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Quantum potential energy and non-locality

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The quantum potential energy, as introduced by David Bohm, is defined and interpreted within symplectic quantum mechanics. It is a form of energy which cannot be localized in space. It represent the energy associated with the spatial curvature of the square-root quantum fidelity.

INTRODUCTION

The concept of a quantum potential energy is central in the de Broglie-Bohm approach to quantum mechanics. Nothing about the de Broglie-Bohm theory will be said in this article. Rather, the quantum potential energy will be addressed from the point of view of symplectic quantum mechanics. This approach has been outlined in more detail elsewhere [1][2]. Here, only a very brief summary is given.

The point of departure is the uncertainty principle. In the symplectic approach, it is understood as a condition on the symplectic capacity c_ϕ of a quantum state ϕ , i.e. a topological restriction on the area of the projection of the state onto the conjugate plane (q, p) [3]. For coherent, or squeezed coherent, states¹, the symplectic capacity is minimized and given by the Gromov width $c_G = h/2$. Thus, in this approach, for general states ϕ , the uncertainty principle is stated as

$$c_\phi(t) \geq c_G. \quad (1)$$

Due to the uncertainty principle, any given pair of states ϕ and ψ can have a non-zero overlap Ω of their symplectic capacities, see Fig.1, where the states are coherent, and where R is the absolute value of the overlap.

The physical quantity which play the central role in symplectic quantum mechanics is quantum fidelity, $F(\Omega)$. It is given by the Born rule as the absolute square of the overlap, i.e.

$$F(\Omega) = |\Omega|^2. \quad (2)$$

It is physically interpreted to describe the probability that the pair of states are mistaken for each other by an observer performing a measurement on the system.

The Schrödinger equation, written as

$$i \frac{h}{2\pi} \frac{\partial \Omega(t)}{\partial t} = \mathcal{H} \Omega(t), \quad (3)$$

is obtained as a direct consequence of requiring that quantum fidelity is conserved in time for isolated systems.

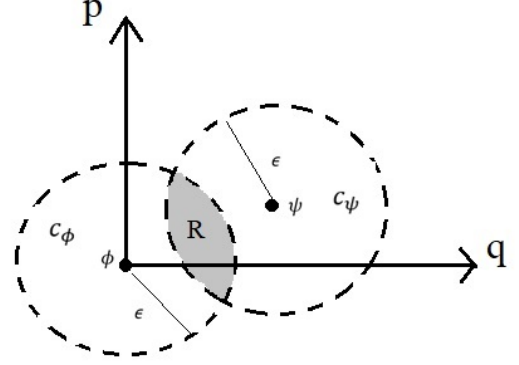


FIG. 1. Overlap between the symplectic capacities c_ϕ and c_ψ for the coherent states ϕ and ψ with quantum resolution $\epsilon = \sqrt{h/2\pi}$.

Thus, the Schrödinger equation is the quantum generalization of the Gibbs-Liouville theorem² to the situation where the Hamiltonian flow is topologically constrained by the uncertainty principle.

SCHRÖDINGER EQUATION IN POLAR FORM

The complex-valued overlap Ω can be represented in polar form by the real-valued phase-space functions R and \mathcal{S} as

$$\Omega = R \cdot e^{2\pi i \mathcal{S}/h}. \quad (4)$$

The Schrödinger equation, when the overlap is polar decomposed, take the form

$$\left(\frac{ih}{2\pi} \frac{\partial R}{\partial t} - R \cdot \frac{\partial \mathcal{S}}{\partial t} \right) \cdot e^{2\pi i \mathcal{S}/h} = \mathcal{H} \left(R \cdot e^{2\pi i \mathcal{S}/h} \right). \quad (5)$$

The system that will be considered is that of a single non-relativistic particle with mass m that propagates along one spatial dimension q in the presence of an external potential $V(q)$ which is not explicitly time-dependent. In this case, the Hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m} + V(q). \quad (6)$$

¹ For coherent states, the uncertainties in q and p are symmetric, whereas this is not the case for squeezed coherent states. Therefore, the symplectic capacities are shaped as circles and ellipses, respectively.

