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# The three dimensional Maxwell equations with strongly anisotropic electric permittivity

Mihai BOSTAN \*

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## Abstract

The subject matter of this work concerns the propagation of the electro-magnetic fields through strongly anisotropic media, in the three dimensional setting. We concentrate on the asymptotic behavior for the solutions of the Maxwell equations when the electric permittivity tensor is strongly anisotropic. We derive limit models and prove their well-posedness. We appeal to the variational framework and study the propagation speed of the solutions. We prove that almost all the electro-magnetic energy concentrates inside the propagation cone of the limit model.

**Keywords:** Maxwell equations, Evolution second order problems, Variational solutions, Asymptotic behavior.

**AMS classification:** 35Q61, 35L45

## 1 Introduction

We study the evolution of the electro-magnetic fields, described by the Maxwell equations [10, 8, 13, 7]

$$\partial_t D - \operatorname{rot} H = 0, \quad \partial_t B + \operatorname{rot} E = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \quad (1)$$

$$\operatorname{div} D = 0, \quad \operatorname{div} B = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (2)$$

Here the vector fields  $D, B$  stand for the electric induction and magnetic induction respectively and the vector fields  $E, H$  are the electric intensity and magnetic intensity respectively. We assume linear constitutive relations between inductions and intensities

$$D = \epsilon E, \quad B = \mu H$$

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where  $\epsilon = \epsilon(x) \in \mathcal{M}_3(\mathbb{R})$  is the electric permittivity and  $\mu > 0$  is the magnetic permeability. We supplement the Maxwell equations (1), (2) by the initial conditions  $(D(0), B(0)) = (D^{\text{in}}, B^{\text{in}})$ . Thanks to the identity

$$\operatorname{div}(E \wedge H) = \operatorname{rot}E \cdot H - \operatorname{rot}H \cdot E$$

we deduce

$$\partial_t D \cdot E + \partial_t B \cdot H + \operatorname{div}(E \wedge H) = 0.$$

When the electric permittivity tensor is symmetric, we obtain the energy conservation

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \{\epsilon^{-1} D \cdot D + \mu^{-1} |B|^2\} dx = \int_{\mathbb{R}^3} \{\partial_t D \cdot E + \partial_t B \cdot H\} dx = \int_{\mathbb{R}^3} \operatorname{div}(H \wedge E) dx = 0$$

provided that the initial energy is finite

$$\frac{1}{2} \int_{\mathbb{R}^3} \{\epsilon^{-1} D^{\text{in}} \cdot D^{\text{in}} + \mu^{-1} |B^{\text{in}}|^2\} dx < +\infty.$$

Moreover, it is well known that if initially  $D^{\text{in}}, B^{\text{in}}$  are divergence free, then  $D(t), B(t)$  are divergence free for any  $t \in \mathbb{R}$ .

In this work we study the asymptotic behavior of (1), (2) when the electric permittivity possesses disparate eigenvalues

$$\epsilon_\delta = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e.$$

Here for any  $x \in \mathbb{R}^3$ ,  $\{e_1(x), e_2(x), e(x)\}$  is a direct orthonormal basis of  $\mathbb{R}^3$  and  $n_1(x), n_2(x), \delta n(x)$  are the medium indexes at the point  $x$  (*i.e.*  $n_1^2, n_2^2, (\delta n)^2$  are the eigenvalues of  $\epsilon_\delta$ ) and  $\delta > 0$  is a small parameter. The relation between the medium indexes characterizes the materials. When the indexes are all different, we talk about biaxial media. When two of them are the same and the other one is different, the medium is called uniaxial. Many important materials are uniaxial : calcite, mica, quartz. We focus on the asymptotic behavior when one of the eigenvalues is negligible with respect to the other eigenvalues, that is, when  $\delta \searrow 0$ . See [3, 1, 2] for similar studies, concerning parabolic or transport problems. The behavior of the solutions for the wave equation whose propagation speed becomes very large along some direction has been studied recently [5]. It was shown that the limit model is a wave equation, coming out through averaging with respect to the characteristic flow along the direction of fast propagation. The motivation concerns the efficient numerical resolution of multi scale problems, involving strong anisotropy [6, 4, 9, 11, 12, 14, 15]. Another application is the study of strongly anisotropic uniaxial and biaxial media. If the case of uniform electric permittivity tensor can be handled by propagating plane waves accordingly to Fresnel relation, see Section 2, in the case of non uniform electric permittivity tensor, an asymptotic analysis based on variational formulations is required, cf. Sections 4, 5. Concerning the Maxwell system, we prove the following weak convergence result, toward a constrained formulation of the Maxwell equations.

### Theorem 1.1

*Assume that the electric permittivity tensor writes*

$$\epsilon_\delta = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e$$

where  $e = \frac{\nabla\varphi}{|\nabla\varphi|}$ ,  $\nabla\varphi(x) \neq 0$  for a.a.  $x \in \mathbb{R}^3$ ,  $\varphi \in C^2(\mathbb{R}^3)$  and  $n_1, n_2, n$  are locally bounded from below and above : for any compact set  $K \subset \mathbb{R}^3$ , there are  $0 < m_K \leq M_K < +\infty$  such that

$$m_K \leq \min\{n_1, n_2, n\} \leq \max\{n_1, n_2, n\} \leq M_K, \quad x \in K.$$

We consider the family of initial conditions  $(D^{\delta,\text{in}}, B^{\delta,\text{in}})_\delta$  verifying

$$\sup_{\delta>0} \int_{\mathbb{R}^3} [\epsilon_\delta^{-1} D^{\delta,\text{in}} \cdot D^{\delta,\text{in}} + \mu^{-1} |B^{\delta,\text{in}}|^2] dx < +\infty, \quad \text{div} D^{\delta,\text{in}} = 0, \quad \text{div} B^{\delta,\text{in}} = 0, \quad \delta > 0.$$

Then there is a sequence  $(\delta_k)_k$  converging to 0 such that the variational solutions  $(D^{\delta_k}, B^{\delta_k})$  of (1), (2) with the electric permittivity  $\epsilon_{\delta_k}$ , corresponding to the initial conditions  $(D^{\delta_k,\text{in}}, B^{\delta_k,\text{in}})$ , converge weakly  $\star$  in  $L^\infty(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)) \times L^\infty(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  toward the solution  $(D, B) \in C(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  of the problem

$$D \cdot \nabla\varphi = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} D \cdot \Phi dx - \int_{\mathbb{R}^3} \mu^{-1} B \cdot \text{rot}\Phi dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Phi \in C_c^1(\mathbb{R}^3)^3 \quad (3)$$

$$\text{div}(B \wedge \nabla\varphi) = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D \cdot \text{rot}\Psi dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3 \quad (4)$$

such that  $\text{div}(\Psi \wedge \nabla\varphi) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot}\Psi \cdot \text{rot}\Psi dx < +\infty$ , with the initial conditions

$$D(0) = D^{\text{in}}, \quad B(0) = P(B^{\text{in}})$$

where  $D^{\delta_k,\text{in}} \rightharpoonup D^{\text{in}}$  weakly in  $L_{\epsilon_1}^2(\mathbb{R}^3)$ ,  $B^{\delta_k,\text{in}} \rightharpoonup B^{\text{in}}$  weakly in  $L_{\mu^{-1}}^2(\mathbb{R}^3)$ . Moreover, we have  $\text{div}D = 0, \text{div}B = 0$ . Here  $P$  is the orthogonal projection in  $L^2(\mathbb{R}^3)^3$  on the closed subspace  $\{\Psi \in L^2(\mathbb{R}^3)^3 : \text{div}(\Psi \wedge \nabla\varphi) = 0\}$  and

$$L_{\epsilon_1}^2(\mathbb{R}^3) = \{D : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ measurable} : \int_{\mathbb{R}^3} \epsilon_1^{-1} D \cdot D dx < +\infty\}$$

$$L_{\mu^{-1}}^2(\mathbb{R}^3) = \{B : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ measurable} : \int_{\mathbb{R}^3} \mu^{-1} |B|^2 dx < +\infty\}.$$

The variational formulation in Theorem 1.1 involves two constraints :  $D \cdot \nabla\varphi = 0$ ,  $\text{div}(B \wedge \nabla\varphi) = 0$ . It would be very interesting to find an equivalent variational formulation after eliminating these constraints. This is the object of the next result. When the vector field  $e$  is uniform, that is  $\varphi(x) = x \cdot e, x \in \mathbb{R}^3$ , for some unitary vector  $e \in \mathbb{R}^3$ , we show that the previous limit model appears like a Maxwell system in which the electric field in the Faraday equation is replaced by its orthogonal projection over the  $L^2$  divergence free vector fields, orthogonal to  $e$ .

### Theorem 1.2

Assume that the electric permittivity tensor writes

$$\epsilon_1 = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + n^2 e \otimes e$$

where  $e$  is a unitary vector in  $\mathbb{R}^3$  and  $n_1, n_2, n$  are bounded from below and above : there are  $0 < m \leq M < +\infty$  such that

$$m \leq \min\{n_1, n_2, n\} \leq \max\{n_1, n_2, n\} \leq M, \quad x \in \mathbb{R}^3.$$

Let us consider  $(D, B) \in C(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu}^2(\mathbb{R}^3))$  the variational solution of the problem

$$D \cdot e = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} D \cdot \Phi \, dx - \int_{\mathbb{R}^3} \mu^{-1} B \cdot \operatorname{rot} \Phi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Phi \in C_c^1(\mathbb{R}^3)^3 \quad (5)$$

$$\operatorname{div}(B \wedge e) = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D \cdot \operatorname{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3 \quad (6)$$

such that  $\operatorname{div}(\Psi \wedge e) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx < +\infty$ , with the initial conditions

$$D(0) = D^{\text{in}} \in L_{\epsilon_1}^2(\mathbb{R}^3), \quad B(0) = B^{\text{in}} \in L_{\mu}^2(\mathbb{R}^3), \quad D^{\text{in}} \cdot e = 0, \quad \operatorname{div}(B^{\text{in}} \wedge e) = 0. \quad (7)$$

Then the problem (5), (6), (7) is equivalent to the variational problem

$$D \cdot e = 0, \quad \partial_t D - \operatorname{rot}(\mu^{-1} B) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \quad (8)$$

$$\operatorname{div}(B \wedge e) = 0, \quad \partial_t B + \operatorname{rot}(Q(\epsilon_1^{-1} D)) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \quad (9)$$

$$D(0) = D^{\text{in}}, \quad B(0) = B^{\text{in}}, \quad D^{\text{in}} \cdot e = 0, \quad \operatorname{div}(B^{\text{in}} \wedge e) = 0 \quad (10)$$

where  $Q$  is the orthogonal projection in  $L^2(\mathbb{R}^3)^3$  on the closed subspace  $\{\xi \in L^2(\mathbb{R}^3)^3 : \operatorname{div} \xi = 0, \xi \cdot e = 0\}$ .

The weak convergence result in Theorem 1.1 becomes a strong convergence result, for well prepared initial conditions. This result relies on the conservation of the electromagnetic energy.

### Theorem 1.3

Assume that the electric permittivity tensor writes

$$\epsilon_{\delta} = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e$$

where  $e = \frac{\nabla \varphi}{|\nabla \varphi|}$ ,  $\nabla \varphi(x) \neq 0$  for a.a.  $x \in \mathbb{R}^3$ ,  $\varphi \in C^2(\mathbb{R}^3)$  and  $n_1, n_2, n$  are locally bounded from below and above : for any compact set  $K \subset \mathbb{R}^3$ , there are  $0 < m_K \leq M_K < +\infty$  such that

$$m_K \leq \min\{n_1, n_2, n\} \leq \max\{n_1, n_2, n\} \leq M_K, \quad x \in K.$$

We suppose that the initial conditions  $(D^{\delta, \text{in}}, B^{\delta, \text{in}})_{\delta > 0}$  are well prepared

$$\lim_{\delta \searrow 0} D^{\delta, \text{in}} = D^{\text{in}} \quad \text{strongly in } L_{\epsilon_1}^2(\mathbb{R}^3), \quad \lim_{\delta \searrow 0} \frac{D^{\delta, \text{in}} \cdot e}{\delta n} = 0 \quad \text{strongly in } L^2(\mathbb{R}^3)$$

$$\lim_{\delta \searrow 0} B^{\delta, \text{in}} = B^{\text{in}} \quad \text{strongly in } L_{\mu}^2(\mathbb{R}^3), \quad \operatorname{div}(B^{\text{in}} \wedge \nabla \varphi) = 0.$$

Then we have the convergences

$$\lim_{\delta \searrow 0} D^{\delta} = D \quad \text{in } L_{\text{loc}}^2(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)), \quad \lim_{\delta \searrow 0} B^{\delta} = B \quad \text{in } L_{\text{loc}}^2(\mathbb{R}; L_{\mu}^2(\mathbb{R}^3))$$

$$\lim_{\delta \searrow 0} \frac{D^{\delta} \cdot e}{\delta n} = 0 \quad \text{in } L_{\text{loc}}^2(\mathbb{R}; L^2(\mathbb{R}^3))$$

where  $(D^{\delta}, B^{\delta})_{\delta > 0}$  are the variational solutions of (1), (2) with the electric permittivity  $(\epsilon_{\delta})_{\delta > 0}$ , corresponding to the initial conditions  $(D^{\delta, \text{in}}, B^{\delta, \text{in}})_{\delta > 0}$ , and  $(D, B)$  is the variational solution of (3), (4), corresponding to the initial condition  $(D^{\text{in}}, B^{\text{in}})$ .

