Fluid models with phase transition for kinetic equations in swarming

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Abstract

We concentrate on kinetic models for swarming with individuals interacting through self-propelling and friction forces, alignment and noise. We assume that the velocity of each individual relaxes to the mean velocity. In our present case, the equilibria depend on the density and the orientation of the mean velocity, whereas the mean speed is not anymore a free parameter and a phase transition occurs in the homogeneous kinetic equation. We analyze the profile of equilibria for general potentials identifying a family of potentials leading to phase transitions. Finally, we derive the fluid equations when the interaction frequency becomes very large.

Keywords: Swarming, Cucker-Smale model, Phase transition.

AMS classification: 92D50, 82C40, 92C10.

1 Introduction

This paper is concerned with the derivation of fluid models for populations of self-propelled individuals, with alignment and noise [31, 23, 28] starting from their kinetic description. The alignment between particles is imposed by relaxing the individuals velocities towards the mean velocity [32, 40, 42, 24, 25, 47, 48] in the presence of an asymptotic convergence to a fixed speed [37]. We refer to [49, 20, 36, 22, 10, 11, 41, 26, 27] and the references therein for a derivation of kinetic equations for collective behavior from microscopic models by mean-field limit approaches. In this work, we mainly deal with a localized version of the classical Cucker-Smale consensus in velocity or flocking model [32] for alignment, see also [4]. This model assumes that all interactions leading to alignment happen only at the present location of the individual, that is, the communication function is a Delta Dirac. Similar kinetic theory approaches to model swarms have been used in the multiscale description of vehicular traffic, crowds, active particles, and multicellular systems, see for instance [7, 8, 21, 9, 2] and the references therein for recent reviews of these perspectives of current research.

We here concentrate on kinetic alignment models with phase transition [53, 33, 38, 3, 34, 4, 45]. Phase transitions are important to be analysed properly since they characterize the

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values of the parameters leading to sudden dramatic changes of their asymptotic behavior. Phase transitions in collective behavior models driven by alignment were first discussed in the seminal paper [53]. They reported a behavioral change in the system from overall coordinated orientation - ordered state by polarization - to total mixing of preferred directions - disordered state - driven by the noise strength in the system, this phenomena is usually called a noise-driven phase transition. This issue is already very challenging even in the homogeneous periodic setting for the so called McKean-Vlasov equations [46]. Classical bifurcation analysis and statistical mechanics viewpoints have been used in this setting [51, 30], leading recently to sufficient conditions for its classification [29] in terms of continuity or discontinuity of the order parameter at the phase transition tipping point. We also refer to [39] for a careful numerical study of the possibly complicated bifurcations depending on the potential shape. Here, we start from a model leading to phase transitions in the homogeneous setting in which the developed theory applies [33, 38, 3, 34, 4, 45] in order to rigorously derive the hydrodynamic system dealing with spatial perturbations around the associated equilibria in the inhomogeneous kinetic problem. This approach has been championed by P. Degond and collaborators for the Vicsek-Fokker-Planck model in a series of papers introducing the novel technique of generalized collision invariants, see [35, 33, 34], see related results by different approaches [17, 1]. These hydrodynamic systems are usually referred as Self-Organized Hydrodynamics (SOH). Our goal in this work is to derive the corresponding SOH system for kinetic inhomogeneous problems of Cucker-Smale type with phase transtion at their corresponding homogeneous Fokker-Planck equation in contrast to [17, 1]. Generalized collision invariants are understood in our present approach in a more classical sense and associated to the elements of the kernel of suitable adjoint operators as in the classical Boltzmann and Fokker-Planck operators.

We denote by $f = f(t, x, v) \geq 0$ the particle density in the phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $d \geq 2$. The self-propulsion and friction mechanism writes $\mathrm{div}_v\{f\nabla_v V(|\,\cdot\,|)\}$, where $v \mapsto V(|v|)$ is a confining potential. When considering $V_{\alpha,\beta}(|v|) = \beta\frac{|v|^4}{4} - \alpha\frac{|v|^2}{2}$, with $\alpha, \beta > 0$, we obtain the term $\mathrm{div}_v\{f(\beta|v|^2 - \alpha)v\}$ see [16, 17] and also [13, 14, 15] for results based on averaging methods in magnetic confinement. The relaxation towards the mean velocity is given by $\mathrm{div}_v\{f(v - u[f])\}$ cf. [35], where for any particle density the notation $u[f]$ stands for the mean velocity

$$u[f] = \frac{\int_{\mathbb{R}^d} f(v)\; v \;\mathrm{d}v}{\int_{\mathbb{R}^d} f(v)\;\mathrm{d}v}.$$

Including noise with respect to the velocity variable, we obtain the Fokker-Planck type equation

$$\partial_t f + v\cdot\nabla_x f = Q(f) := \mathrm{div}_v\{\sigma\nabla_v f + f(v-u[f]) + f\nabla_v V(|\cdot|)\},\;\; (t,x,v) \in \mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d\,. \quad (1)$$

When considering large time and space scales in (1), we are led to the kinetic equation

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}Q(f^\varepsilon),\;\; (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d. \quad (2)$$

We investigate the asymptotic behavior of the family $(f^\varepsilon)_{\varepsilon>0}$, when ε becomes small. We expect that the limit density $f(t, x, \cdot) = \lim_{\varepsilon\searrow 0} f^\varepsilon(t, x, \cdot)$ is an equilibrium for the interaction mechanism

$$Q(f(t, x, \cdot)) = 0,\;\; (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

For any $u \in \mathbb{R}^d$ we introduce the notations

$$\Phi_u(v) = \frac{|v-u|^2}{2} + V(|v|),\;\; Z(\sigma,u) = \int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_u(v)}{\sigma}\right)\,\mathrm{d}v,\;\; M_u(v) = \frac{\exp\left(-\frac{\Phi_u(v)}{\sigma}\right)}{Z(\sigma,u)}.$$

Actually the function Z depends only on σ and $|u|$, see Proposition 2.1, and thus we will write $Z = Z(\sigma, l = |u|)$. Notice that for any smooth particle density f and any $u \in \mathbb{R}^d$ we have

$$\sigma \nabla_v f + f(v - u[f]) + f \nabla_v V(|\cdot|) = \sigma M_u(v) \nabla_v \left(\frac{f}{M_u} \right)$$

leading to the following representation formula

$$Q(f) = \sigma \mathrm{div}_v \left(M_{u[f]} \nabla_v \left(\frac{f}{M_{u[f]}} \right) \right).$$

Multiplying by $f/M_{u[f]}$ and integrating by parts with respect to the velocity imply that any equilibrium satisfies

$$f = \rho[f] M_{u[f]}, \quad \rho[f] = \int_{\mathbb{R}^d} f(v) \, \mathrm{d}v.$$

Recall that $u[f]$ is the mean velocity, and therefore we impose

$$\int_{\mathbb{R}^d} f(t,x,v)(v - u[f(t,x,\cdot)]) \, \mathrm{d}v = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{3}$$

Notice that Φ_u is left invariant by any orthogonal transformation preserving u. Consequently, we deduce (see Proposition 2.1) that $\int_{\mathbb{R}^d} f(v) \, v \, \mathrm{d}v$ is parallel to u, and therefore the constraint (3) fix only the modulus of the mean velocity, and not its orientation (which remains a free parameter).

Our first important observation gives a characterization to find the bifurcation diagram of stationary solutions of $Q(f) = 0$. We prove that M_u is an equilibrium if and only if $l = |u|$ is a critical point of $Z(\sigma, \cdot)$, cf. Proposition 2.1. Moreover, several values for $|u|$, or only one are admissible, depending on the diffusion coefficient σ. In that case we will say that a phase transition occurs. Notice that in this work we do not distinguish between phase transitions and bifurcation points. For any particle density $f = f(v)$, the notation $\Omega[f]$ stands for the orientation of the mean velocity $u[f]$, if $u[f] \neq 0$

$$\Omega[f] = \frac{u[f]}{|u[f]|} = \frac{\int_{\mathbb{R}^d} f(v) \, v \, \mathrm{d}v}{\left| \int_{\mathbb{R}^d} f(v) \, v \, \mathrm{d}v \right|}$$

and any vector in $\mathbb{S}^{d-1}$, if $u[f] = 0$. Here $\mathbb{S}^{d-1}$ is the set of unit vectors in $\mathbb{R}^d$. Notice also that we always have

$$u[f] = |u[f]| \, \Omega[f].$$

Finally, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the limit particle density is a von Mises-Fisher distribution $f(t,x,v) = \rho(t,x) M_{|u|\Omega(t,x)}(v)$ parametrized by the concentration $\rho(t,x) = \rho[f(t,x,\cdot)]$ and the orientation $\Omega(t,x) = \Omega[f(t,x,\cdot)]$. We identify a class of potentials $v \mapsto V(|v|)$ such that a phase transition occurs and we derive the fluid equations satisfied by the macroscopic quantities ρ, Ω. More exactly we assume that the potential $v \mapsto V(|v|)$ satisfies

$$\lim_{|v| \to +\infty} \frac{\frac{|v|^2}{2} + V(|v|)}{|v|} = +\infty \tag{4}$$

(such that Z is well defined) and belongs to the family $\mathcal{V}$ defined by: there exists $\sigma_0 > 0$ verifying

1. For any $0 < \sigma < \sigma_0$ there is $l(\sigma) > 0$ such that $Z(\sigma, l)$ is stricly increasing on $[0, l(\sigma)]$ and strictly decreasing on $[l(\sigma), +\infty[$;

2. For any $\sigma \geq \sigma_0, Z(\sigma, l)$ is strictly decreasing on $[0, +\infty[$.

The first important result in this work shows that potentials in $\mathcal{V}$ have a phase transition at $\sigma = \sigma_0$ as shown in Section 2.

