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ABSTRACT

We extend a recent classical mechanical analog of Bohr's atom consisting of a scalar field coupled to a massive point-like particle [P. Jamet and A. Drezet, "A mechanical analog of Bohr's atom based on de Broglie's double-solution approach," Chaos **31**, 103120 (2021)] by adding and studying the contribution of a uniform weak magnetic field on their dynamics. In doing so, we are able to recover the splitting of the energy levels of the atom called Zeeman's effect within the constraints of our model and in agreement with the semiclassical theory of Sommerfeld. This result is obtained using Larmor's theorem for both the field and the particle, associating magnetic effects with inertial Coriolis forces in a rotating frame of reference. Our work, based on the old "double solution" theory of de Broglie, shows that a dualistic model involving a particle guided by a scalar field can reproduce the normal Zeeman effect.

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Quantum mechanics is generally opposed to classical physics. However, Louis de Broglie already attempted, and partially succeeded, to give a physical spatiotemporal interpretation of wave-particle duality in the 1920s. In the present work, we further develop a classical analog model of the old Bohr–Sommerfeld atom involving a particle coupled to a wave. Here, we introduce a coupling with an external magnetic field and show, using the Larmor theorem, that we can recover the famous Zeeman splitting of energy levels predicted in the old quantum theory. This, we believe, can shed new light on the relations between the quantum and the classical worlds and will provide new ways for observing this relation experimentally.

I. INTRODUCTION

As early as 1897,[1] and following the interests of Michael Faraday that led him to perform his last experiments on this subject in 1862,[2] Pieter Zeeman published a study on the effects of a magnetic field on the characteristics of the light emitted by various sources. He observed at the time that the spectral lines of sodium salts were widened when the substance was placed near a magnetic field. With his experiments, he was able to not only identify this widening as being related only to the microscopic properties of the sources and not on the thermodynamic state of the substance, but also to explain its origin using Lorentz's theory[3] with whom he shared the Nobel prize in 1902. The theoretical explanation at the time was that light was emitted by oscillating or orbiting discrete charges so that their orbital period was increased or decreased through the effect of the magnetic field, and, in turn, the frequency of the light would also be modified accordingly.

Years later in 1913, Bohr proposed a new planetary model[4] for the atom, following more than a decade of similar works on the structure of matter—in particular, by Larmor who wrote himself a planetary model with elliptical orbits explaining Zeeman's observations in late 1897[5]—introducing the hypothesis of quantization for orbital momenta. In this model, negatively charged electrons orbit around a positively charged nucleus—at the time only moving along circular paths but extended in 1916 by Sommerfeld to elliptical ones[6]—but only at fixed radii and energies. In this theory, light is emitted when an electron jumps from one energy level to a lower one, emitting a quantum of light with frequency equal to the difference in the energies of the two orbits divided by Planck's constant h. This is clearly reminiscent of Lorentz, Zeeman, and Larmor's explanations of 1897;[1,3,5] indeed, one can easily recover the same phenomenon of the effect of a magnetic field on the frequency of the electromagnetic waves, relating the orbital period or the angular frequency to that of the waves. Of course, this model also has the advantage of explaining the origin of the line structure in the spectra of atoms and predicts that it is not widening of these lines but rather splitting of a single line into $2n+1$, where n is the orbital angular momentum quantum number characterizing the orbit $L = \oint p\,\mathrm{d}q = nh$. This explanation, however, could not account for all observations. In fact, two phenomena existed,

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the normal Zeeman effect where a spectral line splits into an odd number of subsequent lines, correctly explained by the theory, and an anomalous one with a line dividing into an even number, as yet unexplained. Later still, the development of modern quantum mechanics by de Broglie, Schrödinger, Heisenberg, Born, and many others led to yet another atomic model, which once again predicted this normal Zeeman splitting, but also the anomalous one (related to the existence of the half-integer spin for the electron) characterizing any electronic orbital by a set of four quantum numbers $|n, l, l_z, s_z\rangle$.[6]

More recently, experiments have been performed[7–10] on droplets bouncing on a vibrating liquid bath showing that such droplets can exhibit quantum-like properties (for reviews, see Refs. 11–14). In particular, if the bath itself is rotated, the quantization of the orbital angular momentum of the droplet induces, in turn, quantization of other quantities, such as the radii of the orbits, which were shown to follow the same rules as Zeeman or Landau levels.[9] This phenomenon can be understood as an interaction between the orbital angular momentum of the droplet and an "effective magnetic field," which actually is an inertial force arising from the rotating reference frame.

In this paper, we will go back to the old quantum theory and start from a new atomic model developed recently[15,16] with a mechanical approach related to ideas introduced by de Broglie[17–19] and add to it an external magnetic field to recover the normal Zeeman splitting of the energy levels. As we will see, this model will indeed explain this phenomenon both from a particle and a field point of view, while also gaining insights into some limitations or constraints on such a procedure. After briefly recalling in Sec. II the characteristics of our previously developed model,[15,16] we will, in particular, look first at the dynamics of the particle in Sec. III A and then of the guiding field in Sec. III B, making use of an interesting method for finding the solutions of complex field equations coupled to a magnetic field using Larmor's theorem.[5,20] Moreover, we stress that the general methodology proposed in this work is based on mechanical analogies that can be applied to several fields of physics, including acoustic and optics. We discuss the fundamental and applicative potentialities of our work in the final discussion.

II. DESCRIPTION OF OUR MODEL

In a previous paper[16] (hereafter called JD), we developed a classical analog of a quantum atom based on some ideas first introduced by Louis de Broglie, known as the "double solution," which successfully reproduced the usual quantization rules of Bohr and Sommerfeld. More specifically, this model uses a complex scalar field $u(x)$ defining a physical object in space-time [to be distinguished from the usual wave-function $\Psi(x)$ introduced in quantum mechanics] that is coupled to a particle represented by a point-like mass m_p via a holonomic constraint between the field itself and an internal oscillation $z(\tau)$ for the particle (τ is a relativistic proper time for the particle). We also introduced a four-vector potential $A(x)$ to take into account external electromagnetic forces on both the field and the particle, such as a central electrostatic energy potential $U(r) = eV(r) = -\alpha/r$ [$\alpha = e^2/(4\pi)$ denotes Sommerfeld's fine structure constant, and we use the convention $e = -|e| < 0$]. Here, we will study this system further by adding an external uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$ in order to derive the quantized splitting of the energy levels with respect to the projection of the angular momentum L_z, called the Zeeman effect.

