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1 **SHAPE DERIVATIVE FOR SOME EIGENVALUE FUNCTIONALS IN**  
2 **ELASTICITY THEORY\***

3 FABIEN CAUBET<sup>†</sup>, MARC DAMBRINE<sup>‡</sup>, AND RAJESH MAHADEVAN<sup>§</sup>

4 **Abstract.** This work is the second part of a previous paper which was devoted to scalar  
5 problems. Here we study the shape derivative of eigenvalue problems of elasticity theory for various  
6 kinds of boundary conditions, that is Dirichlet, Neumann, Robin, and Wentzell boundary conditions.  
7 We also study the case of composite materials, having in mind applications in the sensitivity analysis  
8 of mechanical devices manufactured by additive printing.

9 The main idea, which rests on the computation of the derivative of a minimum with respect to  
10 a parameter, was successfully applied in the scalar case in the first part of this paper and is here  
11 extended to more interesting situations in the vectorial case (linear elasticity), with applications in  
12 additive manufacturing. These computations for eigenvalues in the elasticity problem for generalized  
13 boundary conditions and for composite elastic structures constitute the main novelty of this paper.  
14 The results obtained here also show the efficiency of this method for such calculations whereas the  
15 methods used previously even for classical clamped or transmission boundary conditions are more  
16 lengthy or, are based on various simplifying assumptions, such as the simplicity of the eigenvalue or  
17 the existence of a shape derivative.

18 **Key words.** eigenvalues of elasticity operators, shape derivatives, shape sensitivity analysis,  
19 generalized boundary conditions

20 **AMS subject classifications.** 49Q10, 35P15, 49R05

21 **1. Introduction.**

22 **1.1. Motivations and generalities on shape derivatives.** Many problems  
23 ranging from engineering to physics deal with questions of optimal shapes or designs.  
24 An important class of these problems involves eigenvalues of elliptic operators since  
25 they are important in understanding the vibrating modes of a mechanical structure. A  
26 famous example is the so-called *Rayleigh-Faber-Krahn inequality* for the first vibrating  
27 mode of a clamped membrane. In recent years, additive manufacturing, or the so-  
28 called *3D printing*, has been used in the manufacturing of machine parts with complex  
29 geometries or even having a heterogeneous structure. The structural properties of  
30 these parts depend on two important features: the distribution of the materials and  
31 the effect of thin coatings on the boundary of the device. Of course, engineers would  
32 like to optimize the performance of such a printed device by means of an optimal  
33 layout of the materials. One of the criteria to consider in the performance of the  
34 device are its vibrational properties. In this work, we study the shape sensitivity of  
35 eigenvalue problems in linear elasticity for a wide variety of boundary conditions and  
36 for both homogeneous and heterogeneous materials with the above applications in  
37 mind.

38 Let  $\mathcal{O}_{\text{ad}}$  be a family of *admissible open sets* in  $\mathbb{R}^d$ ,  $d = 1, 2, 3$ , which is stable  
39 with respect to a family of diffeomorphisms  $(\mathbf{I} + t\mathbf{V})$ , that is, for a given  $\delta > 0$ , we

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40 have  $(\mathbf{I}+t\mathbf{V})(\Omega) \in \mathcal{O}_{\text{ad}}$  whenever  $\Omega \in \mathcal{O}_{\text{ad}}$  for all  $t \in [0, \delta]$  and for all  $\mathbf{V}$  smooth vector  
 41 fields with compact support in a neighborhood of  $\Omega$ . Previously and in what follows,  $\mathbf{I}$   
 42 denotes the identity vector field. The *semi-derivative* a shape functional  $F : \mathcal{O}_{\text{ad}} \rightarrow \mathbb{R}$ ,  
 43 in the sense of J. Hadamard [17], at  $\Omega \in \mathcal{O}_{\text{ad}}$  in the direction of a vector field  $\mathbf{V}$ , is  
 44 defined as

$$45 \quad (1.1) \quad F'(\Omega; \mathbf{V}) := \lim_{t \rightarrow 0^+} \frac{F(\Omega_t) - F(\Omega)}{t},$$

46 where

$$47 \quad \Omega_t := \Psi_t(\Omega), \quad \text{being } \Psi_t(x) := x + t\mathbf{V}(x),$$

48 whenever the limit in (1.1) exists. This is called a *shape derivative* if it exists and is  
 49 a linear functional with respect to  $\mathbf{V}$ .

50 **1.2. Aim of this work.** In the first part of the present work exposed in [8],  
 51 we have shown how to compute efficiently the semi-derivative of eigenvalue function-  
 52 als in the scalar case by following a procedure developed initially by M. Delfour and  
 53 J.P. Zolesio for dealing with the sensitivity with respect to a parameter in minimiza-  
 54 tion problems. In this paper, we focus on vector case and, to be specific, on the study  
 55 of the semi-derivatives for several families of eigenvalue problems in linear elasticity  
 56 problems for various kinds of boundary conditions. Whether or not this is linear and  
 57 continuous with respect to the vector field will not be addressed here. Indeed, the  
 58 fact that eigenvalues in elasticity problems are not simple makes it very little probable  
 59 that we could go beyond establishing the existence of a semi-derivative.

60 **OUR STRATEGY.** For establishing the existence of a semi-derivative in the prob-  
 61 lems of our interest we shall adopt the following approach. This approach starts with  
 62 the application of the following version of Theorem 2.1, Chapter 10, M. Delfour and  
 63 J.P. Zolesio [14] for proving the existence and for obtaining an initial expression for  
 64 the semi-derivative.

65 **THEOREM 1.1.** *Let  $X$  be a Banach space and let  $G : [0, \delta] \times X \rightarrow \mathbb{R}$  be a given*  
 66 *functional and we set*

$$67 \quad g(t) = \inf_X G(t, x) \quad \text{and} \quad X(t) = \{x \in X : G(t, x) = g(t)\}.$$

68 *If the following hypotheses hold,*

- 69 (H1)  $X(t) \neq \emptyset$  for all  $t \in [0, \delta]$ ,
- 70 (H2)  $\partial_t G(t, x)$  exists in  $[0, \delta]$  at all  $x \in \cup_{t \in [0, \delta]} X(t)$ ,
- 71 (H3) there exists a topology  $\tau$  on  $X$  such that, for every sequence  $\{t_n\} \subset ]0, \delta]$   
 72 tending to 0 and  $x_n \in X(t_n)$ , there exists  $x_0 \in X(0)$  and a subsequence  $\{t_{n_k}\}$   
 73 of  $\{t_n\}$ , for which
  - 74 (i)  $x_{n_k} \rightarrow x_0$  with respect to  $\tau$
  - 75 (ii)  $\liminf_{k \rightarrow \infty} \partial_t G(t_{n_k}, x_{n_k}) \geq \partial_t G(0, x_0)$ ,
- 76 (H4) for all  $x \in X(0)$ , the function  $t \rightarrow \partial_t G(t, x)$  is upper semi-continuous at  
 77  $t = 0$ ,

then we have that

$$g'(0) = \inf_{x \in X(0)} \partial_t G(0, x).$$

78 In our setting, the functional  $G(t, \cdot)$  will be chosen to be the Rayleigh quotient as-  
 79 sociated to the original eigenvalue problem on the perturbed domain  $\Omega_t$  after it is

80 transported back to  $\Omega$ . We then follow the procedure used in the scalar case [8], for  
 81 a step-by-step verification of the hypotheses which guarantee the applicability of the  
 82 theorem. It follows that we break up the different terms which constitute the Rayleigh  
 83 quotient and calculate their contributions to the shape derivative through Proposi-  
 84 tions 3.1 and 3.2 proved below. Then the rather complicated obtained expressions are  
 85 simplified thanks to a systematic choice of test functions in the variational formula-  
 86 tion of the eigenvalue problem. By following this methodical approach, we are able to  
 87 rigorously establish the existence of the Eulerian semi-derivatives in these problems  
 88 and obtain the corresponding boundary representations in a simplified manner.

89 MAIN NOVELTIES OF THIS WORK. Now a word about some existing results for  
 90 such derivatives in the elasticity case. Previously, the shape sensitivity of eigenvalue  
 91 problems of elasticity has been considered for example by J. Sokolowski and J.P.  
 92 Zolesio in [23] and G. Allaire and F. Jouve in [2]. In these works, the computation for  
 93 this shape derivative in the presence of Dirichlet and Neumann boundary conditions  
 94 is given while assuming that the eigenvalue in consideration is simple (and this may  
 95 be the case for certain domains although not true in general). The arguments therein  
 96 are based on a suitable adjoint formulation.

97 In the present work, we avoid the hypothesis of the simplicity of the eigenvalue  
 98 and, by following our unified and systematic approach, we do not only recover the  
 99 earlier results but also are able to extend it with a fair amount of ease to other  
 100 boundary value problems of interest, especially in additive printing, like the so-called  
 101 *Wentzell boundary conditions*. Let us emphasize that such boundary conditions are  
 102 not a mathematical curiosity but appear naturally in the context of linear elasticity  
 103 as soon as the configuration presents discontinuities on the material properties on  
 104 a submanifold (see, e.g., [9] for a crusted body or [20] for an interface problem).  
 105 In particular, the Wentzell boundary conditions, coming from asymptotic analysis  
 106 (see [20, 21, 9] for the mechanical and theoretical justification of such conditions),  
 107 permit to model *coating or membrane effects*. Notice that this approximation of an  
 108 original structure with a thin layer by adhering to another domain with new boundary  
 109 conditions, called *generalized impedance boundary conditions*, is a classical method in  
 110 order to avoid huge difficulties in the theoretical and numerical analysis of a thin  
 111 structure (for instance a mesh refinement adapted to the thickness of the layer).

112 We also underline that we consider two types of eigenvalues problems: the volume  
 113 and surface types. If the volume type is more classical, at least for classical bound-  
 114 ary conditions, the study of shape sensitivity of surface type eigenvalues problem is,  
 115 up to our knowledge, much less studied although this permits to study transmission  
 116 problems. Let us emphasize that the surface eigenvalue problems do not model ei-  
 117 genvalues of thin structures like shells. They have been introduced to justify that the  
 118 asymptotic models derived by M. David, J.J. Marigo and C. Pideri in [20, 21] are well  
 119 posed in the sense that the problems are of Fredholm type (see [4] for the scalar case  
 120 and [5] for the elastic case in dimension two). This is why we also deal with these  
 121 problems in this paper, in order to be as complete as possible.

122 Motivated by structural optimization of multi-phase material, we consider, in a  
 123 second step, the eigenvalue problem for a mixture of two isotropic elastic materials.  
 124 We specify that we use the terminology *composite* to refer to this case. In addition to  
 125 considering such piecewise constant material properties in the interior of the domain,  
 126 the effect of a thin coating is also taken into account by allowing a Robin or Wentzell  
 127 boundary condition.

128 **1.3. Organization of the paper.** To conclude this introduction, the paper is  
 129 organized as follows. The main results of the paper are stated in Section 2. We present  
 130 first the result in the case of a single isotropic elastic material and, then in the case  
 131 of a mixture of two phases. The proofs are gathered in Section 3: we first provide the  
 132 derivatives of the elementary terms arising in Rayleigh quotient in Section 3.2, and  
 133 then give the proof of the main theorems in Sections 3.3 and 3.4. Finally, we recall  
 134 (classical) background and technical results in Appendix A.

135 **2. The results.**

**2.1. Notations.** We consider a bounded open subset  $\Omega$  of  $\mathbb{R}^d$  with a  $\mathcal{C}^{2,1}$  bound-  
 ary  $\partial\Omega$ . Firstly, at each point of  $\partial\Omega$ , we consider an orthonormal frame  $(\boldsymbol{\tau}, \mathbf{n})$  consist-  
 ing of a family of orthonormal tangential vectors, denoted by  $\boldsymbol{\tau}$ , and the unit normal  
 vector, denoted by  $\mathbf{n}$ . Then the tangential projection is given by

$$\Pi_d := \mathbf{I}_d - \mathbf{n} \otimes \mathbf{n}$$

and, in the local frame, has the representation

$$\Pi_d = \begin{pmatrix} \mathbf{I}_{d-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\mathbf{I}_d$  and  $\mathbf{I}_{d-1}$  are respectively the identity matrices of size  $d \times d$  and  $(d-1) \times (d-1)$ .  
 More generally, any  $d \times d$  matrix  $\mathcal{M}$  has the following representation in the frame  $(\boldsymbol{\tau}, \mathbf{n})$ :

$$\begin{pmatrix} \mathcal{M}_{\boldsymbol{\tau}\boldsymbol{\tau}} & \mathcal{M}_{\boldsymbol{\tau}\mathbf{n}} \\ \mathcal{M}_{\mathbf{n}\boldsymbol{\tau}} & \mathcal{M}_{\mathbf{n}\mathbf{n}} \end{pmatrix},$$

136 with the components  $\mathcal{M}_{\boldsymbol{\tau}\boldsymbol{\tau}} := \Pi_d \mathcal{M} \Pi_d$ ,  $\mathcal{M}_{\boldsymbol{\tau}\mathbf{n}} := \Pi_d \mathcal{M} (\mathbf{I}_d - \Pi_d)$ ,  $\mathcal{M}_{\mathbf{n}\boldsymbol{\tau}} := (\mathbf{I}_d -$   
 137  $\Pi_d) \mathcal{M} \Pi_d$  and  $\mathcal{M}_{\mathbf{n}\mathbf{n}} := (\mathbf{I}_d - \Pi_d) \mathcal{M} (\mathbf{I}_d - \Pi_d)$ .

