved in the computation become positive.

#### 3.1 Existence of steady states

First, under assumptions (H1) to (H3), we determine the steady states of (1)-(3) which are given by the solutions of the following equations with nonnegative components:

$$D(S_{\text{in}} - S) - f_1(S, x_2)x_1 - f_2(S, x_1)x_2 = 0 \quad (7)$$

$$(f_1(S, x_2) - D_1)x_1 = 0 \quad (8)$$

$$(f_2(S, x_1) - D_2)x_2 = 0. \quad (9)$$

System (1)-(3) predicts four steady states labeled as below:

- $\mathcal{E}_0$  ( $x_1 = x_2 = 0$ ): the washout of two species.
- $\mathcal{E}_1$  ( $x_1 > 0, x_2 = 0$ ): species  $x_1$  survives while species  $x_2$  is extinct.
- $\mathcal{E}_2$  ( $x_1 = 0, x_2 > 0$ ): species  $x_1$  is extinct while species  $x_2$  survives.
- $\mathcal{E}^*$  ( $x_1 > 0, x_2 > 0$ ): the coexistence of two species.

If  $x_1 = x_2 = 0$ , from (7) one has  $S = S_{\text{in}}$ , then the washout  $\mathcal{E}_0 = (S_{\text{in}}, 0, 0)$  always exists.

From assumption (H2), for  $i = 1, 2$ , the function  $S \mapsto f_i(S, 0)$  is increasing, then the equation  $f_i(S, 0) = \alpha_i D + a_i$  has a unique solution for all  $0 < D < (f_i(+\infty, 0) - a_i)/\alpha_i$ . This solution is called the *break-even concentration* and noted by

$$S = \lambda_i(D). \quad (10)$$

If  $D > (f_i(+\infty, 0) - a_i)/\alpha_i$ , we put  $\lambda_i(D) = +\infty$ . The following result gives the necessary and sufficient conditions for the existence of the boundary steady states  $\mathcal{E}_1 = (\tilde{S}_1, \tilde{x}_1, 0)$  and  $\mathcal{E}_2 = (\tilde{S}_2, 0, \tilde{x}_2)$  of system (1)-(3) according to the operating parameters  $D$  and  $S_{\text{in}}$ . Assume that assumptions (H0)-(H3) hold. For  $i = 1, 2, j = 1, 2, i \neq j$ , let  $\mathcal{E}_i$  the boundary steady state of (1)-(3) with  $x_i > 0$  and  $x_j = 0$ , where

$$\tilde{S}_i = \lambda_i(D) \quad \text{and} \quad \tilde{x}_i = \frac{D}{D_i}(S_{\text{in}} - \lambda_i(D)). \quad (11)$$

It exists if and only if

$$f_i(S_{\text{in}}, 0) > D_i. \quad (12)$$

If it exists, then it is unique. In the following, we will determine the existence condition of the coexistence steady state  $\mathcal{E}^* = (S^*, x_1^*, x_2^*)$  where both populations survive, that is,  $x_1 > 0$  and  $x_2 > 0$ . Therefore, its components must be the solutions of the set of equations

$$D(S_{\text{in}} - S) = D_1 x_1 + D_2 x_2, \quad (13)$$

$$f_1(S, x_2) = D_1, \quad (14)$$

$$f_2(S, x_1) = D_2. \quad (15)$$

From (13), we obtain

$$S^* = S_{\text{in}} - D_1 x_1^*/D - D_2 x_2^*/D. \quad (16)$$

By replacing the expression (16) in (14-15), we allow to say that  $(x_1 = x_1^*, x_2 = x_2^*)$  is a solution of

$$\begin{cases} f_1(S_{\text{in}} - D_1 x_1/D - D_2 x_2/D, x_2) = D_1, \\ f_2(S_{\text{in}} - D_1 x_1/D - D_2 x_2/D, x_1) = D_2. \end{cases} \quad (17)$$

From (16), one see that  $S^*$  is positive if and only if  $D_1 x_1^*/D + D_2 x_2^*/D < S_{\text{in}}$ . Thus, (17) has a positive solution in the interior of the set  $M$  defined by (5).