As shown in JD, our system is characterized by the action

$$\begin{aligned} I = &-\int \left[m_\mathrm{p} - \frac{1}{2} m_\mathrm{p}\sigma \left(|\dot{z}(\tau)|^2 - \Omega_\mathrm{p}^2 |z(\tau)|^2 \right) \right] \mathrm{d}\tau \\ &+ \int \left\{ \mathscr{N}(\tau) \left[z(\tau) - u(x_\mathrm{p}(\tau)) \right]^* \right. \\ &\left. + \mathscr{N}^*(\tau) \left[z(\tau) - u(x_\mathrm{p}(\tau)) \right] \right\} \mathrm{d}\tau \\ &- e \int A(x_\mathrm{p}(\tau)) \dot{x}_\mathrm{p}(\tau)\, \mathrm{d}\tau + T \int (Du)(Du)^* \, \mathrm{d}^4 x, \end{aligned} \tag{1}$$

which contains several physical quantities listed in Table I and discussed in more detail in JD. In our work, we use natural units ($c = \hbar = 1$) and the Minkowski metric $\eta_{\mu\nu}$ with signature $(1, -1, -1, -1)$.

The physical meaning of the different contributions to Eq. (1) must be briefly discussed. The first line in Eq. (1) contains a single particle relativistic Lagrangian where $\mathrm{d}\tau = \sqrt{\eta_{\mu\nu}\, dx_\mathrm{p}^\mu(\tau)\, dx_\mathrm{p}^\nu(\tau)}$ is a Minkowski proper time interval defined along the particle trajectory (repeated indices are implicitly summed over), and m_p is a rest mass for the particle with trajectory $x_\mathrm{p}(\tau)$. The specificity of our model is to generally have a nonconstant mass obtained by adding a harmonic oscillator term in the first line of Eq. (1). This harmonic oscillator is associated with an internal degree of freedom $z(\tau)$ (representing an internal clock for the particle) and a typical pulsation Ω_p. This term plays a key role in the analogy with de Broglie's theory as we show below. We mention that the addition of this term generalizes our non-relativistic mechanical analogy developed in Ref. 15. The second and third lines of the action [Eq. (1)] define a holonomic constraint connecting the wave $u(x)$ and the particle at the position $x = x_\mathrm{p}(\tau)$. The terms $\mathscr{N}(\tau)$, $\mathscr{N}^*(\tau)$ play the role of reaction forces between the wave, the internal clock, and the particle. Finally, the last line in Eq. (1) describes a standard quadratic field Lagrangian density for a d'Alembert scalar wave equation (T is a constant similar to a tension in a elastic string[15]). Here, however, the field $u(x)$ is complex valued, and instead of the usual partial derivative ∂_μ, we have a covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ depending on the external electromagnetic potential $A(x)$ interacting with the

TABLE I. Summary of the different parameters of the model.

Parameters	Physical meaning
$u(x)$	Fundamental field
$A^\mu(x)$	Electromagnetic four-vector potential
$z(\tau)$	Internal oscillator
$\mathscr{N}(\tau)$	Internal reaction force acting upon the particle
$x_\mathrm{p}^\mu(\tau)$	Four-vector position of the particle
T	Tension of the field
Ω_p	Internal oscillator pulsation
m_p	Bare particle mass
σ	Coupling constant
$e = -\lvert e\rvert$	Particle electric charge

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u-wave. The model is thus gauge invariant in a way similar to the Klein–Gordon equation. The last line of Eq. (1) also includes a standard coupling term for the particle and the external electromagnetic potential $A_\mu(x)$. We stress that here, we have used the same electric charge e for both the particle and the covariant derivative D_μ. As explained in JD, this hypothesis could be relaxed but will be considered in the following.

Using a variational principle $\delta I = 0$, Eq. (1) leads to a system of coupled differential equations, from which we can extract a regime called transparency[15,16,21] where the force $\mathcal{N}(\tau)$ vanishes, meaning that the particle and the field are no longer affecting each other. In this case, the equations decouple, apart from the holonomic constraint

$$z(\tau) = u(x_{\rm p}(\tau)), \tag{2}$$

which defines a synchronization condition between the u-wave and the particle. This is reminiscent of the "phase harmony" condition introduced by de Broglie in his Ph.D. thesis.[22] This is fundamental in our model where the particle is guided by the u-wave. The dynamics of $z(\tau)$ then describes a simple harmonic motion

$$z(\tau) = z_0\, e^{-i\Omega_{\rm p}\tau}, \tag{3}$$

while the particle with coordinate $x_{\rm p}(\tau)$ and the u-field follow the equations

$$m_{\rm eff.}\frac{d^2 x_{{\rm p}\mu}(\tau)}{d\tau^2} = eF_{\mu\nu}(x_{\rm p}(\tau))\dot{x}^\nu_{\rm p}, \tag{4}$$

$$D^2 u(x) = 0, \tag{5}$$

with $m_{\rm eff.} = m_{\rm p}\left(1 + \sigma\Omega_{\rm p}^2\,|z_0|^2\right)$ being an effective "dressed" mass that takes into account both the usual inertia $m_{\rm p}$ and the oscillatory motion along z, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the electromagnetic tensor, and $D_\mu = \partial_\mu + ieA_\mu$ being the covariant derivative. Equation (4) describes the usual Lorentz force acting on a point-like particle (with dressed mass), and Eq. (5) describes a field dynamics similar to the Klein–Gordon equation but without a mass term. The coupling between the wave and the particle is fixed by the holonomic constraint [Eq. (2)] and the internal oscillation [Eq. (3)].

Solving Eqs. (3)–(5) may prove difficult because of both the relativistic formalism and the presence of the four-vector potential involving electric and magnetic contributions. As we showed in JD in the presence of a Coulomb electrostatic field, we can obtain a solution of Eq. (5) defining a guiding wave for the particle moving along a circular orbit. The phase harmony condition $z = u$ forces the particle motion to be quantized, and the dynamics reproduces the Bohr–Sommerfeld quantization formula for the energy levels, velocities, and radii of the particle. Inspired by de Broglie's work,[17–19] we found a solution of Eq. (5) where $u(x)$ is a sum of two counter-propagating modes with frequencies $\omega_\pm$ and orbital angular momentum $L_z := m_\pm$ (i.e., projected along the z-axis). This solution induces a supraluminal phase wave having all the properties of the wave introduced by de Broglie in his Ph.D. work.[17] In turn, the particle-wave system fulfills the holonomic constraint $z = u$ along the orbit as it is required in order to induce the quantization of the particle motion. Here, we introduce an additional constant magnetic field, and we will see that it is still possible to give approximate solutions in the non-relativistic and weak magnetic field limits corresponding to the standard Zeeman effect.