It is interesting to estimate the propagation speed of the solutions. While the solutions of the Maxwell equations (1), (2) with the electric permittivity  $\epsilon_\delta$  propagate with the speed  $c_\delta = \mu^{-1/2} \|\epsilon_\delta^{-1/2}\|_\infty = \mathcal{O}(1/\delta)$ , we prove that the solutions of the limit model, when  $\delta \searrow 0$ , propagate with the speed  $c_\infty = \lim_{\delta \rightarrow +\infty} c_\delta = \mu^{-1/2} \lim_{\delta \rightarrow +\infty} \|\epsilon_\delta^{-1/2}\|_\infty$ . Nevertheless, the energy of the solutions  $(D^\delta, B^\delta)$  outside the propagation cone of the limit model, that is  $\{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : |x| \geq R + c_\infty |t|\}$ , becomes negligible, when  $\delta \searrow 0$ .

**Theorem 1.4**

Assume that the hypotheses of Theorem 1.3 hold true, with  $\varphi(x) = |x|, x \in \mathbb{R}^3$ . Let us denote by  $(D^\delta, B^\delta), (D, B)$  the unique variational solutions of the problems (1), (2) with the electric permittivity  $\epsilon_\delta$ , initial conditions  $(D^{\delta, \text{in}}, B^{\delta, \text{in}})$  and (3), (4) with the initial conditions  $(D^{\text{in}}, B^{\text{in}})$ . We assume that there is  $R > 0$  such that

$$D^{\text{in}}(x) = 0, \quad B^{\text{in}}(x) = 0, \quad |x| > R$$

and we denote by  $c_\infty$  the speed  $\mu^{-1/2} \lim_{\delta \rightarrow +\infty} \|\epsilon_\delta^{-1/2}\|_\infty$ . For any  $T \in \mathbb{R}_+$  we have

$$\lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \frac{\epsilon_\delta^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2}{2} \mathbf{1}_{\{|x| > R + c_\infty |t|\}} \, dx dt = 0.$$

Our paper is organized as follows. In Section 2 we discuss the case of uniform electric permittivity tensor and perform the asymptotic analysis for uniaxial and biaxial media. In Section 3 we recall the variational framework of the Maxwell system. In Section 4 we establish the weak convergence result. In Section 5 we investigate the strong convergence. The propagation speed of the solutions is analyzed in Section 6.

## 2 Uniform electric permittivity

The study of the electro-magnetic waves depends on the propagation medium. We distinguish between biaxial media, *i.e.*, indexes all different, and uniaxial media *i.e.*, two equal indexes, but different with respect to the third one. The main issue of this work is to understand the propagation of the electro-magnetic waves when one of the indexes is negligible with respect to the other indexes. We investigate the media characterized by the indexes  $\{n_1, n_2, \delta n\}, \delta \searrow 0$  when  $n_1 \neq n_2$  (biaxial media) and when  $n_1 = n_2$  (uniaxial media). When the electric permittivity tensor is uniform, the analysis is standard : we are looking for plane waves described by vector fields of the form  $A \cos(k \cdot x - \omega t), A, k \in \mathbb{R}^3 \setminus \{0\}, \omega \in \mathbb{R}$ . This approach leads to the dispersion relation (Fresnel equation) relating  $\omega, k$  and the medium indexes. The asymptotic behavior  $\delta \searrow 0$  follows easily, by direct computations, in that case (uniform electric permittivity tensor). We indicate the limit Fresnel equation and the limit electro-magnetic field, when  $\delta \searrow 0$ . The general case (non uniform electric permittivity tensor) is more difficult and cannot be reduced to plane waves. It will be discussed in the next sections, by appealing to variational formulations. We present the limit problem satisfied by the electro-magnetic field when one medium index is negligible with respect to the other indexes. In particular we estimate the propagation speed with respect to the medium indexes, providing a formula which is compatible with the Fresnel equation.

Let us concentrate first on the Maxwell equations with uniform electric permittivity tensor

$$\epsilon = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + n_3^2 e_3 \otimes e_3$$

where  $\{e_1, e_2, e_3\}$  is a fixed direct orthonormal basis of  $\mathbb{R}^3$  and  $n_1, n_2, n_3 > 0$  are constant medium indexes. As usual, for any  $k \in \mathbb{R}^3 \setminus \{0\}$ , we are looking for  $\omega = \omega(k)$  such that there is a non trivial solution for the Maxwell equations of the form

$$D \cos(k \cdot x - \omega t), \quad B \cos(k \cdot x - \omega t), \quad E \cos(k \cdot x - \omega t), \quad H \cos(k \cdot x - \omega t)$$

with  $D, B, E, H \in \mathbb{R}^3$ . In that case, the Maxwell system (1), (2) reduces to

$$\omega D + k \wedge H = 0, \quad \omega B - k \wedge E = 0, \quad D = \epsilon E, \quad B = \mu H.$$

We are not in the framework of solutions with finite total electro-magnetic energy, as considered in Theorems 1.1, 1.3. But we can work in the setting of periodic solutions, with finite electro-magnetic energy over one period domain.

By straightforward manipulations one gets

$$\left( I_3 - \frac{\omega^2 \mu}{|k|^2} \epsilon - \frac{k \otimes k}{|k|^2} \right) E = 0.$$

Using the notations  $\alpha_i = 1 - \frac{\omega^2 \mu}{|k|^2} n_i^2, k_i = k \cdot e_i, i \in \{1, 2, 3\}$ , we obtain the Fresnel equation.

**Proposition 2.1**

*There is  $E \in \mathbb{R}^3 \setminus \{0\}$  such that*

$$\left( I_3 - \frac{\omega^2 \mu}{|k|^2} \epsilon - \frac{k \otimes k}{|k|^2} \right) E = 0 \tag{11}$$

*iff*

$$\frac{k_1^2}{|k|^2} \alpha_2 \alpha_3 + \frac{k_2^2}{|k|^2} \alpha_3 \alpha_1 + \frac{k_3^2}{|k|^2} \alpha_1 \alpha_2 = \alpha_1 \alpha_2 \alpha_3. \tag{12}$$

**Proof.** For the sake of completeness, we give here some proof details. Assume that (11) holds true and let us deduce (12). Taking the scalar product with  $e_1, e_2, e_3$ , the linear system (11) is equivalent to

$$\alpha_i E_i = \frac{k \cdot E}{|k|^2} k_i, \quad E_i = E \cdot e_i, \quad i \in \{1, 2, 3\}. \tag{13}$$

We have

$$\alpha_1 \alpha_2 \alpha_3 (k \cdot E) = \alpha_1 \alpha_2 \alpha_3 \sum_{i=1}^3 (k \cdot e_i) (E \cdot e_i) = (k \cdot E) \left( \frac{k_1^2}{|k|^2} \alpha_2 \alpha_3 + \frac{k_2^2}{|k|^2} \alpha_3 \alpha_1 + \frac{k_3^2}{|k|^2} \alpha_1 \alpha_2 \right).$$

If  $k \cdot E \neq 0$ , we obtain (12). If  $k \cdot E = 0$ , we have  $\alpha_1 E_1 = 0, \alpha_2 E_2 = 0, \alpha_3 E_3 = 0$ . As  $E \in \mathbb{R}^3 \setminus \{0\}$ , we can assume that  $E_1 \neq 0$ , implying that  $\alpha_1 = 0$ . We are done provided that  $k_1 \alpha_2 \alpha_3 = 0$ . Indeed, if  $\alpha_2 \alpha_3 \neq 0$ , then  $E_2 = 0, E_3 = 0$  and  $0 = E \cdot k = E_1 k_1$  implies  $k_1 = 0$ , because  $E_1 \neq 0$ .

Conversely, assume now that (12) holds true, and let us determine  $E \in \mathbb{R}^3 \setminus \{0\}$  satisfying (11), or equivalently (13). If  $\alpha_1\alpha_2\alpha_3 \neq 0$  we take  $E = \sum_{i=1}^3 \frac{k_i}{\alpha_i|k|^2} e_i$ . Thanks to (12), we have  $k \cdot E = \sum_{i=1}^3 \frac{k_i^2}{\alpha_i|k|^2} = 1$  and therefore  $E = \sum_{i=1}^3 E_i e_i \in \mathbb{R}^3 \setminus \{0\}$  satisfies (11). Assume now that  $\alpha_1 = 0, \alpha_2\alpha_3 \neq 0$ . By (12), we deduce that  $k_1 = 0$  and we can take  $E = e_1$ . In the case  $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 \neq 0$  we can take  $E = E_1 e_1 + E_2 e_2$  such that  $(E_1, E_2) \neq (0, 0)$  and  $k_1 E_1 + k_2 E_2 = 0$ . In the case  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ , take any  $E \in \mathbb{R}^3 \setminus \{0\}$  such that  $E \cdot k = 0$ .  $\square$

### Remark 2.1

In the isotropic case,  $n_1 = n_2 = n_3 = n > 0$ ,  $\epsilon = n^2 I_3$ , we have  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1 - \frac{\omega^2 \mu}{|k|^2} n^2 < 1$ . In that case, the Fresnel equation becomes  $\alpha^2 = \alpha^3$  whose unique solution in  $]-\infty, 1[$  is  $\alpha = 0$ , leading to the well known formula  $\omega^2 = \frac{|k|^2}{\mu n^2}$ .

Consider now the strongly anisotropic permittivity tensor

$$\epsilon_\delta = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e, \quad \delta > 0 \quad (14)$$

where  $\{e_1, e_2, e\}$  is a fixed direct orthonormal basis of  $\mathbb{R}^3$  and  $n_1, n_2, n > 0$ . Thanks to Proposition 2.1, the formula for  $\omega_\delta$  comes by

$$\frac{k_1^2}{|k|^2} \alpha_{2\delta} \alpha_{3\delta} + \frac{k_2^2}{|k|^2} \alpha_{3\delta} \alpha_{1\delta} + \frac{k_3^2}{|k|^2} \alpha_{1\delta} \alpha_{2\delta} = \alpha_{1\delta} \alpha_{2\delta} \alpha_{3\delta} \quad (15)$$

where

$$\alpha_{1\delta} = 1 - \frac{\omega_\delta^2 \mu}{|k|^2} n_1^2, \quad \alpha_{2\delta} = 1 - \frac{\omega_\delta^2 \mu}{|k|^2} n_2^2, \quad \alpha_{3\delta} = 1 - \frac{\omega_\delta^2 \mu}{|k|^2} \delta^2 n^2.$$

We investigate the limit of  $\omega_\delta$ , for  $\delta \searrow 0$ , when the electro-magnetic energy is bounded with respect to  $\delta > 0$ , see the hypotheses of Theorem 1.1, and non vanishing when  $\delta \searrow 0$ . By (11) we know that

$$\frac{\omega_\delta^2 \mu}{|k|^2} \epsilon_\delta E^\delta = \left( I_3 - \frac{k \otimes k}{|k|^2} \right) E^\delta$$

implying that  $\epsilon_\delta E^\delta \cdot k = 0$ . Decomposing with respect to the basis  $\{e_1, e_2, e\}$

$$E^\delta = E_1^\delta e_1 + E_2^\delta e_2 + E_3^\delta e, \quad k = k_1 e_1 + k_2 e_2 + k_3 e$$

we obtain

$$n_1^2 E_1^\delta k_1 + n_2^2 E_2^\delta k_2 + \delta^2 n^2 E_3^\delta k_3 = 0, \quad \delta > 0. \quad (16)$$

By the boundedness of the electro-magnetic energy

$$\sup_{\delta > 0} \left( \epsilon_\delta E^\delta \cdot E^\delta + \frac{|B^\delta|^2}{\mu} \right) < +\infty$$

we deduce

$$\sup_{\delta > 0} [n_1^2 (E_1^\delta)^2 + n_2^2 (E_2^\delta)^2 + \delta^2 n^2 (E_3^\delta)^2] < +\infty \quad (17)$$



implying that  $\lim_{\delta \searrow 0}(\delta^2 n^2 E_3^\delta k_3) = 0$  and thus

$$n_1^2 E_1 k_1 + n_2^2 E_2 k_2 = 0, \quad E_1 = \lim_{\delta \searrow 0} E_1^\delta, \quad E_2 = \lim_{\delta \searrow 0} E_2^\delta. \quad (18)$$

By (13) and (16) we obtain

$$\begin{aligned} \left(1 - \frac{\omega_\delta^2 \mu}{|k|^2} n_1^2\right) E_1^\delta &= \alpha_{1\delta} E_1^\delta = \frac{k \cdot E^\delta}{|k|^2} k_1 \\ &= \left[ k_1 E_1^\delta + k_2 E_2^\delta - \frac{n_1^2 k_1 E_1^\delta + n_2^2 k_2 E_2^\delta}{\delta^2 n^2} \right] \frac{k_1}{|k|^2} \\ &= \left[ k_1 E_1^\delta \left(1 - \frac{n_1^2}{\delta^2 n^2}\right) + k_2 E_2^\delta \left(1 - \frac{n_2^2}{\delta^2 n^2}\right) \right] \frac{k_1}{|k|^2}. \end{aligned}$$

Multiplying both sides by  $\delta^2$  and passing to the limit when  $\delta \searrow 0$  yield, thanks to (18)

$$-\lim_{\delta \searrow 0} (\omega_\delta^2 \delta^2) \frac{\mu n_1^2}{|k|^2} E_1 = -\frac{k_1}{|k|^2} \frac{k_1 E_1 n_1^2 + k_2 E_2 n_2^2}{n^2} = 0.$$

We obtain  $E_1 \lim_{\delta \searrow 0} (\omega_\delta^2 \delta^2) = 0$  and similarly  $E_2 \lim_{\delta \searrow 0} (\omega_\delta^2 \delta^2) = 0$ . We assume also that  $(D^\delta = \epsilon_\delta E^\delta, B^\delta)$  are well-prepared, see the hypotheses of Theorem 1.3

$$\lim_{\delta \searrow 0} \frac{D^\delta \cdot e}{\delta n} = \lim_{\delta \searrow 0} \frac{\epsilon_\delta E^\delta \cdot e}{\delta n} = \lim_{\delta \searrow 0} (E^\delta \cdot e) \delta n = n \lim_{\delta \searrow 0} (\delta E_3^\delta) = 0.$$

We deduce that  $E_1^2 + E_2^2 > 0$ , since otherwise, the limit electro-magnetic energy vanishes

$$\lim_{\delta \searrow 0} [n_1^2 (E_1^\delta)^2 + n_2^2 (E_2^\delta)^2 + \delta^2 n^2 (E_3^\delta)^2] = 0.$$

We obtain that  $\lim_{\delta \searrow 0} (\omega_\delta^2 \delta^2) = 0$ . We have to solve (15) in order to determine  $\omega_\delta$ .