Remark 1.1 *The potential* $V(|v|) = \beta\frac{|v|^4}{4} - \alpha\frac{|v|^2}{2}$ *belongs to the family* $\mathcal{V}$ *as shown in [52, 4, 45] in any dimension.*

Theorem 1.1 *Assume that the potential* $v \mapsto V(|v|)$ *satisfies* (4)*, belongs to the family* $\mathcal{V}$ *defined above and that* $0 < \sigma < \sigma_0$*. Let us consider* $(f^\varepsilon)_{\varepsilon>0}$ *satisfying*

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\mathrm{div}_v\{\sigma\nabla_v f^\varepsilon + f^\varepsilon(v - u[f^\varepsilon] + \nabla_v V(|\cdot|)\,)\},\ (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d. \quad (5)$$

Therefore, at any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ *the dominant term in the Hilbert expansion* $f^\varepsilon = f + \varepsilon f^1 + ...$ *is an equilibrium distribution of* Q*, that is* $f(t,x,v) = \rho(t,x)M_{u(t,x)}(v)$*, where*

$$u(t,x) = l(\sigma)\Omega(t,x),\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \quad (6)$$

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \quad (7)$$

$$\partial_t \Omega + l(\sigma)c_\perp\,(\Omega \cdot \nabla_x)\Omega + \frac{\sigma}{l(\sigma)}(I_d - \Omega \otimes \Omega)\frac{\nabla_x \rho}{\rho} = 0,\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (8)$$

The constant $c_\perp$ *is given by*

$$c_\perp = \frac{\int_{\mathbb{R}_+} r^{d+1} \int_0^\pi \cos\theta\, \chi(\cos\theta, r)\, e(\cos\theta, r, l(\sigma)) \sin^{d-1}\theta\ \mathrm{d}\theta\mathrm{d}r}{l(\sigma)\int_{\mathbb{R}_+} r^d \int_0^\pi \chi(\cos\theta, r)\, e(\cos\theta, r, l(\sigma)) \sin^{d-1}\theta\ \mathrm{d}\theta\mathrm{d}r}$$

and the function χ *solves*

$$\begin{aligned} &-\sigma\partial_c\left[r^{d-3}(1-c^2)^{\frac{d-1}{2}} e(c,r,l(\sigma))\partial_c\chi\right] - \sigma\partial_r\left[r^{d-1}(1-c^2)^{\frac{d-3}{2}} e(c,r,l(\sigma))\partial_r\chi\right] \quad (9)\\ &\quad + \sigma(d-2)r^{d-3}(1-c^2)^{\frac{d-5}{2}} e(c,r,l(\sigma))\chi = r^d(1-c^2)^{\frac{d-2}{2}} e(c,r,l(\sigma)),\ (c,r) \in]-1,1[\times\mathbb{R}_+ \end{aligned}$$

where $e(c,r,l) = \exp\left(-\frac{r^2}{2\sigma} + \frac{rcl}{\sigma} - \frac{V(r)}{\sigma}\right)$.

Remark 1.2 *Several considerations regarding the hydrodynamic equations* (6)-(8) *and the asymptotic limit to obtain them are needed:*

- *The asymptotic limit in* (5) *is different from the one analysed in [17] where the friction term is penalized at higher order. The main technical difficulty in [17] compared to our present work is that to solve for the different orders on the expansion in [17] we had to deal with Fokker-Planck equations on the velocity sphere with speed* $\sqrt{\frac{\alpha}{\beta}}$.

- *The hydrodynamic equations* (6)-(8) *in the particular case of the potential* $V(|v|) = \beta\frac{|v|^4}{4} - \alpha\frac{|v|^2}{2}$ *recover the ones obtained in [35, 33, 34, 17] by taking the limit* $\alpha \to \infty$ *with* $\beta/\alpha = O(1)$*. In this limit, the particle density* f *is squeezed to a Dirac on the velocity sphere with speed* $\sqrt{\frac{\alpha}{\beta}}$*. The constants can be computed exactly based on [45] and they converge towards the exact constants obtained in [35, 34, 17]. This is left to the reader for verification.*

- *The hydrodynamic equations* (6)-(8) *have the same structure as the equations derived in [35, 34, 17] just with different constants, and therefore they form a hyperbolic system as shown in [35, Subsection 4.4].*

When $V(|\cdot|)$ belongs to the family $\mathcal{V}$, we know that $|u| \in \{0, l(\sigma)\}$, for any $0 < \sigma < \sigma_0$ and $|u| = 0$ for any $\sigma \geq \sigma_0$. There is no time evolution for $|u|$. But the modulus of the mean velocity evolves in time for other potentials. For example, let us assume that there is $\sigma > 0$, $0 \leq l_1(\sigma) < l_2(\sigma) \leq +\infty$ such that the function $l \mapsto Z(\sigma, l)$ is stricly increasing on $[0, l_1(\sigma)]$, constant on $[l_1(\sigma), l_2(\sigma)[$, and strictly decreasing on $[l_2(\sigma), +\infty[$. In that case, we obtain a balance for $|u|$ as well.

Theorem 1.2 *Assume that the potential $v \mapsto V(|v|)$ satisfies (4) and verifies the above hypothesis for some $\sigma > 0$. Let us consider $(f^\varepsilon)_{\varepsilon>0}$ satisfying*

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathrm{div}_v \{\sigma \nabla_v f^\varepsilon + f^\varepsilon (v - u[f^\varepsilon] + \nabla_v V(|\cdot|)\ )\}, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d.$$

Therefore, at any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ the dominant term in the Hilbert expansion $f^\varepsilon = f + \varepsilon f^1 + ...$ is an equilibrium distribution of Q, that is $f(t, x, v) = \rho(t, x) M_{u(t,x)}(v)$, where

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \tag{10}$$

$$\begin{aligned} \partial_t u + [c_\perp (I_d - \Omega \otimes \Omega) + c_\parallel \Omega \otimes \Omega](u \cdot \partial_x) u + [(c_\perp - 1)(I_d - \Omega \otimes \Omega) + (c_\parallel - 1)\Omega \otimes \Omega] \nabla_x \frac{|u|^2}{2} \\ + \sigma \frac{\nabla_x \rho}{\rho} + c'_\parallel \ \mathrm{div}_x \Omega \ |u| u = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \end{aligned} \tag{11}$$

The constants $c_\perp, c_\parallel, c'_\parallel$ are given by

$$c_\perp = \frac{\int_{\mathbb{R}_+} r^{d+1} \int_0^\pi \cos\theta\, \chi(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d-1}\theta \ \mathrm{d}\theta \mathrm{d}r}{|u| \int_{\mathbb{R}_+} r^d \int_0^\pi \chi(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d-1}\theta \ \mathrm{d}\theta \mathrm{d}r}$$

$$c_\parallel = \frac{\int_{\mathbb{R}_+} r^{d+1} \int_0^\pi \cos^2\theta\, \chi_\Omega(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d-2}\theta \ \mathrm{d}\theta \mathrm{d}r}{2|u| \int_{\mathbb{R}_+} r^d \int_0^\pi \cos\theta \chi_\Omega(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d-2}\theta \ \mathrm{d}\theta \mathrm{d}r}$$

$$c'_\parallel = \frac{\int_{\mathbb{R}_+} r^{d+1} \int_0^\pi \chi_\Omega(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d}\theta \ \mathrm{d}\theta \mathrm{d}r}{(d-1)|u| \int_{\mathbb{R}_+} r^d \int_0^\pi \cos\theta \chi_\Omega(\cos\theta, r)\, e(\cos\theta, r, |u|) \sin^{d-2}\theta \ \mathrm{d}\theta \mathrm{d}r}$$

the function χ solves (9) and the function χ_Ω solves

$$\begin{aligned} -\sigma \partial_c \{r^{d-3}(1-c^2)^{\frac{d-1}{2}} e(c, r, |u|) \partial_c \chi_\Omega\} - \sigma \partial_r \{r^{d-1}(1-c^2)^{\frac{d-3}{2}} e(c, r, |u|) \partial_r \chi_\Omega\} \\ = r^{d-1}(rc - |u|)(1-c^2)^{\frac{d-3}{2}} e(c, r, |u|), \ (c, r) \in]-1, 1[\times]0, +\infty[. \end{aligned}$$

Our paper is organized as follows. In Section 2 we investigate the function Z, whose variations will play a crucial role when determining the equilibria of the interaction mechanism Q. We identify a family of potentials such that a phase transition occurs for some critical diffusion coefficient σ_0. Section 3 is devoted to the study of the linearization of Q and of its formal adjoint. We are led to study the spectral properties of the pressure tensor. The kernel of the adjoint of the linearization of Q is studied in Section 4. These elements will play the role of the collision invariants, when determining the macroscopic equations by the moment method. The main results, Theorem 1.1, 1.2, are proved in Section 5. Some examples are presented in Section 6.