III. SOLUTIONS OF THE EQUATIONS OF MOTION

A. The particle dynamics

Assuming a uniform circular motion in the $(x, y) := (\rho, \varphi)$ plane, let us write Eq. (4) with a specific form of the four electromagnetic potential $\mathrm{R}^2 A = \left(-\frac{e}{4\pi r}, \frac{1}{2}B\rho\hat{\mathbf{e}}_\varphi\right)$,

$$-m_{\rm eff.}\gamma\frac{v^2}{r} = -\frac{\alpha}{r^2} + evB, \tag{6}$$

with v being the particle velocity and $\gamma = (1 - v^2)^{-\frac{1}{2}}$ and where we used the Sommerfeld fine structure constant $\alpha = \frac{e^2}{4\pi}$. Equation (6) describes the classical relativistic mechanics of a point-like particle along a circular orbit in a mixed electro- and magnetostatic external field. Similarly, we can define the integral of motion associated with the angular orbital momentum by using the action variable formalism,

$$J := \oint P{\rm d}l = 2\pi r\left(\gamma m_{\rm eff.}v + \frac{1}{2}eBr\right) = 2\pi n. \tag{7}$$

This notation is similar to the one used by Sommerfeld in his old quantum mechanics.[6] Note first that for simplicity, we identified the spherical radius r with the cylindrical coordinate ρ since we impose a magnetic field along the direction z normal to the plane of motion $z = 0$. We also have introduced a number n defining an integral of motion for the dynamics, i.e., the orbital angular momentum $L_z = n$ of the particle. Knowing that, from now on, we will index all physical quantities related to the particle's motion with this number n. Finally, we will introduce the Larmor frequency defined by $\omega_L = -eB/(2m_{\rm eff.})$, which will be derived later in this article using Larmor's theorem (see Sec. III B).

Equations (6) and (7) allow us to derive the characteristic properties of the motion; in particular, we can get the constant particle velocity along the orbit,

$$v_n = \frac{\alpha}{n}\frac{1}{1 - \frac{m_{\rm eff.}\omega_L r_n^2}{n}}. \tag{8}$$

In the absence of magnetic field, we recover the formula $v_n^{(0)} = \frac{\alpha}{n}$ for the purely Coulombian atom as obtained by Bohr[4] and justified in JD for our model. We also have, in the non-relativistic limit, a fourth order equation for the radius r_n,

$$(m_{\rm eff.}\omega_L)^2 r_n^4 + m_{\rm eff.}\alpha r_n - n^2 = 0. \tag{9}$$

In the case of a weak magnetic field, we can find an approximate solution for this equation with a perturbation around the radius without magnetic field $r_n^{(0)} = n^2/(m_{\rm eff.}\alpha)$ [if $n = 1$, we recover the standard Bohr radius $1/(m_{\rm eff.}\alpha)$]. We obtain

$$r_n \approx \frac{n^2}{m_{\rm eff.}\alpha}\left(1 - (m_{\rm eff.}\omega_L)^2\frac{n^6}{(m_{\rm eff.}\alpha)^4}\right), \tag{10}$$

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which will be equal to $r_n^{(0)}$ at first order in B. We can then use this expression in the velocity, and we obtain

$$v_n \approx \frac{\alpha}{n}\left(1 + (m_{\text{eff.}}\omega_L)\frac{n^3}{(m_{\text{eff.}}\alpha)^2}\right) = \frac{\alpha}{n} + \omega_L r_n^{(0)}. \tag{11}$$

This time, the deviation from $v_n^{(0)} = \alpha/n$ is of order 1 in B. This formula is very intuitive and can be recovered using Larmor's theorem by adding to $v_n^{(0)}$ a velocity $\omega_L r_n^{(0)}$ associated with a change of the reference frame: This is explained in Sec. III B.

In the end, the energy of the particle is

$$E_n \approx m_{\text{eff.}} + \frac{1}{2}m_{\text{eff.}}v_n^2 - \frac{\alpha}{r_n} \approx m_{\text{eff.}}\left(1 - \frac{1}{2}\frac{\alpha^2}{n^2}\right) + n\omega_L, \tag{12}$$

and we recover the usual expression for the Zeeman splitting in the weak field and non-relativistic limits,

$$E_n \approx E_n^{(0)} + n\omega_L = E_n^{(0)} - L_z\frac{eB}{2m_{\text{eff.}}}, \tag{13}$$

where $E_n^{(0)} = m_{\text{eff.}}\left(1 - \frac{1}{2}\frac{\alpha^2}{n^2}\right)$ is the standard Bohr formula for the hydrogen-like atom energy (including a constant mass energy $m_{\text{eff.}}$). Moreover, observe that the previous formulas are valid irrespective of the sign of B and n. If $n > 0$, we have a positive orbital angular momentum $L_z = n$ and the orbit is followed in the anticlockwise direction, whereas if $n = -|n| < 0$, we have a negative orbital angular momentum $L_z = -|n| < 0$ and the orbit is followed in the clockwise direction. Therefore, for a given magnetic field along z, say $B > 0$, we have two solutions for the Zeeman effect corresponding to $E_n \approx E_n^{(0)} \pm |n|\omega_L$: The symmetry between the two travel directions along the orbit existing in the absence of B has been broken. Of course, the same broken symmetry is obtained if instead of considering the two possible travel directions along the orbit, we reverse the magnetic field so as to have $B = -|B| < 0$. This equivalence will be used in Sec. III B dealing with Larmor's theorem.

At this point, we stress that n does not need to be an integer; i.e., we still have a completely classical unquantized motion for the particle. Indeed, the Zeeman effect[1] is not inherently quantum mechanical but can also be found in classical mechanics as it was historically done by Lorentz[3] and Larmor.[20] Moreover, in the following, we will show that this Bohr–Sommerfeld quantization of n is required as soon as we reintroduce the holonomic constraint with the field as was shown previously in JD for the case of the pure Coulombian field.

We also stress that Eq. (13) is a particular case of the formula $E_n = E_n^{(0)} - L_z\frac{eB}{2m_{\text{eff.}}}$, where n and L_z (historically named the magnetic quantum number) are generally different. Indeed, here, we considered a motion in the $z = 0$ plane where L_z is identical to the full orbital angular momentum. In the more general case, we can write $L_z = \cos\theta_0|\mathbf{L}|$, where θ_0 is the angle between the orbital angular momentum pseudovector $\mathbf{L} = m_{\text{eff.}}\mathbf{x}\times\mathbf{v}$ and the z axis oriented along the magnetic field. The effect of the magnetic field perturbation is to induce a slow precession of the electron orbital plane along the z-axis with Larmor's frequency ω_L. Here, we have only considered the case $\theta_0 = 0$ or π corresponding to the anticlockwise and clockwise motions in the $z = 0$ plane around the magnetic field axis. Moreover, we also supposed $|n| = |\mathbf{L}|$ associated with the simplest Bohr-like circular motion. The most general case considered by Sommerfeld[6] allows $|n| \geq |\mathbf{L}|$ corresponding to elliptical orbits with eccentricity $\epsilon = \sqrt{1 - \frac{|\mathbf{L}|^2}{n^2}}$ (the case $n = 0$ is excluded). Here, the restriction $\epsilon = 0$ to a pure circular orbit is imposed by our theoretical model coupling the particle to a wave motion. More precisely, as we will show in Sec. III B, the very demanding holonomic condition [Eq. (3)] requires the velocity to be a constant of motion, which is not compatible with a general elliptical motion.