Secondly, in the whole paper, we use

$$C^{\text{sym}} := \frac{1}{2} (C + {}^t C)$$

to denote the symmetric part of a square matrix  $C$ . For any vector field  $\mathbf{u} =$   
 $(u_i)_{i=1, \dots, d} \in \mathbf{H}^1(\Omega)$ , the *strain tensor*

$$e(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) = (\nabla \mathbf{u})^{\text{sym}}$$

138 is the symmetric part of the Jacobian matrix  $\nabla \mathbf{u}$  whose rows are  ${}^t \nabla u_i$  for  $i = 1, \dots, d$ .

139 We also introduce, for a scalar function  $\phi \in \mathbf{H}^1(\partial\Omega)$ , the *tangential gradient*

140 
$$\nabla_\Gamma \phi := \Pi_d \nabla \phi,$$

141 and, for all vectorial functions  $\boldsymbol{\psi} \in \mathbf{H}^1(\partial\Omega)$ , the *tangential strain*

142 
$$e_\Gamma(\boldsymbol{\psi}) := \frac{1}{2} (\nabla_\Gamma \boldsymbol{\psi} + {}^t \nabla_\Gamma \boldsymbol{\psi}) = (\nabla_\Gamma \boldsymbol{\psi})^{\text{sym}},$$

where the rows of  $\nabla_\Gamma \boldsymbol{\psi}$  are the tangential gradients of the components  $\psi_i$ ,  $i = 1, \dots, d$ .  
 Notice that we have  $\nabla_\Gamma \boldsymbol{\psi} = \nabla \boldsymbol{\psi} \Pi_d$  since

$$\nabla_\Gamma \boldsymbol{\psi} = \begin{pmatrix} {}^t \nabla_\Gamma \psi_1 \\ \vdots \\ {}^t \nabla_\Gamma \psi_d \end{pmatrix} = \begin{pmatrix} {}^t \nabla \psi_1 \Pi_d \\ \vdots \\ {}^t \nabla \psi_d \Pi_d \end{pmatrix} = \nabla \boldsymbol{\psi} \Pi_d,$$

and thus

$$e_{\Gamma}(\boldsymbol{\psi}) = \Pi_d e(\boldsymbol{\psi}) \Pi_d.$$

Therefore, in the local frame, it is of the form  $\begin{pmatrix} e_{\tau\tau} & 0 \\ 0 & 0 \end{pmatrix}$ . Then, we denote by

$$\operatorname{div}_{\Gamma} \boldsymbol{\psi} := \operatorname{Tr}(e_{\Gamma}(\boldsymbol{\psi}))$$

the *surface or tangential divergence*. The tangential divergence of a matrix field  $C$  will be the vector field obtained by taking the tangential divergence of the rows of  $C$ , that is, for all  $i = 1, \dots, d$ ,

$$(\operatorname{div}_{\Gamma} C)_i := \operatorname{div}_{\Gamma} C_i.$$

143 *Remark 2.1.* We should keep in mind the following basic differences with respect  
 144 to the scalar case. On the one hand, in the scalar case, we have  $\nabla_{\Gamma} \psi = \Pi_d \nabla \psi$ ,  
 145 whereas, on the other hand, in the case of elasticity, we have  $\nabla_{\Gamma} \boldsymbol{\psi} = \nabla \boldsymbol{\psi} \Pi_d$ . Also,  
 146 in this case, the tangential strain  $e_{\Gamma}(\boldsymbol{\psi})$  is obtained by reducing the strain  $e(\boldsymbol{\psi})$  to  
 147 the tangent space by multiplying by the projection  $\Pi_d$  on either side. This leads to  
 148 substantial differences in the formulae for shape derivatives in the scalar case and in  
 149 the case of elasticity.

Then we introduce the *signed distance* to the boundary  $\partial\Omega$  defined by

$$b(x) := \begin{cases} d(x, \partial\Omega), & \text{if } x \in \Omega, \\ -d(x, \partial\Omega), & \text{if } x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

and the *mean curvature* at any point on  $\partial\Omega$ , defined by

$$H := \operatorname{div}_{\Gamma} \mathbf{n}.$$

150 Finally, as mentioned in the introduction, given a  $\mathcal{C}^{2,1}$  vector field  $\mathbf{V}$  with compact  
 151 support in a neighborhood of  $\Omega$  and a (small) real number  $\delta > 0$ , we consider the one  
 152 parameter family of deformations

$$153 \quad (2.1) \quad \Psi_t := \mathbf{I} + t\mathbf{V},$$

154 for all  $t \in [0, \delta]$ , which are in fact diffeomorphisms if  $\delta$  is sufficiently small. Then we  
 155 define the *perturbed domain* by

$$156 \quad (2.2) \quad \Omega_t := \Psi_t(\Omega).$$

We also use the following notation for the normal component of the vector field  $\mathbf{V}$ :

$$V_{\mathbf{n}} := \mathbf{V} \cdot \mathbf{n}.$$

**2.2. Shape derivative for eigenvalue problems of linear elasticity-single phase isotropic materials.** We assume that  $\Omega$  is an elastic body and we consider an isotropic elastic medium with *Lamé coefficients*  $\mu > 0$  and  $\lambda > 0$ , and associated *elastic or Hooke tensor*  $A$  given by

$$A\xi := 2\mu\xi + \lambda\operatorname{Tr}(\xi)\mathbf{I}_d, \quad \text{for all symmetric matrices } \xi.$$

157 We also assume that the body  $\Omega$  is surrounded by a thin layer with an elasticity tensor  
 158 given by

$$159 \quad (2.3) \quad A_c \xi := 2\mu_c \xi + \lambda_c \operatorname{Tr}(\xi) \Pi_d,$$

160 where  $\mu_c > 0$  and  $\lambda_c > 0$  are some (modified) Lamé constants which correspond to  
 161 a coating (the thin layer). Then, given  $\alpha, \beta \geq 0$  two real numbers, we are interested  
 162 in the following kinds of eigenvalues problems: of *volume type*, where the spectral  
 163 parameter is in the domain,

$$164 \quad (2.4) \quad \begin{cases} -\operatorname{div}(Ae(\mathbf{u})) & = \Lambda_\Omega(\Omega)\mathbf{u} & \text{in } \Omega, \\ -\beta \operatorname{div}_\Gamma(A_c e_\Gamma(\mathbf{u})) + \alpha \mathbf{u} + Ae(\mathbf{u})\mathbf{n} & = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

165 and of *surface type*, where the spectral parameter is on the boundary,

$$166 \quad (2.5) \quad \begin{cases} -\operatorname{div}(Ae(\mathbf{u})) & = \mathbf{0} & \text{in } \Omega, \\ -\beta \operatorname{div}_\Gamma(A_c e_\Gamma(\mathbf{u})) + \alpha \mathbf{u} + Ae(\mathbf{u})\mathbf{n} & = \Lambda_{\partial\Omega}(\Omega)\mathbf{u} & \text{on } \partial\Omega. \end{cases}$$

167 For different regimes of the parameter, we have different kinds of eigenvalue prob-  
 168 lems. For the choice  $\beta = 0$  and  $\alpha = 0$  in (2.4), we obtain *Neumann* (pure traction)  
 169 eigenvalues. The *Dirichlet* (clamped) eigenvalue problem is obtained from (2.4) in  
 170 the limiting case  $\alpha \rightarrow +\infty$ . The *Robin* eigenvalue problem is obtained from (2.4) by  
 171 taking  $\beta = 0$ . If we take  $\beta = 0$  and  $\alpha = 0$  in (2.5), we obtain the *Steklov* eigenvalue  
 172 problem. Finally, for the choice  $\beta > 0$ , we have the *Wentzell* eigenvalue problem  
 173 (see [9] for the model and the derivation of the Wentzell boundary conditions in the  
 174 elasticity case).

175 These eigenvalues problems arise as minimization of the associated *Rayleigh quo-*  
 176 *tient* given by

$$178 \quad (2.6) \quad \Lambda_\Omega(\Omega) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} \left\{ \frac{1}{\int_\Omega |\mathbf{u}|^2 dx} \left( \int_\Omega Ae(\mathbf{u}) : e(\mathbf{u}) dx \right. \right. \\ \left. \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 d\zeta(x) + \beta \int_{\partial\Omega} A_c e_\Gamma \mathbf{u} : e_\Gamma \mathbf{u} d\zeta(x) \right) \right\},$$

181 and

$$183 \quad (2.7) \quad \Lambda_{\partial\Omega}(\Omega) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} \left\{ \frac{1}{\int_{\partial\Omega} |\mathbf{u}|^2 dx} \left( \int_\Omega Ae(\mathbf{u}) : e(\mathbf{u}) dx \right. \right. \\ \left. \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 d\zeta(x) + \beta \int_{\partial\Omega} A_c e_\Gamma \mathbf{u} : e_\Gamma \mathbf{u} d\zeta(x) \right) \right\}.$$

186 Notice that, in the various eigenvalue problems, an appropriate choice of a subspace  
 187 of  $\mathbf{H}^1(\Omega)$  has to be made for  $\mathcal{H}(\Omega)$ . For example, in the case of the first Dirichlet  
 188 eigenvalue, we may choose  $\beta = 0$  and  $\mathcal{H}(\Omega) = \mathbf{H}_0^1(\Omega)$  in (2.6). In the case of the  
 189 first non-trivial Neumann or Steklov eigenvalue, we may choose  $\alpha = 0$  and  $\beta = 0$   
 190 and take  $\mathcal{H}(\Omega)$  to be the quotient space of  $\mathbf{H}^1(\Omega)$  modulo the rigid transformations.  
 191 In the case of the Wentzell eigenvalue problem, we are in the situation where  $\beta > 0$   
 192 and we need to choose  $\mathcal{H}(\Omega)$  to be  $\{\mathbf{u} \in \mathbf{H}^1(\Omega); \mathbf{u}|_{\partial\Omega} \in \mathbf{H}^1(\partial\Omega)\}$  with the associated  
 193 norm  $\left( \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2 + \|\mathbf{u}\|_{\mathbf{H}^1(\partial\Omega)}^2 \right)^{1/2}$  (quotiented over the subspace of rigid transfor-  
 194 mations if  $\alpha = 0$ ).

195 We now state the results for these problems.

196 THEOREM 2.2. *Given a  $\mathcal{C}^{2,1}$  domain  $\Omega$  and  $\mathbf{V}$  a smooth vector field, the semi-*  
 197 *derivative  $\Lambda'_\Omega(\Omega; \mathbf{V})$  of  $\Lambda_\Omega(\Omega)$  in the direction of the vector field  $\mathbf{V}$  exists and is given*  
 198 *by*  
 199

$$\begin{aligned}
 200 \quad \Lambda'_\Omega(\Omega; \mathbf{V}) = \inf & \left\{ \int_{\partial\Omega} \left( A e(\mathbf{u}) : e(\mathbf{u}) - 4 A e(\mathbf{u}) \mathbf{n} \cdot \Pi_d e(\mathbf{u}) \mathbf{n} \right. \right. \\
 201 & \quad \left. \left. + \alpha \mathbf{u} \cdot (\mathbf{H} \mathbf{u} + 2 \partial_n \mathbf{u} - 4 \Pi_d e(\mathbf{u}) \mathbf{n}) \right. \right. \\
 202 & \quad \left. \left. + \beta \left( \mathbf{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d \right) \right. \right. \\
 203 & \quad \left. \left. + 2\beta \left( A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) - 2 A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n}) \right) - \Lambda_\Omega(\Omega) |\mathbf{u}|^2 \right) V_n \, d\zeta(x) \right\}. \\
 204
 \end{aligned}$$

205 *In the above, the inf is taken with respect to all functions  $\mathbf{u} \in \mathcal{H}(\Omega)$  for which the*  
 206 *value  $\Lambda_\Omega(\Omega)$  is attained in (2.6).*

207 THEOREM 2.3. *Given a  $\mathcal{C}^{2,1}$  domain  $\Omega$  and  $\mathbf{V}$  a smooth vector field, the semi-*  
 208 *derivative  $\Lambda'_{\partial\Omega}(\Omega; \mathbf{V})$  of  $\Lambda_{\partial\Omega}(\Omega)$  in the direction of the vector field  $\mathbf{V}$  exists and is*  
 209 *given by*  
 210

$$\begin{aligned}
 211 \quad \Lambda'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf & \left\{ \int_{\partial\Omega} \left( A e(\mathbf{u}) : e(\mathbf{u}) - 4 A e(\mathbf{u}) \mathbf{n} \cdot \Pi_d e(\mathbf{u}) \mathbf{n} \right. \right. \\
 212 & \quad \left. \left. + \alpha \mathbf{u} (\mathbf{H} \mathbf{u} + 2 \partial_n \mathbf{u} - 4 \Pi_d e(\mathbf{u}) \mathbf{n}) \right. \right. \\
 213 & \quad \left. \left. + \beta \left( \mathbf{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d \right) \right. \right. \\
 214 & \quad \left. \left. + 2\beta \left( A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) - 2 A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n}) \right) \right. \right. \\
 215 & \quad \left. \left. - \Lambda_{\partial\Omega}(\Omega) \mathbf{u} \cdot (\mathbf{H} \mathbf{u} + 2 \partial_n \mathbf{u} - 4 \Pi_d e(\mathbf{u}) \mathbf{n}) \right) V_n \, d\zeta(x) \right\}, \\
 216
 \end{aligned}$$

217 *where the inf is taken with respect to all functions  $\mathbf{u} \in \mathcal{H}(\Omega)$  for which the value*  
 218  *$\Lambda_{\partial\Omega}(\Omega)$  is attained in (2.7).*

219 Apart from the case of Dirichlet and Neumann boundary conditions for the volume  
 220 case stated in Theorem 2.2, the remaining others results are, completely new to our  
 221 best knowledge, new even in the case of a simple eigenvalue. We finally underline  
 222 that, even if it is not the same expression, the given formula given above coincides  
 223 with the known expression in the corresponding to Dirichlet and Neumann case: this  
 224 can be done with checked by direct a computation.