2 Phase transitions and Potentials: Properties of Equilibria

For any $u \in \mathbb{R}^d$ we denote by $\mathcal{T}_u$ the family of orthogonal transformations of $\mathbb{R}^d$ preserving u. Notice that $\mathcal{T}_0$ is the family of all orthogonal transformations of $\mathbb{R}^d$.

Remark 2.1 *The functions on $\mathbb{R}^d$ which are left invariant by the family $\mathcal{T}_0$ are those depending only on $|v|$. The functions on $\mathbb{R}^d$ which are left invariant by the family $\mathcal{T}_u, u \neq 0$, are those depending on $v \cdot u$ and $|v|$.*

Lemma 2.1 *Let u be a vector in $\mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ be a integrable vector field on $\mathbb{R}^d$, which is left invariant by the family $\mathcal{T}_u$ i.e.,*

$$a({}^t\mathcal{O}v) = {}^t\mathcal{O}a(v), \quad v \in \mathbb{R}^d, \quad \mathcal{O} \in \mathcal{T}_u.$$

Then $\int_{\mathbb{R}^d} a(v)\, \mathrm{d}v \in \mathbb{R}u$.

Proof. For any $\mathcal{O} \in \mathcal{T}_u$, we have

$$\int_{\mathbb{R}^d} a(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} a({}^t\mathcal{O}v')\, \mathrm{d}v' = {}^t\mathcal{O} \int_{\mathbb{R}^d} a(v')\, \mathrm{d}v'.$$

For any $\xi \in \mathbb{S}^{d-1} \cap (\mathbb{R}u)^\perp$, we consider $\mathcal{O}_\xi = I_d - 2\xi \otimes \xi \in \mathcal{T}_u$, and thus we obtain

$$\int_{\mathbb{R}^d} a(v)\, \mathrm{d}v = (I_d - 2\xi \otimes \xi) \int_{\mathbb{R}^d} a(v')\, \mathrm{d}v',$$

or equivalently $\xi \cdot \int_{\mathbb{R}^d} a(v)\, \mathrm{d}v = 0$. Therefore, we have $\int_{\mathbb{R}^d} a(v)\, \mathrm{d}v \in ((\mathbb{R}u)^\perp)^\perp = \mathbb{R}u$. □

We assume that

$$\lim_{|v| \to +\infty} \frac{\frac{|v|^2}{2} + V(|v|)}{|v|} = +\infty. \tag{12}$$

Observe that

$$\begin{aligned} Z(\sigma, u) &= \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{\frac{|v|^2}{2} + V(|v|)}{\sigma} + \frac{v \cdot u}{\sigma}\right) \mathrm{d}v \\ &\leq \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}^d} \exp\left[-\frac{|v|}{\sigma}\left(\frac{\frac{|v|^2}{2} + V(|v|)}{|v|} - |u|\right)\right] \mathrm{d}v, \end{aligned}$$

and therefore, under the hypothesis (12), it is easily seen that $Z(\sigma, u)$ is finite for any $\sigma > 0$ and $u \in \mathbb{R}^d$. Similarly we check that for any $\sigma > 0$ and $u \in \mathbb{R}^d$, all the moments of M_u are finite

$$\int_{\mathbb{R}^d} |v|^p M_u(v)\, \mathrm{d}v < +\infty, \quad p \in \mathbb{N}.$$

For further developments, we recall the formula

$$\int_{\mathbb{R}^d} \chi\left(\frac{v \cdot \Omega}{|v|}, |v|\right) \mathrm{d}v = |\mathbb{S}^{d-2}| \int_{\mathbb{R}_+} r^{d-1} \int_0^\pi \chi(\cos\theta, r) \sin^{d-2}\theta\, \mathrm{d}\theta \mathrm{d}r, \tag{13}$$

for any non negative measurable function $\chi = \chi(c, r) :]-1, 1[\times \mathbb{R}_+^\star \to \mathbb{R}$, any $\Omega \in \mathbb{S}^{d-1}$ and $d \geq 2$. Here $|\mathbb{S}^{d-2}|$ is the surface of the unit sphere in $\mathbb{R}^{d-1}$, for $d \geq 3$, and $|\mathbb{S}^0| = 2$ for $d = 2$.

Proposition 2.1 *Assume that the potential $v \mapsto V(|v|)$ satisfies (12). Then the following statements hold true :*

1. *The function $Z(\sigma, u)$ depends only on σ and $|u|$. We will simply write*

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v)}{\sigma}\right) \mathrm{d}v = Z(\sigma, l = |u|).$$

2. *For any $u \in \mathbb{R}^d$, we have $\int_{\mathbb{R}^d} M_u(v) v \, \mathrm{d}v \in \mathbb{R}_+ u$ and obviously, $\int_{\mathbb{R}^d} M_0(v) v \, \mathrm{d}v = 0$.*

3. *The von Mises-Fisher distribution M_u is an equilibrium if and only if $\partial_l Z(\sigma, l) = 0$. For any $\sigma > 0, M_0(v) = Z^{-1}(\sigma, 0) \exp\left(-\Phi_0(v)/\sigma\right)$ is an equilibrium.*

Proof.
1. Applying formula (13) with $\Omega = u/|u|$, if $u \neq 0$, and any $\Omega \in \mathbb{S}^{d-1}$ if $u = 0$, we obtain

$$\begin{aligned} Z &= \int_{\mathbb{R}^d} \exp\left(-\frac{|v|^2}{2\sigma} - \frac{|u|^2}{2\sigma} + \frac{v \cdot u}{\sigma} - \frac{V(|v|)}{\sigma}\right) \mathrm{d}v \\ &= |\mathbb{S}^{d-2}| \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}_+} \exp\left(-\frac{r^2}{2\sigma} - \frac{V(r)}{\sigma}\right) r^{d-1} \int_0^\pi \exp\left(\frac{r|u|\cos\theta}{\sigma}\right) \sin^{d-2}\theta \, \mathrm{d}\theta \mathrm{d}r, \end{aligned}$$

and therefore Z depends only on σ and $|u|$.
2. We consider the integrable vector field $a(v) = M_u(v)v, v \in \mathbb{R}^d$. It is easily seen that for any $\mathcal{O} \in \mathcal{T}_u$, we have

$$\Phi_u({}^t\mathcal{O}v) = \Phi_u(v), \quad M_u({}^t\mathcal{O}v) = M_u(v), \quad v \in \mathbb{R}^d,$$

and therefore the vector field a is left invariant by $\mathcal{T}_u$. Our conclusion follows by Lemma 2.1. It remains to check that $\int_{\mathbb{R}^d} M_u(v)(v \cdot u) \, \mathrm{d}v > 0$, when $u \neq 0$. Indeed, we have

$$Z \int_{\mathbb{R}^d} M_u(v)(v \cdot u) \, \mathrm{d}v = \int_{v \cdot u > 0} \left[\exp\left(-\frac{\Phi_u(v)}{\sigma}\right) - \exp\left(-\frac{\Phi_u(-v)}{\sigma}\right)\right] (v \cdot u) \, \mathrm{d}v,$$

and we are done observing that for any v such that $v \cdot u > 0$ we have

$$-\Phi_u(v) = -\frac{|v|^2}{2} + v \cdot u - \frac{|u|^2}{2} - V(|v|) > -\frac{|v|^2}{2} - v \cdot u - \frac{|u|^2}{2} - V(|v|) = -\Phi_u(-v).$$