B. Larmor's theorem and the solutions for the *u* field

Solving the full wave equation for the field in the presence of the electromagnetic potential

$$\left(\partial_t - i\frac{\alpha}{r}\right)^2 u - \left(\nabla - \frac{1}{2}ieB\rho\hat{\mathbf{e}}_\varphi\right)^2 u = 0 \tag{14}$$

is not an easy task. It is, however, possible to find an approximate solution by emulating the magnetic field with an inertial Coriolis force, as is explained by Larmor's theorem.

Consider first the following wave equation:

$$\left(\partial_{t'} - i\frac{\alpha}{r'}\right)^2 u' - \left(\nabla'\right)^2 u' = 0, \tag{15}$$

defined in an inertial reference frame $\mathscr{R}'$ where the scalar u-field reads $u'(t', \mathbf{x}')$. When this wave equation is watched from the point of view of an accelerated reference frame $\mathscr{R}$ rotating uniformly around the $z = z'$ axis, this wave equation will be modified to include some inertial force terms. For this purpose, we use the following cylindrical coordinate transformation (sketched in Fig. 1):

$$\begin{aligned} t' &= t, \\ \rho' &= \rho, \\ z' &= z, \\ \varphi' &= \varphi - \omega_L t, \end{aligned} \tag{16}$$

where t, ρ, φ, z are coordinates in the accelerated reference frame and ω_L is the constant rotation frequency of $\mathscr{R}'$ relatively to $\mathscr{R}$ (the rotation is supposedly nonrelativistic).

This transformation, in turn, modifies the temporal derivative

$$\partial_{t'} = \partial_t + \omega_L\partial_{\varphi'}, \tag{17}$$

but otherwise, it allows the spatial derivatives unchanged: $\nabla = \nabla'$.

Applying this transformation to Eq. (15) leads (after some calculations detailed in Appendix A) to

$$\left(\partial_t - i\frac{\alpha}{r}\right)^2 u - \nabla^2 u - 2i\omega_L m_{\text{eff.}}\partial_\varphi u + \mathscr{O}(\omega_L^2) \approx 0. \tag{18}$$

On the other hand, going back to Eq. (14), the spatial operator can also be expanded as

$$\left(\nabla - \frac{1}{2}ieB\rho\hat{\mathbf{e}}_\varphi\right)^2 u = \nabla^2 u - ieB\partial_\varphi u + \mathscr{O}(B^2), \tag{19}$$

and Eq. (14) reduces to

$$\left(\partial_t - i\frac{\alpha}{r}\right)^2 u - \nabla^2 u + ieB\partial_\varphi u + \mathscr{O}(B^2) \approx 0. \tag{20}$$

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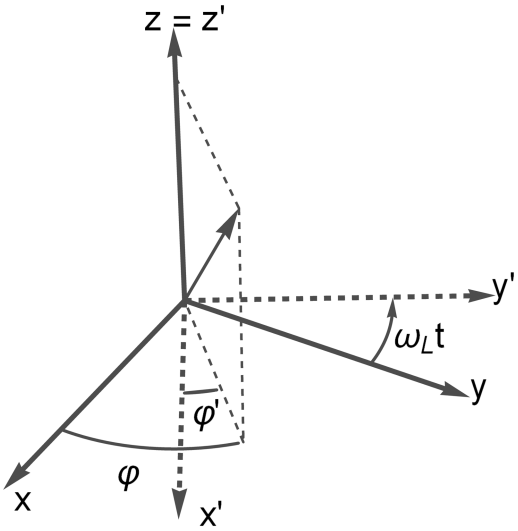

FIG. 1. Illustration of the change of coordinates following Larmor's theorem. The new coordinates x', y', z' (or, equivalently, ρ', φ', z') are associated with a rotation of the original coordinate system x, y, z (or, equivalently, ρ, φ, z) by an angle $\omega_L t$ around the z axis.

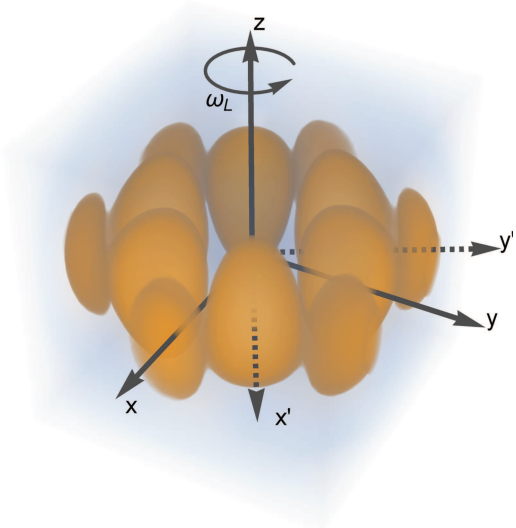

FIG. 2. Rotation of the total u-field following Larmor's theorem. The density of the modulus of the total field $|u(0, \mathbf{x})|$ in the case $n = 1$, $m_+ = 4$, $m_- = 2$ is represented in color.

Comparing Eqs. (18) and (20), we see that the two are identical [i.e., neglecting second order terms $\mathscr{O}(B^2) \sim \mathscr{O}(\omega_L^2)$] if we carefully impose $\omega_L := -eB/(2m_{\text{eff.}})$ defining the Larmor frequency. In other words, a change of coordinates from the laboratory frame $\mathscr{R}$ to a new virtual frame $\mathscr{R}'$ rotating with the Larmor frequency ω_L exactly cancels out the effects of the magnetic field at the first order in B. This means that if we know the solution u' without a magnetic field in a frame $\mathscr{R}'$, we can obtain an approximate solution $u = u'$ in the laboratory frame $\mathscr{R}$ where a weak magnetic field is present by performing the rotation $\varphi = \varphi' + \omega_L t'$ as shown in Fig. 1.