With the notations  $\lambda_\delta = \frac{\omega_\delta^2 \mu}{|k|^2}$ ,  $n_3 = \delta n$ , we obtain

$$\begin{aligned} &k_1^2 (1 - \lambda_\delta n_2^2) (1 - \lambda_\delta n_3^2) + k_2^2 (1 - \lambda_\delta n_3^2) (1 - \lambda_\delta n_1^2) + k_3^2 (1 - \lambda_\delta n_1^2) (1 - \lambda_\delta n_2^2) \\ &= |k|^2 (1 - \lambda_\delta n_1^2) (1 - \lambda_\delta n_2^2) (1 - \lambda_\delta n_3^2). \end{aligned}$$

The above equation also writes

$$\lambda_\delta (a_\delta \lambda_\delta^2 + b_\delta \lambda_\delta + c_\delta) = 0, \quad \delta > 0$$

with

$$\begin{aligned} a_\delta &= n_1^2 n_2^2 n_3^2 |k|^2 = \delta^2 n_1^2 n_2^2 n^2 |k|^2 \\ b_\delta &= -\sum (k_1^2 + k_2^2) n_1^2 n_2^2 = -[(k_1^2 + k_2^2) n_1^2 n_2^2 + \delta^2 (k_2^2 + k_3^2) n_2^2 n^2 + \delta^2 (k_3^2 + k_1^2) n^2 n_1^2] \\ c_\delta &= \sum_{i=1}^3 k_i^2 n_i^2 = k_1^2 n_1^2 + k_2^2 n_2^2 + \delta^2 k_3^2 n^2. \end{aligned}$$

The non trivial solutions are given by

$$\lambda_\delta = \frac{-b_\delta \pm \sqrt{b_\delta^2 - 4a_\delta c_\delta}}{2a_\delta}.$$

Recall that we are searching for solutions satisfying  $\lim_{\delta \searrow 0}(\omega_\delta^2 \delta^2) = 0$ , or equivalently  $\lim_{\delta \searrow 0}(\lambda_\delta \delta^2) = 0$ . Therefore the sign in front of the square root of the discriminant should be minus

$$\lambda_\delta = \frac{-b_\delta - \sqrt{b_\delta^2 - 4a_\delta c_\delta}}{2a_\delta} = \frac{2c_\delta}{-b_\delta + \sqrt{b_\delta^2 - 4a_\delta c_\delta}}.$$

Letting  $\delta \searrow 0$  implies

$$\lim_{\delta \searrow 0} \frac{\omega_\delta^2 \mu}{|k|^2} = \frac{k_1^2 n_1^2 + k_2^2 n_2^2}{(k_1^2 + k_2^2) n_1^2 n_2^2}, \quad k_1^2 + k_2^2 > 0$$

saying that  $(\omega_\delta^2)_{\delta > 0}$  converges when  $\delta \searrow 0$  and  $\omega^2 = \lim_{\delta \searrow 0} \omega_\delta^2$  is given by

$$\frac{\omega^2}{|k|^2} = \frac{k_1^2 n_1^2 + k_2^2 n_2^2}{\mu(k_1^2 + k_2^2) n_1^2 n_2^2}, \quad k_1^2 + k_2^2 > 0. \quad (19)$$

It is easily seen that the wave propagation speed is bounded by

$$\frac{|\omega|}{|k|} \leq \mu^{-1/2} \max \left\{ \frac{1}{n_1}, \frac{1}{n_2} \right\}. \quad (20)$$

Introducing the formula of  $\frac{\omega^2}{|k|^2}$  cf. (19) in the expressions for

$$\alpha_1 = \lim_{\delta \searrow 0} \alpha_{1\delta} = 1 - \frac{\omega^2 \mu}{|k|^2} n_1^2, \quad \alpha_2 = \lim_{\delta \searrow 0} \alpha_{2\delta} = 1 - \frac{\omega^2 \mu}{|k|^2} n_2^2$$

one gets

$$\alpha_1 = \frac{k_1^2}{k_1^2 + k_2^2} \frac{n_2^2 - n_1^2}{n_2^2}, \quad \alpha_2 = \frac{k_2^2}{k_1^2 + k_2^2} \frac{n_1^2 - n_2^2}{n_1^2}. \quad (21)$$

**The limit uniaxial model**  $n_1 = n_2 > 0, k_1^2 + k_2^2 > 0$

First of all (19) becomes  $\frac{\omega^2}{|k|^2} = \lim_{\delta \searrow 0} \frac{\omega_\delta^2}{|k|^2} = \frac{1}{\mu n_1^2} = \frac{1}{\mu n_2^2}$ . As  $n_1 = n_2 > 0$ , we have by (18)  $E_1 k_1 + E_2 k_2 = 0$ . Coming back to (13), we know that

$$\alpha_{1\delta} E_1^\delta = \frac{k \cdot E^\delta}{|k|^2} k_1, \quad \alpha_{2\delta} E_2^\delta = \frac{k \cdot E^\delta}{|k|^2} k_2, \quad \alpha_{3\delta} E_3^\delta = \frac{k \cdot E^\delta}{|k|^2} k_3$$

where  $\lim_{\delta \searrow 0} \alpha_{1\delta} = \alpha_1 = 0$ ,  $\lim_{\delta \searrow 0} \alpha_{2\delta} = \alpha_2 = 0$ , thanks to (21) with  $n_1 = n_2$ , and  $\lim_{\delta \searrow 0} \alpha_{3\delta} = 1$ . Letting  $\delta \searrow 0$  we obtain, by using  $E_1 k_1 + E_2 k_2 = 0$

$$0 = \frac{k_1}{|k|^2} \lim_{\delta \searrow 0} (k_3 E_3^\delta), \quad 0 = \frac{k_2}{|k|^2} \lim_{\delta \searrow 0} (k_3 E_3^\delta), \quad \lim_{\delta \searrow 0} E_3^\delta = \frac{k_3}{|k|^2} \lim_{\delta \searrow 0} (k_3 E_3^\delta).$$

Finally, since  $k_1^2 + k_2^2 > 0$ , we have  $E_1 k_1 + E_2 k_2 = 0, E_3 = 0$ . For the electric and magnetic inductions we write

$$D = \lim_{\delta \searrow 0} \epsilon_\delta E^\delta = n_1^2 E_1 e_1 + n_2^2 E_2 e_2 = \epsilon_1 E, \quad \omega B = \lim_{\delta \searrow 0} (k \wedge E^\delta) = k \wedge E.$$

It is easily seen that the vector fields  $D \cos(k \cdot x - \omega t)$ ,  $B \cos(k \cdot x - \omega t)$  with  $\omega^2 = \frac{|k|^2}{\mu n_1^2} = \frac{|k|^2}{\mu n_2^2}$  solve the limit model in Theorem 1.2 (notice that the tangent vector field

$\epsilon_1^{-1}D \cos(k \cdot x - \omega t)$  is divergence free, because  $k \cdot E = 0$ , and thus  $Q(\epsilon_1^{-1}D \cos(k \cdot x - \omega t)) = \epsilon_1^{-1}D \cos(k \cdot x - \omega t)$ .

**The limit biaxial model**  $n_1 \neq n_2, n_1 > 0, n_2 > 0, k_1^2 + k_2^2 > 0$

As before we have  $\epsilon_\delta E^\delta \cdot k = 0$ , implying

$$n_1^2 E_1^\delta k_1 + n_2^2 E_2^\delta k_2 + \delta^2 n^2 E_3^\delta k_3 = 0, \quad \delta > 0.$$

Passing to the limit when  $\delta > 0$ , one gets, thanks to (17)

$$n_1^2 E_1 k_1 + n_2^2 E_2 k_2 = 0, \quad E_1 = \lim_{\delta \searrow 0} E_1^\delta, \quad E_2 = \lim_{\delta \searrow 0} E_2^\delta.$$

Letting  $\delta \searrow 0$  in (13) we obtain

$$\alpha_1 E_1 = \frac{k_1}{|k|^2} \lim_{\delta \searrow 0} (k \cdot E^\delta), \quad \alpha_2 E_2 = \frac{k_2}{|k|^2} \lim_{\delta \searrow 0} (k \cdot E^\delta), \quad E_3 = \lim_{\delta \searrow 0} E_3^\delta = \frac{k_3}{|k|^2} \lim_{\delta \searrow 0} (k \cdot E^\delta)$$

where  $\alpha_1, \alpha_2$  are given in (21). The electric induction writes

$$D = \lim_{\delta \searrow 0} \epsilon_\delta E^\delta = \lim_{\delta \searrow 0} (n_1^2 E_1^\delta e_1 + n_2^2 E_2^\delta e_2 + \delta^2 n^2 E_3^\delta e) = n_1^2 E_1 e_1 + n_2^2 E_2 e_2 = \epsilon_1 (E_1 e_1 + E_2 e_2)$$

and the magnetic induction is

$$\omega B = \lim_{\delta \searrow 0} (k \wedge E^\delta) = k \wedge E.$$

We claim that the vector fields  $D \cos(k \cdot x - \omega t), B \cos(k \cdot x - \omega t)$ , with  $\omega$  satisfying (19), solve the limit model in Theorem 1.2. Clearly we have  $D \cos(k \cdot x - \omega t) \cdot e = 0$  and

$$\begin{aligned} \operatorname{div}(B \cos(k \cdot x - \omega t) \wedge e) &= \operatorname{rot}(B \cos(k \cdot x - \omega t)) \cdot e \\ &= -\sin(k \cdot x - \omega t) (k \wedge B) \cdot e \\ &= -\frac{\sin(k \cdot x - \omega t)}{\omega} [k \wedge (k \wedge E)] \cdot e. \end{aligned}$$

Notice that

$$\begin{aligned} -[k \wedge (k \wedge E)] \cdot e &= -|k|^2 \left( \frac{k \otimes k}{|k|^2} - I_3 \right) E \cdot e \\ &= \omega^2 \mu (n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2) E \cdot e = 0 \end{aligned}$$

and therefore both constraints of the limit model are satisfied. The Ampère law in (8) comes easily. In order to check the Faraday law in (9), we need to compute the orthogonal projection of  $\epsilon_1^{-1}D \cos(k \cdot x - \omega t) = (E_1 e_1 + E_2 e_2) \cos(k \cdot x - \omega t)$  on the subspace of tangent, that is orthogonal to  $e$ , free divergence vector fields. We introduce the tangent vector field  $\xi$ , given by

$$\begin{aligned} (E_1 e_1 + E_2 e_2) \cos(k \cdot x - \omega t) &= \xi(t, x) + \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} \cos(k \cdot x - \omega t) (k_1 e_1 + k_2 e_2) \\ &= \xi(t, x) + \partial_{x_1} \left[ \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} \sin(k \cdot x - \omega t) \right] e_1 + \partial_{x_2} \left[ \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} \sin(k \cdot x - \omega t) \right] e_2. \end{aligned}$$

A direct computation shows that  $\xi$  is free divergence, implying that

$$Q(\epsilon_1^{-1} D \cos(k \cdot x - \omega t)) = \frac{E_1 k_2 - E_2 k_1}{k_1^2 + k_2^2} (k_2 e_1 - k_1 e_2) \cos(k \cdot x - \omega t).$$

We are done provided that

$$\omega B - k \wedge \frac{E_1 k_2 - E_2 k_1}{k_1^2 + k_2^2} (k_2 e_1 - k_1 e_2) = 0.$$

As  $\omega B = k \wedge E$  we have to check that  $E - \frac{E_1 k_2 - E_2 k_1}{k_1^2 + k_2^2} (k_2 e_1 - k_1 e_2)$  is parallel to  $k$ . Indeed, observe that

$$\begin{aligned} E - \frac{E_1 k_2 - E_2 k_1}{k_1^2 + k_2^2} (k_2 e_1 - k_1 e_2) &= k_1 \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} e_1 + k_2 \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} e_2 + E_3 e_3 \\ &= \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} k \end{aligned}$$

where we have used the equality

$$E_3 = k_3 \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2}$$

coming from the formula, see (21)

$$\frac{\frac{k_1^2}{\alpha_1} + \frac{k_2^2}{\alpha_2}}{k_1^2 + k_2^2} = 1.$$

Actually we have

$$k_3 \frac{E_1 k_1 + E_2 k_2}{k_1^2 + k_2^2} = k_3 \frac{\frac{k_1^2}{|k|^2 \alpha_1} + \frac{k_2^2}{|k|^2 \alpha_2}}{k_1^2 + k_2^2} \lim_{\delta \searrow 0} (k \cdot E^\delta) = \frac{k_3}{|k|^2} \lim_{\delta \searrow 0} (k \cdot E^\delta) = E_3.$$

### 3 The Maxwell equations

We recall briefly the well posedness of the Maxwell system. We appeal to variational formulations cf. [8, 13, 7]. We assume that  $\epsilon$  is a measurable field of symmetric definite positive matrices, locally bounded from below and above : for any compact set  $K \subset \mathbb{R}^3$  there are  $0 < m_K \leq M_K < +\infty$  such that

$$m_K I_3 \leq \epsilon^{1/2}(x) \leq M_K I_3, \quad x \in K.$$

We consider the Hilbert spaces

$$L_{\epsilon^{-1}}^2(\mathbb{R}^3) = \{D = (D_1, D_2, D_3) \text{ measurable} : \int_{\mathbb{R}^3} \epsilon^{-1} D \cdot D \, dx < +\infty\}$$

$$L_{\mu^{-1}}^2(\mathbb{R}^3) = \{B = (B_1, B_2, B_3) \text{ measurable} : \int_{\mathbb{R}^3} \mu^{-1} |B|^2 \, dx < +\infty\}$$

endowed with the scalar products

$$(D, \tilde{D})_{L^2_{\epsilon^{-1}}} = \int_{\mathbb{R}^3} \epsilon^{-1} D \cdot \tilde{D} \, dx, \quad D, \tilde{D} \in L^2_{\epsilon^{-1}}(\mathbb{R}^3)$$

$$(B, \tilde{B})_{L^2_{\mu^{-1}}} = \int_{\mathbb{R}^3} \mu^{-1} B \cdot \tilde{B} \, dx, \quad B, \tilde{B} \in L^2_{\mu^{-1}}(\mathbb{R}^3).$$

Notice that for any compact  $K \subset \mathbb{R}^3$  we have

$$\begin{aligned} \int_K |D| \, dx &= \int_K |\epsilon^{1/2} \epsilon^{-1/2} D| \, dx \leq M_K \int_K |\epsilon^{-1/2} D| \, dx \\ &\leq M_K \left( \int_K dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \epsilon^{-1} D \cdot D \, dx \right)^{1/2}, \quad D \in L^2_{\epsilon^{-1}}(\mathbb{R}^3) \end{aligned}$$

and

$$\int_K |B| \, dx = \mu^{1/2} \int_K \mu^{-1/2} |B| \, dx \leq \mu^{1/2} \left( \int_K dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \mu^{-1} |B|^2 \, dx \right)^{1/2}, \quad B \in L^2_{\mu^{-1}}(\mathbb{R}^3)$$

saying that  $L^2_{\epsilon^{-1}}(\mathbb{R}^3) \subset L^1_{\text{loc}}(\mathbb{R}^3)^3$ ,  $L^2_{\mu^{-1}}(\mathbb{R}^3) \subset L^1_{\text{loc}}(\mathbb{R}^3)^3$ . Therefore the following variational formulation makes sense.