² The authors perspective on the Gibbs-Liouville theorem can be found in reference [4].

Given that momentum generate infinitesimal spatial translations, i.e.

$$p \equiv i \frac{h}{2\pi} \frac{\partial}{\partial q}, \quad (7)$$

the Hamiltonian becomes

$$\mathcal{H} = - \left(\frac{h}{2\pi} \right)^2 \frac{1}{2m} \frac{\partial^2}{\partial q^2} + V(q). \quad (8)$$

With this Hamiltonian, the Schrödinger equation is split into an imaginary-valued part and a real-valued part. The imaginary part, given by

$$\frac{\partial R}{\partial t} = - \frac{1}{2m} \left[2 \frac{\partial R}{\partial q} \frac{\partial \mathcal{S}}{\partial q} + R \cdot \frac{\partial^2 \mathcal{S}}{\partial q^2} \right], \quad (9)$$

can be shown to describe the continuity of quantum fidelity. However, in this article, it is the real-valued part, given by

$$\frac{\partial \mathcal{S}}{\partial t} = - \left[\frac{1}{2m} \left(\frac{\partial \mathcal{S}}{\partial q} \right)^2 + V - \left(\frac{h}{2\pi} \right)^2 \frac{1}{2m} \frac{1}{R} \frac{\partial^2 R}{\partial q^2} \right], \quad (10)$$

which is of interest.

QUANTUM POTENTIAL ENERGY

Let the phase-space function Q be defined by

$$Q \equiv - \left(\frac{h}{2\pi} \right)^2 \frac{1}{2m} \frac{1}{R} \frac{\partial^2 R}{\partial q^2}. \quad (11)$$

Then, the real-part of the Schrödinger equation becomes

$$\frac{\partial \mathcal{S}}{\partial t} = - \left[\frac{1}{2m} \left(\frac{\partial \mathcal{S}}{\partial q} \right)^2 + V + Q \right]. \quad (12)$$

If $Q = 0$, then the equation reduces to the Hamilton-Jacobi equation in classical mechanics with the phase-function \mathcal{S} representing the classical action. The function Q is thus interpreted to give the quantum correction, seeing that it depend on the quantum resolution $\sqrt{h/2\pi}$, to the classical motion of the particle. The equation has the units of energy, and hence Q must represent a form of energy. It is the quantum potential energy as introduced by David Bohm [5][6].

Due to the quantum potential energy being inversely proportional to the mass of the particle, the more massive the particle is, the more classical it will behave.

OVERLAP CURVATURE

What general statements can be made about the overlap amplitude R as the spatial distance q between ϕ and ψ change? The conditions that seem plausible are the following:

- i The largest possible overlap amplitude that can be attained, by changing the spatial distance q , happen when $q \rightarrow 0$, i.e.

$$R_{max}(q) = \lim_{q \rightarrow 0} R(q). \quad (13)$$

Whether or not $R_{max} = c_\phi$ depend not only on the spatial separation q but also on the momentum separation between ϕ and ψ .

- ii The smallest possible overlap amplitude is attained when $q \rightarrow \infty$, i.e.

$$R_{min}(q) = \lim_{q \rightarrow \infty} R(q) = 0. \quad (14)$$

For coherent states, there is a finite distance at which the overlap vanishes. This distance might be referred to as the classical distance, denoted by q_c , since beyond this distance the quantum uncertainty associated with the states are no longer of importance. However, for squeezed coherent states, where it might be that the uncertainty in p is squeezed to an infinitesimally small width, such that the uncertainty in q is extended towards infinity, it is necessary to consider the limit $q \rightarrow q_c = \infty$ to ensure that the overlap vanishes.