In JD, we previously obtained the solution of the wave equation $D^2 u(x) = 0$ in the presence of a Coulombian potential $eV(r) = -\alpha/r$. In particular, we showed that in order to fulfill the holonomic condition $z(\tau) = u(x_n(\tau))$, the u-field must be written as $u(x) = u_+(x) + u_-(x)$ involving two counterpropagating modes $u_\pm$. This is necessary in order to obtain the correct de Broglie "phase harmony" condition between the field and the oscillation z of the particle. Moreover, along the circular orbit of radius r_n, we have

$$u_\pm\left(t, r_n, \frac{\pi}{2}, \varphi\right) = \frac{1}{2} u_0 \mathrm{e}^{\mathrm{i}\left(\pm k_\pm r_n \varphi - \omega'_\pm t\right)}, \tag{21}$$

where we introduced the two quantized wave vectors $k_\pm = \frac{m_\pm}{r_{\mathrm{p}}}$ ($m_\pm \in \mathbb{N}$) and the mode frequencies $\omega'_\pm := \omega_\pm^{(0)}$ (the complete expression for the field is given in JD and in Appendix B). To go further, we first write $u'_\pm(x')$ in the $\mathscr{R}'$ frame where Eq. (15) holds true. If we use Larmor's theorem in the laboratory frame $\mathscr{R}$, we obtain $u_\pm(t, r, \theta, \varphi) = u'_\pm(t', r', \theta', \varphi')$ defining our new solutions. Along the orbit, we thus obtain

$$u_\pm\left(t, r_n, \frac{\pi}{2}, \varphi\right) = \frac{1}{2} u_0\, \mathrm{e}^{\mathrm{i}\left(\pm k_\pm r_n \varphi - (\omega'_\pm \pm m_\pm \omega_L) t\right)}, \tag{22}$$

where the shifted frequencies $\omega'_\pm \pm m_\pm \omega_L$ appear. Note that in the first order approximation, $r_n \simeq r_n^{(0)}$ as justified in Sec. III A. Moreover, along the orbit, the total field $u(x) = u_+(x) + u_-(x)$ is

$$\begin{aligned} u\left(t, r_n, \frac{\pi}{2}, \varphi\right) &= u_0\, e^{i(\tilde{n}\varphi - (\omega'_n + \tilde{n}\omega_L))t} \\ &\quad \cdot \cos\left[(\omega'_n + \tilde{n}\omega_L + \varepsilon)\left(r_n \varphi - \frac{k_n - \eta}{\omega'_n + \tilde{n}\omega_L + \varepsilon} t\right)\right], \end{aligned} \tag{23}$$

with

$$k_n = \frac{k_+ - k_-}{2}, \quad \omega_n = \frac{\omega_+ + \omega_-}{2}, \quad \omega'_n = \frac{\omega'_+ + \omega'_-}{2} \tag{24}$$

and where we introduced

$$\eta = \frac{\varepsilon_+ - \varepsilon_-}{2}, \quad \varepsilon = \frac{\varepsilon_+ + \varepsilon_-}{2}. \tag{25}$$

Moreover, the total field in the 3D space is obtained by summing the two modes, and we illustrate in Fig. 2 the case of a simple combination of modes $m_+ = 4$, $m_- = 2$ leading to the energy level $n = 1$. The wave is rotating (precessing) with the frequency ω_L in agreement with Larmor's theorem. Importantly, in Eq. (23), we have inserted a quantum number $\tilde{n} = \frac{m_+ - m_-}{2} \in \mathbb{Z}/2$. Thus, the mean frequency in the laboratory frame ω_n is related to the mean frequency in the rotating frame ω'_n by

$$\omega_n = \omega'_n + \tilde{n}\omega_L = \omega_n^{(0)} + \tilde{n}\omega_L. \tag{26}$$

This looks like the expression [Eq. (13)] for Zeeman's energy-shift when applying a weak magnetic field.

In order for this analogy to become an identity, we must first have $\tilde{n} = n$. This is justified in our approach[15,16] if, in agreement with

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de Broglie,[17,22] we identify the group velocity of the u-wave in Eq. (23),

$$v_g = \frac{k-\eta}{\omega+\varepsilon}, \tag{27}$$

to the velocity of the particle along the circular trajectory,

$$v_n = \frac{P_n - eA_\varphi}{E_n - eV}, \tag{28}$$

with the canonical momentum $P_n = \gamma m_{\text{eff.}} v_n + eA_\varphi = \gamma m_{\text{eff.}} v_n + \frac{1}{2} eBr_n$. The equality $v_n = v_g$ implies that the particle is constantly in phase with the guiding wave. With this condition, the term $r_n\varphi - v_g t$ becomes a constant in Eq. (23). In other words, this corresponds to synchronization of the internal clock of the particle with the phase of the u-wave as required by the holonomic condition $z = u$. This motivates the fundamental de Broglie relations,

$$P_n = bk_n \text{ and } E_n = b\omega_n, \tag{29}$$

with b an *a priori* arbitrary constant. In our previous work JD, we showed that a self-consistent model is easily obtained if we impose $b = 1$ as a condition. In the following, we will thus accept the requirement $b = 1$ for simplicity, but we emphasize that in principle, our model could be generalized to $b \neq 1$ as explained in JD.

The condition $P_n = k_n = \frac{\tilde{n}}{r_n}$ is clearly reminiscent of Eq. (7), which can be written as

$$\oint P_n dl = 2\pi r_n \oint k_n \, d\varphi = 2\pi \tilde{n}. \tag{30}$$

This is indeed the Bohr–Sommerfeld quantization condition if $n = \tilde{n}$ as is also required to identify Eqs. (13) and (26).

Moreover, the condition $v_n = v_g$ together with Eq. (29) leads to

$$\varepsilon_\pm = \frac{\alpha}{r_n} \pm \frac{1}{2} eBr_n, \tag{31}$$

and therefore, from $k_\pm = \frac{n_\pm}{r_n} = \omega_\pm + \varepsilon_\pm$, we deduce

$$E_n = \omega_n = \frac{N}{r_n} - \varepsilon = \frac{N}{r_n} - \frac{\alpha}{r_n}, \tag{32}$$

where we defined $N = \frac{m_+ + m_-}{2}$. Inserting E_n and $P_n = n/r_n$ into Eq. (28), we then easily find that

$$\begin{aligned} v_p = v_g &\approx \frac{n}{N}\left(1 + \frac{m_{\text{eff.}}\omega_L}{(m_{\text{eff.}}\alpha)^2} n^3\right) + \mathcal{O}(B^2) \\ &= \frac{n}{N}\left(1 + \frac{m_{\text{eff.}}\omega_L r_n^{(0)}}{n}\right) + \mathcal{O}(B^2), \end{aligned} \tag{33}$$