Under assumptions (H0)-(H3), we need the following lemmas in order to solve (17) in the interior of  $M$ . (1). Assume that  $f_1(S_{\text{in}}, 0) > D_1$ . Let  $\tilde{x}_1$  be the unique solution of the equation  $f_1(S_{\text{in}} - D_1 x_1/D, 0) = D_1$  and  $\tilde{x}_2$  be the unique solution of the equation  $f_1(S_{\text{in}} - D_2 x_2/D, 0) = D_1$ . The equation  $f_1(S_{\text{in}} - D_1 x_1/D - D_2 x_2/D, x_2) = D_1$  defines a decreasing function

$$\begin{aligned} F_1 : [0, \tilde{x}_1] &\longrightarrow [0, \tilde{x}_2] \\ x_1 &\longmapsto F_1(x_1) = x_2, \end{aligned}$$

such that  $F_1(\tilde{x}_1) = 0$ ,  $F_1(0) = \bar{x}_2$  and

$$F_1'(x_1) = -\frac{D_1E}{D_2E+DG} < 0, \quad \text{for all } x_1 \in [0, \tilde{x}_1]. \quad (18)$$

Furthermore, the graph  $\gamma_1$  of  $F_1$  lies in the interior of  $M$  for all  $x_1 \in (0, \tilde{x}_1)$  (see Figure 3 (a)).

(2). Assume that  $f_2(S_{in}, 0) > D_2$ . Let  $\tilde{x}_2$  be the unique solution of the equation  $f_2(S_{in} - D_2x_2/D, 0) = D_2$  and  $\bar{x}_1$  be the unique solution of the equation  $f_2(S_{in} - D_1x_1/D, x_1) = D_2$ . The equation  $f_2(S_{in} - D_1x_1/D - D_2x_2/D, x_1) = D_2$  defines a decreasing function

$$\begin{aligned} F_2 : [0, \bar{x}_1] &\longrightarrow [0, \tilde{x}_2] \\ x_1 &\longmapsto F_2(x_1) = x_2, \end{aligned}$$

such that  $F_2(\bar{x}_1) = 0$ ,  $F_2(0) = \tilde{x}_2$  and

$$F_2'(x_1) = -\frac{D_1F+DH}{D_2F} < 0, \quad \text{for all } x_1 \in [0, \bar{x}_1]. \quad (19)$$

In addition, the graph  $\gamma_2$  of  $F_2$  lies in the interior of  $M$  for all  $x_1 \in (0, \bar{x}_1)$  (see Figure 3 (b)).

Using the derivatives (18) and (19) of functions  $F_1$  and  $F_2$ , respectively, a straightforward calculation shows the following result. The difference between the slopes of  $F_1$  and  $F_2$  is

$$F_1'(x_1) - F_2'(x_1) = \frac{D(DGH+D_1FG+D_2EH)}{D_2F(D_2E+DG)} > 0, \quad \text{for all } x_1 < \min(\bar{x}_1, \tilde{x}_1).$$

Hence, if there is a positive intersection  $(x_1, x_2)$  of  $\gamma_1$  and  $\gamma_2$  exists, then the tangent of  $\gamma_1$  at point  $(x_1, x_2)$  is always above the tangent of  $\gamma_2$  at point  $(x_1, x_2)$ . Therefore, if  $F_1(0) = \bar{x}_2 > F_2(0) = \tilde{x}_2$ , there cannot be any intersection between  $\gamma_1$  and  $\gamma_2$ , and  $\gamma_1$  intersects the horizontal axis on the right of  $\gamma_2$ , hence  $\tilde{x}_1 > \bar{x}_1$ .

Recall that, for  $i = 1, 2, j = 1, 2, i \neq j$ ,  $\tilde{x}_i$  and  $\bar{x}_i$  are respectively the solutions of

$$f_i(S_{in} - D_ix_i/D, 0) = D_i \quad \text{and} \quad f_j(S_{in} - D_ix_i/D, x_i) = D_j.$$

These quantities represent the coordinates of the intersections of the curves  $\gamma_1$  and  $\gamma_2$  with the coordinates axes. The relative positions of  $\tilde{x}_i$  and  $\bar{x}_i$ ,  $i = 1, 2$ , play a major role in the behavior of the system. Thus, from Lemma 3.1, there exists three cases that must be distinguished (see Figure 1):

$$\begin{aligned} \text{Case 1 : } &\bar{x}_1 < \tilde{x}_1 \quad \text{and} \quad \tilde{x}_2 < \bar{x}_2. \\ \text{Case 2 : } &\bar{x}_1 < \tilde{x}_1 \quad \text{and} \quad \bar{x}_2 < \tilde{x}_2. \\ \text{Case 3 : } &\tilde{x}_1 < \bar{x}_1 \quad \text{and} \quad \bar{x}_2 < \tilde{x}_2. \end{aligned} \quad (20)$$