3. The von Mises-Fisher distribution M_u is an equilibrium if and only if $\int_{\mathbb{R}^d} M_u(v)(v-u) \, \mathrm{d}v = 0$. By the previous statement we know that $\int_{\mathbb{R}^d} M_u(v) v \, \mathrm{d}v \in \mathbb{R}u$ and therefore M_u is an equilibrium iff $\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega - |u|) \, \mathrm{d}v = 0$, where $\Omega = \frac{u}{|u|}$ if $u \neq 0$ and Ω is any vector in $\mathbb{S}^{d-1}$ if $u = 0$. But we have

$$\begin{aligned} \partial_l Z(\sigma, |u|) &= |\mathbb{S}^{d-2}| \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}_+} \exp\left(-\frac{r^2}{2\sigma} - \frac{V(r)}{\sigma}\right) r^{d-1} \\ &\quad \times \int_0^\pi \exp\left(\frac{r|u|\cos\theta}{\sigma}\right) \frac{r\cos\theta - |u|}{\sigma} \sin^{d-2}\theta \, \mathrm{d}\theta \mathrm{d}r \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v)}{\sigma}\right) \frac{v \cdot \Omega - |u|}{\sigma} \, \mathrm{d}v \\ &= \frac{Z(\sigma, |u|)}{\sigma} \int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega - |u|) \, \mathrm{d}v, \end{aligned} \tag{14}$$

and therefore M_u is an equilibrium if and only if $l = |u|$ is a critical point of $Z(\sigma, \cdot)$. □

Remark 2.2 *As Z depends only on $\sigma, |u|$, we can write*

$$\begin{aligned} Z(\sigma,|u|) &= \int_{\mathbb{R}^d} \exp\left(-\frac{|v-\Omega|u||^2}{2\sigma} - \frac{V(|v|)}{\sigma}\right) \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{|v|^2}{2\sigma} + \frac{(v\cdot\Omega)|u|}{\sigma} - \frac{|u|^2}{2\sigma} - \frac{V(|v|)}{\sigma}\right) \mathrm{d}v, \end{aligned}$$

for any $\Omega \in \mathbb{S}^{d-1}$ and $u \in \mathbb{R}^d$. We deduce that for any $\Omega \in \mathbb{S}^{d-1}$ and $u \in \mathbb{R}^d$, we have

$$\begin{aligned} \partial_l Z(\sigma,|u|) &= \int_{\mathbb{R}^d} \exp\left(-\frac{|v|^2}{2\sigma} + \frac{(v\cdot\Omega)|u|}{\sigma} - \frac{|u|^2}{2\sigma} - \frac{V(|v|)}{\sigma}\right) \frac{v\cdot\Omega - |u|}{\sigma} \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_{|u|\Omega}(v)}{\sigma}\right) \frac{v\cdot\Omega - |u|}{\sigma} \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v)}{\sigma}\right) \frac{(v-u)\cdot\Omega[u]}{\sigma} \mathrm{d}v \end{aligned}$$

and

$$\begin{aligned} \partial^2_{ll} Z(\sigma,|u|) &= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_{|u|\Omega}(v)}{\sigma}\right) \frac{[v\cdot\Omega - |u|\,]^2 - \sigma}{\sigma^2} \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v)}{\sigma}\right) \frac{[(v-u)\cdot\Omega[u]\,]^2 - \sigma}{\sigma^2} \mathrm{d}v, \end{aligned}$$

where $\Omega = \frac{u}{|u|}$ if $u \neq 0$ and Ω is any vector in $\mathbb{S}^{d-1}$ if $u = 0$ (compare with (14), established for $\Omega = u/|u|$, if $u \neq 0$).

At this point, we know that for any $\sigma > 0$, the equilibria are related to the critical points of $Z(\sigma,\cdot)$. In order to find possible bifurcation points of the disordered state $u = 0$, let us analyze the variations of $Z(\sigma,\cdot)$ for small σ. We assume the following hypothesis on the potential

$$V(|\cdot|) \in C^2(\mathbb{R}^d), \quad v \mapsto \frac{|v|^2}{2} + V(|v|) \text{ is strictly convex on } \mathbb{R}^d. \tag{15}$$

For such a potential, we can minimize $\Phi_u(v)$ with respect to $v \in \mathbb{R}^d$, for any $u \in \mathbb{R}^d$. Indeed, the function Φ_u is convex, continuous on $\mathbb{R}^d$ and

$$\begin{aligned} \Phi_u(v) &= \frac{|v-u|^2}{2} + V(|v|) = \frac{|v|^2}{2} + V(|v|) - v\cdot u + \frac{|u|^2}{2} \\ &= |v| \left(\frac{\frac{|v|^2}{2} + V(|v|)}{|v|} - \frac{v\cdot u}{|v|}\right) + \frac{|u|^2}{2} \\ &\geq |v| \left(\frac{\frac{|v|^2}{2} + V(|v|)}{|v|} - |u|\right) + \frac{|u|^2}{2}. \end{aligned}$$

By (12) we deduce that $\lim_{|v|\to+\infty} \Phi_u(v) = +\infty$ and therefore Φ_u has a minimum point $\overline{v} \in \mathbb{R}^d$. This minimum point is unique (use $\overline{v} - u + (\nabla_v V(|\cdot|))(\overline{v}) = 0$ and the strict convexity of $v \mapsto \frac{|v|^2}{2} + V(|v|)$). We intend to analyze the sign of $\partial_l Z(\sigma,|u|)$ for small σ.

Performing the change of variable $v = \overline{v} + \sqrt{\sigma} w$ leads to

$$\partial_l Z(\sigma, |u|)\sigma^{1-d/2} \exp\left(\frac{\Phi_u(\overline{v})}{\sigma}\right) = \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v) - \Phi_u(\overline{v}) - \nabla_v \Phi_u(\overline{v}) \cdot (v - \overline{v})}{\sigma}\right) \tag{16}$$

$$\times \frac{(v-u)\cdot\Omega[u]}{\sigma^{d/2}}\,\mathrm{d}v$$

$$= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(\overline{v} + \sqrt{\sigma}w) - \Phi_u(\overline{v}) - \sqrt{\sigma}\nabla_v \Phi_u(\overline{v}) \cdot w}{\sigma}\right) (\overline{v} + \sqrt{\sigma}w - u)\cdot\Omega[u]\,\mathrm{d}w$$

$$= \int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_0(\overline{v} + \sqrt{\sigma}w) - \Phi_0(\overline{v}) - \sqrt{\sigma}\nabla_v \Phi_0(\overline{v}) \cdot w}{\sigma}\right) (\overline{v} + \sqrt{\sigma}w - u)\cdot\Omega[u]\,\mathrm{d}w.$$

We need to determine the sign of $(\overline{v} - u)\cdot\Omega[u]$, where $\overline{v}$ is the minimum point of Φ_u. As $V(|\cdot|) \in C^1(\mathbb{R}^d)$, we have $V'(0) = 0$. We assume that $V(\cdot)$ possesses another critical point $r_0 > 0$ and

$$V'(r) < 0 \ \text{ for any } 0 < r < r_0 \ \text{ and } V'(r) > 0 \ \text{ for any } r > r_0. \tag{17}$$

Notice that this is the case for $V_{\alpha,\beta}(r) = \beta\frac{r^4}{4} - \alpha\frac{r^2}{2}, \alpha, \beta > 0$, with $r_0 = \sqrt{\alpha/\beta}$.

Proposition 2.2 *Assume that* (12), (15), (17) *hold true. Then*

1. *The function $r \mapsto r + V'(r)$ is strictly increasing on $\mathbb{R}_+$ and maps $[0, r_0]$ to $[0, r_0]$, and $]r_0, +\infty[$ to $]r_0, +\infty[$.*

2. *We have*

$$(\overline{v} - u)\cdot\Omega[u] > 0 \ \text{ for any } 0 < |u| < r_0, \ \inf_{\delta \le |u| \le r_0 - \delta} (\overline{v} - u)\cdot\Omega[u] > 0, \ \ 0 < \delta < \frac{r_0}{2}$$

and

$$(\overline{v} - u)\cdot\Omega[u] < 0 \ \text{ for any } |u| > r_0, \ \inf_{|u| \ge r_0 + \delta} (u - \overline{v})\cdot\Omega[u] > 0, \ \ \delta > 0.$$