from which we can deduce an interesting condition on the value of the fine structure constant α by comparing to Eq. (11),

$$\alpha \approx \frac{n^2}{N} + \mathcal{O}(B^2) \equiv \alpha_0 + \mathcal{O}(B^2), \tag{34}$$

where $\alpha_0 = n^2/N$ is the value for an isolated atom, i.e., without a magnetic field as studied in JD. This relation is fundamental since it provides a selection rule for the different possible values of n and N or, equivalently, $m_\pm$. As we showed in JD if α_0^{-1} is an integer (e.g., $\alpha_0^{-1} = 137$), then the allowed values for n are necessarily limited to integer numbers $|n| = 1, 2, \ldots$. As justified in JD, this is central to recover the Bohr–Sommerfeld quantization postulate since *a priori* our number $n \equiv \tilde{n} = \frac{m_+ - m_-}{2} \in \mathbb{Z}/2$ is not necessarily an integer: Without this selection rule, it would also allow half-integers $1/2, 3/2, \ldots$. While half-quanta are not *a priori* useless in connection with fermionic spin (that was the initial reasoning of Heisenberg as a student of Sommerfeld[6]), we prefer here to be rather conservative and follow the usual approach without half-quanta (reserving the possibility to come back to this issue in a future work).

One final important relation is obtained by writing the holonomic constraint

$$z(t) = z_0 \, e^{-i\Omega_p \sqrt{1-v_n^2} t} = u(t, x_p(t)) = u_0 \, e^{i\mathcal{L}_p t}, \tag{35}$$

where $\mathcal{L}_p$ is the particle Lagrangian,

$$\begin{aligned} \mathcal{L}_p &= -m_{\text{eff.}}\sqrt{1-v_n^2} - eV(r_n) + eA_\varphi(r_n)v_n \\ &= -m_{\text{eff.}}\sqrt{1-v_n^2} + \frac{\alpha}{r_n} + \frac{1}{2} eBr_n v_n, \end{aligned} \tag{36}$$

which is, of course, the classical Lagrangian for a relativistic particle in an electromagnetic field. The presence of the Lagrangian comes from the fact that the phase of $u(t, x_p(t))$ can be written as $(k_n v_n - \omega_n)t = (P_n v_n - E_n)t$, which by definition is $\mathcal{L}_p t$. Therefore, as required by the holonomic condition, we have equal amplitudes $z_0 = u_0$ as well as equal phases,

$$\begin{aligned} \Omega_p &= m_{\text{eff.}} - \left(\frac{\alpha}{r_n} + \frac{1}{2} eBr_n v_n\right) \frac{1}{\sqrt{1-v_n^2}} \\ &\approx m_{\text{eff.}}\left(1 - \frac{\alpha^2}{n^2}\right) + \omega_L n \left(1 - \frac{\alpha^2}{2n^2}\right), \end{aligned} \tag{37}$$

which, remembering that $m_{\text{eff.}} = m_p(1 + \sigma\Omega_p^2 |z_0|^2)$, completely determines the amplitude z_0 and, therefore, all the characteristics of the motion using the different parameters of our model.

Before concluding, a few points must be emphasized in order to show the consistency of our model: (i) First, in JD, the frequencies $\omega_\pm^{(0)}$ of the modes were calculated to be

$$\begin{aligned} \omega_\pm^{(0)} &= \frac{m_{\text{eff.}}}{\sqrt{1-\frac{\alpha^2}{n^2}}}\left(1 \pm \frac{\alpha}{n} - \frac{\alpha^2}{n^2}\right) \\ &\approx m_{\text{eff.}}\left(1 \pm \frac{\alpha}{n} - \frac{\alpha^2}{2n^2}\right). \end{aligned} \tag{38}$$

The dominant contribution is the effective mass $m_{\text{eff.}}$, and this is not modified if we include the first order magnetic perturbation so that $\omega_\pm = \omega_\pm^{(0)} \pm m_\pm \omega_L \sim m_{\text{eff.}}$. This plays a role in self-consistency of our method (see Appendix A).

(ii) Second, we stress that the Larmor theorem is usually presented for particles and not for waves. Moreover, the derivation of the particle dynamics given in Sec. III A could be done directly using the Larmor theorem as it was proposed by Larmor himself.[20] The derivation is straightforward: Starting from the inertial reference frame $\mathcal{R}'$ where the nonrelativistic particle dynamics is described

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by a Newtonian equation

$$m_{\text{eff.}}\frac{d}{dt'}\mathbf{v}'_p(t') = -\frac{\alpha\hat{\mathbf{r}}'}{r'^2} \tag{39}$$

that involves a Coulombian (electrostatic) force, and using the coordinate transformation Eq. (16) to the accelerated reference frame $\mathscr{R}$ uniformly rotating around the z axis with an angular velocity $\omega_L\hat{\mathbf{z}}$, we obtain

$$m_{\text{eff.}}\frac{d}{dt}\mathbf{v}_p(t) \approx -\frac{\alpha\hat{\mathbf{r}}}{r^2} - 2m_{\text{eff.}}\mathbf{v}_p(t)\times\omega_L\hat{\mathbf{z}} + \mathscr{O}(\omega_L^2). \tag{40}$$

Up to second order terms in ω_L, this equation is similar to the dynamics of a particle submitted to both a central Coulombian force and to the Laplace–Ampere–Lorentz force in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$; i.e.,

$$m_{\text{eff.}}\frac{d}{dt}\mathbf{v}_p(t) = -\frac{\alpha\hat{\mathbf{r}}}{r^2} + e\mathbf{v}_p(t)\times B\hat{\mathbf{z}}. \tag{41}$$

The identity occurs if we assume $\omega_L = -\frac{eB}{2m_{\text{eff.}}}$ that constitutes the Larmor formula. Importantly, the particle velocity in the $\mathscr{R}$ reference frame is given by $\mathbf{v}_p(t) = \mathbf{v}'_p(t') + \omega_L\hat{\mathbf{z}}\times\mathbf{r}$; i.e.,

$$v_\varphi = v'_\varphi + \omega_L r := \frac{\alpha}{n} + \omega_L r_n, \tag{42}$$

which is equivalent to Eq. (11), $v = \frac{\alpha}{n} + \omega_L r_n^{(0)}$ with $r_n \simeq r_n^{(0)}$. In the end, the nonrelativistic particle energy $E' - m_{\text{eff.}} = \frac{1}{2}m_{\text{eff.}}v'^2 - \frac{\alpha}{r'}$ in $\mathscr{R}'$ transforms into the energy $E - m_{\text{eff.}} = \frac{1}{2}m_{\text{eff.}}v^2 - \frac{\alpha}{r} = E' + m_{\text{eff.}}\omega_L r_n v_n + \mathscr{O}(\omega_L^2)$ in $\mathscr{R}'$. Using the formula for $r_n \simeq n^2/m_{\text{eff.}}\alpha$ and v_n leads to