All these previous results lead us to the following proposition which gives the necessary and sufficient condition for the existence and uniqueness of positive steady state  $\mathcal{E}^*$ . Assume that assumptions (H0)-(H3) and condition (12) hold for  $i = 1, 2$ . A positive steady state  $\mathcal{E}^*$  exists if and only if the curves  $\gamma_1$  and  $\gamma_2$  have a positive intersection in the interior of  $M$ . More precisely,  $(x_1 = x_1^*, x_2 = x_2^*)$  is a positive solution of equation

$$x_2 = F_1(x_1) \quad \text{and} \quad x_2 = F_2(x_1). \quad (21)$$

and  $S^*$  is given by (16). Furthermore,  $\mathcal{E}^*$  exists if and only if Case 2 of (20) holds. If it exists, then it is unique.

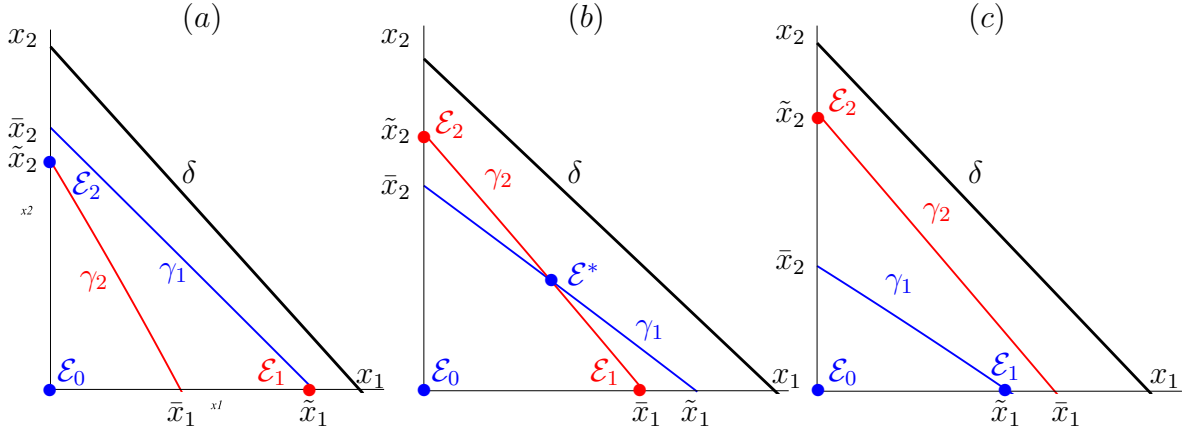


Figure 1: Steady states of (1)–(3) according to the cases of (20): (a) Case 1, (b) Case 2, (c) Case 3. We have chosen the red color for LES steady state and the blue color for unstable steady state.

### 3.2 Local stability

In the following, we will investigate the local stability of all steady states of system (1)–(3). Let computing the Jacobian matrix of (1)–(3) at  $(S, x_1, x_2)$  which takes the form

$$J = \begin{pmatrix} -D - x_1E - x_2F & -f_1(S, x_2) + x_2H & x_1G - f_2(S, x_1) \\ x_1E & f_1(S, x_2) - D_1 & -x_1G \\ x_2F & -x_2H & f_2(S, x_1) - D_2 \end{pmatrix}$$

where  $E, F, G$  and  $H$  are given by (6). The following proposition determines the local stability conditions of the boundary steady states. (1).  $\mathcal{E}_0$  is LES if and only if  $f_i(S_{\text{in}}, 0) < D_i$ , for  $i = 1, 2$ .

(2). For  $i = 1, 2$ ,  $\mathcal{E}_i$  is LES if and only if  $\bar{x}_i < \tilde{x}_i$ . Now if a positive steady state  $\mathcal{E}^*$  exists, we analytically characterize their stability by using the Routh–Hurwitz stability criterion. For  $\mathcal{E}^*$ , the characteristic polynomial is

$$P(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

where the coefficients  $c_i, i = 1, 2, 3$  are given in Table 1.