### Definition 3.1

We say that  $(D, B) \in C(\mathbb{R}; L^2_{\epsilon^{-1}}(\mathbb{R}^3)) \times C(\mathbb{R}; L^2_{\mu^{-1}}(\mathbb{R}^3))$  is a variational solution of (1), (2) if for any  $(\Phi, \Psi) \in C_c^1(\mathbb{R}^3)^3 \times C_c^1(\mathbb{R}^3)^3$  we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} D(t, x) \cdot \Phi(x) \, dx - \int_{\mathbb{R}^3} \mu^{-1} B(t, x) \cdot \text{rot} \Phi \, dx = 0 \text{ in } \mathcal{D}'(\mathbb{R})$$

$$\frac{d}{dt} \int_{\mathbb{R}^3} B(t, x) \cdot \Psi(x) \, dx + \int_{\mathbb{R}^3} \epsilon^{-1} D(t, x) \cdot \text{rot} \Psi \, dx = 0 \text{ in } \mathcal{D}'(\mathbb{R}).$$

Notice that  $\epsilon^{-1} D$  is locally integrable, since

$$\int_K |\epsilon^{-1} D| \, dx \leq \int_K |\epsilon^{-1/2}| |\epsilon^{-1/2} D| \, dx \leq \frac{1}{m_K} \left( \int_K dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \epsilon^{-1} D \cdot D \, dx \right)^{1/2}$$

or by observing that  $\epsilon^{-1}$  is locally bounded and  $D$  is locally integrable. Thus the above definition makes sense.

The uniqueness of the variational solution comes by the following standard result.

### Proposition 3.1

Let  $\theta = \theta(t, x)$  be a function in  $C_c^1(\mathbb{R} \times \mathbb{R}^3)$ . Then any variational solution of (1), (2) satisfies in  $\mathcal{D}'(\mathbb{R})$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \theta [\epsilon^{-1}(x) D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2] \, dx &= \int_{\mathbb{R}^3} (\epsilon^{-1} D \wedge \mu^{-1} B) \cdot \nabla \theta \, dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} \partial_t \theta [\epsilon^{-1}(x) D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2] \, dx. \end{aligned}$$

In particular we have the energy conservation

$$\frac{d}{dt} \int_{\mathbb{R}^3} [\epsilon^{-1}(x)D(t, x) \cdot D(t, x) + \mu^{-1}|B(t, x)|^2] dx = 0.$$

For any  $(D^{\text{in}}, B^{\text{in}}) \in L_{\epsilon^{-1}}^2(\mathbb{R}^3) \times L_{\mu^{-1}}^2(\mathbb{R}^3)$ , there is at most one variational solution  $(D, B) \in C(\mathbb{R}; L_{\epsilon^{-1}}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  satisfying  $(D, B)(0) = (D^{\text{in}}, B^{\text{in}})$ .

For the existence of variational solution we appeal to the following well known result [13, 7].

**Theorem 3.1**

Let us consider two separable Hilbert spaces  $V, H$  and  $i : V \rightarrow H$  a bounded linear injective application, whose image  $i(V)$  is dense in  $H$ . Consider also a bounded bilinear symmetric application  $a : V \times V \rightarrow \mathbb{R}$ , which is coercive on  $V$  with respect to  $H$ , that is, there are  $\alpha > 0, C \geq 0$  such that for any  $v \in V$

$$a(v, v) + C|i(v)|_H^2 \geq \alpha\|v\|_V^2$$

and a bounded linear application  $l : V \rightarrow \mathbb{R}$ . Then for any  $u^0 \in V, u^1 \in H$  there is a unique function  $u \in C(\mathbb{R}, V)$  with  $u' \in C(\mathbb{R}, H)$  such that for any  $v \in V$

$$\frac{d}{dt}(u'(t), i(v))_H + a(u(t), v) + l(v) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R})$$

and  $u(0) = u^0, u'(0) = u^1$ . Moreover we have the conservation

$$\frac{1}{2}|u'(t)|_H^2 + \frac{1}{2}a(u(t), u(t)) + l(u(t)) = \frac{1}{2}|u^1|_H^2 + \frac{1}{2}a(u^0, u^0) + l(u^0), \quad t \in \mathbb{R}.$$

**Remark 3.1**

The notation  $u'$  stands for the time derivation of  $t \rightarrow iu(t)$  i.e.,

$$\frac{d}{dt}(iu(t), z)_H = (u'(t), z)_H \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \text{for any } z \in H.$$

As a direct consequence of Theorem 3.1, we obtain the existence of variational solution for (1), (2).

**Theorem 3.2**

For any  $(D^{\text{in}}, B^{\text{in}}) \in L_{\epsilon^{-1}}^2(\mathbb{R}^3) \times L_{\mu^{-1}}^2(\mathbb{R}^3)$  there is a unique variational solution  $(D, B) \in C(\mathbb{R}; L_{\epsilon^{-1}}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  for (1), (2). This solution satisfies the energy conservation

$$\int_{\mathbb{R}^3} (\epsilon^{-1}D(t, x) \cdot D(t, x) + \mu^{-1}|B(t, x)|^2) dx = \int_{\mathbb{R}^3} (\epsilon^{-1}D^{\text{in}} \cdot D^{\text{in}} + \mu^{-1}|B^{\text{in}}|^2) dx, \quad t \in \mathbb{R}.$$

If  $D^{\text{in}}, B^{\text{in}}$  are divergence free, so are  $D(t), B(t)$  for any  $t \in \mathbb{R}$ .

**Proof.**

We apply Theorem 3.1 with the separable Hilbert spaces

$$H = L^2(\mathbb{R}^3)^3, \quad V = \{\Psi \in L^2(\mathbb{R}^3)^3 : \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx < +\infty\}$$

endowed with the scalar products

$$(\Psi, \tilde{\Psi})_H = \int_{\mathbb{R}^3} \Psi \cdot \tilde{\Psi} \, dx, \quad \Psi, \tilde{\Psi} \in H, \quad ((\Psi, \tilde{\Psi}))_V = \int_{\mathbb{R}^3} (\Psi \cdot \tilde{\Psi} + \epsilon^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \tilde{\Psi}) \, dx, \quad \Psi, \tilde{\Psi} \in V.$$

We consider the imbedding  $i : V \rightarrow H, i(\Psi) = \Psi, \Psi \in V$ , the bilinear form

$$a : V \times V \rightarrow \mathbb{R}, \quad a(\Psi, \tilde{\Psi}) = \mu^{-1} \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \tilde{\Psi} \, dx, \quad \Psi, \tilde{\Psi} \in V$$

and the linear form

$$l : V \rightarrow \mathbb{R}, \quad l(\Psi) = \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot \operatorname{rot} \Psi \, dx, \quad \Psi \in V.$$

The fact that  $\epsilon^{1/2}$  is locally bounded from above allows us to establish that  $V$  is a Hilbert space. As  $\epsilon^{-1/2}$  is also locally bounded from above, we deduce that  $C_c^1(\mathbb{R}^3)^3 \subset V$  and  $C_c^1(\mathbb{R}^3)^3 = i(C_c^1(\mathbb{R}^3)^3) \subset H$ , implying that  $i(V)$  is dense in  $H$ . Obviously  $a(\cdot, \cdot)$  is a bounded bilinear symmetric application. It is also coercive on  $V$  with respect to  $H$

$$a(\Psi, \Psi) + \mu^{-1} |i(\Psi)|_H^2 = \mu^{-1} \|\Psi\|_V^2, \quad \Psi \in V.$$

Observe that  $l(\cdot)$  is a bounded linear application

$$\begin{aligned} |l(\Psi)| &\leq \left( \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot D^{\text{in}} \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx \right)^{1/2} \\ &\leq \left( \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot D^{\text{in}} \, dx \right)^{1/2} \|\Psi\|_V, \quad \Psi \in V. \end{aligned}$$

Therefore, by Theorem 3.1, there is a unique function  $U \in C(\mathbb{R}, V)$ , with  $\partial_t U \in C(\mathbb{R}, H)$  verifying for any  $\Psi \in V$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \partial_t U(t, x) \cdot \Psi(x) \, dx + \mu^{-1} \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} U(t, x) \cdot \operatorname{rot} \Psi(x) \, dx + \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot \operatorname{rot} \Psi \, dx = 0$$

in  $\mathcal{D}'(\mathbb{R})$  and  $U(0) = 0 \in V, \partial_t U(0) = B^{\text{in}} \in H$ . We claim that  $D = D^{\text{in}} + \mu^{-1} \operatorname{rot} U \in C(\mathbb{R}; L_{\epsilon^{-1}}^2(\mathbb{R}^3)), B = \partial_t U \in C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  is a variational solution for (1), (2). Indeed, for any  $\Phi \in C_c^1(\mathbb{R}^3)^3$  we have in  $\mathcal{D}'(\mathbb{R})$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} D(t, x) \cdot \Phi(x) \, dx &= \frac{d}{dt} \int_{\mathbb{R}^3} \mu^{-1} \operatorname{rot} U \cdot \Phi \, dx = \frac{d}{dt} \int_{\mathbb{R}^3} \mu^{-1} U \cdot \operatorname{rot} \Phi \, dx \\ &= \int_{\mathbb{R}^3} \mu^{-1} \partial_t U \cdot \operatorname{rot} \Phi \, dx = \int_{\mathbb{R}^3} \mu^{-1} B \cdot \operatorname{rot} \Phi \, dx. \end{aligned}$$

For any  $\Psi \in V$  we obtain in  $\mathcal{D}'(\mathbb{R})$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} B(t, x) \cdot \Psi(x) \, dx &= \frac{d}{dt} \int_{\mathbb{R}^3} \partial_t U \cdot \Psi \, dx = -\mu^{-1} \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} U \cdot \operatorname{rot} \Psi \, dx \\ &\quad - \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot \operatorname{rot} \Psi \, dx \\ &= - \int_{\mathbb{R}^3} \epsilon^{-1} (D^{\text{in}} + \mu^{-1} \operatorname{rot} U) \cdot \operatorname{rot} \Psi \, dx \\ &= - \int_{\mathbb{R}^3} \epsilon^{-1} D \cdot \operatorname{rot} \Psi \, dx. \end{aligned}$$

We know that

$$\frac{1}{2} \frac{d}{dt} \{ |\partial_t U|_H^2 + a(U(t), U(t)) + 2l(U(t)) \} = 0, \quad t \in \mathbb{R}$$

implying that

$$\frac{d}{dt} \left\{ \int_{\mathbb{R}^3} |B|^2 \, dx + \mu^{-1} \int_{\mathbb{R}^3} \epsilon^{-1} \operatorname{rot} U \cdot \operatorname{rot} U \, dx + 2 \int_{\mathbb{R}^3} \epsilon^{-1} D^{\text{in}} \cdot \operatorname{rot} U \, dx \right\} = 0.$$

We deduce the energy conservation

$$\frac{d}{dt} \left\{ \int_{\mathbb{R}^3} [\mu^{-1} |B(t, x)|^2 + \epsilon^{-1} D(t, x) \cdot D(t, x)] \, dx \right\} = 0, \quad t \in \mathbb{R}.$$

Using the Ampère equation (1) with  $\Phi = \nabla \theta, \theta \in C_c^2(\mathbb{R}^3)$  yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} D(t, x) \cdot \nabla \theta \, dx = \int_{\mathbb{R}^3} \mu^{-1} B(t, x) \cdot \operatorname{rot} \nabla \theta \, dx = 0$$

and thus  $\operatorname{div}(D(t) - D^{\text{in}}) = 0$ . Similarly, thanks to the Faraday equation (1) one gets  $\operatorname{div}(B(t) - B^{\text{in}}) = 0$ . If  $D^{\text{in}}, B^{\text{in}}$  are divergence free, so are  $D(t), B(t)$  for any  $t \in \mathbb{R}$ . The uniqueness of the solution  $(D, B)$  follows by Proposition 3.1 or thanks to the uniqueness part in Theorem 3.1.  $\square$

## 4 Weak convergence result

We analyze the behavior of the variational solutions of (1), (2) when the medium indexes, appearing in the electric permittivity tensor take disparate values

$$\epsilon_\delta = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e.$$

We assume that the unitary vector field  $e$  writes

$$e = \frac{\nabla \varphi}{|\nabla \varphi|} \tag{22}$$

for some function  $\varphi \in C^2(\mathbb{R}^3)$ ,  $\nabla \varphi(x) \neq 0$  a.a.  $x \in \mathbb{R}^3$ . We prove now the weak convergence result for the variational solutions of (1), (2) when  $\delta \searrow 0$ . We expect that the limit, when  $\delta \searrow 0$ , of the solutions  $(D^\delta, B^\delta)_{\delta > 0}$ , satisfies another variational problem, see Theorem 1.1.