**2.3. Shape derivative for eigenvalue problems of linear elasticity - composite materials.** Consider now a subset  $\Omega_1$  of  $\Omega$  with a  $\mathcal{C}^{2,1}$  boundary and set  $\Omega_2 := \Omega \setminus \overline{\Omega_1}$ . We assume that there exists  $\rho > 0$  such that  $\|x - y\| \geq \rho$  for all  $x \in \Omega_1$  and  $y \in \partial\Omega$ . We consider two isotropic elastic materials, with elasticity tensors  $A_1 \neq A_2$  given by (for  $i = 1, 2$ )

$$A_i \xi := 2 \mu_i \xi + \lambda_i \text{Tr}(\xi) \text{I}_d,$$

with Lamé coefficients  $\mu_i > 0$  and  $\lambda_i > 0$ , which occupy respectively the domains  $\Omega_1$  and  $\Omega_2$  with respective densities  $\rho_1 > 0$  and  $\rho_2 > 0$  (with  $\rho_1 \neq \rho_2$ ). We set

$$\rho := \rho_1 \chi_{\Omega_1} + \rho_2 \chi_{\Omega_2} \quad \text{and} \quad A := A_1 \chi_{\Omega_1} + A_2 \chi_{\Omega_2}.$$



As previously,  $\mathbf{n}$  denotes the exterior unit normal to  $\partial\Omega$ . Moreover  $\Gamma$  stands for the interface between  $\Omega_1$  and  $\Omega_2$ , that is

$$\Gamma := \partial\Omega_1 \cap \partial\Omega_2 = \partial\Omega_1,$$

and, on  $\Gamma$ , the notation  $\mathbf{n}$  will represent the unit normal pointing outward from  $\Omega_1$ , that is

$$\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$$

(where  $\mathbf{n}_i$ ,  $i = 1, 2$ , represent the exterior unit normal to  $\partial\Omega_i$ ). We summarize the notations in Figure 1. We also use the notation  $[\cdot]$  in order to represent the jump on the interface  $\Gamma$ , that is, for a function  $u$  and a point  $x \in \Gamma$ ,

$$[u](x) := \lim_{\varepsilon \rightarrow 0^+} (u(x - \varepsilon\mathbf{n}(x)) - u(x + \varepsilon\mathbf{n}(x))) = u_1 - u_2.$$

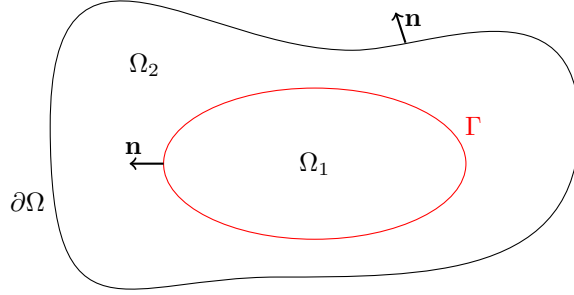


FIG. 1. Notations

225 We consider the eigenvalue problem of *volume type*

$$226 \quad (2.8) \quad \begin{cases} -\operatorname{div}(A(x)e(\mathbf{u})) = \mathfrak{M}_\Omega(\Omega)\rho(x)\mathbf{u} & \text{in } \Omega, \\ -\beta\operatorname{div}_\Gamma(A_c e_\Gamma(\mathbf{u})) + \alpha\mathbf{u} + Ae(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

227 and of *surface type*

$$228 \quad (2.9) \quad \begin{cases} -\operatorname{div}(A(x)e(\mathbf{u})) = \mathbf{0} & \text{in } \Omega, \\ -\beta\operatorname{div}_\Gamma(A_c e_\Gamma(\mathbf{u})) + \alpha\mathbf{u} + Ae(\mathbf{u})\mathbf{n} = \mathfrak{M}_{\partial\Omega}(\Omega)\mathbf{u} & \text{on } \partial\Omega. \end{cases}$$

229 As previously, for different regimes of the parameters  $\alpha$  and  $\beta$ , we obtain different  
230 kinds of boundary conditions and the eigenvalues are associated to minimization of  
231 the Rayleigh quotients

$$232 \quad (2.10) \quad \mathfrak{M}_\Omega(\Omega) = \inf_{\mathbf{u} \in \mathfrak{H}(\Omega)} \left\{ \frac{1}{\int_\Omega \rho |\mathbf{u}|^2} \left( \int_\Omega A(x)e(\mathbf{u}) : e(\mathbf{u}) \, dx \right. \right. \\ 233 \quad \left. \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 \, d\zeta(x) + \beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) \, d\zeta(x) \right) \right\},$$

235

236 and

237

$$(2.11) \quad \mathfrak{M}_{\partial\Omega}(\Omega) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} \left\{ \frac{1}{\int_{\partial\Omega} |\mathbf{u}|^2 d\zeta(x)} \left( \int_{\Omega} A(x)e(\mathbf{u}) : e(\mathbf{u}) dx \right. \right. \\ \left. \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 d\zeta(x) + \beta \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) d\zeta(x) \right) \right\},$$

239

240

241 where  $\mathcal{H}(\Omega)$  is an appropriate subspace of  $\mathbf{H}^1(\Omega)$  as discussed above.

242

We now state the results for these problems.

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245

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THEOREM 2.4. *Let  $\Omega$  be a  $\mathcal{C}^{2,1}$  domain and  $\mathbf{V}$  a smooth vector field. Let  $\mathbf{u}$  be a normalized eigenfunction corresponding to  $\mathfrak{M}_{\Omega}(\Omega)$ . Then the semi-derivative  $\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})$  of  $\mathfrak{M}_{\Omega}(\Omega)$  in the direction of the vector field  $\mathbf{V}$  exists and is given by*

247

$$\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) = \inf \left\{ \int_{\partial\Omega_1} \left( [Ae(\mathbf{u}) : e(\mathbf{u})] - 2Ae(\mathbf{u})\mathbf{n} \cdot [\partial_n \mathbf{u}] - \mathfrak{M}_{\Omega}(\Omega)[\rho]|\mathbf{u}|^2 \right) V_n d\zeta(x) \right.$$

248

$$+ \int_{\partial\Omega} \left( Ae(\mathbf{u}) : e(\mathbf{u}) - 4Ae(\mathbf{u})\mathbf{n} \cdot \Pi_d e(\mathbf{u})\mathbf{n} + \alpha \mathbf{u} \cdot (\mathbf{H}\mathbf{u} + 2\partial_n \mathbf{u} - 4\Pi_d e(\mathbf{u})\mathbf{n}) \right.$$

249

$$+ \beta (\mathbf{H}A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d)$$

250

$$\left. + 2\beta \left( A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) - 2A_c e_{\Gamma}(\mathbf{u}) : \nabla_{\Gamma} (\Pi_d e(\mathbf{u})\mathbf{n}) \right) - \mathfrak{M}_{\Omega}(\Omega)\rho_2 |\mathbf{u}|^2 \right) V_n d\zeta(x) \Big\},$$

251

252

253

where the inf is taken over all functions  $\mathbf{u} \in \mathcal{H}(\Omega)$  for which the value  $\mathfrak{M}_{\Omega}(\Omega)$  is attained in (2.10).

254

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THEOREM 2.5. *Let  $\Omega$  be a  $\mathcal{C}^{2,1}$  domain and  $\mathbf{V}$  a smooth vector field. Let  $\mathbf{u}$  be a normalized eigenfunction corresponding to  $\mathfrak{M}_{\partial\Omega}$ . Then the semi-derivative  $\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V})$  of  $\mathfrak{M}_{\partial\Omega}(\Omega)$  in the direction of the vector field  $\mathbf{V}$  exists and is given by*

258

$$\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf \left\{ \int_{\partial\Omega_1} \left( [Ae(\mathbf{u}) : e(\mathbf{u})] - 2Ae(\mathbf{u})\mathbf{n} \cdot [\partial_n \mathbf{u}] \right) V_n d\zeta(x) \right.$$

259

$$+ \int_{\partial\Omega} \left( Ae(\mathbf{u}) : e(\mathbf{u}) - 4Ae(\mathbf{u})\mathbf{n} \cdot \Pi_d e(\mathbf{u})\mathbf{n} + \alpha \mathbf{u} \cdot (\mathbf{H}\mathbf{u} + 2\partial_n \mathbf{u} - 4\Pi_d e(\mathbf{u})\mathbf{n}) \right.$$

260

$$+ \beta (\mathbf{H}A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d)$$

261

$$+ 2\beta \left( A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) - 2A_c e_{\Gamma}(\mathbf{u}) : \nabla_{\Gamma} (\Pi_d e(\mathbf{u})\mathbf{n}) \right)$$

262

$$\left. - \mathfrak{M}_{\partial\Omega}(\Omega)\rho_2 \mathbf{u} \cdot (\mathbf{H}\mathbf{u} + 2\partial_n \mathbf{u} - 4\Pi_d e(\mathbf{u})\mathbf{n}) \right) V_n d\zeta(x) \Big\},$$

263

264

265

where the inf is taken over all  $\mathbf{u} \in \mathcal{H}(\Omega)$  for which the value  $\mathfrak{M}_{\partial\Omega}(\Omega)$  is attained in (2.11).

266

267

Obviously, Theorem 2.2 (respectively Theorem 2.3) can be obtained as a particular case of Theorem 2.4 (respectively Theorem 2.5) by letting  $A_1 = A_2 = A$

268 and  $\rho_1 = \rho_2 = 1$ . Even so, we present the proofs of Theorem 2.2 (respectively The-  
 269 orem 2.3) since the main ideas can be illustrated more clearly in these particular  
 270 cases.

271 *Remark 2.6.* We can notice that the formulae of the above Theorems 2.2, 2.3, 2.4  
 272 and 2.5 are a little bit different and more complicated than the scalar case exposed  
 273 in [8]. Indeed, in the scalar case, some simplifications occur, which is not the case in  
 274 the elasticity case: this is linked with the differences underlined in Remark 2.1 (see  
 275 also Remark 3.4 below).

276 **3. Proofs.** The shape derivative results stated in the previous section will be  
 277 established in this section in the framework of Theorem 1.1 by following a general  
 278 strategy which we employed in the scalar problems (see [8]) and is recalled below for  
 279 the benefit of the reader.

280 **3.1. General strategy.** The first step is to reformulate the eigenvalue problem  
 281 for the perturbed domain  $\Omega_t$ , which is obtained by the minimization of a Rayleigh  
 282 quotient, as a minimization problem for a functional  $G(t, \cdot)$  in a space  $\mathcal{H}(\Omega)$  which is  
 283 independent of the parameter  $t$ .

284 The next step consists in verifying that the assumptions of Theorem 1.1 are  
 285 satisfied. For verifying the hypothesis (H3), in the class of eigenvalue problems, we  
 286 usually need to show the  $\Gamma$ -convergence (see Appendix A for some reminders on this  
 287 notion) of  $G(t, \cdot)$  to  $G(0, \cdot)$  as  $t \rightarrow 0^+$  in the weak topology of  $\mathcal{H}(\Omega)$  and later the  
 288 strong convergence of a sequence of minimizers.

289 Then, Theorem 1.1 allows us to immediately calculate the shape derivative by  
 290 evaluating  $\inf_{\mathbf{u} \in \mathbf{X}(0)} \partial_t G(0, \mathbf{u})$  where  $\mathbf{X}(0)$  is, generally, an eigenspace for the problem  
 291 on  $\Omega$ . In the case of a simple eigenfunction, it is enough to evaluate at a normalized  
 292 eigenfunction. An initial expression for  $\partial_t G(0, \mathbf{u})$  is obtained by using the propositions  
 293 given in the following subsection and this gives an integral over the domain  $\Omega$ .

294 As a last step, we transform and simplify the initial calculation of  $\partial_t G(0, \mathbf{u})$ , to get  
 295 a boundary expression for  $\partial_t G(0, \mathbf{u})$ . This can be usually achieved by choosing  $-\nabla \mathbf{u} \mathbf{V}$   
 296 as a test function in the governing equation, provided that it has enough regularity.