Table 1: Routh–Hurwitz coefficients for  $\mathcal{E}^*$  where  $E, F, G$  and  $H$  are given by (6) and evaluated at  $\mathcal{E}^*$ .

$$\begin{aligned} c_1 &= D + Ex_1^* + Fx_2^* \\ c_2 &= D_1Ex_1^* + D_2Fx_2^* - (GH + FG + EH)x_1^*x_2^* \\ c_3 &= -(DGH + D_1FG + D_2EH)x_1^*x_2^* \\ \hline c_4 &= c_1c_2 - c_3 \end{aligned}$$

From Table 1, one can see that the coefficient  $c_3$  is always negative. According to the Routh–Hurwitz criterion,  $\mathcal{E}^*$  is unstable. Thus, we can state the following result. If a positive steady state  $\mathcal{E}^*$  exists, then it is always unstable. The necessary and sufficient conditions of existence

and local stability of all steady states of (1)–(3) are summarized in the following Table 2.

From Table 2, we can deduce the following result.

- $\mathcal{E}_0$  is LES if and only if  $\mathcal{E}_1$  and  $\mathcal{E}_2$  do not exist.
- A positive steady state  $\mathcal{E}^*$  exists if and only if  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are LES.

Table 2: Existence and stability conditions of steady states of (1)–(3).

	Existence	Local stability
$\mathcal{E}_0$	Always exists	$f_i(S_{\text{in}}, 0) < D_i, i = 1, 2$
$\mathcal{E}_1$	$f_1(S_{\text{in}}, 0) > D_1$	$\bar{x}_1 < \tilde{x}_1$
$\mathcal{E}_2$	$f_2(S_{\text{in}}, 0) > D_2$	$\bar{x}_2 < \tilde{x}_2$
$\mathcal{E}^*$	Case 2 of (20) holds	unstable whenever it exists

## IV NUMERICAL SIMULATIONS

This section is dedicated to present some numerical simulations which illustrate our findings. To this end, we use the following growth rates satisfying assumptions (H0)–(H3):

$$f_1(S, x_2) = \frac{m_1 S}{K_1 + S} \frac{1}{1 + x_2/L_1}, \quad f_2(S, x_1) = \frac{m_2 S}{K_2 + S} \frac{1}{1 + x_1/L_2}, \quad (22)$$

where  $m_1, m_2$  denote the maximum growth rates;  $K_1$  and  $K_2$  denote the Michaelis-Menten constants;  $L_1$  (respectively  $L_2$ ) represents the inhibition factor due to  $x_2$  (respectively  $x_1$ ) for the growth of the species  $x_1$  (respectively the species  $x_2$ ). The parameter values used for numerical simulations are provided in Table 3.

Figure 2 illustrates the trajectories over time in three-dimensional space  $(S, x_1, x_2)$  for several positive initial conditions in the three cases of (20). In Case 1 of (20), Figure 2 (a) shows the convergence towards the steady state  $\mathcal{E}_1 \simeq (0.14835, 0.94627, 0)$  for any initial positive condition in the interior of  $M$ . In Case 3 of (20) (see Figure 1 (c)), the solutions of model (1)–(3) converges toward the extinction of the first species  $\mathcal{E}_2 \simeq (0.2, 0, 0.8)$  for several positive initial conditions as shown in Figure 2 (c).

In the Case 2 of (20) (see Figure 1 (b)), from Table 2 and Proposition 3.2, system (1)–(3) exhibits a bi-stability where at most one species wins the competition according to the initial condition. In Figure 2 (b), we show the bi-stability with two basins of attraction, with either convergence to  $\mathcal{E}_1 \simeq (0.22413, 0.59681, 0)$  or  $\mathcal{E}_2 \simeq (0.22448, 0, 0.705)$ . These two basins are separated by the stable manifold of saddle point  $\mathcal{E}^*$ .