$$E_n \simeq E'_n + n\omega_L + \mathscr{O}(\omega_L^2) \tag{43}$$

recovering the Zeeman splitting of Eq. (13). Therefore, we see that the two applications of the Larmor theorem to, respectively, the wave and the particle are self consistent. The coordinate transformation [Eq. (16)] from the reference frame $\mathscr{R}'$ to $\mathscr{R}$ leads to a circular particle orbit with a modified velocity $v_n = \frac{\alpha}{n} + \omega_L r_n^{(0)}$, whereas the u-field is also modified accordingly to obey Eq. (23). In the $\mathscr{R}'$ reference frame, the u-field is given by the unperturbed wave field associated with the pure Coulombian solution studied in JD. The Larmor theorem applied to both the field and the particle allows us to obtain the new dynamics in the presence of a weak constant magnetic field B valid in the laboratory reference frame $\mathscr{R}$.

(iii) As a final remark, we must return to our comment at the end of Sec. III A concerning the limitation of our model to uniform circular motion. For this purpose, it is enough to consider the regime without an external magnetic field, i.e., $B = 0$, and to use once more the holonomic condition [Eq. (35)] with the Lagrangian given by Eq. (36). This leads to

$$(\Omega_{\text{p}} - m_{\text{eff.}})\gamma_n^{-1} = -\alpha/r_n = E_n - m_{\text{eff.}}\gamma_n, \tag{44}$$

where we used the definition of the constant energy $E_n = m_{\text{eff.}}\gamma_n - \alpha/r_n$ (with $\gamma_n = 1/\sqrt{1-v_n^2}$). Solving Eq. (44) gives

$$\gamma_n = \frac{E_n}{2m_{\text{eff.}}} + \frac{1}{2m_{\text{eff.}}}\sqrt{(E_n^2 - 4m_{\text{eff.}}(\Omega_{\text{p}} - m_{\text{eff.}}))}, \tag{45}$$

which corresponds to a constant velocity. This clearly shows the intrinsic limitation of our model associated with the constant value of the internal frequency Ω_{p}.

IV. DISCUSSION AND PERSPECTIVES

In this paper, we have extended our 3D mechanical atomic model presented in JD by adding an external uniform magnetic field. We recovered the mathematics of the normal Zeeman splitting of the energy levels in the atom in the weak field limit by making use of an equivalence between a constant and uniform magnetic field and inertial forces of a rotating frame of reference.

Going back to the results of the present work, we showed that the quantum Zeeman effect arises naturally in our model. This extends the regime of applicability of the analogy discussed in our previous article JD. There are, however, some limitations that need to be considered, mainly the difficulties of our model to introduce elliptical orbits or the case of s-orbitals with $n = 0$ that are part of quantum mechanics. Moreover, here, we limited our analysis to the transparency regime where $\mathscr{N} = 0$. Going beyond would imply to consider transitions between orbits (e.g., to study the stability of trajectories or the chaotic nature of the dynamics). Such a regime clearly needs further considerations.

In JD, we suggested ways of experimentally reproducing the Bohr–Sommerfeld quantum condition [Eq. (7)] as implementations of our atomic model, in particular, using optical vortices with definite orbital angular momenta to trap small particles on quantized orbits (following the method developed by Ashkin[23] for optical tweezers). This same system could naturally be extended to include a transverse magnetic field, which, by interacting with the orbital angular momentum, should also reproduce the results shown in this paper, i.e., the modification of all orbital quantities following Zeeman's results. The most important feature in the optical tweezers analogy is the holonomic condition $z(\tau) = u(x_{\text{p}}(\tau))$ that can be seen as a particular weak resonance condition for a nanoparticle or a nano-antenna in an optical[24] or plasmonic field[25] (that is, if we can neglect scattering). Further studies should be done in order to understand this interesting analogy in both the 2D and 3D regime. Potential applications in biology or micro/nano technology could ultimately be considered. More generally, if one were to produce an experimental demonstration of this system, introducing inertial forces would suffice at first order to mimic the effects of an electromagnetic field, as was hinted in other experimental and theoretical works on hydrodynamical analogs.[9,26] This opens interesting perspectives for discussing analogies with the Zeeman effect using mechanical or acoustical systems. This was the original motivation of our nonrelativistic model[15] where a particle sliding on a vibrating string was guided by the phase wave generated by two counterpropagating plane waves $u_\pm$. Clearly, this shows the strong transdisciplinarity of our analogy and method. Furthermore, the different mathematical tools used in our model coupling a wave to a guided particle also stress the links between methods applied to classical mechanics of point-like objects and the dynamics of waves (in full agreement with the goal of de Broglie in his double solution theory[19]). We believe that this constitutes strength of our general methodology that can be applied to different fields.

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Moreover, at a fundamental level, it should be noted that this equivalence between a constant and uniform magnetic field and inertial forces of a rotating frame of reference could be taken one step further by using a general relativity formalism and interpreting the whole electromagnetic field as a curvature in the metric. The idea is to find some kind of an equivalence between the two covariant derivatives in general relativity and our formalism,

$$\partial_\mu + ieA_\mu \sim \partial_\mu + \tilde{\Gamma}_\mu, \tag{46}$$

with $\tilde{\Gamma}_\mu$ being some contraction or operation on the Levi-Civita connection $\Gamma^\rho_{\mu\,\nu}$ defined in general relativity by

$$\Gamma^\rho_{\mu\,\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right), \tag{47}$$

with $g_{\mu\nu}$ being the metric tensor and ∂_μ the derivative with respect to the coordinate x^μ. This of course cannot be done in the general case, and the equivalence (46) is only here to represent schematically the goal of the analogy. However, it was done, for example, in the case of weak static fields[26,27]—which is our case of interest—using the electromagnetic tensor $F_{\mu\nu}$ and part of the affine connection $\Gamma^\rho_{0\,\nu}$,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &\equiv \eta_{\mu\rho}\Gamma^\rho_{0\,\nu} = \frac{1}{2}\eta_{\mu\rho} g^{\rho\lambda}\left(\partial_0 g_{\lambda\nu} + \partial_\nu g_{0\lambda} - \partial_\lambda g_{0\nu}\right), \end{aligned} \tag{48}$$

with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ being the Minkowski metric. On top of being interesting analogies in and of themselves, these ideas of emulating or equating "real" forces with inertial ones can prove computationally useful as was shown here by our solving of the covariant wave equation. This general relativity formalism has, of course, the added benefit of being able to account for both electric and magnetic forces with a combination of Coriolis and centrifugal forces.