Proof.
1. By (15) we know that Φ_0 is strictly convex on $\mathbb{R}^d$ and we deduce that $r \mapsto \frac{r^2}{2} + V(r)$ is strictly convex on $\mathbb{R}_+$. Therefore the function $r \mapsto r + V'(r)$ is strictly increasing on $\mathbb{R}_+$ and maps $[0, r_0]$ to $[0, r_0]$. It remains to check that it is unbounded when $r \to +\infty$. Suppose that there is a constant C such that $r + V'(r) \le C, r \in \mathbb{R}_+$. After integration with respect to r, one gets

$$\frac{r^2}{2} + V(r) \le V(0) + Cr, \ \ r \in \mathbb{R}_+,$$

implying that

$$\frac{\frac{r^2}{2} + V(r)}{r} \le \frac{V(0)}{r} + C, \ \ r \in \mathbb{R}_+,$$

which contradicts (12).
2. Let us consider $0 < |u| < r_0$. Therefore, $\overline{v} \neq 0$ and

$$\left(|\overline{v}| + V'(|\overline{v}|)\right)\frac{\overline{v}}{|\overline{v}|} = u,$$

implying that $|\overline{v}| + V'(|\overline{v}|) = |u| \in]0, r_0[$. By the previous statement we obtain $0 < |\overline{v}| < r_0$, $\Omega[\overline{v}] = \frac{\overline{v}}{|\overline{v}|} = \frac{u}{|u|} = \Omega[u]$, and thus

$$(\overline{v} - u)\cdot\Omega[u] = -V'(|\overline{v}|)\frac{\overline{v}}{|\overline{v}|}\cdot\Omega[u] = -V'(|\overline{v}|) > 0.$$

Clearly, for any $0 < \delta < r_0/2$, we have

$$\inf_{\delta \leq |u| \leq r_0 - \delta} (\overline{v} - u) \cdot \Omega[u] = \inf_{\delta \leq |u| \leq r_0 - \delta} (-V'(|\overline{v}|)) > 0.$$

Similarly, for any $|u| > r_0$, we have $|\overline{v}| > r_0$ and

$$(\overline{v} - u) \cdot \Omega[u] = -V'(|\overline{v}|) \frac{\overline{v}}{|\overline{v}|} \cdot \Omega[u] = -V'(|\overline{v}|) < 0.$$

As before, for any $\delta > 0$, we obtain

$$\inf_{|u| \geq r_0 + \delta} (u - \overline{v}) \cdot \Omega[u] = \inf_{|u| \geq r_0 + \delta} V'(|\overline{v}|) > 0.$$

□

The previous arguments allow us to complete the analysis of the variations of $Z(\sigma, |u|)$, when σ is small. The convergence when $\sigma \searrow 0$ in (16) can be handled by dominated convergence, provided that $w \mapsto |w| \exp\left(-\frac{\partial_v^2 \Phi_0(\overline{v}) w \cdot w}{2}\right)$ belongs to $L^1(\mathbb{R}^d)$. We assume that there is $\lambda < 1$ such that

$$v \mapsto V_\lambda(|v|) := \lambda \frac{|v|^2}{2} + V(|v|) \text{ is convex on } \mathbb{R}^d. \tag{18}$$

The potentials $V_{\alpha,\beta}(|v|) = \beta \frac{|v|^4}{4} - \alpha \frac{|v|^2}{2}, 0 < \alpha < 1, \beta > 0$ satisfy the above hypothesis. Under (18), we write

$$\Phi_0(v) = (1 - \lambda) \frac{|v|^2}{2} + V_\lambda(|v|), \quad v \in \mathbb{R}^d,$$

and therefore

$$\partial_v^2 \Phi_0(v) = (1 - \lambda) I_d + \partial_v^2 V_\lambda(|\cdot|) \geq (1 - \lambda) I_d, \quad v \in \mathbb{R}^d,$$

implying that

$$\int_{\mathbb{R}^d} |w| \exp\left(-\frac{\partial_v^2 \Phi_0(\overline{v}) w \cdot w}{2}\right) \, \mathrm{d}w \leq \int_{\mathbb{R}^d} |w| \exp\left(-\frac{(1 - \lambda)|w|^2}{2}\right) \, \mathrm{d}w < +\infty.$$

Notice that (18) guarantees (12) and (15). Indeed, the function $v \mapsto V_\lambda(|v|)$ being convex, it is bounded from below by a linear function

$$\exists \, (v_\lambda, C_\lambda) \in \mathbb{R}^d \times \mathbb{R} \text{ such that } V_\lambda(|v|) \geq (v \cdot v_\lambda) + C_\lambda, \quad v \in \mathbb{R}^d,$$

and therefore

$$\frac{\Phi_0(v)}{|v|} = \frac{(1 - \lambda)\frac{|v|^2}{2} + V_\lambda(|v|)}{|v|} \geq (1 - \lambda)\frac{|v|}{2} - |v_\lambda| + \frac{C_\lambda}{|v|} \to +\infty, \quad \text{as } |v| \to +\infty.$$

Obviously, Φ_0 is strictly convex, as sum between the strictly convex function $v \mapsto (1 - \lambda)\frac{|v|^2}{2}$ and the convex function $v \mapsto V_\lambda(|v|)$.

In order to conclude the study of the variations of Z for small $\sigma > 0$, we consider potentials V satisfying $V(|\cdot|) \in C^2(\mathbb{R}^d)$, (17) and (18). We come back to (16). Notice that

$$\begin{aligned} \Phi_0(\overline{v} + \sqrt{\sigma} w) - \Phi_0(\overline{v}) - \sqrt{\sigma} \nabla_v \Phi_0(\overline{v}) \cdot w \geq (1 - \lambda) \frac{|\overline{v} + \sqrt{\sigma} w|^2}{2} - (1 - \lambda) \frac{|\overline{v}|^2}{2} \\ - (1 - \lambda) \sqrt{\sigma} \overline{v} \cdot w = (1 - \lambda) \sigma \frac{|w|^2}{2}, \end{aligned}$$

implying that, for any $0 < \sigma \leq 1$

$$\left|\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)(\overline{v}+\sqrt{\sigma}w-u)\cdot\Omega[u]\right|$$
$$\leq \exp\left(-(1-\lambda)\frac{|w|^2}{2}\right)\left[|(\overline{v}-u)\cdot\Omega[u]|+|w|\right].$$

As the function $w \mapsto \exp\left(-(1-\lambda)\frac{|w|^2}{2}\right)\left[|(\overline{v}-u)\cdot\Omega[u]|+|w|\right]$ belongs to $L^1(\mathbb{R}^d)$, we deduce by dominated convergence that

$$\lim_{\sigma\searrow 0}\left\{\partial_l Z(\sigma,|u|)\sigma^{1-d/2}\exp\left(\frac{\Phi_u(\overline{v})}{\sigma}\right)\right\} = (\overline{v}-u)\cdot\Omega[u]\int_{\mathbb{R}^d}\exp\left(-\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\right)\,\mathrm{d}w.$$

As we know, cf. Proposition 2.2, that $\inf_{|u|\in[\delta,r_0-\delta]\cup[r_0+\delta,+\infty[}|(\overline{v}-u)\cdot\Omega[u]|>0, 0<\delta<r_0/2$, we deduce that for any $\delta\in]0,r_0/2[$, there is $\sigma_\delta>0$ such that

$$\partial_l Z(\sigma,|u|)>0 \ \text{ for any } 0<\sigma<\sigma_\delta,\ \ \delta\leq|u|\leq r_0-\delta$$

and

$$\partial_l Z(\sigma,|u|)<0 \ \text{ for any } 0<\sigma<\sigma_\delta,\ \ |u|\geq r_0+\delta.$$

Motivated by the above behavior of the function Z, we assume that the potential $v\mapsto V(|v|)$ satisfies (12) (such that Z is well defined) and belongs to the family $\mathcal{V}$ defined by: there exists $\sigma_0>0$ verifying

1. For any $0<\sigma<\sigma_0$ there is $l(\sigma)>0$ such that $Z(\sigma,l)$ is stricly increasing on $[0,l(\sigma)]$ and strictly decreasing on $[l(\sigma),+\infty[$;
2. For any $\sigma\geq\sigma_0, Z(\sigma,l)$ is strictly decreasing on $[0,+\infty[$.

In fact, the critical diffusion coefficient σ_0 vanishes the second order derivative of Z with respect to l, at $l=0$, as shown next.