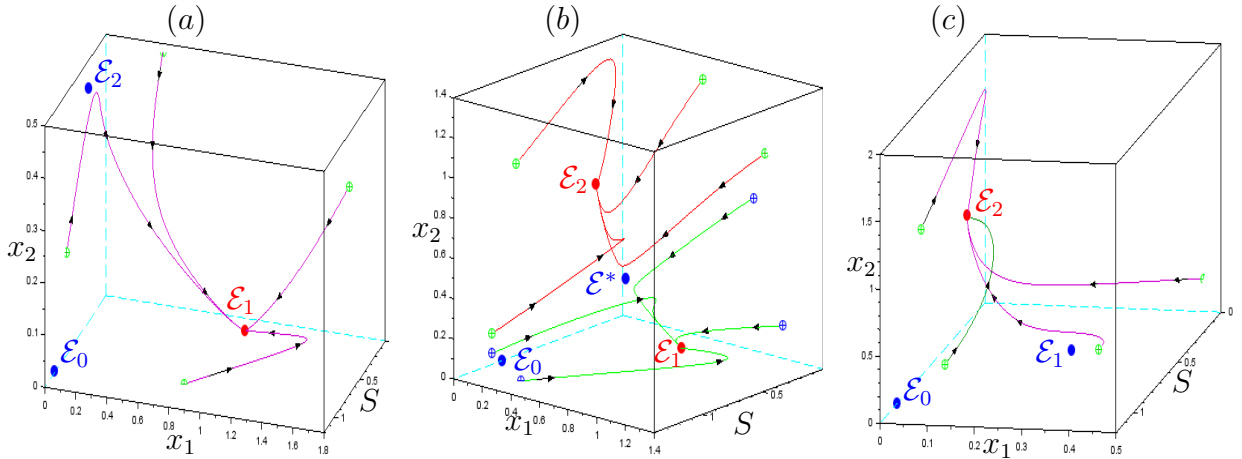


Figure 2: The trajectories of system (1)–(3) in three-dimensional space  $(S, x_1, x_2)$ : (a) Convergence to  $\mathcal{E}_1$ , (b) bi-stability of  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , (c) convergence to  $\mathcal{E}_2$ .

## V CONCLUSION

In this work, we have considered the mathematical model describing the competition of two species for a single growth-limiting resource in a chemostat with a mutual inhibition relation-



ship between species proposed by El Hajji [9]. In the absence of mortality species, it was proved that the system predicts the CEP. For general interspecific density-dependent growth functions and different dilution rates, our mathematical analysis proves that the outcome of competition satisfies the CEP which predicts that only one species can exist. Using the nullcline method [7, 13, 15], we demonstrate that a positive steady state is unique and unstable if it exists. The positive steady state exists if and only if both boundary steady states are LES. Therefore, the model exhibits the bi-stability.

Our study is the first attempt to highlight the effect of species mortality on the behavior of a two-species interspecific density-dependent model in the chemostat. Our theoretical message explains that the addition of mortality could not lead to the coexistence around a stable positive steady state.

In the existing literature [6, 7], it was proved that the simultaneous presence of the inter- and intraspecific interference in the competition for a single substrate can explain the coexistence of species. Furthermore, for small enough interspecific interference terms, there is a stable persistence of two species which can coexist for any positive initial condition. However, if these terms are large enough, the system exhibits bi-stability with the competitive exclusion of one species according to the initial condition. These results are numerically shown in the paper of Lobry et al. [3].

In the other hand, it was proved that it possible to get coexistence of species, see [4], in the case when the authors considered only the intraspecific interference in a competition model in the chemostat. Hence, the intraspecific interference is necessary to ensure the coexistence of species in the competition models. Moreover, it has a significant impact to explain the coexistence around a stable positive steady state.

## VI ACKNOWLEDGEMENTS

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## A PROOFS

**Proof of Proposition 3.1:** For  $i = 1, 2$ ,  $j = 1, 2$ ,  $i \neq j$ , a boundary steady state  $\mathcal{E}_i$  is a solution of (7)–(9) with  $\tilde{x}_i > 0$  and  $x_j = 0$ . That is, the components  $\tilde{S}_i$  and  $\tilde{x}_i$  are the solutions of equations

$$D(S_{\text{in}} - \tilde{S}_i) = D_i \tilde{x}_i \quad (23)$$

$$f_i(\tilde{S}_i, 0) = D_i. \quad (24)$$

Using definition (10) of  $\lambda_i$ , from (24) we have  $\tilde{S}_i = \lambda_i(D)$ . Thus, by replacing the expression of  $\tilde{S}_i$  in (23), we obtain  $\tilde{x}_i = \frac{D}{D_i}(S_{\text{in}} - \lambda_i(D))$  which is positive if and only if (12) holds. This completes the proof.  $\square$