**Proof.** (of Theorem 1.1)

Notice that for any  $0 < \delta \leq 1$  we have

$$\epsilon_\delta^{-1} = \frac{e_1 \otimes e_1}{n_1^2} + \frac{e_2 \otimes e_2}{n_2^2} + \frac{e \otimes e}{\delta^2 n^2} \geq \frac{e_1 \otimes e_1}{n_1^2} + \frac{e_2 \otimes e_2}{n_2^2} + \frac{e \otimes e}{n^2} = \epsilon_1^{-1}$$

implying that

$$\sup_{0 < \delta \leq 1} \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^{\delta, \text{in}} \cdot D^{\delta, \text{in}} + \mu^{-1} |B^{\delta, \text{in}}|^2] dx \leq \sup_{0 < \delta \leq 1} \int_{\mathbb{R}^3} [\epsilon_\delta^{-1} D^{\delta, \text{in}} \cdot D^{\delta, \text{in}} + \mu^{-1} |B^{\delta, \text{in}}|^2] dx < +\infty.$$

The boundedness of the family  $(D^{\delta, \text{in}}, B^{\delta, \text{in}})_{0 < \delta \leq 1}$  in  $L_{\epsilon_1}^2(\mathbb{R}^3) \times L_{\mu^{-1}}^2(\mathbb{R}^3)$  allows us to extract a sequence  $(\delta_k)_k \subset ]0, 1]$ , converging toward 0 such that

$$D^{\delta_k, \text{in}} \rightharpoonup D^{\text{in}} \text{ weakly in } L_{\epsilon_1}^2(\mathbb{R}^3)$$

$$B^{\delta_k, \text{in}} \rightharpoonup B^{\text{in}} \text{ weakly in } L_{\mu^{-1}}^2(\mathbb{R}^3).$$

We also have

$$\sup_k \frac{1}{\delta_k^2} \int_{\mathbb{R}^3} \frac{(D^{\delta_k, \text{in}} \cdot e)^2}{n^2} dx < +\infty$$

and therefore, for any  $\theta \in C_c(\mathbb{R}^3)$  we obtain

$$\int_{\mathbb{R}^3} (D^{\text{in}} \cdot e) \theta(x) dx = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^3} (D^{\delta_k, \text{in}} \cdot e) \theta(x) dx = 0$$

saying that  $D^{\text{in}} \cdot e = 0$ . By the energy conservation we have

$$\begin{aligned} & \sup_{0 < \delta \leq 1, t \in \mathbb{R}} \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2] dx \\ & \leq \sup_{0 < \delta \leq 1, t \in \mathbb{R}} \int_{\mathbb{R}^3} [\epsilon_\delta^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2] dx \\ & = \sup_{0 < \delta \leq 1} \int_{\mathbb{R}^3} [\epsilon_\delta^{-1} D^{\delta, \text{in}}(x) \cdot D^{\delta, \text{in}}(x) + \mu^{-1} |B^{\delta, \text{in}}(x)|^2] dx < +\infty. \end{aligned} \tag{23}$$

After a new extraction, there is a sequence, still denoted by  $(\delta_k)_k \subset ]0, 1]$  such that

$$D^{\delta_k} \rightharpoonup D \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)), \quad B^{\delta_k} \rightharpoonup B \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$$

when  $k \rightarrow +\infty$ . By (23) we have

$$\sup_{t \in \mathbb{R}, k} \frac{1}{\delta_k^2} \int_{\mathbb{R}^3} \frac{(D^{\delta_k}(t, x) \cdot e)^2}{n^2} dx < +\infty$$

and therefore we obtain

$$D \cdot e = 0.$$

As  $\text{div} D^{\delta_k, \text{in}} = 0$ ,  $\text{div} B^{\delta_k, \text{in}} = 0$ , we also have  $\text{div} D^{\delta_k}(t) = 0$ ,  $\text{div} B^{\delta_k}(t) = 0$ ,  $t \in \mathbb{R}$ , and thus

$$\text{div} D = 0, \quad \text{div} B = 0.$$

For any  $\Phi \in C_c^1(\mathbb{R}^3)^3$  and  $k \in \mathbb{N}$  we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} D^{\delta_k} \cdot \Phi \, dx - \int_{\mathbb{R}^3} \mu^{-1} B^{\delta_k} \cdot \operatorname{rot} \Phi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (24)$$

After passing to the limit when  $k \rightarrow +\infty$  one gets

$$\frac{d}{dt} \int_{\mathbb{R}^3} D \cdot \Phi \, dx - \int_{\mathbb{R}^3} \mu^{-1} B \cdot \operatorname{rot} \Phi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R})$$

saying that

$$\partial_t D - \mu^{-1} \operatorname{rot} B = 0. \quad (25)$$

In order to pass to the limit in the Faraday equation (1) we consider  $\Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$  such that  $\operatorname{div}(\Psi \wedge \nabla \varphi) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx < +\infty$ . The variational formulation of (2) writes

$$\frac{d}{dt} \int_{\mathbb{R}^3} B^{\delta_k}(t, x) \cdot \Psi(x) \, dx + \int_{\mathbb{R}^3} \epsilon_{\delta_k}^{-1} D^{\delta_k}(t, x) \cdot \operatorname{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (26)$$

By the formula  $\operatorname{rot} \Psi \cdot \nabla \varphi = \operatorname{div}(\Psi \wedge \nabla \varphi) = 0$ , we deduce that  $\frac{e \otimes e}{n^2} D^{\delta_k} \cdot \operatorname{rot} \Psi = 0$ . Therefore we can write  $\epsilon_{\delta_k}^{-1} D^{\delta_k}(t, x) \cdot \operatorname{rot} \Psi = \epsilon_1^{-1} D^{\delta_k}(t, x) \cdot \operatorname{rot} \Psi$  and (26) becomes

$$\frac{d}{dt} \int_{\mathbb{R}^3} B^{\delta_k}(t, x) \cdot \Psi(x) \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D^{\delta_k}(t, x) \cdot \operatorname{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (27)$$

Passing to the limit when  $k \rightarrow +\infty$ , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} B(t, x) \cdot \Psi(x) \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D(t, x) \cdot \operatorname{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (28)$$

for any  $\Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$  such that  $\operatorname{div}(\Psi \wedge \nabla \varphi) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx < +\infty$ . Actually the previous formula holds true for any  $\Psi \in L^2(\mathbb{R}^3)^3$  such that  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi \, dx < +\infty$  and  $\operatorname{div}(\Psi \wedge \nabla \varphi) = 0$ . We claim that  $\operatorname{div}(B \wedge \nabla \varphi) = 0$ . Coming back to the Ampère equation (25), we obtain for any  $\chi \in C_c^1(\mathbb{R}^3)$

$$\frac{d}{dt} \int_{\mathbb{R}^3} (D(t, x) \cdot \nabla \varphi) \chi(x) \, dx - \int_{\mathbb{R}^3} \mu^{-1} B(t, x) \cdot \operatorname{rot}(\chi \nabla \varphi) \, dx = 0.$$

But we know that  $D \cdot e = 0$ , implying that

$$\int_{\mathbb{R}^3} \nabla \chi \cdot (B \wedge \nabla \varphi) \, dx = - \int_{\mathbb{R}^3} B(t, x) \cdot (\nabla \chi \wedge \nabla \varphi) \, dx = 0, \quad \chi \in C_c^1(\mathbb{R}^3)$$

and therefore  $\operatorname{div}(B \wedge \nabla \varphi) = 0$ . By standard arguments we check that  $(D, B)$  have traces in  $C(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$ . We concentrate now on the initial conditions. For any  $\Phi \in C_c^1(\mathbb{R}^3)^3$ ,  $t \in \mathbb{R}$  we have by (24)

$$\int_{\mathbb{R}^3} D^{\delta_k}(t, x) \cdot \Phi(x) \, dx - \int_{\mathbb{R}^3} D^{\delta_k, \text{in}}(x) \cdot \Phi(x) \, dx - \int_0^t \int_{\mathbb{R}^3} \mu^{-1} B^{\delta_k}(s, x) \cdot \operatorname{rot} \Phi \, dx ds = 0.$$

Letting  $k \rightarrow +\infty$  yields

$$\int_{\mathbb{R}^3} D(t, x) \cdot \Phi(x) \, dx - \int_{\mathbb{R}^3} D^{\text{in}}(x) \cdot \Phi(x) \, dx - \int_0^t \int_{\mathbb{R}^3} \mu^{-1} B(s, x) \cdot \text{rot} \Phi \, dx ds = 0$$

implying that

$$\int_{\mathbb{R}^3} D^{\text{in}}(x) \cdot \Phi(x) \, dx = \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} D(t, x) \cdot \Phi(x) \, dx = \int_{\mathbb{R}^3} D(0, x) \cdot \Phi(x) \, dx, \quad \Phi \in C_c^1(\mathbb{R}^3)^3.$$

We deduce that  $D(0) = D^{\text{in}}$ . Similarly, by (27), we write for any  $\Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$ ,  $\text{div}(\Psi \wedge \nabla \varphi) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \Psi \cdot \text{rot} \Psi \, dx < +\infty$

$$\int_{\mathbb{R}^3} B^{\delta k}(t, x) \cdot \Psi(x) \, dx - \int_{\mathbb{R}^3} B^{\delta k, \text{in}}(x) \cdot \Psi(x) \, dx + \int_0^t \int_{\mathbb{R}^3} \epsilon_1^{-1} D^{\delta k}(s, x) \cdot \text{rot} \Psi \, dx ds = 0.$$

Letting  $k \rightarrow +\infty$  gives

$$\int_{\mathbb{R}^3} B(t, x) \cdot \Psi(x) \, dx - \int_{\mathbb{R}^3} B^{\text{in}}(x) \cdot \Psi(x) \, dx + \int_0^t \int_{\mathbb{R}^3} \epsilon_1^{-1} D(s, x) \cdot \text{rot} \Psi \, dx ds = 0.$$

Recall that  $P$  stands for the orthogonal projection in  $L^2(\mathbb{R}^3)^3$  on  $\{\tilde{\Psi} \in L^2(\mathbb{R}^3)^3 : \text{div}(\tilde{\Psi} \wedge \nabla \varphi) = 0\}$  and therefore we obtain

$$\int_{\mathbb{R}^3} P(B^{\text{in}}) \cdot \Psi(x) \, dx = \int_{\mathbb{R}^3} B^{\text{in}}(x) \cdot \Psi(x) \, dx = \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} B(t, x) \cdot \Psi(x) \, dx = \int_{\mathbb{R}^3} B(0, x) \cdot \Psi(x) \, dx.$$

As we know that  $\text{div}(B(0) \wedge \nabla \varphi) = 0$ , it follows that  $B(0) = P(B^{\text{in}})$ .  $\square$

#### Remark 4.1

The well posedness of the model (3), (4) follows by Theorem 3.1 when considering the separable Hilbert spaces

$$\mathcal{H} = \{\Psi \in L^2(\mathbb{R}^3)^3 : \text{div}(\Psi \wedge \nabla \varphi) = 0\}$$

$$\mathcal{V} = \{\Psi \in L^2(\mathbb{R}^3)^3 : \int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \Psi \cdot \text{rot} \Psi \, dx < +\infty, \text{div}(\Psi \wedge \nabla \varphi) = 0\}$$

endowed with the scalar products

$$(\Psi, \tilde{\Psi})_{\mathcal{H}} = \int_{\mathbb{R}^3} \Psi \cdot \tilde{\Psi} \, dx, \quad \Psi, \tilde{\Psi} \in \mathcal{H}$$

$$((\Psi, \tilde{\Psi}))_{\mathcal{V}} = \int_{\mathbb{R}^3} \Psi \cdot \tilde{\Psi} \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \Psi \cdot \text{rot} \tilde{\Psi} \, dx, \quad \Psi, \tilde{\Psi} \in \mathcal{V}.$$

We consider the imbedding  $i : \mathcal{V} \rightarrow \mathcal{H}$ ,  $i(\Psi) = \Psi$ ,  $\Psi \in \mathcal{V}$ , the bilinear form

$$\mathbf{a} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}, \quad \mathbf{a}(\Psi, \tilde{\Psi}) = \mu^{-1} \int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \Psi \cdot \text{rot} \tilde{\Psi} \, dx, \quad \Psi, \tilde{\Psi} \in \mathcal{V}$$

and the linear form

$$l : \mathcal{V} \rightarrow \mathbb{R}, \quad l(\Psi) = \int_{\mathbb{R}^3} \epsilon_1^{-1} D^{\text{in}} \cdot \text{rot} \Psi \, dx, \quad \Psi \in \mathcal{V}.$$

It is easily seen that  $\mathbf{a}(\cdot, \cdot)$  is a bounded bilinear symmetric application, coercive on  $\mathcal{V}$  with respect to  $\mathcal{H}$

$$\mathbf{a}(\Psi, \Psi) + \mu^{-1} |i(\Psi)|_{\mathcal{H}}^2 = \mu^{-1} \|\Psi\|_{\mathcal{V}}^2, \quad \Psi \in \mathcal{V}$$

and  $l(\cdot)$  is a bounded linear application on  $\mathcal{V}$ . Appealing to Theorem 3.1 there is a unique function  $\mathcal{U} \in C(\mathbb{R}, \mathcal{V})$  with  $\partial_t \mathcal{U} \in C(\mathbb{R}, \mathcal{H})$  verifying for any  $\Psi \in \mathcal{V}$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \partial_t \mathcal{U} \cdot \Psi \, dx + \mu^{-1} \int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \mathcal{U} \cdot \text{rot} \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D^{\text{in}} \cdot \text{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R})$$

and  $\mathcal{U}(0) = 0 \in \mathcal{V}$ ,  $\partial_t \mathcal{U}(0) = P(B^{\text{in}}) \in \mathcal{H}$ . As before we check that  $D = D^{\text{in}} + \mu^{-1} \text{rot} \mathcal{U} \in C(\mathbb{R}; L_{\epsilon_1^{-1}}^2(\mathbb{R}^3))$  and  $B = \partial_t \mathcal{U} \in C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  give a variational solution for (3), (4).