Proposition 2.3 *Let $V(|\cdot|)\in\mathcal{V}$ be a potential satisfying* (12). *Then we have*

$$\partial_{ll}^2 Z(\sigma,0)\geq 0,\ \ 0<\sigma<\sigma_0,\ \ \partial_{ll}^2 Z(\sigma_0,0)=0,\ \ \partial_{ll}^2 Z(\sigma,0)\leq 0,\ \ \sigma>\sigma_0$$

and

$$\partial_{ll}^2 Z(\sigma,l(\sigma))\leq 0,\ \ 0<\sigma<\sigma_0.$$

Proof. By Remark 2.2 we know that $Z(\sigma,\cdot)$ possesses a second order derivative with respect to l. As $\partial_l Z(\sigma,0)=0$, we write

$$\frac{1}{2}\partial_{ll}^2 Z(\sigma,0)=\lim_{l\searrow 0}\frac{Z(\sigma,l)-Z(\sigma,0)-l\partial_l Z(\sigma,0)}{l^2}=\lim_{l\searrow 0}\frac{Z(\sigma,l)-Z(\sigma,0)}{l^2}.$$

We deduce that $\partial_{ll}^2 Z(\sigma,0)\geq 0$ for any $0<\sigma\leq\sigma_0$ and $\partial_{ll}^2 Z(\sigma,0)\leq 0$ for any $\sigma\geq\sigma_0$. In particular $\partial_{ll}^2 Z(\sigma_0,0)=0$. For any $0<\sigma<\sigma_0$, the function $Z(\sigma,\cdot)$ possesses a maximum at $l=l(\sigma)>0$ and therefore $\partial_{ll}^2 Z(\sigma,l(\sigma))\leq 0$. □

It is also easily seen that $\lim_{\sigma\nearrow\sigma_0} l(\sigma)=0$. Indeed, assume that there is $\eta>0$ and a sequence $(\sigma_n)_{n\geq 1}\nearrow\sigma_0$ such that $0<\sigma_n<\sigma_0, l(\sigma_n)\geq\eta$ for any $n\geq 1$. We have

$$Z(\sigma_n,l(\sigma_n))\geq Z(\sigma_n,\eta)>Z(\sigma_n,0),\ \ n\geq 1.$$

After passing to the limit when $n\to+\infty$, we obtain a contradiction

$$Z(\sigma_0,\eta)\geq Z(\sigma_0,0)>Z(\sigma_0,\eta)$$

and therefore $\lim_{\sigma\nearrow\sigma_0} l(\sigma)=0$. We have proved that $\sigma\mapsto l(\sigma)$ is continuous.

Remark 2.3 *Given a potential $V(|\cdot|) \in \mathcal{V}$, then the unique bifurcation point from the disordered state happens at σ_0. In fact, if we define the function*

$$H(\sigma, l) = \int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega - l)\, \mathrm{d}v\,,$$

as in [4]. Then by (14), we get $\sigma \partial_l Z(\sigma, l) = Z(\sigma, l) H(\sigma, l)$. By taking the derivative with respect to l, we obtain

$$\partial_l H = \sigma \left(\frac{\partial^2_{ll} Z}{Z} - \frac{(\partial_l Z)^2}{Z^2} \right).$$

Therefore, for the curve $l(\sigma)$ such that $H(\sigma, l(\sigma)) = 0$, we get $\partial_l H(\sigma_0, 0) = 0$. Using implicit differentiation and the continuity of the curves and the functions involved, it is also easy to check that $\partial_\sigma H(\sigma_0, 0) = 0$. Therefore, to clarify the behavior of the two curves at σ_0, one needs to work more to compute the $\lim_{\sigma \nearrow \sigma_0} l'(\sigma)$. In any case, this shows that σ_0 is the only bifurcation point from the manifold of disorder states $u = 0$ for potentials $V(|\cdot|) \in \mathcal{V}$ without the need of applying the Crandall-Rabinowitz bifurcation theorem. It would be interesting to use Crandall-Rabinowitz for general potentials to identify more general conditions for bifurcations.

In the last part of this section, we explore some properties of the potentials V in the class $\mathcal{V}$. We show that under the hypothesis (18), we retrieve a weaker version of (17).

Proposition 2.4 *Let $V(|\cdot|) \in \mathcal{V}$ be a potential satisfying (12). The application $\sigma \mapsto l(\sigma)$ is continuous on $\mathbb{R}_+^\star$. Moreover, if $V(|\cdot|) \in C^2(\mathbb{R}^d)$ verifies (18) and there is the limit $\lim_{\sigma \searrow 0} l(\sigma) = r_0 > 0$, then*

$$V'(r) \leq 0 \ \text{ for any } 0 < r \leq r_0 \ \text{ and } V'(r) \geq 0 \ \text{ for any } r \geq r_0.$$

Proof. We are done if we check the continuity ant any $\sigma \in]0, \sigma_0[$. Assume that there is a sequence $(\sigma_n)_{n \geq 1} \subset]0, \sigma_0[$, $\lim_{n \to +\infty} \sigma_n = \sigma \in]0, \sigma_0[$ and $\eta > 0$ such that $l(\sigma_n) > l(\sigma) + \eta$ for any $n \geq 1$. We have

$$Z(\sigma_n, l(\sigma_n)) > Z(\sigma_n, l(\sigma) + \eta) > Z(\sigma_n, l(\sigma_n)), \quad n \geq 1,$$

leading to the contradiction

$$Z(\sigma, l(\sigma) + \eta) \geq Z(\sigma, l(\sigma)) > Z(\sigma, l(\sigma) + \eta).$$

Similarly, assume that there is a sequence $(\sigma_n)_{n \geq 1} \subset]0, \sigma_0[$, $\lim_{n \to +\infty} \sigma_n = \sigma \in]0, \sigma_0[$ and $\eta \in]0, l(\sigma)[$ such that $l(\sigma_n) < l(\sigma) - \eta$ for any $n \geq 1$. We have

$$Z(\sigma_n, l(\sigma_n)) \geq Z(\sigma_n, l(\sigma) - \eta) > Z(\sigma_n, l(\sigma)),$$

leading to the contradiction

$$Z(\sigma, l(\sigma) - \eta) \geq Z(\sigma, l(\sigma)) > Z(\sigma, l(\sigma) - \eta).$$

Therefore $\lim_{n \to +\infty} l(\sigma_n) = l(\sigma)$ for any sequence $(\sigma_n)_{n \geq 1}$, $\lim_{n \to +\infty} \sigma_n = \sigma \in]0, \sigma_0[$.

Assume now that $\lim_{\sigma \searrow 0} l(\sigma) = r_0 > 0$. For any $l \in]0, r_0[$, we have $0 < l < l(\sigma)$ for $\sigma \in]0, \sigma_0[$ small enough. As $Z(\sigma, \cdot)$ is strictly increasing on $[0, l(\sigma)]$, we deduce that $\partial_l Z(\sigma, l) > 0$ for σ small enough, and by (16) it comes that

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_0(\overline{v} + \sqrt{\sigma} w) - \Phi_0(\overline{v}) - \sqrt{\sigma} \nabla_v \Phi_0(\overline{v}) \cdot w}{\sigma} \right) (-V'(|\overline{v}|) + \sqrt{\sigma} w \cdot \Omega)\, \mathrm{d}w > 0,$$

where $\overline{v}$ is the minimum point of $\Phi_{l\Omega}$, that is $\overline{v} = |\overline{v}|\Omega, |\overline{v}| + V'(|\overline{v}|) = l$. Passing to the limit when $\sigma \searrow 0$ yields

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\right)\,\mathrm{d}w\; V'(|\overline{v}|) \le 0,$$

and therefore $V'(|\overline{v}|) \le 0$. As before, (18) implies (15) and therefore $r \mapsto r + V'(r)$ is strictly increasing on $\mathbb{R}_+$. We have $l - |\overline{v}| = V'(|\overline{v}|) \le 0$ and $l = |\overline{v}| + V'(|\overline{v}|) \ge l + V'(l)$ saying that $V'(l) \le 0$ for any $l \in]0, r_0[$, and also for $l = r_0$.

Consider now $l > r_0$. For $\sigma \in]0, \sigma_0[$ small enough we have $l > l(\sigma)$ and therefore $\partial_l Z(\sigma, l) < 0$. As before, (16) leads to $l - |\overline{v}| = V'(|\overline{v}|) \ge 0$ and we have $l = |\overline{v}| + V'(|\overline{v}|) \le l + V'(l)$ saying that $V'(l) \ge 0$ for any $l > r_0$, and also for $l = r_0$. In particular r_0 is a critical point of V. □

In the next result we analyze the behavior of $l(\sigma)$ for σ small.

Proposition 2.5 *Let $V(|\cdot|) \in \mathcal{V}$ be a potential satisfying (12), (18). If $V(|\cdot|) \in C_b^3(\mathbb{R}^d)$ and there is the limit $\lim_{\sigma\searrow 0} l(\sigma) = r_0 > 0$, then we have for any $\Omega \in \mathbb{S}^{d-1}$*

$$V''(r_0)\lim_{\sigma\searrow 0}\frac{l(\sigma)-r_0}{\sigma} = -\frac{1+V''(r_0)}{6}\frac{\int_{\mathbb{R}^d}(w\cdot\Omega)\partial_v^3\Phi_0(r_0\Omega)(w,w,w)\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}{\int_{\mathbb{R}^d}\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}$$

where $\partial_v^3\Phi_0(r_0\Omega)(w,w,w) = \sum_{1\le i,j,k\le d}\frac{\partial^3\Phi_0}{\partial_{v_k}\partial_{v_j}\partial_{v_i}}(r_0\Omega)w_kw_jw_i$.