**Proof of Lemma 3.1:** We will prove item 1 of Proposition and item 2 will be obtained in the same fashion. Let  $l_1$  be a fixed line defined by  $x_1 = b_1$  that intersects the line  $\delta$  at point  $x_2 = c_2 = DS_{\text{in}}/D_2 - D_1b_1/D_2$  (see Fig. 3 (a)). From assumptions (H1)-(H2), we see that the function  $x_2 \mapsto f_1(S_{\text{in}} - D_1b_1/D - D_2x_2/D, x_2)$  is decreasing from  $f_1(S_{\text{in}} - D_1b_1/D, 0)$  for  $x_2 = 0$  to 0 for  $x_2 = c_2$ . Therefore, there exists a unique  $b_2 \in [0, c_2)$  such that  $f_1(S_{\text{in}} - D_1b_1/D - D_2b_2/D, b_2) = D_1$  if and only if

$$f_1(S_{\text{in}} - D_1b_1/D, 0) \geq D_1,$$

which is equivalent to  $b_1 \leq \tilde{x}_1$ . Indeed, for  $i = 1$  from (23), one has  $S = S_{\text{in}} - D_1x_1/D$ . By replacing this expression in (24), one can see that  $\tilde{x}_1$  is the unique solution of  $f_1(S_{\text{in}} - D_1x_1/D, 0) = D_1$  since the function  $x_1 \mapsto f_1(S_{\text{in}} - D_1x_1/D, 0)$  is decreasing from  $f_1(S_{\text{in}}, 0)$  for  $x_1 = 0$  to 0 for  $x_1 = DS_{\text{in}}/D_1$ .

Therefore, we have shown that for all  $b_1 \in [0, \tilde{x}_1]$  there exists a unique  $b_2 \in [0, c_2)$  such that  $f_1(S_{\text{in}} - D_1b_1/D - D_2b_2/D, b_2) = D_1$ , that is, each line  $l_1$  meets the set  $f_1(S_{\text{in}} - D_1x_1/D - D_2x_2/D, x_2) = D_1$  exactly once if

$0 \leq b_1 \leq \tilde{x}_1$  and not at all if  $b_1 > \tilde{x}_1$ . Thus, we define the function  $F_1$  by  $b_2 = F_1(b_1)$ . The graph  $\gamma_1$  of this function lies in  $M$  (see Fig. 3 (a)).

By the implicit function theorem, the function  $F_1$  is decreasing. Indeed, using assumptions (H2) and (H3) and notations (6), it follows that

$$F_1'(x_1) = -\frac{\frac{\partial f_1}{\partial x_1}(x_1, F_1(x_1))}{\frac{\partial f_1}{\partial x_2}(x_1, F_1(x_1))} = \frac{\frac{D_1}{D} \frac{\partial f_1}{\partial S}}{-\frac{D_2}{D} \frac{\partial f_1}{\partial S} + \frac{\partial f_1}{\partial x_2}} = \frac{-D_1 E}{D_2 E + D G} < 0$$

Furthermore, if  $x_2 = 0$  then  $F_1(x_1) = 0$  which equivalent to solve the equation  $f_1(S_{\text{in}} - D_1 x_1/D, 0) = D_1$ , that is,  $x_1 = \tilde{x}_1$  the unique solution as it was shown above. Moreover, if  $x_1 = 0$ , then

$$F_1(0) = x_2 \Leftrightarrow f_1(S_{\text{in}} - D_2 x_2/D, x_2) = D_1.$$

From (H3), we see that the function  $x_2 \mapsto f_1(S_{\text{in}} - D_2 x_2/D, x_2)$  is decreasing from  $f_1(S_{\text{in}}, 0)$  for  $x_2 = 0$  to 0 for  $x_2 = DS_{\text{in}}/D_2$ . Consequently, the equation  $f_1(S_{\text{in}} - D_2 x_2/D, x_2) = D_1$  has a solution  $\bar{x}_2 \in (0, S_{\text{in}})$  if and only if condition (12) hold for  $i = 1$ . If such an  $\bar{x}_2$  exists then it is unique.  $\square$