The conservation

$$\frac{1}{2} \frac{d}{dt} \{ |\partial_t \mathcal{U}|_{\mathcal{H}}^2 + \mathbf{a}(\mathcal{U}(t), \mathcal{U}(t)) + 2l(\mathcal{U}(t)) \} = 0, \quad t \in \mathbb{R}$$

leads to the energy conservation

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \{ \epsilon_1^{-1} D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2 \} \, dx = 0, \quad t \in \mathbb{R}.$$

The uniqueness of the solution of (3), (4) comes by the uniqueness part in Theorem 3.1. It is easily seen that the constraint  $D \cdot e = 0$  is propagated in time. For any  $\chi \in C_c^1(\mathbb{R}^3)$  one gets, thanks to the constraint  $\text{div}(B \wedge \nabla \varphi) = 0$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} (D \cdot \nabla \varphi) \chi(x) \, dx &= \int_{\mathbb{R}^3} \mu^{-1} B \cdot \text{rot}(\chi \nabla \varphi) \, dx \\ &= \int_{\mathbb{R}^3} \mu^{-1} B \cdot (\nabla \chi \wedge \nabla \varphi) \, dx \\ &= - \int_{\mathbb{R}^3} \mu^{-1} \nabla \chi \cdot (B \wedge \nabla \varphi) \, dx = 0 \end{aligned}$$

saying that  $(D(t) \cdot e) = (D(0) \cdot e) = 0, t \in \mathbb{R}$ . The divergence constraints are preserved as well  $\text{div} D(t) = \text{div} D(0), \text{div} B(t) = \text{div} B(0), t \in \mathbb{R}$ .

Solving for the variational formulation (3), (4) while testing against vector fields  $\Psi$  satisfying the constraint  $\text{div}(\Psi \wedge \nabla \varphi) = 0$  is a difficult task. Instead, we investigate equivalent formulations, by getting rid of this constraint, cf. Theorem 1.2. We prove now Theorem 1.2, assuming that  $\varphi(x) = x \cdot e, x \in \mathbb{R}^3$ , for some unitary vector  $e \in \mathbb{R}^3$ . We establish several preliminary lemmas.

#### Lemma 4.1

Let  $e \in \mathbb{R}^3$  be a unitary vector. We consider the vector fields  $\xi, \Psi \in L^2(\mathbb{R}^3)^3$  satisfying

$$\text{div} \xi = 0, \quad \xi \cdot e = 0, \quad \text{div}(\Psi \wedge e) = 0.$$

Then we have  $\int_{\mathbb{R}^3} \xi(x) \cdot \Psi(x) \, dx = 0$ .

**Proof.**

We fix a direct orthonormal basis  $\{\tilde{e}_1, \tilde{e}_2, e\}$  of  $\mathbb{R}^3$ . For any vector field  $\eta$  on  $\mathbb{R}^3$ , we denote  $\tilde{\eta}$  the vector field given by

$$\tilde{\eta}(\tilde{y}_1, \tilde{y}_2, z) = ( (\eta(x) \cdot \tilde{e}_1), (\eta(x) \cdot \tilde{e}_2), (\eta(x) \cdot e) ), \quad x = \tilde{y}_1 \tilde{e}_1 + \tilde{y}_2 \tilde{e}_2 + ze, \quad (\tilde{y}_1, \tilde{y}_2, z) \in \mathbb{R}^3.$$

Obviously we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\tilde{\xi}|^2(\tilde{y}_1, \tilde{y}_2, z) \, d(\tilde{y}_1, \tilde{y}_2) dz &= \int_{\mathbb{R}^3} |\xi|^2(x) \, dx < +\infty \\ \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\tilde{\Psi}|^2(\tilde{y}_1, \tilde{y}_2, z) \, d(\tilde{y}_1, \tilde{y}_2) dz &= \int_{\mathbb{R}^3} |\Psi|^2(x) \, dx < +\infty \end{aligned}$$

implying that  $(\tilde{\xi}_1, \tilde{\xi}_2)(\cdot, \cdot, z), (\tilde{\Psi}_1, \tilde{\Psi}_2)(\cdot, \cdot, z) \in L^2(\mathbb{R}^2)^2$  for almost all  $z \in \mathbb{R}$ . We claim that  $(\tilde{\xi}_1, \tilde{\xi}_2)(\cdot, \cdot, z)$  is a divergence free vector field in  $\mathbb{R}^2$ . Let us consider  $\tilde{\alpha} = \tilde{\alpha}(z) \in C_c^1(\mathbb{R}), \tilde{\theta} = \tilde{\theta}(\tilde{y}_1, \tilde{y}_2) \in C_c^1(\mathbb{R}^2)$  and we introduce  $\alpha(x) = \tilde{\alpha}(x \cdot e), \theta(x) = \tilde{\theta}(x \cdot \tilde{e}_1, x \cdot \tilde{e}_2)$ . Obviously we have  $\alpha\theta \in C_c^1(\mathbb{R}^3)$  and therefore

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \xi(x) \cdot \nabla(\alpha\theta) \, dx \\ &= \int_{\mathbb{R}} \tilde{\alpha}(z) \int_{\mathbb{R}^2} [\tilde{\xi}_1(\tilde{y}_1, \tilde{y}_2, z) \partial_{\tilde{y}_1} \tilde{\theta}(\tilde{y}_1, \tilde{y}_2) + \tilde{\xi}_2(\tilde{y}_1, \tilde{y}_2, z) \partial_{\tilde{y}_2} \tilde{\theta}(\tilde{y}_1, \tilde{y}_2)] \, d(\tilde{y}_1, \tilde{y}_2) dz \end{aligned}$$

saying that  $\operatorname{div}_{\tilde{y}}(\tilde{\xi}_1, \tilde{\xi}_2)(\cdot, \cdot, z) = 0$ . Similarly we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} (\Psi(x) \wedge e) \cdot \nabla(\alpha\theta) \, dx \\ &= \int_{\mathbb{R}^3} [-(\Psi(x) \cdot \tilde{e}_1) \tilde{e}_2 + (\Psi(x) \cdot \tilde{e}_2) \tilde{e}_1] \cdot [\partial_z \tilde{\alpha} \tilde{\theta} e + \tilde{\alpha}(\partial_{\tilde{y}_1} \tilde{\theta} \tilde{e}_1 + \partial_{\tilde{y}_2} \tilde{\theta} \tilde{e}_2)] \, dx \\ &= \int_{\mathbb{R}} \tilde{\alpha}(z) \int_{\mathbb{R}^2} [\tilde{\Psi}_2(\tilde{y}_1, \tilde{y}_2, z) \partial_{\tilde{y}_1} \tilde{\theta} - \tilde{\Psi}_1(\tilde{y}_1, \tilde{y}_2, z) \partial_{\tilde{y}_2} \tilde{\theta}] \, d(\tilde{y}_1, \tilde{y}_2) dz \end{aligned}$$

and thus  $\operatorname{div}_{\tilde{y}}(\tilde{\Psi}_2, -\tilde{\Psi}_1)(\cdot, \cdot, z) = 0$ . Combining  $\operatorname{div}_{\tilde{y}}(\tilde{\xi}_1, \tilde{\xi}_2)(\cdot, \cdot, z) = 0, \operatorname{div}_{\tilde{y}}(\tilde{\Psi}_2, -\tilde{\Psi}_1)(\cdot, \cdot, z) = 0$  we deduce that

$$\int_{\mathbb{R}^2} (\tilde{\xi}_1 \tilde{\Psi}_1 + \tilde{\xi}_2 \tilde{\Psi}_2)(\tilde{y}_1, \tilde{y}_2, z) \, d(\tilde{y}_1, \tilde{y}_2) = 0, \quad z \in \mathbb{R}$$

and by noticing that  $\tilde{\xi}_3 = 0$ , we obtain

$$\int_{\mathbb{R}^3} \xi(x) \cdot \Psi(x) \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\tilde{\xi}_1 \tilde{\Psi}_1 + \tilde{\xi}_2 \tilde{\Psi}_2)(\tilde{y}_1, \tilde{y}_2, z) \, d(\tilde{y}_1, \tilde{y}_2) dz = 0.$$

□

**Lemma 4.2**

Let us consider the set  $\mathcal{C}_e = \{\Psi \in L^2(\mathbb{R}^3)^3 : \operatorname{div}(\Psi \wedge e) = 0\} \subset L^2(\mathbb{R}^3)^3$ , where  $e \in \mathbb{R}^3$  is a unitary vector. The orthogonal of  $\mathcal{C}_e$  in  $L^2(\mathbb{R}^3)^3$  is given by

$$\mathcal{C}_e^\perp = \{\xi \in L^2(\mathbb{R}^3)^3 : \operatorname{div} \xi = 0, \xi \cdot e = 0\}.$$

**Proof.**

By Lemma 4.1 we know that

$$\{\xi \in L^2(\mathbb{R}^3)^3 : \operatorname{div}\xi = 0, \xi \cdot e = 0\} \subset \mathcal{C}_e^\perp. \quad (29)$$

Consider now  $\xi \in \mathcal{C}_e^\perp$ . For any function  $u \in C_c^2(\mathbb{R}^3)$ , we have

$$\nabla u \in L^2(\mathbb{R}^3)^3, \operatorname{div}(\nabla u \wedge e) = \operatorname{rot}\nabla u \cdot e = 0$$

saying that  $\nabla u \in \mathcal{C}_e$ . Therefore  $\int_{\mathbb{R}^3} \xi \cdot \nabla u \, dx = 0$ , that is  $\operatorname{div}\xi = 0$ . For any  $\alpha \in C_c^1(\mathbb{R}^3)$  we have

$$\alpha e \in L^2(\mathbb{R}^3)^3, \operatorname{div}(\alpha e \wedge e) = 0$$

saying that  $\alpha e \in \mathcal{C}_e$ . We deduce that  $\int_{\mathbb{R}^3} \alpha(x) e \cdot \xi(x) \, dx = 0$  and thus  $\xi \cdot e = 0$ . We obtain

$$\mathcal{C}_e^\perp \subset \{\xi \in L^2(\mathbb{R}^3)^3 : \operatorname{div}\xi = 0, \xi \cdot e = 0\}. \quad (30)$$

Our conclusion follows by (29), (30).  $\square$

We denote by  $P$  the orthogonal projection in  $L^2(\mathbb{R}^3)^3$  over the closed subspace  $\mathcal{C}_e$ . For further developments we need the following regularity result.

**Lemma 4.3**

Let  $\psi$  be a vector field in  $L^2(\mathbb{R}^3)^3$  and  $\Psi \in \mathcal{C}_e, \xi \in \mathcal{C}_e^\perp$  such that  $\psi = \Psi + \xi$ . If  $\operatorname{rot}\psi \in L^2(\mathbb{R}^3)^3$ , then  $\operatorname{rot}\Psi, \operatorname{rot}\xi \in L^2(\mathbb{R}^3)^3$  and we have

$$\|\operatorname{rot}\psi\|_{L^2}^2 = \|\operatorname{rot}\Psi\|_{L^2}^2 + \|\operatorname{rot}\xi\|_{L^2}^2.$$

**Proof.**

We consider  $\rho \in C_c^\infty(\mathbb{R}^3), \rho \geq 0, \int_{\mathbb{R}^3} \rho(x) \, dx = 1$  and  $\rho_k = k^3 \rho(k \cdot), k \in \mathbb{N}^*$ . Let us introduce the vector fields

$$\psi_k = \psi \star \rho_k, \quad \Psi_k = \Psi \star \rho_k, \quad \xi_k = \xi \star \rho_k, \quad k \in \mathbb{N}^*.$$

We have

$$\psi_k = \Psi_k + \xi_k, \quad \Psi_k \in \mathcal{C}_e, \quad \xi_k \in \mathcal{C}_e^\perp, \quad \operatorname{rot}\psi_k = \operatorname{rot}\psi \star \rho_k \in L^2(\mathbb{R}^3)^3$$

$$\operatorname{rot}\Psi_k = \int_{\mathbb{R}^3} \nabla \rho_k(\cdot - y) \wedge \Psi(y) \, dy \in L^2(\mathbb{R}^3)^3, \quad \operatorname{rot}\xi_k = \int_{\mathbb{R}^3} \nabla \rho_k(\cdot - y) \wedge \xi(y) \, dy \in L^2(\mathbb{R}^3)^3.$$