Proof. We fix $\Omega \in \mathbb{S}^{d-1}$. For any $\sigma \in]0, \sigma_0[$ we have $\partial_l Z(\sigma, l(\sigma)) = 0$, and (16) implies

$$\int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w) - \Phi_0(\overline{v}) - \sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)(-V'(|\overline{v}|) + \sqrt{\sigma}w\cdot\Omega)\,\mathrm{d}w = 0, \quad (19)$$

where $\overline{v}$ is the minimum point of $\Phi_{l(\sigma)\Omega}$, that is $\overline{v} = |\overline{v}|\Omega, |\overline{v}| + V'(|\overline{v}|) = l(\sigma)$. As the function $r \mapsto r + V'(r)$ is strictly increasing on $\mathbb{R}_+$, when $\sigma \searrow 0$, we have $l(\sigma) \to r_0$ and $|\overline{v}|$ converges toward the reciprocal image of r_0, through the function $r \mapsto r + V'(r)$, which is r_0. We deduce

$$\frac{l(\sigma)-r_0}{\sigma} = \frac{|\overline{v}|-r_0}{\sigma} + \frac{V'(|\overline{v}|)-V'(r_0)}{|\overline{v}|-r_0}\,\frac{|\overline{v}|-r_0}{\sigma},$$

implying that

$$\lim_{\sigma\searrow 0}\frac{l(\sigma)-r_0}{\sigma} = (1+V''(r_0))\lim_{\sigma\searrow 0}\frac{|\overline{v}|-r_0}{\sigma}.$$

We will compute

$$\lim_{\sigma\searrow 0}\frac{V'(|\overline{v}|)}{\sigma} = V''(r_0)\lim_{\sigma\searrow 0}\frac{|\overline{v}|-r_0}{\sigma}.$$

Thanks to (19) we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w\;\lim_{\sigma\searrow 0}\frac{V'(|\overline{v}|)}{\sigma} \qquad (20)\\ &= \lim_{\sigma\searrow 0}\int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)\frac{w\cdot\Omega}{\sqrt{\sigma}}\,\mathrm{d}w.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)\frac{w\cdot\Omega}{\sqrt{\sigma}}\,\mathrm{d}w \qquad (21)\\ &\quad= \int_{\mathbb{R}^d}\left[\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)\right.\\ &\qquad\left.-\exp\left(-\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\right)\right]\frac{w\cdot\Omega}{\sqrt{\sigma}}\,\mathrm{d}w\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\sigma\searrow 0}\frac{1}{\sqrt{\sigma}}\left[\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)-\exp\left(-\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\right)\right]\\
&\quad=-\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\\
&\qquad\times\lim_{\sigma\searrow 0}\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w-\frac{\sigma}{2}\,\partial_v^2\Phi_0(\overline{v})w\cdot w}{\sigma^{3/2}}\\
&\quad=-\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\lim_{\sigma\searrow 0}\frac{1}{\sqrt{\sigma}}\int_0^1(1-t)[\partial_v^2\Phi_0(\overline{v}+t\sqrt{\sigma}w)-\partial_v^2\Phi_0(\overline{v})]w\cdot w\,\mathrm{d}t\\
&\quad=-\frac{1}{6}\partial_v^3\Phi_0(r_0\Omega)(w,w,w)\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right).
\end{aligned}$$

Recall that, thanks to (18), we have $\partial_v^2\Phi_0(v)\geq(1-\lambda)I_d, v\in\mathbb{R}^d$, implying that

$$\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}=\int_0^1(1-t)\partial_v^2\Phi_0(\overline{v}+t\sqrt{\sigma}w)w\cdot w\,\mathrm{d}t\geq(1-\lambda)\frac{|w|^2}{2}$$

and

$$\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\geq(1-\lambda)\frac{|w|^2}{2},\quad w\in\mathbb{R}^d.$$

Therefore the integrand of the right hand side in (21) can be bounded, uniformly with respect to $\sigma>0$ by a L^1 function

$$\begin{aligned}
&\left|\exp\left(-\frac{\Phi_0(\overline{v}+\sqrt{\sigma}w)-\Phi_0(\overline{v})-\sqrt{\sigma}\nabla_v\Phi_0(\overline{v})\cdot w}{\sigma}\right)-\exp\left(-\frac{\partial_v^2\Phi_0(\overline{v})w\cdot w}{2}\right)\right|\frac{|(w\cdot\Omega)|}{\sqrt{\sigma}}\\
&\leq\exp\left(-(1-\lambda)\frac{|w|^2}{2}\right)\frac{|(w\cdot\Omega)|}{\sqrt{\sigma}}\int_0^1(1-t)[\partial_v^2\Phi_0(\overline{v}+t\sqrt{\sigma}w)-\partial_v^2\Phi_0(\overline{v})]w\cdot w\,\mathrm{d}t\\
&\leq\|V(|\cdot|)\|_{C_b^3(\mathbb{R}^d)}|w|^2\exp\left(-(1-\lambda)\frac{|w|^2}{2}\right),\quad w\in\mathbb{R}^d.
\end{aligned}$$

Combining (20), (21), we obtain by dominated convergence

$$\begin{aligned}
V''(r_0)\lim_{\sigma\searrow 0}\frac{|\overline{v}|-r_0}{\sigma}&=\lim_{\sigma\searrow 0}\frac{V'(|\overline{v}|)}{\sigma}\\
&=-\frac{\int_{\mathbb{R}^d}(w\cdot\Omega)\partial_v^3\Phi_0(r_0\Omega)(w,w,w)\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}{6\int_{\mathbb{R}^d}\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}
\end{aligned}$$

and therefore

$$\begin{aligned}
V''(r_0)\lim_{\sigma\searrow 0}\frac{l(\sigma)-r_0}{\sigma}&=(1+V''(r_0))\,V''(r_0)\lim_{\sigma\searrow 0}\frac{|\overline{v}|-r_0}{\sigma}\\
&=-\frac{1+V''(r_0)}{6}\frac{\int_{\mathbb{R}^d}(w\cdot\Omega)\partial_v^3\Phi_0(r_0\Omega)(w,w,w)\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}{\int_{\mathbb{R}^d}\exp\left(-\frac{\partial_v^2\Phi_0(r_0\Omega)w\cdot w}{2}\right)\,\mathrm{d}w}.
\end{aligned}$$

□

3 Linearization of the interaction mechanism

We intend to investigate the asymptotic behavior of (2) when $\varepsilon \searrow 0$. We introduce the formal development

$$f^\varepsilon = f + \varepsilon f^1 + ...$$

and we expect that $Q(f) = 0$ and

$$\partial_t f + v \cdot \nabla_x f = \lim_{\varepsilon \searrow 0} \frac{Q(f^\varepsilon) - Q(f)}{\varepsilon} = \mathrm{d}Q_f(f^1) =: \mathcal{L}_f(f^1). \tag{22}$$

As seen before, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the individual density $f(t, x, \cdot)$ is a von Mises-Fisher distribution

$$f(t, x, v) = \rho(t, x) M_{|u|\Omega(t,x)}(v), \quad v \in \mathbb{R}^d$$

where $|u|$ is a critical point of $Z(\sigma, \cdot)$, that is

$$|u| \in \{0, l(\sigma)\} \text{ if } 0 < \sigma < \sigma_0 \text{ and } |u| = 0 \text{ if } \sigma \geq \sigma_0.$$

It remains to determine the fluid equations satisfied by the macroscopic quantities ρ, Ω. When $|u| = 0$, the continuity equation leads to $\partial_t \rho = 0$. In the sequel we concentrate on the case $|u| = l(\sigma), 0 < \sigma < \sigma_0$ (that is, the modulus of the mean velocity is given, as a function of σ). We follow the strategy in [17, 1]. We consider

$$L^2_{M_u} = \{\chi : \mathbb{R}^d \to \mathbb{R} \text{ measurable }, \int_{\mathbb{R}^d} (\chi(v))^2 M_u(v)\, \mathrm{d}v < +\infty\}$$

and

$$H^1_{M_u} = \{\chi : \mathbb{R}^d \to \mathbb{R} \text{ measurable }, \int_{\mathbb{R}^d} [\, (\chi(v))^2 + |\nabla_v \chi|^2\,] M_u(v)\, \mathrm{d}v < +\infty\}.$$

We introduce the usual scalar products

$$(\chi, \theta)_{M_u} = \int_{\mathbb{R}^d} \chi(v)\theta(v) M_u(v)\, \mathrm{d}v, \quad \chi, \theta \in L^2_{M_u},$$

$$((\chi, \theta))_{M_u} = \int_{\mathbb{R}^d} (\chi(v)\theta(v) + \nabla_v \chi \cdot \nabla_v \theta) M_u(v)\, \mathrm{d}v, \quad \chi, \theta \in H^1_{M_u}$$

and we denote by $|\cdot|_{M_u}, \|\cdot\|_{M_u}$ the associated norms. Moreover we need a Poincaré inequality. This comes from the equivalence between the Fokker-Planck and Schrödinger operators. As described in [12], we can write it as

$$-\frac{\sigma}{\sqrt{M_u}} \mathrm{div}_v \left(M_u \nabla_v \left(\frac{g}{\sqrt{M_u}} \right) \right) = -\sigma \Delta_v g + \left[\frac{1}{4\sigma} |\nabla_v \Phi_u|^2 - \frac{1}{2} \Delta_v \Phi_u \right] g.$$

The operator $\mathcal{H}_u = -\sigma \Delta_v + \left[\frac{1}{4\sigma} |\nabla_v \Phi_u|^2 - \frac{1}{2} \Delta_v \Phi_u \right]$ is defined in the domain

$$D(\mathcal{H}_u) = \left\{ g \in L^2(\mathbb{R}^d),\ \left[\frac{1}{4\sigma} |\nabla_v \Phi_u|^2 - \frac{1}{2} \Delta_v \Phi_u \right] g \in L^2(\mathbb{R}^d),\ \ \Delta_v g \in L^2(\mathbb{R}^d) \right\}.$$

We have a spectral decomposition of the operator $\mathcal{H}_u$ under suitable confining assumptions (cf. Theorem XIII.67 in [50]).