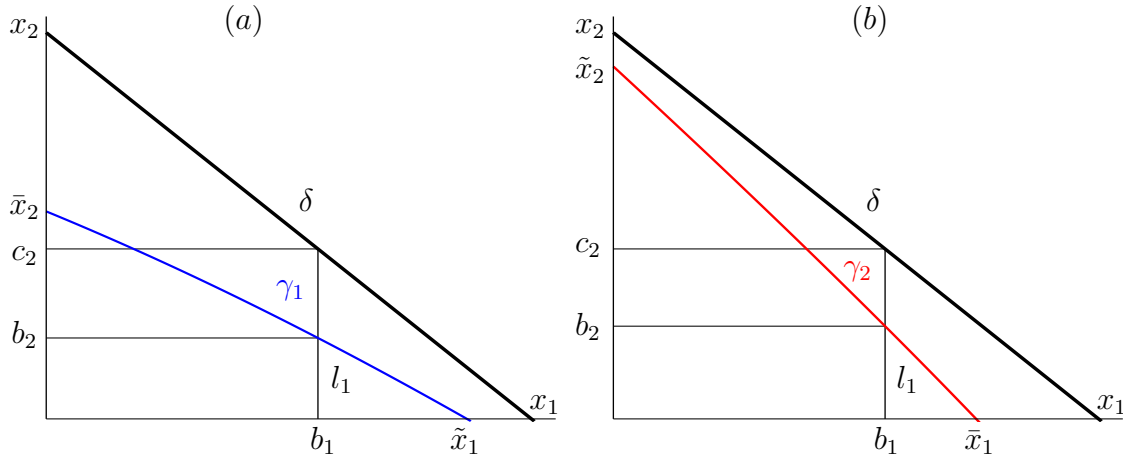


Figure 3: The nullclines of (1)–(3): (a) Definition of  $F_1$ . (b) Definition of  $F_2$ .

### Proof of Proposition 3.1:

By (17) and Lemma 3.1, a positive steady state exists if and only if the curves  $\gamma_1$  and  $\gamma_2$  have a positive intersection  $(x_1^*, x_2^*)$ . If the intersection exists, then it lies in the interior of  $M$ . Consequently, the component  $S^*$  which defined by (16), is positive.

The necessary and sufficient condition of the intersection given by Case 2 of 20, and the uniqueness of the positive steady state are illustrated in Figure 1 (b).  $\square$

**Proof of Proposition 3.2:** (1). At  $\mathcal{E}_0 = (S_{\text{in}}, 0, 0)$ , the characteristic polynomial is

$$P(\lambda) = (-D - \lambda)(f_1(S_{\text{in}}, 0) - D_1 - \lambda)(f_2(S_{\text{in}}, 0) - D_2 - \lambda)$$

Therefore,  $\mathcal{E}_0$  is LES if and only if  $f_i(S_{\text{in}}, 0) < D_i$ , for  $i = 1, 2$ . Using Proposition 3.1, we deduce that these conditions hold if and only if the boundary steady states  $\mathcal{E}_i$ ,  $i = 1, 2$  do not exist.

(2). At  $\mathcal{E}_1 = (\lambda_1(D), \tilde{x}_1, 0)$ , the characteristic polynomial is

$$P(\lambda) = (f_2(S_{\text{in}} - D_1 \tilde{x}_1/D, \tilde{x}_1) - D_2 - \lambda)(\lambda^2 + \beta_1 \lambda + \beta_2)$$

where  $\beta_1 = D + \tilde{x}_1 E$  and  $\beta_2 = D_1 \tilde{x}_1 E$ . Since  $\beta_1 > 0$  and  $\beta_2 > 0$ , the real parts of the roots of the quadratic factor are negative. Therefore,  $\mathcal{E}_1$  is LES if and only if  $f_2(S_{\text{in}} - D_1 \tilde{x}_1/D, \tilde{x}_1) < D_2$ , that is,  $\bar{x}_1 < \tilde{x}_1$ .

Similarly, at  $\mathcal{E}_2 = (\lambda_2(D), 0, \tilde{x}_2)$ , the characteristic polynomial is

$$P(\lambda) = (f_1(S_{\text{in}} - D_2 \tilde{x}_2/D, \tilde{x}_2) - D_1 - \lambda)(\lambda^2 + \delta_1 \lambda + \delta_2)$$

where  $\delta_1 = D + \tilde{x}_2 F$  and  $\delta_2 = D_2 \tilde{x}_2 F$ . Since  $\delta_1 > 0$  and  $\delta_2 > 0$ , the real parts of the roots of the quadratic factor are negative. Therefore,  $\mathcal{E}_2$  is LES if and only if  $f_1(S_{\text{in}} - D_2 \tilde{x}_2/D, \tilde{x}_2) < D_1$ , that is,  $\bar{x}_2 < \tilde{x}_2$ .  $\square$

Table 3: Parameter values used for model (1)-(3) when the growth rates  $f_1$  and  $f_2$  are given by (22).