Notice that

$$\operatorname{div}\operatorname{rot}\Psi_k = 0, \quad \operatorname{rot}\Psi_k \cdot e = \operatorname{div}(\Psi_k \wedge e) = 0$$

saying that  $\operatorname{rot}\Psi_k \in \mathcal{C}_e^\perp$ . The notation  $\partial\xi_k$  stands for the jacobian matrix of the vector field  $\xi_k$ . Observing that

$$\operatorname{rot}\xi_k \wedge e = (\partial\xi_k - {}^t\partial\xi_k)e = (e \cdot \nabla)\xi_k - \nabla(e \cdot \xi_k) = (e \cdot \nabla)\xi_k$$

we obtain

$$\operatorname{div}(\operatorname{rot}\xi_k \wedge e) = \operatorname{div}((e \cdot \nabla)\xi_k) = (e \cdot \nabla)\operatorname{div}\xi_k = 0$$

saying that  $\operatorname{rot}\xi_k \in \mathcal{C}_e$ . We deduce that  $\int_{\mathbb{R}^3} \operatorname{rot}\Psi_k \cdot \operatorname{rot}\xi_l \, dx = 0, k, l \in \mathbb{N}^*$  and therefore

$$\int_{\mathbb{R}^3} |\operatorname{rot}\psi_k - \operatorname{rot}\psi_l|^2 \, dx = \int_{\mathbb{R}^3} |\operatorname{rot}\Psi_k - \operatorname{rot}\Psi_l|^2 \, dx + \int_{\mathbb{R}^3} |\operatorname{rot}\xi_k - \operatorname{rot}\xi_l|^2 \, dx, \quad k, l \in \mathbb{N}^*.$$

As  $(\text{rot}\psi_k)_k$  converges toward  $\text{rot}\psi$  in  $L^2(\mathbb{R}^3)^3$ , we deduce that  $(\text{rot}\Psi_k)_k, (\text{rot}\xi_k)_k$  are Cauchy sequences, and thus convergent sequences in  $L^2(\mathbb{R}^3)^3$ . Therefore we obtain

$$\text{rot}\Psi = \lim_{k \rightarrow +\infty} \text{rot}\Psi_k \in L^2(\mathbb{R}^3)^3, \quad \text{rot}\xi = \lim_{k \rightarrow +\infty} \text{rot}\xi_k \in L^2(\mathbb{R}^3)^3$$

and

$$\|\text{rot}\psi\|_{L^2}^2 = \lim_{k \rightarrow +\infty} \|\text{rot}\psi_k\|_{L^2}^2 = \lim_{k \rightarrow +\infty} \{\|\text{rot}\Psi_k\|_{L^2}^2 + \|\text{rot}\xi_k\|_{L^2}^2\} = \|\text{rot}\Psi\|_{L^2}^2 + \|\text{rot}\xi\|_{L^2}^2.$$

□

In the next lemma we observe that the rot operator maps  $\mathcal{C}_e$  to  $\mathcal{C}_e^\perp$  and  $\mathcal{C}_e^\perp$  to  $\mathcal{C}_e$ .

**Lemma 4.4**

1. For any vector field  $\Psi \in \mathcal{C}_e$  such that  $\text{rot}\Psi \in L^2(\mathbb{R}^3)^3$ , we have  $\text{rot}\Psi \in \mathcal{C}_e^\perp$ .
2. For any vector field  $\xi \in \mathcal{C}_e^\perp$  such that  $\text{rot}\xi \in L^2(\mathbb{R}^3)^3$ , we have  $\text{rot}\xi \in \mathcal{C}_e$ .

**Proof.**

1. If  $\Psi \in \mathcal{C}_e$  such that  $\text{rot}\Psi \in L^2(\mathbb{R}^3)^3$ , then we have  $\text{rot}\Psi \cdot e = \text{div}(\Psi \wedge e) = 0$ ,  $\text{div}\text{rot}\Psi = 0$ , saying that  $\text{rot}\Psi \in \mathcal{C}_e^\perp$ .
2. Consider a vector field  $\xi \in \mathcal{C}_e^\perp$  such that  $\text{rot}\xi \in L^2(\mathbb{R}^3)^3$ . For any compactly supported smooth function  $\theta \in C_c^\infty(\mathbb{R}^3)$  we have

$$\begin{aligned} \langle \text{div}(\text{rot}\xi \wedge e), \theta \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} &= - \int_{\mathbb{R}^3} (\text{rot}\xi \wedge e) \cdot \nabla \theta \, dx \\ &= \int_{\mathbb{R}^3} (\nabla \theta \wedge e) \cdot \text{rot}\xi \, dx \\ &= \int_{\mathbb{R}^3} \xi \cdot \text{rot}(\nabla \theta \wedge e) \, dx \\ &= \int_{\mathbb{R}^3} [\nabla(e \cdot \nabla \theta) - \Delta \theta e] \cdot \xi \, dx \\ &= 0 \end{aligned} \tag{31}$$

since  $\text{div}\xi = 0$  and  $\xi \cdot e = 0$ . Therefore we obtain  $\text{div}(\text{rot}\xi \wedge e) = 0$ , saying that  $\text{rot}\xi \in \mathcal{C}_e$ . □

We are ready to prove Theorem 1.2.

**Proof.** (of Theorem 1.2)

We show the equivalence between (6) and (9). The operator  $Q : L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3$  is the orthogonal projection on  $\mathcal{C}_e^\perp$  cf. Lemma 4.2. Therefore we have  $Q = Id - P$ , where  $P$  is the orthogonal projection on  $\mathcal{C}_e$ . Assume that (9) holds true. For any  $\Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$  such that

$$\text{div}(\Psi \wedge e) = 0, \quad \int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot}\Psi \cdot \text{rot}\Psi \, dx < +\infty$$

we have

$$\text{rot}\Psi \cdot e = \text{div}(\Psi \wedge e) = 0, \quad \text{div}\text{rot}\Psi = 0, \quad \text{rot}\Psi = \epsilon_1^{1/2}(\epsilon_1^{-1/2} \text{rot}\Psi) \in L^2(\mathbb{R}^3)^3$$

saying that  $\text{rot}\Psi \in \mathcal{C}_e^\perp$ . We deduce that  $Q(\text{rot}\Psi) = \text{rot}\Psi$ . Noticing that  $\epsilon_1^{-1}D = \epsilon_1^{-1/2}(\epsilon_1^{-1/2}D)$  belongs to  $L^2(\mathbb{R}^3)^3$ , we obtain in  $\mathcal{D}'(\mathbb{R})$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1}D \cdot \text{rot}\Psi \, dx &= \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1}D \cdot Q(\text{rot}\Psi) \, dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\Psi \, dx \\ &= 0. \end{aligned}$$

Thus (6) holds true. Conversely, assume now that (6) holds true and let us check that (9) holds also true. Consider  $\psi \in L^2(\mathbb{R}^3)^3$  such that  $\text{rot}\psi \in L^2(\mathbb{R}^3)^3$ . By Lemma 4.3, we have

$$\psi = \Psi + \xi, \quad \Psi = P\psi \in \mathcal{C}_e, \quad \xi = Q\psi \in \mathcal{C}_e^\perp, \quad \text{rot}\Psi, \text{rot}\xi \in L^2(\mathbb{R}^3)^3.$$

We are done if we establish (9) when using the test vector fields  $\Psi$  and  $\xi$ . Concerning the vector field  $\Psi$  we have in  $\mathcal{D}'(\mathbb{R})$ , thanks to (6), and by noticing that  $\text{rot}\Psi \in L^2(\mathbb{R}^3)^3$  belongs to  $\mathcal{C}_e^\perp$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\Psi \, dx &= \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1}D \cdot Q(\text{rot}\Psi) \, dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1}D \cdot \text{rot}\Psi \, dx \\ &= 0. \end{aligned}$$

When considering the vector field  $\xi$ , observe that the constraint  $\text{div}(B \wedge e) = 0$  implies, by Lemma 4.1

$$\int_{\mathbb{R}^3} B \cdot \xi \, dx = 0. \quad (32)$$

By the second statement in Lemma 4.4 we know that  $\text{rot}\xi \in L^2(\mathbb{R}^3)^3$  belongs to  $\mathcal{C}_e$ . By the orthogonality of the elements in  $\mathcal{C}_e, \mathcal{C}_e^\perp$ , we deduce that

$$\int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\xi \, dx = 0 \quad (33)$$

and combining (32), (33) yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \xi \, dx + \int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\xi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

□

#### Remark 4.2

If the initial conditions  $(D^{\text{in}}, B^{\text{in}})$  satisfy the constraints in (10), then these constraints propagate in time. Indeed, let us consider a vector field  $\xi \in \mathcal{C}_e^\perp$  such that  $\text{rot}\xi \in L^2(\mathbb{R}^3)^3$ . By the second statement in Lemma 4.4 we know that  $\text{rot}\xi \in \mathcal{C}_e$ . Therefore we have  $\int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\xi \, dx = 0$  and thanks to Faraday equation in (9), one gets

$$\frac{d}{dt} \int_{\mathbb{R}^3} B(t, x) \cdot \xi(x) \, dx = \frac{d}{dt} \int_{\mathbb{R}^3} B(t, x) \cdot \xi(x) \, dx + \int_{\mathbb{R}^3} Q(\epsilon_1^{-1}D) \cdot \text{rot}\xi \, dx = 0.$$



We deduce that  $\int_{\mathbb{R}^3} (B(t, x) - B^{\text{in}}(x)) \cdot \xi(x) \, dx = 0$  for any  $\xi \in \mathcal{C}_e^\perp$  such that  $\text{rot} \xi \in L^2(\mathbb{R}^3)^3$ . Actually the previous equality holds true for any  $\xi \in \mathcal{C}_e^\perp$ , implying that  $B(t) - B^{\text{in}} \in \mathcal{C}_e$ . Therefore we have

$$\text{div}(B(t) \wedge e) = \text{div}(B^{\text{in}} \wedge e) = 0, \quad t \in \mathbb{R}.$$

Using now Ampère equation in (8) with the vector field  $\Phi(x) = \alpha(x)e, \alpha \in C_c^1(\mathbb{R}^3)$  yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} D(t, x) \cdot e \, \alpha(x) \, dx &= \int_{\mathbb{R}^3} \mu^{-1} B(t, x) \cdot (\nabla \alpha \wedge e) \, dx \\ &= -\mu^{-1} \int_{\mathbb{R}^3} (B \wedge e) \cdot \nabla \alpha \, dx \\ &= \mu^{-1} \langle \text{div}(B \wedge e), \alpha \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = 0. \end{aligned}$$

We deduce that  $D(t) \cdot e = D^{\text{in}} \cdot e = 0, t \in \mathbb{R}$ .

## 5 Strong convergence result

As usual, when the initial conditions are well prepared, strong convergences occur. The key point is to combine weak convergences with the energy conservation. The following standard results will be used.

### Lemma 5.1

Let  $(A^\delta)_{\delta>0}, (B^\delta)_{\delta>0}$  be two families of real numbers and  $A, B \in \mathbb{R}$  such that

$$\limsup_{\delta \searrow 0} (A^\delta + B^\delta) \leq A + B, \quad A \leq \liminf_{\delta \searrow 0} A^\delta, \quad B \leq \liminf_{\delta \searrow 0} B^\delta.$$

Then we have

$$\lim_{\delta \searrow 0} A^\delta = A, \quad \lim_{\delta \searrow 0} B^\delta = B.$$

### Proof.

We have the inequalities

$$A + B \geq \limsup_{\delta \searrow 0} (A^\delta + B^\delta) \geq \limsup_{\delta \searrow 0} A^\delta + \liminf_{\delta \searrow 0} B^\delta \geq \limsup_{\delta \searrow 0} A^\delta + B$$

saying that

$$A \leq \liminf_{\delta \searrow 0} A^\delta \leq \limsup_{\delta \searrow 0} A^\delta \leq A.$$

Therefore  $(A^\delta)_{\delta>0}$  converges toward  $A$  as  $\delta \searrow 0$ . Similarly,  $(B^\delta)_{\delta>0}$  converges toward  $B$  as  $\delta \searrow 0$ .  $\square$

The next proposition is well known. It allows us to deduce strong convergence when the sequence of the norms is bounded by the norm of the weak limit. Its proof is left to the reader.

### Proposition 5.1

Assume that  $A : \mathbb{R}^3 \rightarrow \mathcal{M}_3(\mathbb{R})$  is a measurable field of symmetric non negative matrices and that  $(A^{1/2}w^\delta)_\delta$  converges weakly in  $L^2([-T, T]; L^2(\mathbb{R}^3)^3)$  toward  $A^{1/2}w^0$ , when  $\delta \searrow 0$ , where  $w^\delta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \delta > 0$  and  $w^0 : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are measurable vector fields.

1. Then we have

$$\int_{-T}^T \int_{\mathbb{R}^3} A(x) w^0(t, x) \cdot w^0(t, x) \, dx dt \leq \liminf_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} A(x) w^\delta(t, x) \cdot w^\delta(t, x) \, dx dt.$$

2. If

$$\limsup_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} A(x) w^\delta(t, x) \cdot w^\delta(t, x) \, dx dt \leq \int_{-T}^T \int_{\mathbb{R}^3} A(x) w^0(t, x) \cdot w^0(t, x) \, dx dt$$

then the family  $(A^{1/2} w^\delta)_\delta$  converges strongly in  $L^2([-T, T]; L^2(\mathbb{R}^3)^3)$  toward  $A^{1/2} w^0$ , when  $\delta \searrow 0$ .