297 **3.2. Preliminary computations.** Before computing the shape derivatives, we  
 298 first prove some preliminary results. We compute the separate contributions of the  
 299 different terms of the Rayleigh quotient to the derivatives  $\partial_t G(0, \mathbf{u})$  in the various  
 300 problems. For this, we rely on the classical formulae in the calculation of shape  
 301 derivatives which are recalled in Lemma A.1 and Lemma A.2 in the appendix.

302 PROPOSITION 3.1. For  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ , we have

$$304 \quad (3.1) \quad \partial_t \left( \int_{\Omega_t} A e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) \, dx \right) \Big|_{t=0} \\
 305 \quad \quad \quad = \int_{\partial\Omega} A e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + 2 \int_{\Omega} A e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx, \\
 306 \\
 307$$

$$308 \quad (3.2) \quad \partial_t \left( \int_{\Omega_t} |(\mathbf{u} \circ \Psi_t^{-1})|^2 \, dx \right) \Big|_{t=0} = \int_{\partial\Omega} |\mathbf{u}|^2 V_n \, d\zeta(x) + 2 \int_{\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx$$

309 and  
 310

$$311 \quad (3.3) \quad \partial_t \left( \int_{\partial\Omega_t} |(\mathbf{u} \circ \Psi_t^{-1})|^2 \, dx \right) \Big|_{t=0} =$$

$$\int_{\partial\Omega} \left( \mathbb{H} |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x).$$

*Proof.* The above formulae are obtained by a straightforward application of the formulae for derivatives of domain and boundary integrals given in Lemma A.1 and Lemma A.2 in the appendix and the fact that  $\partial_t(\mathbf{u} \circ \Psi_t^{-1})|_{t=0} = -\nabla \mathbf{u} \mathbf{V}$  since  $\partial_t(\Psi_t^{-1})|_{t=0} = -\mathbf{V}$  (see [19, equation (5.7)]).  $\square$

PROPOSITION 3.2. For  $\mathbf{u} \in \mathbf{H}^2(\Omega)$ , we have

$$\begin{aligned} (3.4) \quad & \partial_t \left( \int_{\partial\Omega_t} A_c e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1}) : e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1}) \, dx \right) \Big|_{t=0} \\ &= \int_{\partial\Omega} \left( \mathbb{H} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) + 2A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) \right. \\ &\quad \left. - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\ &\quad + 2 \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : \left( e_{\Gamma}(-\nabla \mathbf{u} \mathbf{V}) + \Pi_d e(\mathbf{u})(\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) \right. \\ &\quad \left. + (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x). \end{aligned}$$

*Proof.* By applying the classical derivation formula recalled in Lemma A.2, we get

$$\begin{aligned} & \partial_t \left( \int_{\partial\Omega_t} A_c e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1}) : e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1}) \, dx \right) \Big|_{t=0} \\ &= \int_{\partial\Omega} \left( \mathbb{H} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) + \partial_n (A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u})) \right) V_n \, d\zeta(x) \\ &\quad + 2 \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : \partial_t (e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1})) \Big|_{t=0} \, d\zeta(x). \end{aligned}$$

We conclude using the fact (see respectively Lemma A.3 and Lemma A.4)

$$\partial_n (A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u})) = 2A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d,$$

and

$$\begin{aligned} & \partial_t (e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1})) \Big|_{t=0} = e_{\Gamma}(-\nabla \mathbf{u} \mathbf{V}) \\ & \quad + \Pi_d e(\mathbf{u})(\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) + (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d. \quad \square \end{aligned}$$

We also need the following proposition concerning the case of eigenvalue problems for composites. Let  $\Omega_1$  and  $\Omega_2$  be a subdivision of  $\Omega$  as presented in Section 2.3 with the corresponding notations for normal vectors and for the jumps of functions. Then we define following perturbed elasticity tensor and density

$$A_t := A_1 \chi_{\Omega_{1,t}} + A_2 \chi_{\Omega_{2,t}} \quad \text{and} \quad \rho_t := \rho_1 \chi_{\Omega_{1,t}} + \rho_2 \chi_{\Omega_{2,t}}$$

with

$$\Omega_{1,t} := \Psi_t(\Omega_1) \quad \text{and} \quad \Omega_{2,t} := \Psi_t(\Omega_2).$$

340 PROPOSITION 3.3. For  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ , we have

341

$$\begin{aligned}
342 \quad (3.5) \quad & \partial_t \left( \int_{\Omega_t} A_t e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) \, dx \right) \Big|_{t=0} \\
343 \quad &= \int_{\Gamma} [Ae(\mathbf{u}) : e(\mathbf{u})] V_n \, d\zeta(x) + 2 \int_{\Omega_1} A_1 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx \\
344 \quad &+ 2 \int_{\Omega_2} A_2 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) \\
345 \quad &
\end{aligned}$$

346 and

347

$$\begin{aligned}
348 \quad (3.6) \quad & \partial_t \left( \int_{\Omega_t} \rho_t |(\mathbf{u} \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} = \int_{\Gamma} [\rho |\mathbf{u}|^2] V_n \, d\zeta(x) \\
349 \quad &+ 2 \int_{\Omega_1} \rho_1 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx + 2 \int_{\Omega_2} \rho_2 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\partial\Omega} \rho_2 |\mathbf{u}|^2 V_n \, d\zeta(x). \\
350 \quad &
\end{aligned}$$

351 *Proof.* The above formulae are obtained by an application of Lemma A.1 to each  
352 of the terms on the right hand side after writing

353

$$\begin{aligned}
354 \quad & \int_{\Omega_t} A_t e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) \, dx \\
355 \quad &= \int_{\Omega_{1,t}} A_1 e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) \, dx + \int_{\Omega_{2,t}} A_2 e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) \, dx \quad \square \\
356 \quad &
\end{aligned}$$

and

$$\int_{\Omega_t} \rho_t |(\mathbf{u} \circ \Psi_t^{-1})|^2 dx = \int_{\Omega_{1,t}} \rho_1 |(\mathbf{u} \circ \Psi_t^{-1})|^2 dx + \int_{\Omega_{2,t}} \rho_2 |(\mathbf{u} \circ \Psi_t^{-1})|^2 dx.$$

357

### 3.3. Semi-derivatives for single phase isotropic materials.

358

359 **3.3.1. Proof of Theorem 2.2.** The considered eigenvalue functional on the  
360 perturbed domain is

361

$$\begin{aligned}
361 \quad (3.7) \quad \Lambda_{\Omega}(\Omega_t) = \inf_{\mathbf{v} \in \mathcal{H}(\Omega_t)} & \left\{ \frac{1}{\int_{\Omega_t} |\mathbf{v}|^2 \, dx} \left( \int_{\Omega_t} Ae(\mathbf{v}) : e(\mathbf{v}) \, dx + \alpha \int_{\partial\Omega_t} |\mathbf{v}|^2 \, d\zeta(x) \right. \right. \\
362 \quad & \left. \left. + \beta \int_{\partial\Omega_t} A_c e_{\Gamma}(\mathbf{v}) : e_{\Gamma}(\mathbf{v}) \, d\zeta(x) \right) \right\}, \\
363 \quad &
\end{aligned}$$

where  $\mathcal{H}(\Omega_t)$  is a suitable subspace of  $\mathbf{H}^1(\Omega_t)$  as discussed in Section 2.2. Since the  
function space  $\mathcal{H}(\Omega_t)$  gets mapped to a function space  $\mathcal{H}(\Omega)$  which is independent  
of  $t$  under the isomorphism  $\mathbf{v} \mapsto \mathbf{v} \circ \Psi_t$ , the above functional can be obtained as a  
minimization problem over  $\mathcal{H}(\Omega)$  as follows

$$\Lambda_{\Omega}(\Omega_t) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} G_{\Omega}(t, \mathbf{u})$$

364

where the functional  $G_{\Omega}$  is defined by

365

$$\begin{aligned}
366 \quad (3.8) \quad G_\Omega(t, \mathbf{u}) &:= \frac{1}{\int_{\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 dx} \left( \int_{\Omega_t} A e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) dx \right. \\
367 \quad &+ \alpha \int_{\partial\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 d\zeta(x) + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) : e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) d\zeta(x) \Big). \\
368
\end{aligned}$$

369 EXISTENCE OF THE SEMI-DERIVATIVE. First, we check that the assumptions of  
370 Theorem 1.1 are satisfied for the above functional  $G_\Omega$ .

371 Let us start by Assumption (H1). The arguments to show that the set of mini-  
372 mizers of (3.7) is non-empty for each  $t$  is classical and is based on the direct method  
373 of calculus of variations. In fact, the functional is lower semi-continuous for the weak  
374 topology on  $\mathcal{H}(\Omega_t)$ , since the numerator is convex and continuous for the strong  
375 topology on  $\mathcal{H}(\Omega_t)$  (and therefore weakly lower semi-continuous), and since the de-  
376 nominator is continuous due to the compact inclusion of  $\mathbf{H}^1(\Omega_t)$  into  $\mathbf{L}^2(\Omega_t)$ . As  
377 concerns the coercivity of the functional for given  $t$ , it is enough to show that the  
378 numerator dominates square of the norm or a quotient norm on  $\mathcal{H}(\Omega_t)$ . In the case of  
379 Dirichlet eigenvalue problem, this can be obtained from the coercivity of the tensor  $A$   
380 and by the use of *Korn's inequality* (see, e.g., [1, Lemma 2.25] or [16, Theorem 3.1]).  
381 In the case of the first non-trivial Neumann eigenvalue problem, one uses the coerciv-  
382 ity of the tensor  $A$  and the generalized Korn's inequality, that is, Korn's inequality  
383 modulo rigid transformations. When  $\alpha > 0$  it is enough, once again, to use Korn's  
384 inequality without quotienting. The set  $\mathbf{X}(t)$ , defined in Theorem 1.1, of minimizers  
385 for  $G_\Omega(t, \cdot)$  is obtained by transporting the minimizers in (3.7) to  $\Omega$  by composition  
386 with  $\Psi_t$ . Therefore Assumption (H1) is satisfied.

Let us now check Assumption (H2). Since  $\nabla(\mathbf{u} \circ \Psi_t^{-1}) = (\nabla \mathbf{u} \, D\Psi_t^{-1}) \circ \Psi_t^{-1}$ , we have

$$e(\mathbf{u} \circ \Psi_t^{-1}) = (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1}.$$

and

$$e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) = (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1} (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t),$$

387 where  $\mathbf{n}_t$  the normal vector field on  $\partial\Omega_t$ . Therefore,

388

$$389 \quad G_\Omega(t, \mathbf{u})$$

$$\begin{aligned}
390 \quad &= \frac{1}{\int_{\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 dx} \left( \int_{\Omega_t} A (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1} : (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1} dx \right. \\
391 \quad &+ \alpha \int_{\partial\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 d\zeta(x) + \beta \int_{\partial\Omega_t} A_c (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1} (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) \\
392 \quad &\quad : (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} \circ \Psi_t^{-1} (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) d\zeta(x) \Big). \\
393
\end{aligned}$$

394 Then, by a change of variables, this can be written as

395

$$396 \quad (3.9) \quad G_\Omega(t, \mathbf{u}) = \frac{1}{\int_\Omega |\mathbf{u}|^2 j(t) dx} \left( \int_\Omega A (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} : (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} j(t) dx \right)$$

$$\begin{aligned}
& + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} A_c(\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \\
& \quad : (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \omega(t) d\zeta(x) \Big),
\end{aligned}$$

where

$$j(t) := \det(D\Psi_t(x)) \quad \text{and} \quad \omega(t) := \det(D\Psi_t(x)) \|\mathbf{D}\Psi_t^{-1}(x)\mathbf{n}(x)\|$$

are respectively the Jacobian and the surface Jacobian and where

$$\boldsymbol{\nu}_t := \mathbf{n}_t \circ \Psi_t.$$

Clearly, by the definition (2.1),  $\Psi_t$  depends smoothly on  $t$  and, for  $t$  small enough,  $\Psi_t$  is a diffeomorphism by which we have that  $j(t)$  and  $\omega(t)$  are smooth functions of  $t$ . Also, since  $\partial\Omega$  is smooth, it follows that  $\Omega_t$  has the same smoothness of  $\Omega$  and therefore,  $\mathbf{n}_t$  is differentiable with respect to  $t$  for  $t$  small enough (see, e.g., [19, Proposition 5.4.14]). Therefore, we are able to conclude from the previous expression (3.9) that  $G_\Omega(\cdot, \mathbf{u})$  is derivable for all  $t$  small enough, for all  $\mathbf{u} \in \mathcal{H}(\Omega)$ , and this gives the hypothesis (H2) of Theorem 1.1.

Before proving Assumption (H3), let us focus briefly on Assumption (H4). The derivative  $\partial_t G_\Omega(\cdot, \mathbf{u})$  may be obtained by deriving under the integral sign in the previous equation and since all the integrands are  $\mathcal{C}^1$  functions of  $t$ , it follows that  $\partial_t G_\Omega(\cdot, \mathbf{u})$  is also continuous with respect to  $t$ , for  $t$  small enough. This gives Assumption (H4).