Parameter	$m_1$	$K_1$	$L_1$	$m_2$	$K_2$	$L_2$	$D$	$S_{in}$	$\alpha_1$	$\alpha_2$	$a_1$	$a_2$
Fig. 3								3	0.5	0.25	0.8	0.5
Figs. 1(a) and 2(a)	5	1.5	1	3	1	1	0.5	1	0.1	0.5	0.4	0.5
Figs. 1(b) and 2(b)								1	0.1	0.7	0.6	0.2
Figs. 1(c) and 2(c)								1	0.6	0.1	0.8	0.45

## B PARAMETER VALUES USED FOR NUMERICAL SIMULATIONS

All the values of the parameters used in the numerical simulations are provided in Table 3.

## REFERENCES

- [1] S. R. Hansen and S. P. Hubbel. “Single-nutrient microbial competition : Qualitative agreement between experimental and theoretically forecast outcomes”. In: *Science* (1980), pages 1491–1493.
- [2] H. Smith and P. Waltman. *The Theory of the Chemostat, Dynamics of Microbial Competition*. Cambridge University Press, 1995.
- [3] C. Lobry and J. Harmand. “A new hypothesis to explain the coexistence of n species in the presence of a single resource”. In: *C. R. Biol.* 329 (2006), pages 40–46.
- [4] C. Lobry, A. Rapaport, and F. Mazenc. “Sur un modèle densité-dépendent de compétition sur une ressource”. In: *C. R. Biol.* 329 (2006), pages 63–70.
- [5] M. Elhajji and A. Rapaport. “How mutual inhibition confirms competitive exclusion principle”. In: *Proceedings of the 5th Conference on Trends in Applied Mathematics in Tunisia, Algeria, Morocco (TAMTAM’11)*. 2011, pages 443–449.
- [6] N. Abdellatif, R. Fekih-Salem, and T. Sari. “Competition for a single resource and coexistence of several species in the chemostat”. In: *Math. Biosci. Eng.* 13 (2016), pages 631–652.
- [7] R. Fekih-Salem, C. Lobry, and T. Sari. “A density-dependent model of competition for one resource in the chemostat”. In: *Math. Biosci.* 268 (2017), pages 104–122.
- [8] J. Harmand, C. Lobry, A. Rapaport, and T. Sari. *The Chemostat: Mathematical Theory of Microorganism Cultures, vol. 1*. Wiley, New York: Chemical Eng. Ser., Chemostat Bioprocesses Set, 2017.
- [9] M. Elhajji. “How can inter-specific interferences explain coexistence or confirm the competitive exclusion principle in a chemostat?” In: *Int. J. Biomath.* 11 (2018).
- [10] M. Elhajji, J. Harmand, H. Chaker, and C. Lobry. “Association between competition and obligate mutualism in a chemostat”. In: *J. Biol. Dyn.* 3 (2018), pages 635–647.
- [11] R. Fekih-Salem and T. Sari. “Properties of the chemostat model with aggregated biomass and distinct removal rates”. In: *SIAM J. Appl. Dyn. Syst. (SIADS)* 18 (2019), pages 481–509.
- [12] T. Mtar, R. Fekih-Salem, and T. Sari. “Effect of the mortality on a density-dependent model with a predator-prey relationship”. In: *CARI 2020–15th African Conference on Research in Computer Science and Applied Mathematics*. Thiès, Senegal, Oct. 2020.
- [13] T. Mtar, R. Fekih-Salem, and T. Sari. “Interspecific density-dependent model of predator-prey relationship in the chemostat”. In: *Int. J. Biomath.* 14 (2020).
- [14] S. Alsahafi and S. Woodcock. “Mutual inhibition in presence of a virus in continuous culture”. In: *Math. Biosci. Eng.* 4 (2021), pages 3258–3273.

- [15] T. Mtar, R. Fekih-Salem, and T. Sari. “Mortality can produce limit cycles in density-dependent models with a predator-prey relationship”. In: *to appear in Discrete Contin. Dyn. Syst. Ser. B* (2021).