- iii The overlap should have no stationary points in the interval $q \in [0, q_c]$, i.e.

$$\frac{\partial R}{\partial q} \neq 0, \quad q \in [0, q_c]. \quad (15)$$

This is to ensure that the quantum potential energy, Q , does not become zero anywhere in the interval.

- iv The overlap should decrease with distance, i.e. the gradient should be negative,

$$\frac{\partial R}{\partial q} < 0. \quad (16)$$

- v The gradient should become zero in the limit $q \rightarrow \infty$, i.e.

$$\lim_{q \rightarrow \infty} \frac{\partial R}{\partial q} = 0. \quad (17)$$

- vi The gradient should increase towards zero in the interval $q \in [0, q_c]$, i.e.

$$\frac{\partial^2 R}{\partial q^2} > 0. \quad (18)$$

This is because the further away the states ϕ and ψ are, the smaller will the incremental changes in the gradient be for equal displacements along q . The gradient cannot be constant since that would imply that $Q = 0$.

The simplest possible ansatz, satisfying the above conditions, for the general form of $R(q)$ is given by

$$R(q) = R_{max} \cdot e^{-\sqrt{\kappa} \cdot q}, \quad (19)$$

where the curvature of the overlap amplitude, κ , is defined by

$$\kappa \equiv \frac{1}{R} \frac{\partial^2 R}{\partial q^2}. \quad (20)$$

Due to conditions (iv) and (vi), the curvature cannot fall off with distance. Thus, if it varies in space, it should fluctuate about an averaged value $\bar{\kappa}$ ³, see Fig.2.

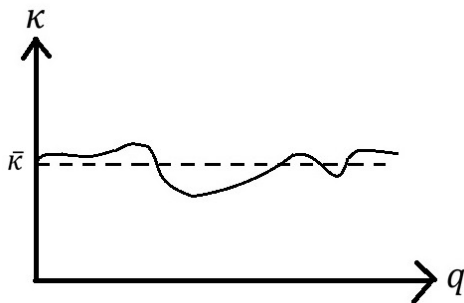


FIG. 2. The overlap curvature varying throughout space about its average value.

NON-LOCALITY

In terms of the overlap curvature, which is always positive, the quantum potential energy is written as

$$Q(\kappa) = - \left(\frac{\hbar}{2\pi} \right)^2 \frac{\kappa}{2m} < 0 \quad (21)$$

Since the curvature do not fall off with distance, neither does the quantum potential energy. Hence, it cannot be localized to a specific region within space. It is a non-local form of energy. All the other known forms of potential energy are local. Thus, it should not be thought of as a potential energy at all.

Since it is the non-zeroeness of the quantum potential energy which distinguish the quantum realm from its classical counterpart, it is tempting to state that the key fundamental manifestation, or prediction, of quantum mechanics is non-locality. This point of view is not

original. It has been proposed by Bohm and Hiley [7]. The originality in this article, if there is any, as compared to previous studies, is the physical appearance and interpretation of the quantum potential energy within symplectic quantum mechanics.

CONCLUSION

The quantum potential energy has the units of energy but it does not share the characteristic properties of neither potential nor kinetic energies as understood in classical physics. What, then, type of energy does it describe? It is an entirely new form of energy, which appear in quantum systems, that is non-local in space. Since the overlap amplitude is related to the quantum fidelity by

$$R = \sqrt{F}, \quad (22)$$

where \sqrt{F} is the so-called square-root quantum fidelity, the quantum potential energy can be written as

$$Q = - \left(\frac{\hbar}{2\pi} \right)^2 \frac{1}{2m} \frac{1}{\sqrt{F}} \frac{\partial^2 \sqrt{F}}{\partial q^2}. \quad (23)$$

The quantum potential energy is thus a measure for the spatial curvature of the square-root quantum fidelity.

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³ The origin of the fluctuations in the curvature, and some of their consequences, will be addressed in an upcoming article.