We now proceed to show that the hypothesis (H3) holds for the strong topology on  $\mathcal{H}(\Omega)$ . This will be achieved through the following steps. First, we show that  $G_\Omega(t, \cdot)$  converges, in the sense of  $\Gamma$ -limit, to  $G_\Omega(0, \cdot)$  as  $t \rightarrow 0^+$ , in the weak topology on  $\mathcal{H}(\Omega)$  (see Definition A.6 and Proposition A.7 in the Appendix for some reminders on this notion; also refer to [11]).

(i) Consider a sequence  $\{\mathbf{u}^t\}$  which converges weakly to a  $\mathbf{u}$  in  $\mathcal{H}(\Omega)$ . We obtain the estimate

$$G_\Omega(t, \mathbf{u}^t) = G_\Omega(0, \mathbf{u}^t) + (G_\Omega(t, \mathbf{u}^t) - G_\Omega(0, \mathbf{u}^t)) \geq G_\Omega(0, \mathbf{u}^t) + O(t).$$

Indeed, since any weakly convergent sequence  $\{\mathbf{u}^t\}$  is bounded in  $\mathcal{H}(\Omega)$  and the coefficients in both the numerator and denominator of  $G_\Omega$  given by (3.9) are continuous in  $t$ , we obtain that  $G_\Omega(t, \mathbf{u}^t) - G_\Omega(0, \mathbf{u}^t)$  is  $O(t)$  (that is, goes to 0 as  $t \rightarrow 0^+$ ). Then, to conclude the  $\Gamma$ -liminf inequality of Definition A.6, it is enough to use the already observed fact that  $G_\Omega(0, \cdot)$  is lower semi-continuous for the weak topology on  $\mathcal{H}(\Omega)$ .

(ii) The  $\Gamma$ -limsup inequality of Definition A.6 is obtained by taking the constant sequence  $\mathbf{u}$ , for any given  $\mathbf{u} \in \mathcal{H}(\Omega)$ , and observing as previously that  $G_\Omega(t, \mathbf{u}) \rightarrow G_\Omega(0, \mathbf{u})$  as  $t \rightarrow 0^+$ .

Having obtained the  $\Gamma$ -convergence of  $G_\Omega(t, \cdot)$ , Proposition A.7 ensures that  $\Lambda_\Omega(\Omega_t) \rightarrow \Lambda_\Omega(\Omega)$  as  $t \rightarrow 0^+$  since the minimum of  $G_\Omega(t, \cdot)$  converges to the minimum of  $G_\Omega(0, \cdot)$ . Moreover, the 0-homogeneity of the Rayleigh quotients  $G_\Omega(t, \cdot)$  means that, for each  $t$ , it is enough to consider a minimizer  $\mathbf{u}^t$  for which the denominator is 1. Under this normalization, we have the equi-coercivity of  $G_\Omega(t, \cdot)$  using the coercivity of the tensor  $A$  and Korn's inequality by the same arguments used during the verification of the hypothesis (H1): there exists constant a positive constant  $C$  such that

$$C \|\mathbf{u}^t\|_{\mathcal{H}(\Omega)}^2 \leq G_\Omega(t, \mathbf{u}^t), \quad \text{for all } t.$$

This implies, by Proposition A.7, that  $\{\mathbf{u}^t\}$  converges weakly in  $\mathcal{H}(\Omega)$  to a minimizer  $\mathbf{u}$  of  $G_\Omega(0, \cdot)$ . To conclude this part, we will prove the strong convergence of  $\{\mathbf{u}^t\}$  to  $\mathbf{u}$  in  $\mathcal{H}(\Omega)$ . The equi-coercivity can be used once again to give us the following inequality:

$$C\|\mathbf{u}^t - \mathbf{u}\|_{\mathcal{H}(\Omega)}^2 \leq G_\Omega(t, \mathbf{u}^t - \mathbf{u}).$$

425 It remains to prove that  $G_\Omega(t, \mathbf{u}^t - \mathbf{u}) \rightarrow 0$  when  $t \rightarrow 0$ . Expanding the quadratic  
426 function  $G_\Omega(t, \cdot)$  on the right hand side we get

427

$$428 \quad G_\Omega(t, \mathbf{u}^t - \mathbf{u}) = G_\Omega(t, \mathbf{u}^t) + G_\Omega(t, \mathbf{u})$$

$$429 \quad - 2 \left( \int_\Omega A (\nabla \mathbf{u}^t \mathbf{D} \Psi_t^{-1})^{\text{sym}} : (\nabla \mathbf{u} \mathbf{D} \Psi_t^{-1})^{\text{sym}} j(t) dx + \alpha \int_{\partial\Omega} \mathbf{u}^t : \mathbf{u} \omega(t) d\zeta(x) \right.$$

$$430 \quad \left. + \beta \int_{\partial\Omega} A_c (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u}^t \mathbf{D} \Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \right.$$

$$431 \quad \left. : (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} \mathbf{D} \Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \omega(t) d\zeta(x) \right).$$

432

433 Then we use the uniform convergence of the coefficients, the weak convergence of  $\{\mathbf{u}^t\}$   
434 to  $\mathbf{u}$  and the convergence of  $\Lambda_\Omega(\Omega_t)$  to  $\Lambda_\Omega(\Omega)$  to obtain that

$$435 \quad G_\Omega(t, \mathbf{u}^t - \mathbf{u}) \longrightarrow \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega)$$

$$436 \quad - 2 \left\{ \int_\Omega A e(\mathbf{u}) : e(\mathbf{u}) dx + \alpha \int_{\partial\Omega} 2|\mathbf{u}|^2 d\zeta(x) + \beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) d\zeta(x) \right\}$$

$$437 \quad = \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega) - 2\Lambda_\Omega(\Omega) = 0.$$

438 Hence  $\{\mathbf{u}^t\}$  converges strongly to  $\mathbf{u}$  in  $\mathcal{H}(\Omega)$  and, since we have seen that  $\partial_t G_\Omega(\cdot, \mathbf{u})$   
439 is continuous with respect to  $t$ , this proves hypothesis (H3).

440 The existence of the semi-derivative  $\Lambda'_\Omega(\Omega; \mathbf{V})$  follows from Theorem 1.1 since we  
441 have proved above that the four assumptions of the theorem are satisfied for  $G_\Omega$ .

COMPUTATION OF THE DIRECTIONAL SHAPE DERIVATIVES. We want to obtain a suitable expression for  $\partial_t G_\Omega(0, \mathbf{u})$  whenever  $\mathbf{u}$  is a normalized eigenfunction for  $\Lambda_\Omega(\Omega)$  since, by the theorem,

$$\Lambda'_\Omega(\Omega; \mathbf{V}) = \inf\{\partial_t G_\Omega(0, \mathbf{u}); \Lambda_\Omega(\Omega) \text{ is attained at } \mathbf{u}\}.$$

442 First, using the expressions (3.1)-(3.4) evaluated at  $t = 0$ , we get

443

$$444 \quad \partial_t G(0, \mathbf{u}) = \int_{\partial\Omega} A e(\mathbf{u}) : e(\mathbf{u}) V_n d\zeta(x) + 2 \int_\Omega A e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) dx$$

$$445 \quad + \alpha \left( \int_{\partial\Omega} \left( \mathbb{H} |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) d\zeta(x) \right)$$

$$446 \quad + \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2 A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right.$$

$$447 \quad \left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} \mathbf{D}^2 b + \mathbf{D}^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n d\zeta(x)$$

$$448 \quad + 2\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) + \Pi_d e(\mathbf{u}) (\mathbf{n} \otimes \nabla_\Gamma V_n + \nabla_\Gamma V_n \otimes \mathbf{n}) \right)$$



$$\begin{aligned}
& + (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \Big) d\zeta(x) \\
& - \Lambda_{\Omega}(\Omega) \left( \int_{\partial\Omega} |\mathbf{u}|^2 V_n d\zeta(x) + 2 \int_{\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) dx \right).
\end{aligned}$$

Using  $-\nabla \mathbf{u} \mathbf{V}$  as a test function in (2.4), we observe that

$$\begin{aligned}
& \int_{\Omega} A e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) + \alpha \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) + \beta \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(-\nabla \mathbf{u} \mathbf{V}) \\
& = \Lambda_{\Omega}(\Omega) \int_{\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}).
\end{aligned}$$

Notice that the function  $-\nabla \mathbf{u} \mathbf{V}$  belongs to  $\mathbf{H}^1(\Omega)$ . Indeed  $\mathbf{V}$  is assumed to be smooth and the boundary  $\partial\Omega$  has a  $\mathcal{C}^{2,1}$  regularity and then  $\mathbf{u} \in \mathbf{H}^2(\Omega)$  by usual a priori estimates (see [5, Theorem 1.1 and its proof]. Also, observe that for symmetric matrix  $B$  and any square matrix  $C$ , we have  $B:C = B: {}^t C$  and choose  $B = A_c e_{\Gamma}(\mathbf{u})$  along with  $C = (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d$ . We use these to get that

$$\begin{aligned}
(3.10) \quad \partial_t G(0, \mathbf{u}) &= \int_{\partial\Omega} A e(\mathbf{u}) : e(\mathbf{u}) V_n d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n d\zeta(x) \\
&+ \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) + 2A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) \right. \\
&- \left. A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d \right) V_n d\zeta(x) \\
&+ 4\beta \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : \left( (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) d\zeta(x) - \Lambda_{\Omega}(\Omega) \int_{\partial\Omega} |\mathbf{u}|^2 V_n d\zeta(x).
\end{aligned}$$

*Remark 3.4.* Notice that we have a factor 4 as compared to 2 in the corresponding term in the scalar case owing to the fact that the tangential strain is obtained by multiplying the strain by  $\Pi_d$  on either side and while deriving with respect to  $t$ , we obtain an additional term as observed in Remark A.5 (see also Remark 2.1).

We also observe that

$$(3.11) \quad A_c e_{\Gamma}(\mathbf{u}) : (\mathbf{n} \otimes \nabla_{\Gamma} V_n) e(\mathbf{u}) \Pi_d = 0.$$

Indeed, after setting,  $\mathcal{A} := A_c e_{\Gamma}(\mathbf{u})$ ,  $\mathcal{B} := \mathbf{n} \otimes \nabla_{\Gamma} V_n$ ,  $\mathcal{C} := e(\mathbf{u})$  and  $\mathcal{D} := \Pi_d$ , and writing these in the local frame, we obtain

$$\begin{aligned}
& A_c e_{\Gamma}(\mathbf{u}) : \mathbf{n} \otimes \nabla_{\Gamma} V_n e(\mathbf{u}) \Pi_d = \mathcal{A} : \mathcal{B} \mathcal{C} \mathcal{D} \\
& = \begin{pmatrix} \mathcal{A}_{\tau\tau} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ \mathcal{B}_{n\tau} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C}_{\tau\tau} & \mathcal{C}_{\tau n} \\ \mathcal{C}_{n\tau} & \mathcal{C}_{nn} \end{pmatrix} \begin{pmatrix} \mathbb{I}_{d-1} & 0 \\ 0 & 0 \end{pmatrix} = 0.
\end{aligned}$$

Next we remark that  $C : ((\mathbf{v} \otimes \mathbf{w}) B) = \mathbf{v} \cdot (C {}^t B \mathbf{w})$  (for any matrices  $C$  and  $B$  and any vectors  $\mathbf{v}$  and  $\mathbf{w}$ ) and apply this to  $A_c e_{\Gamma}(\mathbf{u}) : \nabla_{\Gamma} V_n \otimes \mathbf{n} e(\mathbf{u}) \Pi_d$ . We also remark that  $A_c e_{\Gamma}(\mathbf{u}) \Pi_d e(\mathbf{u}) \mathbf{n}$  is a tangential vector since  $\Pi_d$  commutes with  $A_c e_{\Gamma}(\mathbf{u})$  and so we can apply the tangential Stokes formula without any curvature term (see, e.g., [14, Equation (5.27)]) and obtain that

$$(3.12) \quad \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : (\nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d d\zeta(x) = \int_{\partial\Omega} \nabla_{\Gamma} V_n \cdot (A_c e_{\Gamma}(\mathbf{u}) \Pi_d e(\mathbf{u}) \mathbf{n}) d\zeta(x)$$

$$\begin{aligned}
&= - \int_{\partial\Omega} \operatorname{div}_\Gamma (A_c e_\Gamma(\mathbf{u}) \Pi_d e(\mathbf{u}) \mathbf{n}) V_n \, d\zeta(x) \\
&= - \int_{\partial\Omega} (\operatorname{div}_\Gamma (A_c e_\Gamma(\mathbf{u})) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} + A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n})) V_n \, d\zeta(x).
\end{aligned}$$

489

490 Therefore, inserting (3.11) and (3.12) in (3.10), we get

491

$$\begin{aligned}
\partial_t G(0, \mathbf{u}) &= \int_{\partial\Omega} A e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
&+ \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\
&- A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \left. \right) V_n \, d\zeta(x) \\
&- 4\beta \int_{\partial\Omega} (\operatorname{div}_\Gamma (A_c e_\Gamma(\mathbf{u})) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} + A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n})) V_n \, d\zeta(x) \\
&- \Lambda_\Omega(\Omega) \int_{\partial\Omega} |\mathbf{u}|^2 V_n \, d\zeta(x).
\end{aligned}$$