**Proof.** (of Theorem 1.3)

We already know by Theorem 1.1 that

$$D^\delta \rightharpoonup D \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}, L_{\epsilon_1}^2(\mathbb{R}^3)) \text{ and weakly in } L_{\text{loc}}^2(\mathbb{R}, L_{\epsilon_1}^2(\mathbb{R}^3))$$

$$B^\delta \rightharpoonup B \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}, L_{\mu^{-1}}^2(\mathbb{R}^3)) \text{ and weakly in } L_{\text{loc}}^2(\mathbb{R}, L_{\mu^{-1}}^2(\mathbb{R}^3))$$

where  $(D, B) \in C(\mathbb{R}; L_{\epsilon_1}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  is the unique variational solution of the problem (3), (4), satisfying the initial condition

$$(D(0), B(0)) = (D^{\text{in}}, P(B^{\text{in}})) = (D^{\text{in}}, B^{\text{in}}).$$

By the energy conservation we obtain for any  $t \in \mathbb{R}$

$$\begin{aligned} & \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2] \, dx + (\delta^{-2} - 1) \int_{\mathbb{R}^3} \frac{(D^\delta(t, x) \cdot e)^2}{n^2} \, dx \\ &= \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(0, x) \cdot D^\delta(0, x) + \mu^{-1} |B^\delta(0, x)|^2] \, dx + (\delta^{-2} - 1) \int_{\mathbb{R}^3} \frac{(D^\delta(0, x) \cdot e)^2}{n^2} \, dx \end{aligned}$$

and

$$\int_{\mathbb{R}^3} [\epsilon_1^{-1} D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2] \, dx = \int_{\mathbb{R}^3} [\epsilon_1^{-1} D(0, x) \cdot D(0, x) + \mu^{-1} |B(0, x)|^2] \, dx.$$

We have the uniform convergences with respect to  $t \in \mathbb{R}$

$$\begin{aligned} & \lim_{\delta \searrow 0} \left\{ \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2] \, dx + (\delta^{-2} - 1) \int_{\mathbb{R}^3} \frac{(D^\delta(t, x) \cdot e)^2}{n^2} \, dx \right\} \\ &= \lim_{\delta \searrow 0} \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(0, x) \cdot D^\delta(0, x) + \mu^{-1} |B^\delta(0, x)|^2] \, dx \\ &= \int_{\mathbb{R}^3} [\epsilon_1^{-1} D(0, x) \cdot D(0, x) + \mu^{-1} |B(0, x)|^2] \, dx \\ &= \int_{\mathbb{R}^3} [\epsilon_1^{-1} D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2] \, dx. \end{aligned}$$

We deduce for any  $T \in \mathbb{R}_+$

$$\begin{aligned}
& \limsup_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2] dx dt \\
& \leq \lim_{\delta \searrow 0} \int_{-T}^T \left\{ \int_{\mathbb{R}^3} [\epsilon_1^{-1} D^\delta \cdot D^\delta + \mu^{-1} |B^\delta|^2] dx + (\delta^{-2} - 1) \int_{\mathbb{R}^3} \frac{(D^\delta \cdot e)^2}{n^2} dx \right\} dt \\
& = \int_{-T}^T \int_{\mathbb{R}^3} [\epsilon_1^{-1} D(t, x) \cdot D(t, x) + \mu^{-1} |B(t, x)|^2] dx dt.
\end{aligned} \tag{34}$$

By weak convergence we also have

$$\begin{aligned}
\int_{-T}^T \int_{\mathbb{R}^3} \epsilon_1^{-1} D(t, x) \cdot D(t, x) dx dt & \leq \liminf_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) dx dt \\
\int_{-T}^T \int_{\mathbb{R}^3} \mu^{-1} |B(t, x)|^2 dx dt & \leq \liminf_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \mu^{-1} |B^\delta(t, x)|^2 dx dt
\end{aligned}$$

and by Lemma 5.1 we deduce

$$\begin{aligned}
\lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \epsilon_1^{-1} D^\delta(t, x) \cdot D^\delta(t, x) dx dt & = \int_{-T}^T \int_{\mathbb{R}^3} \epsilon_1^{-1} D(t, x) \cdot D(t, x) dx dt \\
\lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \mu^{-1} |B^\delta(t, x)|^2 dx dt & = \int_{-T}^T \int_{\mathbb{R}^3} \mu^{-1} |B(t, x)|^2 dx dt.
\end{aligned}$$

Thanks to Proposition 5.1 we obtain the strong convergences, when  $\delta \searrow 0$

$$D^\delta \rightarrow D \text{ in } L_{\text{loc}}^2(\mathbb{R}, L_{\epsilon_1^{-1}}^2(\mathbb{R}^3)), \quad B^\delta \rightarrow B \text{ in } L_{\text{loc}}^2(\mathbb{R}, L_{\mu^{-1}}^2(\mathbb{R}^3)).$$

Coming back to (34), we deduce as well the convergence

$$\lim_{\delta \searrow 0} \frac{1}{\delta^2} \int_{-T}^T \int_{\mathbb{R}^3} \frac{(D^\delta(t, x) \cdot e)^2}{n^2} dx dt = 0, \quad T \in \mathbb{R}_+.$$

□

## 6 Propagation speed

It is well known that the solutions of the Maxwell equations (1), (2) propagate with finite speed  $c = \mu^{-1/2} \|\epsilon^{-1/2}\|_\infty$ . When the electric permittivity possesses disparate eigenvalues

$$\epsilon_\delta = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + \delta^2 n^2 e \otimes e$$

we obtain  $c_\delta = \mu^{-1/2} \|\max\{1/n_1, 1/n_2, 1/(\delta n)\}\|_\infty = \mathcal{O}(1/\delta)$  when  $\delta \searrow 0$ . Nevertheless, the solutions of the limit model (5), (6) propagate with finite speed, not depending on  $\delta > 0$ . We prove that this speed is given by

$$c_\infty = \lim_{\delta \rightarrow +\infty} c_\delta = \mu^{-1/2} \lim_{\delta \rightarrow +\infty} \|\epsilon_\delta^{-1/2}\|_\infty = \mu^{-1/2} \|\max\{1/n_1, 1/n_2\}\|_\infty.$$

Moreover we establish that the energy of the solutions  $(D^\delta, B^\delta)$ , outside the propagation cone associated to the limit model, is negligible when  $\delta \searrow 0$ , that is, almost all energy of  $(D^\delta, B^\delta)$  concentrates inside the propagation cone of speed  $c_\infty$ . For simplifying our computations, we consider  $\varphi(x) = |x|, e = \frac{x}{|x|}, x \in \mathbb{R}^3 \setminus \{0\}$ .

**Proposition 6.1**

Assume that the electric permittivity tensor writes

$$\epsilon_1 = n_1^2 e_1 \otimes e_1 + n_2^2 e_2 \otimes e_2 + n^2 e \otimes e, \quad e = \frac{x}{|x|}, x \in \mathbb{R}^3 \setminus \{0\}$$

and  $n_1, n_2, n$  are locally bounded from below and above : for any compact set  $K \subset \mathbb{R}^3$ , there are  $0 < m_K \leq M_K < +\infty$  such that

$$m_K \leq \min\{n_1, n_2, n\} \leq \max\{n_1, n_2, n\} \leq M_K, \quad x \in K.$$

Let us consider  $(D, B) \in C(\mathbb{R}; L_{\epsilon_1^{-1}}^2(\mathbb{R}^3)) \times C(\mathbb{R}; L_{\mu^{-1}}^2(\mathbb{R}^3))$  the unique variational solution of the problem

$$D \cdot e = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} D \cdot \Phi \, dx - \int_{\mathbb{R}^3} \mu^{-1} B \cdot \text{rot} \Phi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Phi \in C_c^1(\mathbb{R}^3)^3 \quad (35)$$

$$\text{div}(B \wedge e) = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} B \cdot \Psi \, dx + \int_{\mathbb{R}^3} \epsilon_1^{-1} D \cdot \text{rot} \Psi \, dx = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad \Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3 \quad (36)$$

such that  $\text{div}(\Psi \wedge e) = 0$ ,  $\int_{\mathbb{R}^3} \epsilon_1^{-1} \text{rot} \Psi \cdot \text{rot} \Psi \, dx < +\infty$ , with the initial conditions

$$D(0) = D^{\text{in}} \in L_{\epsilon_1^{-1}}^2(\mathbb{R}^3), \quad B(0) = B^{\text{in}} \in L_{\mu^{-1}}^2(\mathbb{R}^3), \quad D^{\text{in}} \cdot e = 0, \quad \text{div}(B^{\text{in}} \wedge e) = 0.$$

Assume that for some  $R > 0$ , the initial conditions satisfy

$$D^{\text{in}}(x) = 0, \quad B^{\text{in}}(x) = 0, \quad |x| > R.$$

Then we have

$$D(t, x) = 0, \quad B(t, x) = 0, \quad |x| > R + c_\infty |t|$$

where  $c_\infty = \mu^{-1/2} \|\max\{\frac{1}{n_1}, \frac{1}{n_2}\}\|_\infty$ .

**Proof.**

Pick a non decreasing function  $\theta \in C_b^1(\mathbb{R})$  such that  $\theta(r) = 0$  if  $r \leq 0$ ,  $\theta(r) > 0$  if  $r > 0$ . Using (35) with  $\Phi = \theta(|x| - R - c_1 |t|) \epsilon_1^{-1} D$  and (36) with  $\Psi = \theta(|x| - R - c_1 |t|) \mu^{-1} B$ , we obtain in  $\mathcal{D}'(\mathbb{R})$ , after standard manipulations (including regularization)

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \theta(|x| - R - c_1 |t|) \frac{\epsilon_1^{-1} D \cdot D + \mu^{-1} |B|^2}{2} \, dx \\ & + \text{sgn}(t) \int_{\mathbb{R}^3} \theta'(|x| - R - c_1 |t|) \left[ \frac{\epsilon_1^{-1} D \cdot D + \mu^{-1} |B|^2}{2} c_1 - (\epsilon_1^{-1} D \wedge \mu^{-1} B) \cdot e \, \text{sgn}(t) \right] \, dx = 0. \end{aligned}$$

Notice that  $\Psi = \theta(|x| - R - c_1 |t|) \mu^{-1} B$  is allowed as test vector field in (36) since

$$\text{div}(\theta(|x| - R - c_1 |t|) B \wedge e) = \theta(|x| - R - c_1 |t|) \text{div}(B \wedge e) + \theta'(|x| - R - c_1 |t|) \left( \frac{x}{|x|} \wedge B \right) \cdot e = 0.$$

It is easily seen that

$$\begin{aligned} (\epsilon_1^{-1} D \wedge \mu^{-1} B) \cdot e \, \text{sgn}(t) & \leq |\epsilon_1^{-1} D| |\mu^{-1} B| \leq \|\epsilon_1^{-1/2}\| \mu^{-1/2} |\epsilon_1^{-1/2} D| |\mu^{-1/2} B| \\ & \leq c_1 \frac{\epsilon_1^{-1} D \cdot D + \mu^{-1} |B|^2}{2}. \end{aligned}$$

Therefore we obtain

$$\operatorname{sgn}(t) \frac{d}{dt} \int_{\mathbb{R}^3} \theta(|x| - R - c_1|t|) \frac{\epsilon_1^{-1} D \cdot D + \mu^{-1} |B|^2}{2} dx \leq 0$$

implying that

$$\int_{\mathbb{R}^3} \theta(|x| - R - c_1|t|) \frac{\epsilon_1^{-1} D \cdot D + \mu^{-1} |B|^2}{2} dx \leq \int_{\mathbb{R}^3} \theta(|x| - R) \frac{\epsilon_1^{-1} D^{\text{in}} \cdot D^{\text{in}} + \mu^{-1} |B^{\text{in}}|^2}{2} dx$$

and thus, for any  $|x| > R + c_1|t|$  we have  $D(t, x) = 0$ ,  $B(t, x) = 0$ .

As  $D \cdot e = 0$ , we have  $\epsilon_1^{-1} D = \epsilon_\delta^{-1} D$ . Similarly, for any  $\Psi \in C^1(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$  such that  $\operatorname{div}(\Psi \wedge e) = \operatorname{rot} \Psi \cdot e = 0$ , we have

$$\int_{\mathbb{R}^3} \epsilon_\delta^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi dx = \int_{\mathbb{R}^3} \epsilon_1^{-1} \operatorname{rot} \Psi \cdot \operatorname{rot} \Psi dx.$$

Therefore  $(D, B)$  solves (35), (36) with  $\epsilon_1^{-1}$  replaced by  $\epsilon_\delta^{-1}$ , for any  $\delta > 0$ . By the previous arguments, we obtain

$$D(t, x) = 0, \quad B(t, x) = 0, \quad |x| > R + c_\delta|t|, \quad c_\delta = \mu^{-1/2} \|\epsilon_\delta^{-1/2}\|_\infty, \quad \delta > 0$$

implying that

$$D(t, x) = 0, \quad B(t, x) = 0, \quad |x| > R + c_\infty|t|$$

where

$$c_\infty = \mu^{-1/2} \lim_{\delta \rightarrow +\infty} \|\epsilon_\delta^{-1/2}\|_\infty = \mu^{-1/2} \|\max\{1/n_1, 1/n_2\}\|_\infty.$$

□

We investigate now how much energy of the solutions  $(D^\delta, B^\delta)$  concentrates inside the propagation cone of the limit model. Although, when  $\delta \searrow 0$ , the propagation cone  $\{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : |x| < R + c_\delta|t|\}$  is much larger than  $\{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : |x| < R + c_1|t|\}$ , we will see that almost all the energy of the solution  $(D^\delta, B^\delta)$  lies inside the propagation cone of the limit model.

**Proof.** (of Theorem 1.4)

By Theorem 1.3 we have

$$\lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \frac{\epsilon_1^{-1} (D^\delta - D) \cdot (D^\delta - D) + \mu^{-1} |B^\delta - B|^2}{2} dx dt = 0$$

$$\lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \left( \frac{1}{\delta^2} - 1 \right) \left( \frac{D^\delta \cdot e}{n} \right)^2 dx dt = 0.$$

As we know, by Proposition 6.1, that  $D(t, x) = 0, B(t, x) = 0$  for  $|x| > R + c_\infty|t|$ , we obtain

$$\begin{aligned} & \lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \frac{\epsilon_\delta^{-1} D^\delta(t, x) \cdot D^\delta(t, x) + \mu^{-1} |B^\delta(t, x)|^2}{2} \mathbf{1}_{\{|x| > R + c_\infty|t|\}} dx dt \\ &= \lim_{\delta \searrow 0} \int_{-T}^T \int_{\mathbb{R}^3} \left[ \frac{\epsilon_1^{-1} D^\delta \cdot D^\delta + \mu^{-1} |B^\delta|^2}{2} + \frac{1}{2} \left( \frac{1}{\delta^2} - 1 \right) \left( \frac{D^\delta \cdot e}{n} \right)^2 \right] \mathbf{1}_{\{|x| > R + c_\infty|t|\}} dx dt \\ &= 0. \end{aligned}$$

□

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