In the end, the formalism developed in the present work allows us to consider very general cases of wave-particle duality with various external potentials, such as a central electrostatic potential or a uniform magnetic field, with the two main assumptions, i.e., the existence of a periodic phenomenon in the particle and its holonomic coupling with the field, being at the root of all quantum phenomena. These are ideas that clearly deserve to be studied extensively and using various approaches (e.g., pilot-wave theories, double solution theories, Bohmian mechanics, hydrodynamic analogs, . . .[19,28–30]), as they are able to provide us with insights into the nature of quantum mechanics.

AUTHOR DECLARATIONS

Conflict of Interest

The authors declare they have no conflict of interest to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: LARMOR'S THEOREM FOR THE WAVE EQUATION

The application of the transformation from Eq. (16) to (15) leads to

$$\begin{aligned} \left(\partial_t - i\frac{\alpha}{r'}\right)^2 &= \left(\partial_t - i\frac{\alpha}{r} + \omega_L \partial_\varphi\right)^2 \\ &= \left(\partial_t - i\frac{\alpha}{r}\right)^2 + 2\omega_L \partial_\varphi \left(\partial_t - i\frac{\alpha}{r}\right) + \mathscr{O}(\omega_L^2), \end{aligned} \tag{A1}$$

where we used $r = \sqrt{\rho^2 + z^2} = \sqrt{\rho'^2 + z'^2} = r'$. Moreover, assuming a harmonic solution for the u-field $u'(t', \mathbf{x}') := u(t, \mathbf{x}) = f(\mathbf{x})\, e^{-i\omega t}$, and if we neglect the radial term $\alpha/r \ll \omega$, we can write

$$\left(\partial_t - i\frac{\alpha}{r}\right) f(\mathbf{x})\, e^{-i\omega t} \approx -i\omega f(\mathbf{x})\, e^{-i\omega t}, \tag{A2}$$

which finally gives us for the temporal part of the d'Alembert operator when we identify the energy ω with an equivalent mass $m_{\text{eff.}}$,

$$\left(\partial_t - i\frac{\alpha}{r} + \omega_L \partial_\varphi\right)^2 \approx \left(\partial_t - i\frac{\alpha}{r}\right)^2 u - 2i\omega_L m_{\text{eff.}} \partial_\varphi u + \mathscr{O}(\omega_L^2). \tag{A3}$$

Moreover, we justify the identification $\omega \sim m_{\text{eff.}}$ used in Eq. (A2) afterward. Indeed, from Eq. (38), we see that the dominant contribution in the mode frequencies $\omega_\pm^{(0)}$ is the effective mass $m_{\text{eff.}}$, and this is not modified if we include the first order magnetic perturbation so that $\omega_\pm = \omega_\pm^{(0)} \pm m_\pm \omega_L \sim m_{\text{eff.}}$. This, in turn, justifies why it was judicious in Eq. (A3) to identify the frequency ω with the mass term $m_{\text{eff.}}$ (a similar method is used for deriving Schrödinger's equation from Klein–Gordon's one). Therefore, the self-consistency of our derivation of Larmor's theorem for the wave equation and of the related dispersion relation given by Eq. (B5) is *a posteriori* justified.

APPENDIX B: SOLUTIONS OF THE WAVE EQUATION

For the present problem writing the two solutions $u'_\pm(x')$ in the $\mathscr{R}'$ frame where Eq. (15) holds true, we have (using spherical coordinates)

$$u'_\pm(t', r', \theta', \varphi') = A_\pm R_{\tilde{l}_\pm}(r') P_{l_\pm}^{\pm m_\pm}(\cos\theta')\, e^{i(\pm m_\pm \varphi' - \omega'_\pm t)}, \tag{B1}$$

where $l_\pm \geq 0$ and $m_\pm \leq l_\pm$ are positive integers labeling spherical harmonics $P_{l_\pm}^{\pm m_\pm}(\cos\theta')\, e^{\pm i m_\pm \varphi'}$ and where the radial contribution $R_{l'_\pm}(r')$ (with $\tilde{l}_\pm = \frac{-1}{2} + \sqrt{((l_\pm + \frac{1}{2})^2 - \alpha^2)}$) is a complex function,

$$R_{\tilde{l}_\pm}(r') = e^{i\omega'_\pm r'} r'^{\tilde{l}_\pm} M(\tilde{l}_\pm + 1 - i\alpha, 2\tilde{l}_\pm + 2, -2i\omega_\pm r'), \tag{B2}$$

defined using the Kummer confluent hypergeometric function $M(a, b, c) = {}_1F_1(a, b, c)$.

Moreover, using Larmor's theorem in the laboratory frame $\mathscr{R}$, we obtain $u_\pm(t, r, \theta, \varphi) = u'_\pm(t', r', \theta', \varphi')$,

$$u_\pm(t, r, \theta, \varphi) = A_\pm R_{\tilde{l}_\pm}(r) P_{l_\pm}^{\pm m_\pm}(\cos\theta)\, e^{i(\pm m_\pm \varphi - (\omega'_\pm \pm m_\pm \omega_L) t)}. \tag{B3}$$

This equation clearly shows that the mode frequencies $\omega_\pm$ in the $\mathscr{R}$ reference frame are, according to Larmor's theorem, given by the

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formula

$$\omega_{\pm} = \omega'_{\pm} \pm m_{\pm}\omega_L, \tag{B4}$$

which is actually reminiscent of the Zeeman spectrum given in Eq. (13). Therefore, in Eq. (B4), $\omega'_{\pm} = \omega^{(0)}_{\pm}$ can be interpreted as the zero-order solution in the absence of a magnetic field,

$$\omega_{\pm} = \omega^{(0)}_{\pm} \pm m_{\pm}\omega_L. \tag{B5}$$

To go further, we recall from our previous article JD that along the circular path of the particle the two modes $u_{\pm}$ read

$$u_{\pm}\left(t, r_n, \frac{\pi}{2}, \varphi\right) = \frac{1}{2}u_0 e^{i(\pm k_{\pm} r_n \varphi - \omega_{\pm} t)}, \tag{B6}$$

with the wave vector condition $k_{\pm} = \frac{m_{\pm}}{r_p}$ ($m_{\pm} \in \mathbb{N}$). By definition of the modes $u_{\pm}$, we have $A_{\pm}R_{\bar{l}_{\pm}}(r_n)P^{\pm m_{\pm}}_{l_{\pm}}(0) = \frac{u_0}{2}$ imposing a specific condition on the amplitudes of the two modes u_+ and u_-.

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