497

498 Then using the boundary condition in (2.4), to replace the term  $\operatorname{div}_\Gamma (A_c e_\Gamma(\mathbf{u}))$ , we  
499 obtain

500

$$\begin{aligned}
\partial_t G(0, \mathbf{u}) &= \int_{\partial\Omega} A e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
&+ \beta \int_{\partial\Omega} (\mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d) V_n \, d\zeta(x) \\
&+ 2\beta \int_{\partial\Omega} (A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) - 2A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n})) V_n \, d\zeta(x) \\
&+ 4 \int_{\partial\Omega} (-\alpha \mathbf{u} - A e(\mathbf{u}) \mathbf{n}) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} V_n \, d\zeta(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |\mathbf{u}|^2 V_n \, d\zeta(x).
\end{aligned}$$

505

506 This may be further rearranged to obtain the expression announced in Theorem 2.2.

507 **3.3.2. Proof of Theorem 2.3.** The eigenvalue functional over the perturbed  
508 domain reads

509

$$(3.13) \quad \Lambda_{\partial\Omega}(\Omega_t) = \inf_{\mathbf{v} \in \mathcal{H}(\Omega_t)} \left\{ \frac{1}{\int_{\partial\Omega_t} |\mathbf{v}|^2 \, dx} \left( \int_{\Omega_t} A e(\mathbf{v}) : e(\mathbf{v}) \, dx + \alpha \int_{\partial\Omega_t} |\mathbf{v}|^2 \, d\zeta(x) \right. \right. \\
\left. \left. + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{v}) : e_\Gamma(\mathbf{v}) \, d\zeta(x) \right) \right\},$$

511

512

and this may be reformulated over a function space  $\mathcal{H}(\Omega)$  which is independent of  $t$  using the isomorphism  $\mathbf{v} \mapsto \mathbf{v} \circ \Psi_t$  and setting  $\mathbf{u} = \mathbf{v} \circ \Psi_t$  as

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} G_{\partial\Omega}(t, \mathbf{u})$$

513 where  
514

$$515 \quad (3.14) \quad G_{\partial\Omega}(t, \mathbf{u}) := \frac{1}{\int_{\partial\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 dx} \left( \int_{\Omega_t} A e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) dx \right. \\ 516 \quad \left. + \alpha \int_{\partial\Omega_t} |\mathbf{u} \circ \Psi_t^{-1}|^2 d\zeta(x) + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) : e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) d\zeta(x) \right). \\ 517$$

518 EXISTENCE OF DIRECTIONAL SHAPE DERIVATIVES. We go over the main argu-  
519 ments needed for applying Theorem 1.1 to problem (2.7) to obtain the existence of  
520 the directional shape derivative.

521 Let us start by Assumption (H1). The arguments are exactly the same as in the previ-  
522 ous case concerning  $G_\Omega$ , except that which is needed for the continuity of the denom-  
523 inator. In this case, it is enough to use the compact inclusion of  $\mathbf{H}^1(\Omega_t)$  into  $\mathbf{L}^2(\partial\Omega_t)$   
524 (for which we refer, e.g., to [3]). Then, as in the previous case, the set  $X(t)$  of minimiz-  
525 ers for  $G_{\partial\Omega}(t, \cdot)$  is obtained by transporting the minimizers in (3.13) to the domain  $\Omega$   
526 by composition with  $\Psi_t$ . Therefore Assumption (H1) is satisfied.

527 Concerning Assumption (H2), we first get the following expression for  $G_{\partial\Omega}$   
528

$$(3.15) \\ 529 \quad G_{\partial\Omega}(t, \mathbf{u}) = \frac{1}{\int_{\partial\Omega} |\mathbf{u}|^2 \omega(t) d\zeta(x)} \left( \int_{\Omega} A (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} : (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} j(t) dx \right. \\ 530 \quad \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} A_c (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \right. \\ 531 \quad \left. : (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} \, D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \omega(t) d\zeta(x) \right). \\ 532$$

533 Due to the smooth dependence in  $t$  of the coefficients appearing in (3.15), we con-  
534 clude that  $G_{\partial\Omega}(\cdot, \mathbf{u})$  is derivable with respect to  $t$  for all  $\mathbf{u} \in \mathcal{H}(\Omega)$ , which gives  
535 Assumption (H2).

536 As in the previous case of  $G_\Omega$ , the derivative of the individual terms may be  
537 obtained by deriving under the integrals which lead to the fact that  $\partial_t G_{\partial\Omega}(\cdot, \mathbf{u})$  is  
538 also continuous with respect to  $t$ , for all  $\mathbf{u} \in \mathcal{H}(\Omega)$ , due to the  $\mathcal{C}^1$  nature of the  
539 coefficients. This gives Assumption (H4).

540 Finally we prove that assumption (H3) is also satisfied by showing, as in the case  
541 of  $G_\Omega$ , that  $G_{\partial\Omega}(t, \cdot)$  converges to  $G_{\partial\Omega}(0, \cdot)$  as  $t \rightarrow 0$  in the sense of  $\Gamma$ -limit in the  
542 weak topology on  $\mathcal{H}(\Omega)$  and that the minimizers converge in the strong topology.

543 Thus the existence of the semi-derivative  $\Lambda'_{\partial\Omega}(\Omega; \mathbf{V})$  follows from Theorem 1.1.

COMPUTATION OF DIRECTIONAL SHAPE DERIVATIVES. We only need to get a  
suitable expression for  $\partial_t G_{\partial\Omega}(0, \mathbf{u})$  whenever  $\mathbf{u}$  is a normalized eigenfunction for  
 $\Lambda_{\partial\Omega}(\Omega)$  since, by the theorem,

$$\Lambda'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf \{ \partial_t G_{\partial\Omega}(0, \mathbf{u}); \Lambda_{\partial\Omega}(\Omega) \text{ is attained at } \mathbf{u} \}.$$

544 Using the expressions (3.1)-(3.4) evaluated at  $t = 0$ , we get  
545

$$\begin{aligned}
546 \quad \partial_t G_{\partial\Omega}(0, \mathbf{u}) &= \int_{\partial\Omega} Ae(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + 2 \int_{\Omega} Ae(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx \\
547 \quad &+ \alpha \left( \int_{\partial\Omega} \left( \mathbb{H} |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x) \right) \\
548 \quad &+ \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) + 2 A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) \right. \\
549 \quad &\left. - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\
550 \quad &+ 2\beta \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : \left( e_{\Gamma}(-\nabla \mathbf{u} \mathbf{V}) + \Pi_d e(\mathbf{u}) (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) \right. \\
551 \quad &\left. + (\mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x) \\
552 \quad &- \Lambda_{\partial\Omega}(\Omega) \left( \int_{\partial\Omega} \left( \mathbb{H} |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x) \right).
\end{aligned}$$

553

554 Now, given  $u$  an eigenfunction in (2.5) whose  $L^2$  norm is 1, we use  $-\nabla \mathbf{u} \mathbf{V}$  as a test  
555 function in (2.5) since  $\mathbf{u} \in \mathbf{H}^2(\Omega)$  by usual a priori estimates (see [5, Theorem 1.1  
556 and its proof] and we observe that

557

$$\begin{aligned}
558 \quad &\int_{\Omega} Ae(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx + \alpha \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x) \\
559 \quad &+ \beta \int_{\partial\Omega} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x) = \Lambda_{\partial\Omega}(\Omega) \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx,
\end{aligned}$$

560

561 and then arguing as in the previous subsection while using the boundary condition  
562 in (2.5), we get

563

$$\begin{aligned}
564 \quad \partial_t G(0, \mathbf{u}) &= \int_{\partial\Omega} Ae(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
565 \quad &+ \beta \int_{\partial\Omega} (\mathbb{H} A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u}) - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d) V_n \, d\zeta(x) \\
566 \quad &+ 2\beta \int_{\partial\Omega} (A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) - 2A_c e_{\Gamma}(\mathbf{u}) : \nabla_{\Gamma} (\Pi_d e(\mathbf{u}) \mathbf{n})) V_n \, d\zeta(x) \\
567 \quad &+ 4 \int_{\partial\Omega} (-\alpha \mathbf{u} - Ae(\mathbf{u}) \mathbf{n}) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} V_n \, d\zeta(x) \\
568 \quad &- \Lambda_{\partial\Omega}(\Omega) \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u} - 4\mathbf{u} \cdot \Pi_d e(\mathbf{u}) \mathbf{n}) V_n \, d\zeta(x),
\end{aligned}$$

569

570 and by rearranging the terms we get the desired expression.

571

### 3.4. Shape derivative for eigenvalue problems for composite materials.

572

572 We recall that we use the following notation:  $A_t = A_1 \chi_{\Omega_{1,t}} + A_2 \chi_{\Omega_{2,t}}$  and  $\rho_t =$   
573  $\rho_1 \chi_{\Omega_{1,t}} + \rho_2 \chi_{\Omega_{2,t}}$ , with  $\Omega_t = \Psi_t(\Omega)$ ,  $\Omega_{1,t} = \Psi_t(\Omega_1)$  and  $\Omega_{2,t} = \Psi_t(\Omega_2)$ .

574

3.4.1. Proof of Theorem 2.4. The considered perturbed problem on  $\Omega_t$  reads

575

$$\begin{aligned}
576 \quad \mathfrak{M}_\Omega(\Omega_t) &= \inf_{\mathbf{v} \in \mathcal{H}(\Omega_t)} \left\{ \frac{1}{\int_{\Omega_t} \rho_t |\mathbf{v}|^2 dx} \left( \int_{\Omega_t} A_t(x) e(\mathbf{v}) : e(\mathbf{v}) dx + \alpha \int_{\partial\Omega_t} |\mathbf{v}|^2 d\zeta(x) \right. \right. \\
577 \quad &\quad \left. \left. + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{v}) : e_\Gamma(\mathbf{v}) d\zeta(x) \right) \right\}.
\end{aligned}$$

578

579 The above can be formulated as

$$580 \quad (3.16) \quad \mathfrak{M}_\Omega(\Omega_t) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} G_\Omega(t, \mathbf{u}),$$

581 with

$$\begin{aligned}
582 \quad G_\Omega(t, \mathbf{u}) &:= \frac{1}{\int_{\Omega_t} \rho_t |(\mathbf{u} \circ \Psi_t^{-1})|^2 dx} \left( \int_{\Omega_t} A_t(x) e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) dx \right. \\
583 \quad &\quad \left. + \alpha \int_{\partial\Omega_t} |(\mathbf{u} \circ \Psi_t^{-1})|^2 d\zeta(x) + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) : e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) d\zeta(x) \right).
\end{aligned}$$

585

586 We proceed as in the proofs of the previous theorems.

587 **EXISTENCE OF DIRECTIONAL SHAPE DERIVATIVES.** The existence of the semi-  
588 derivative  $\mathfrak{M}'_\Omega(\Omega; \mathbf{V})$  follows from Theorem 1.1 once the hypothesis of the theorem  
589 are verified.

590 The verification of the hypothesis (H1) is like in the previous subsections due to  
591 the coercivity of the tensor  $A_t$ .

592 The differentiability of  $G_\Omega(\cdot, \mathbf{u})$  with respect to  $t$  for any  $\mathbf{u} \in \mathcal{H}(\Omega)$  is seen once  
593 we use a change of variables to rewrite  $G_\Omega(t, \mathbf{u})$  as  
594

$$\begin{aligned}
595 \quad (3.17) \quad G_\Omega(t, \mathbf{u}) &= \frac{1}{\int_{\Omega} |\mathbf{u}|^2 j(t) dy} \left( \int_{\Omega} C_t(y) (\nabla \mathbf{u} D\Psi_t^{-1})^{\text{sym}} : (\nabla \mathbf{u} D\Psi_t^{-1})^{\text{sym}} j(t) dy \right. \\
596 \quad &\quad \left. + \alpha \int_{\partial\Omega} |\mathbf{u}|^2 \omega(t) d\zeta(y) + \beta \int_{\partial\Omega} A_c (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \right. \\
597 \quad &\quad \left. : (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) (\nabla \mathbf{u} D\Psi_t^{-1})^{\text{sym}} (\mathbf{I}_d - \boldsymbol{\nu}_t \otimes \boldsymbol{\nu}_t) \omega(t) d\zeta(y) \right),
\end{aligned}$$

598

while observing that

$$C_t(y) := A_t(\Psi_t(y)) = A_1 \chi_{\Omega_{1,t}}(\Psi_t(y)) + A_2 \chi_{\Omega_{2,t}}(\Psi_t(y)) = A_1 \chi_{\Omega_1}(y) + A_2 \chi_{\Omega_2}(y)$$

599 is independent of  $t$ . The differentiability with respect to  $t$ , that is hypothesis (H2),  
600 then follows due to the smooth dependence of the coefficients with respect to  $t$ , and  
601 also the hypothesis (H4) follows.

602 The hypothesis (H3) is proved by showing, similarly as in the subsection 2.2,  
603 that  $G_\Omega(t, \cdot)$  converges to  $G_\Omega(0, \cdot)$  as  $t \rightarrow 0$  in the sense of  $\Gamma$ -limit in the weak  
604 topology on  $\mathcal{H}(\Omega)$  and that the minimizers converge in the strong topology.

COMPUTATION OF DIRECTIONAL SHAPE DERIVATIVES. Thus, we now only need to get a suitable expression for  $\partial_t G_\Omega(0, \mathbf{u})$  given any normalized eigenfunction  $\mathbf{u}$  for  $\mathfrak{M}_\Omega(\Omega)$  since, by the theorem,

$$\mathfrak{M}'_\Omega(\Omega; \mathbf{V}) = \inf\{\partial_t G_\Omega(0, \mathbf{u}); \mathfrak{M}_\Omega(\Omega) \text{ is attained at } \mathbf{u}\}.$$

605 Using the calculated expressions in (3.5), (3.6), (3.3) and (3.4), we get  
606

$$\begin{aligned} 607 \quad \partial_t G_\Omega(0, \mathbf{u}) &= \int_\Gamma [Ae(\mathbf{u}) : e(\mathbf{u})] V_n \, d\zeta(x) + 2 \int_{\Omega_1} A_1 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx \\ 608 \quad &+ 2 \int_{\Omega_2} A_2 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) \\ 609 \quad &+ \alpha \left( \int_{\partial\Omega} \left( \mathbb{H} |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x) \right) \\ 610 \quad &+ \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2 A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\ 611 \quad &\left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\ 612 \quad &+ 2\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) + \Pi_d e(\mathbf{u}) (\mathbf{n} \otimes \nabla_\Gamma(V_n) + \nabla_\Gamma(V_n) \otimes \mathbf{n}) \right. \\ 613 \quad &\left. + (\mathbf{n} \otimes \nabla_\Gamma(V_n) + \nabla_\Gamma(V_n) \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x) \\ 614 \quad &- \mathfrak{M}_\Omega(\Omega) \left( \int_\Gamma [\rho |\mathbf{u}|^2] V_n \, d\zeta(x) + 2 \int_{\Omega_1} \rho_1 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx \right. \\ 615 \quad &\left. + 2 \int_{\Omega_2} \rho_2 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\partial\Omega} \rho_2 |\mathbf{u}|^2 V_n \, d\zeta(x) \right). \\ 616 \end{aligned}$$

617 Notice that the eigenmode  $\mathbf{u}$  does not belong to  $\mathbf{H}^2(\Omega)$  due to the jumps on the  
618 interface. Therefore the function  $-\nabla \mathbf{u} \mathbf{V}$  does not belong anymore to  $\mathbf{H}^1(\Omega)$  and  
619 hence cannot be used as test function directly. However the restriction of  $\mathbf{u}$  to each  $\Omega_i$ ,  
620 for  $i = 1, 2$ , belongs to  $\mathbf{H}^2(\Omega_i)$  thanks to regularity assumptions on both the outer  
621 boundary and the interface. Then, multiplying (2.8) by  $-\nabla \mathbf{u} \mathbf{V} \in \mathbf{H}^1(\Omega_i)$  in each  $\Omega_i$   
622 and integrating by part on  $\Omega_i$ , for  $i = 1, 2$ , we obtain that  
623

$$\begin{aligned} 624 \quad &\int_{\Omega_1} A_1 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\Omega_2} A_2 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx \\ 625 \quad &+ \int_{\partial\Omega_1} [Ae(\mathbf{u}) \mathbf{n} \cdot (-\nabla \mathbf{u} \mathbf{V})] \, d\zeta(x) + \alpha \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) + \beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) \\ 626 \quad &= \mathfrak{M}_\Omega(\Omega) \left( \int_{\Omega_1} \rho_1 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\Omega_2} \rho_2 \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, dx \right). \\ 627 \end{aligned}$$

628 Then, noticing that  $\nabla_\Gamma \mathbf{u}$  has a continuous trace on  $\partial\Omega_1$  as also  $Ae(\mathbf{u}) \mathbf{n}$  we obtain  
629

$$\begin{aligned} 630 \quad &-\int_{\partial\Omega_1} [Ae(\mathbf{u}) \mathbf{n} \cdot (-\nabla \mathbf{u} \mathbf{V})] \, d\zeta(x) = \int_{\partial\Omega_1} Ae(\mathbf{u}) \mathbf{n} \cdot [(\nabla \mathbf{u} \mathbf{V})] \, d\zeta(x) \\ 631 \quad &= \int_{\partial\Omega_1} Ae(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] V_n \, d\zeta(x), \end{aligned}$$

632

633 Using the above, we get

634

$$\begin{aligned}
635 \quad \partial_t G_\Omega(0, \mathbf{u}) &= \int_\Gamma [Ae(\mathbf{u}) : e(\mathbf{u})] V_n \, d\zeta(x) - 2 \int_{\partial\Omega_1} Ae(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
636 \quad &+ \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbf{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
637 \quad &+ \beta \int_{\partial\Omega} \left( \mathbf{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\
638 \quad &\left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\
639 \quad &+ 4\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( (\mathbf{n} \otimes \nabla_\Gamma V_n + \nabla_\Gamma V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x) \\
640 \quad &\quad - \mathfrak{M}_\Omega(\Omega) \left( \int_\Gamma [\rho |\mathbf{u}|^2] V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |\mathbf{u}|^2 V_n \, d\zeta(x) \right). \\
641
\end{aligned}$$

642 An argument which shows that  $A_c e_\Gamma(\mathbf{u}) : \mathbf{n} \otimes \nabla_\Gamma V_n e(\mathbf{u}) \Pi_d = 0$  (see (3.11)), then leads  
643 to

644

$$\begin{aligned}
645 \quad \partial_t G_\Omega(0, \mathbf{u}) &= \int_{\partial\Omega_1} [Ae(\mathbf{u}) : e(\mathbf{u})] V_n - 2 \int_{\partial\Omega_1} Ae(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
646 \quad &+ \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbf{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
647 \quad &+ \beta \int_{\partial\Omega} \left( \mathbf{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\
648 \quad &\left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\
649 \quad &+ 4\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( (\nabla_\Gamma V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x) \\
650 \quad &\quad - \mathfrak{M}_\Omega(\Omega) \left( \int_{\partial\Omega_1} [\rho] |\mathbf{u}|^2 V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |\mathbf{u}|^2 V_n \, d\zeta(x) \right). \\
651
\end{aligned}$$

652 Then, applying the tangential Stokes formula, similarly as in (3.12), we get

653

$$\begin{aligned}
654 \quad \partial_t G_\Omega(0, \mathbf{u}) &= \int_{\partial\Omega_1} [Ae(\mathbf{u}) : e(\mathbf{u})] V_n - 2 \int_{\partial\Omega_1} Ae(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
655 \quad &+ \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbf{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
656 \quad &+ \beta \int_{\partial\Omega} \left( \mathbf{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\
657 \quad &+ 2\beta \int_{\partial\Omega} \left( A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) - 2A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma (\Pi_d e(\mathbf{u}) \mathbf{n}) \right) V_n \, d\zeta(x) \\
658 \quad &+ 4 \int_{\partial\Omega} (-\alpha \mathbf{u} - Ae(\mathbf{u}) \mathbf{n}) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} V_n \, d\zeta(x) \\
659 \quad &\quad - \mathfrak{M}_\Omega(\Omega) \left( \int_{\partial\Omega_1} [\rho] |\mathbf{u}|^2 V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |\mathbf{u}|^2 V_n \, d\zeta(x) \right) \\
660
\end{aligned}$$

661 and after rearranging the terms we get the announced expression.

662 **3.4.2. Proof of Theorem 2.5.** We will now calculate the sensitivity of  $\mathfrak{M}_{\partial\Omega}(\Omega)$   
 663 with respect to variations of the domain  $\Omega$  and of the interface  $\Gamma$ . The perturbed  
 664 problem then reads  
 665

$$666 \quad \mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{\mathbf{v} \in \mathcal{H}(\Omega_t)} \left\{ \frac{1}{\int_{\partial\Omega_t} |\mathbf{v}|^2 d\zeta(x)} \left( \int_{\Omega_t} A_t(x) e(\mathbf{v}) : e(\mathbf{v}) dx + \alpha \int_{\partial\Omega_t} |\mathbf{v}|^2 d\zeta(x) \right. \right. \\ 667 \quad \left. \left. + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{v}) : e_\Gamma(\mathbf{v}) d\zeta(x) \right) \right\}.$$

669 The above can be formulated as

$$670 \quad (3.18) \quad \mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{\mathbf{u} \in \mathcal{H}(\Omega)} G_{\partial\Omega}(t, \mathbf{u}),$$

671 with  
 672

$$673 \quad G_{\partial\Omega}(t, \mathbf{u}) := \frac{1}{\int_{\partial\Omega_t} |(\mathbf{u} \circ \Psi_t^{-1})|^2 d\zeta(x)} \left( \int_{\Omega_t} A_t(x) e(\mathbf{u} \circ \Psi_t^{-1}) : e(\mathbf{u} \circ \Psi_t^{-1}) dx \right. \\ 674 \quad \left. + \alpha \int_{\partial\Omega_t} |(\mathbf{u} \circ \Psi_t^{-1})|^2 d\zeta(x) + \beta \int_{\partial\Omega_t} A_c e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) : e_\Gamma(\mathbf{u} \circ \Psi_t^{-1}) d\zeta(x) \right).$$

676 The verification of the hypotheses of Theorem 1.1 which guarantee the existence  
 677 of the semi-derivative  $\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V})$  can be shown using arguments from the previous  
 678 subsections. We now get a suitable expression for  $\partial_t G_{\partial\Omega}(0, \mathbf{u})$  given a normalized  
 679 eigenfunction  $\mathbf{u}$  for  $\mathfrak{M}_{\partial\Omega}(\Omega)$ . To begin with we have  
 680

$$681 \quad \partial_t G_{\partial\Omega}(0, \mathbf{u}) = \int_{\Gamma} [Ae(\mathbf{u}) : e(\mathbf{u})] V_n d\zeta(x) + 2 \int_{\Omega_1} A_1 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) dx \\ 682 \quad + 2 \int_{\Omega_2} A_2 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) dx + \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n d\zeta(x) \\ 683 \quad + \alpha \left( \int_{\partial\Omega} \left( H |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) d\zeta(x) \right) \\ 684 \quad + \beta \int_{\partial\Omega} \left( H A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2 A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\ 685 \quad \left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n d\zeta(x) \\ 686 \quad + 2\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) + \Pi_d e(\mathbf{u}) (\mathbf{n} \otimes \nabla_\Gamma(V_n) + \nabla_\Gamma(V_n) \otimes \mathbf{n}) \right. \\ 687 \quad \left. + (\mathbf{n} \otimes \nabla_\Gamma(V_n) + \nabla_\Gamma(V_n) \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) d\zeta(x) \\ 688 \quad - \mathfrak{M}_{\partial\Omega}(\Omega) \left( \int_{\partial\Omega} \left( H |\mathbf{u}|^2 + \partial_n |\mathbf{u}|^2 \right) V_n d\zeta(x) + 2 \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) d\zeta(x) \right).$$

690 Then, multiplying (2.9) by  $-\nabla \mathbf{u} \mathbf{V}$  in each  $\Omega_i$ , for  $i = 1, 2$ , we observe that  
 691



$$\begin{aligned}
692 \quad & \int_{\Omega_1} A_1 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx + \int_{\Omega_2} A_2 e(\mathbf{u}) : e(-\nabla \mathbf{u} \mathbf{V}) \, dx - \int_{\partial\Omega_1} A e(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
693 \quad & + \alpha \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) + \beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) = \mathfrak{M}_{\partial\Omega}(\Omega) \int_{\partial\Omega} \mathbf{u} \cdot (-\nabla \mathbf{u} \mathbf{V}) \, d\zeta(x). \\
694
\end{aligned}$$

695 Using the above we get

$$\begin{aligned}
696 \quad & \partial_t G_\Omega(0, \mathbf{u}) = \int_\Gamma [A e(\mathbf{u}) : e(\mathbf{u})] V_n \, d\zeta(x) - 2 \int_{\partial\Omega_1} A e(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
697 \quad & + \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
698 \quad & + \beta \int_{\partial\Omega} \left( \mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) + 2A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) \right. \\
699 \quad & \left. - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d \right) V_n \, d\zeta(x) \\
700 \quad & + 4\beta \int_{\partial\Omega} A_c e_\Gamma(\mathbf{u}) : \left( (\mathbf{n} \otimes \nabla_\Gamma(V_n) + \nabla_\Gamma(V_n) \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d \right) \, d\zeta(x) \\
701 \quad & - \mathfrak{M}_\Omega(\Omega) \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x). \\
702 \\
703
\end{aligned}$$

704 Then, continuing as in the proof of Theorem 2.4, we obtain

$$\begin{aligned}
705 \quad & \partial_t G_\Omega(0, \mathbf{u}) = \int_{\partial\Omega_1} [A e(\mathbf{u}) : e(\mathbf{u}) V_n] - 2 \int_{\partial\Omega_1} A e(\mathbf{u}) \mathbf{n} \cdot [\partial_n \mathbf{u}] \, d\zeta(x) \\
706 \quad & + \int_{\partial\Omega} A_2 e(\mathbf{u}) : e(\mathbf{u}) V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot \partial_n \mathbf{u}) V_n \, d\zeta(x) \\
707 \quad & + \beta \int_{\partial\Omega} (\mathbb{H} A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u}) - A_c e_\Gamma(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b^t \nabla \mathbf{u}) \Pi_d) V_n \, d\zeta(x) \\
708 \quad & + 2\beta \int_{\partial\Omega} \left( A_c e_\Gamma(\partial_n \mathbf{u}) : e_\Gamma(\mathbf{u}) - 2A_c e_\Gamma(\mathbf{u}) : \nabla_\Gamma(\Pi_d e(\mathbf{u}) \mathbf{n}) \right) V_n \, d\zeta(x) \\
709 \quad & + 4 \int_{\partial\Omega} (-\alpha \mathbf{u} - A e(\mathbf{u}) \mathbf{n}) \cdot \Pi_d e(\mathbf{u}) \mathbf{n} V_n \, d\zeta(x) \\
710 \quad & - \mathfrak{M}_\Omega(\Omega) \int_{\partial\Omega} (\mathbb{H} |\mathbf{u}|^2 + 2\mathbf{u} \cdot (\partial_n \mathbf{u} - 2\Pi_d e(\mathbf{u}) \mathbf{n})) V_n \, d\zeta(x). \\
711 \\
712
\end{aligned}$$

### 713 Appendix A. Auxiliary results on shape derivatives.

714 The purpose of this subsection is to recall some auxiliary results or notions used  
715 in the calculations of the shape sensitivities.

#### 716 A.1. Classical derivative formulæ with respect to the domain.

LEMMA A.1 (See, e.g., [19]). *Let  $\delta > 0$ . Let a vector field  $\mathbf{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$  and let*

$$\Psi : t \in [0, \delta) \mapsto \Psi_t = \mathbf{I} + t\mathbf{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d).$$

*Let a bounded Lipschitz open set  $\Omega$  in  $\mathbb{R}^d$  and let  $\Omega_t := \Psi_t(\Omega)$  for all  $t \in [0, \delta)$ . We consider a function  $f$  such that  $t \in [0, \delta) \mapsto f(t) \in L^1(\mathbb{R}^d)$  is differentiable at 0 with  $f(0) \in W^{1,1}(\mathbb{R}^d)$ . Then the function*

$$t \in [0, \delta) \mapsto F(t) = \int_{\Omega_t} f(t, x) \, dx$$

is differentiable at 0 (we say that  $F$  admits a semi-derivative) and we have

$$F'(0) = \int_{\partial\Omega} f(0, x) V_n \, d\zeta(x) + \int_{\Omega} f'(0, x) \, dx,$$

717 where  $V_n = \mathbf{V} \cdot \mathbf{n}$ .

LEMMA A.2 (See, e.g., [19]). Let  $\delta > 0$ . Let a vector field  $\mathbf{V} \in \mathbf{C}^{1,\infty}(\mathbb{R}^d)$  and let

$$\Psi : t \in [0, \delta) \mapsto \Psi_t = \mathbf{I} + t\mathbf{V} \in \mathbf{C}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded open set  $\Omega$  in  $\mathbb{R}^d$  of classe  $\mathcal{C}^2$  and let  $\Omega_t := \Psi_t(\Omega)$  for all  $t \in [0, \delta)$ . We consider a function  $g$  such that  $t \in [0, \delta) \mapsto g(t) \circ \Psi_t \in \mathbf{W}^{1,1}(\Omega)$  is differentiable at 0 with  $g(0) \in \mathbf{W}^{2,1}(\Omega)$ . Then the function

$$t \in [0, \delta) \mapsto G(t) = \int_{\partial\Omega_t} g(t, x) \, dx$$

is differentiable at 0 (we say that  $G$  admits a semi-derivative), the function  $t \in [0, \delta) \mapsto g(t)|_{\omega} \in \mathbf{W}^{1,1}(\omega)$  is differentiable at 0 for all open set  $\omega \subset \bar{\omega} \subset \Omega$  and the derivative  $g'(0)$  belongs to  $\mathbf{W}^{1,1}(\Omega)$  and we have

$$G'(0) = \int_{\partial\Omega} (g'(0, x) + (\mathbf{H}g(0, x) + \partial_n g) V_n) \, d\zeta(x),$$

718 where  $V_n = \mathbf{V} \cdot \mathbf{n}$  and where  $\mathbf{H}$  is the mean curvature function on  $\partial\Omega$ .

719 **A.2. Decomposition formulæ.**

720 LEMMA A.3. Given a bounded open set  $\Omega$  in  $\mathbb{R}^d$  of class  $\mathcal{C}^2$  and  $\mathbf{u} \in \mathbf{H}^2(\mathbb{R}^d)$  we  
721 have

$$722 \quad \partial_n (A_c e_{\Gamma}(\mathbf{u}) : e_{\Gamma}(\mathbf{u})) = 2A_c e_{\Gamma}(\partial_n \mathbf{u}) : e_{\Gamma}(\mathbf{u}) - A_c e_{\Gamma}(\mathbf{u}) : \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d,$$

723 where  $b$  is the signed distance to the boundary  $\partial\Omega$ .

*Proof.* Let us first notice that  $\partial_n \Pi_d = 0$  and that  $\Pi_d D^2 b = D^2 b \Pi_d = D^2 b$ , and we underline the fact that  $\nabla_{\Gamma} \mathbf{u} = \nabla \mathbf{u} \Pi_d$  since

$$\nabla_{\Gamma} \mathbf{u} = \begin{pmatrix} {}^t \nabla_{\Gamma} u_1 \\ \vdots \\ {}^t \nabla_{\Gamma} u_d \end{pmatrix} = \begin{pmatrix} {}^t \nabla u_1 \Pi_d \\ \vdots \\ {}^t \nabla u_d \Pi_d \end{pmatrix} = \nabla \mathbf{u} \Pi_d.$$

Then we have

$$\partial_n (\nabla_{\Gamma} \mathbf{u}) = \partial_n (\nabla \mathbf{u} \Pi_d) = \partial_n (\nabla \mathbf{u}) \Pi_d \quad \text{and} \quad \partial_n (\nabla \mathbf{u}) = D^2 \mathbf{u} \mathbf{n}.$$

Thus  $\nabla (\partial_n \mathbf{u}) = \nabla (\nabla \mathbf{u} \mathbf{n}) = D^2 \mathbf{u} \mathbf{n} + \nabla \mathbf{u} \nabla \mathbf{n} = \partial_n (\nabla \mathbf{u}) + \nabla \mathbf{u} D^2 b$ . Hence we obtain

$$\nabla_{\Gamma} (\partial_n \mathbf{u}) = \nabla (\partial_n \mathbf{u}) \Pi_d = \partial_n (\nabla \mathbf{u}) \Pi_d + \nabla \mathbf{u} D^2 b \Pi_d = \partial_n (\nabla_{\Gamma} \mathbf{u}) + \nabla \mathbf{u} D^2 b.$$

We also obtain, noticing that  $D^2 b$  is symmetric,

$${}^t \nabla_{\Gamma} (\partial_n \mathbf{u}) = \partial_n ({}^t \nabla_{\Gamma} \mathbf{u}) + D^2 b {}^t \nabla \mathbf{u}.$$

724 We deduce from the previous computation that

$$\begin{aligned}
725 \quad \partial_n (e_\Gamma(\mathbf{u})) &= \frac{1}{2} \partial_n (\Pi_d (\nabla_\Gamma u + {}^t \nabla_\Gamma \mathbf{u}) \Pi_d) = \frac{1}{2} \partial_n (\Pi_d \nabla_\Gamma u + {}^t \nabla_\Gamma \mathbf{u} \Pi_d) \\
726 \quad &= \frac{1}{2} (\Pi_d \partial_n (\nabla_\Gamma u) + \partial_n ({}^t \nabla_\Gamma \mathbf{u}) \Pi_d) \\
727 \quad &= \frac{1}{2} (\Pi_d (\nabla_\Gamma (\partial_n \mathbf{u}) - \nabla \mathbf{u} D^2 b) + ({}^t \nabla_\Gamma (\partial_n \mathbf{u}) - D^2 b {}^t \nabla \mathbf{u}) \Pi_d) \\
728 \quad &= e_\Gamma (\partial_n \mathbf{u}) - \frac{1}{2} (\Pi_d \nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u} \Pi_d) \\
729 \quad &= e_\Gamma (\partial_n \mathbf{u}) - \frac{1}{2} \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d.
\end{aligned}$$

730 Therefore, we obtain that

$$\begin{aligned}
731 \quad \partial_n (A_c e_\Gamma(\mathbf{u}) : e_\Gamma(\mathbf{u})) &= 2A_c \partial_n (e_\Gamma(\mathbf{u})) : e_\Gamma(\mathbf{u}) \\
732 \quad &= 2A_c \left( e_\Gamma (\partial_n \mathbf{u}) - \frac{1}{2} \Pi_d (\nabla \mathbf{u} D^2 b + D^2 b {}^t \nabla \mathbf{u}) \Pi_d \right) : e_\Gamma(\mathbf{u}), \\
733 \quad & \\
734 \quad &
\end{aligned}$$

735 which concludes the proof.  $\square$

736 LEMMA A.4. *Given a bounded open set  $\Omega$  in  $\mathbb{R}^d$  of class  $\mathcal{C}^2$ ,  $\mathbf{V} \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$*   
737 *and  $\mathbf{u} \in \mathbf{H}^2(\mathbb{R}^d)$ , we have*

$$\begin{aligned}
738 \quad \partial_t (e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1}))|_{t=0} &= e_\Gamma(-\nabla \mathbf{u} \mathbf{V}) \\
739 \quad &+ \Pi_d e(\mathbf{u})(\mathbf{n} \otimes \nabla_\Gamma V_n + \nabla_\Gamma V_n \otimes \mathbf{n}) + (\mathbf{n} \otimes \nabla_\Gamma V_n + \nabla_\Gamma V_n \otimes \mathbf{n}) e(\mathbf{u}) \Pi_d, \\
740 \quad &
\end{aligned}$$

742 where  $V_n = \mathbf{V} \cdot \mathbf{n}$ .

743 *Proof.* We first recall that, since  $\partial_t \mathbf{n}_t|_{t=0} = -\nabla_\Gamma V_n$ , we have  $\partial_t \Pi_d|_{t=0} = \mathbf{n} \otimes$   
744  $\nabla_\Gamma V_n + \nabla_\Gamma V_n \otimes \mathbf{n}$ . Hence we obtain the result noticing that  $e_\Gamma(\mathbf{u}) = \Pi_d e(\mathbf{u}) \Pi_d$ .  $\square$

745 *Remark A.5.* As compared to the scalar case dealt with in our previous paper,  
746 since  $e_\Gamma(\mathbf{u})$  is obtained by multiplying  $e(\mathbf{u})$  on both sides by  $\Pi_d$ , when we derive  
747  $e_{\Gamma_t}(\mathbf{u} \circ \Psi_t^{-1})$  with respect to  $t$  we obtain an extra term.

748 **A.3.  $\Gamma$ -convergence.** For the convenience of the reader, we recall the definition  
749 and the main property of the  $\Gamma$ -convergence. For further details we refer to Dal  
750 Maso [11].

751 DEFINITION A.6. (*Sequential  $\Gamma$ -convergence*) *A family of functionals  $\{F_t\}_{t>0}$  de-*  
752 *finied on a topological space  $X$  is said to be sequentially  $\Gamma$ -convergent to a functional  $F$*   
753 *as  $t \rightarrow 0^+$  if the two following statements hold.*

754 (i)  $\Gamma$ -lim inf inequality. *For every sequence  $\{x_t\}$  converging to  $x \in X$ , we have:*

$$755 \quad (A.1) \quad \liminf_{t \rightarrow 0^+} F_t(x_t) \geq F(x).$$

756 (ii)  $\Gamma$ -lim sup inequality. *For every  $x \in X$ , there exists a sequence  $\{x_t\}$  converging*  
757 *to  $x$  such that*

$$758 \quad (A.2) \quad \limsup_{t \rightarrow 0^+} F_t(x_t) \leq F(x).$$

759 *When properties (i) and (ii) are satisfied, we write  $F = \Gamma - \lim_{t \rightarrow 0^+} F_t$ .*

760 PROPOSITION A.7. Let  $F_t : X \rightarrow \mathbb{R}$  be a sequence of functionals on a topological  
761 space such that:

762 (i)  $F = \Gamma - \lim_{t \rightarrow 0^+} F_t$ ,

763 (ii)  $\sup_t F_t(x_t) < +\infty \Rightarrow \{x_t\}$  is sequentially relatively compact in  $X$ .

764 Then we have the convergence:  $\inf F_t \rightarrow \inf F$  as  $t \rightarrow 0^+$  and, every cluster point of a  
765 minimizing sequence  $\{x_t\}$  (i.e. such that  $F_t(x_t) = \inf_{x \in X} F_t(x)$ ) achieves the minimum  
766 of  $F$ .

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