Lemma 3.1 *Assume that the function* $v \mapsto \frac{1}{4\sigma}|\nabla_v \Phi_u|^2 - \frac{1}{2}\Delta_v \Phi_u$ *belongs to* $L^1_{\rm loc}(\mathbb{R}^d)$*, is bounded from below and is coercive i.e.*

$$\lim_{|v|\to+\infty} \left[\frac{1}{4\sigma}|\nabla_v \Phi_u|^2 - \frac{1}{2}\Delta_v \Phi_u\right] = +\infty.$$

Then $\mathcal{H}_u^{-1}$ *is a self adjoint compact operator in* $L^2(\mathbb{R}^d)$ *and* $\mathcal{H}_u$ *admits a spectral decomposition, that is, a nondecreasing sequence of real numbers* $(\lambda_u^n)_{n\in\mathbb{N}}$*,* $\lim_{n\to+\infty} \lambda_u^n = +\infty$*, and a* $L^2(\mathbb{R}^d)$*-orthonormal basis* $(\psi_u^n)_{n\in\mathbb{N}}$ *such that* $\mathcal{H}_u \psi_u^n = \lambda_u^n \psi_u^n, n \in \mathbb{N}$*,* $\lambda_u^0 = 0$*,* $\lambda_u^1 > 0$*.*

Therefore, under the hypotheses in Lemma 3.1, for any $u \in \mathbb{R}^d$ there is $\lambda_u > 0$ such that for any $\chi \in H^1_{M_u}$ we have

$$\sigma \int_{\mathbb{R}^d} |\nabla_v \chi|^2 M_u(v)\, \mathrm{d}v \geq \lambda_u \int_{\mathbb{R}^d} \left|\chi(v) - \int_{\mathbb{R}^d} \chi(v') M_u(v')\, \mathrm{d}v'\right|^2 M_u(v)\, \mathrm{d}v. \tag{23}$$

The fluid equations are obtained by taking the scalar product of (22) with elements in the kernel of the (formal) adjoint of $\mathcal{L}_f$, that is with functions $\psi = \psi(v)$ such that

$$\int_{\mathbb{R}^d} (\mathcal{L}_f g)(v)\psi(v)\, \mathrm{d}v = 0, \quad \text{for any function } \ g = g(v),$$

see also [5, 6, 18, 19, 43, 44]. For example, $\psi = 1$ belongs to the kernel of $\mathcal{L}_f^\star$

$$\int_{\mathbb{R}^d} (\mathcal{L}_f g)(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} \lim_{\varepsilon \searrow 0} \frac{Q(f+\varepsilon g) - Q(f)}{\varepsilon}\, \mathrm{d}v = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \{Q(f+\varepsilon g) - Q(f)\}\, \mathrm{d}v = 0,$$

and we obtain the continuity equation (7)

$$\partial_t \int_{\mathbb{R}^d} f\, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} fv\, \mathrm{d}v = \int_{\mathbb{R}^d} \mathcal{L}_f(f^1)\, \mathrm{d}v = 0.$$

In the sequel we determine the formal adjoint of the linearization of the collision operator Q around its equilibria.

Proposition 3.1 *Let* $f = f(v)$ *be an equilibrium with non vanishing mean velocity*

$$f = \rho M_u, \ \ \rho = \rho[f], \ \ u = |u|\Omega[f], \ \ |u| = l(\sigma), \ \ 0 < \sigma < \sigma_0.$$

1. *The linearization* $\mathcal{L}_f = \mathrm{d}Q_f$ *is given by*

$$\mathcal{L}_f g = \mathrm{div}_v \left\{\sigma \nabla_v g + g \nabla_v \Phi_u - M_u \int_{\mathbb{R}^d} (v' - u) g(v')\, \mathrm{d}v'\right\}.$$

2. *The formal adjoint of* $\mathcal{L}_f$ *is*

$$\mathcal{L}_f^\star \psi = \sigma \frac{\mathrm{div}_v(M_u \nabla_v \psi)}{M_u} + (v-u)\cdot W[\psi], \ \ W[\psi] := \int_{\mathbb{R}^d} M_u(v) \nabla_v \psi\, \mathrm{d}v.$$

3. *We have the identity*

$$\mathcal{L}_f(f(v-u)) = \sigma \nabla_v f - \mathrm{div}_v\left(f \mathcal{M}_u\right), \ \ \mathcal{M}_u := \int_{\mathbb{R}^d} M_u(v')(v'-u) \otimes (v'-u)\, \mathrm{d}v'.$$

Proof.
1. We have

$$\mathcal{L}_f g = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} Q(f+sg) = \mathrm{div}_v \left\{ \sigma \nabla_v g + g \nabla_v \Phi_u - f \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} u[f+sg] \right\}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} u[f+sg] = \frac{\int_{\mathbb{R}^d} (v-u[f]) g(v)\ \mathrm{d}v}{\int_{\mathbb{R}^d} f(v)\ \mathrm{d}v}.$$

Therefore we obtain

$$\mathcal{L}_f g = \mathrm{div}_v \left\{ \sigma \nabla_v g + g \nabla_v \Phi_u - -M_u \int_{\mathbb{R}^d} (v' - u[f]) g(v')\ \mathrm{d}v' \right\}.$$

2. We have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\mathcal{L}_f g)(v)\psi(v)\ \mathrm{d}v = -\int_{\mathbb{R}^d} \left\{ \sigma \nabla_v g + g \nabla_v \Phi_u - M_u(v) \int_{\mathbb{R}^d} (v' - u[f]) g(v')\ \mathrm{d}v' \right\} \cdot \nabla_v \psi\ \mathrm{d}v \\
&= \int_{\mathbb{R}^d} g(v) \left(\sigma \mathrm{div}_v \nabla_v \psi - \nabla_v \psi \cdot \nabla_v \Phi_u \right)\ \mathrm{d}v + \int_{\mathbb{R}^d} g(v')(v' - u[f])\ \mathrm{d}v' \cdot \int_{\mathbb{R}^d} M_u(v) \nabla_v \psi\ \mathrm{d}v \\
&= \int_{\mathbb{R}^d} g(v) \left(\sigma \mathrm{div}_v \nabla_v \psi - \nabla_v \psi \cdot \nabla_v \Phi_u + (v - u[f]) \cdot W[\psi] \right)\ \mathrm{d}v
\end{aligned}$$

implying

$$\mathcal{L}_f^\star \psi = \sigma \frac{\mathrm{div}_v (M_u \nabla_v \psi)}{M_u} + (v - u[f]) \cdot W[\psi].$$

3. For any $i \in \{1, ..., d\}$ we have

$$\begin{aligned}
\mathcal{L}_f (f(v-u)_i) &= \mathrm{div}_v \left[(v-u)_i \underbrace{(\sigma \nabla_v f + f \nabla_v \Phi_u)}_{=0} + \sigma f e_i - M_u \int_{\mathbb{R}^d} (v'-u)_i (v'-u) f(v')\ \mathrm{d}v' \right] \\
&= \sigma \partial_{v_i} f - \mathrm{div}_v \left(f \int_{\mathbb{R}^d} (v'-u) \otimes (v'-u) M_u(v')\ \mathrm{d}v' \right)_i
\end{aligned}$$

and therefore

$$\mathcal{L}_f (f(v-u)) = \sigma \nabla_v f - \mathrm{div}_v (f \mathcal{M}_u).$$

□

We identify now the kernel of $\mathcal{L}_f^\star$.

Lemma 3.2 *Let $f = \rho M_u > 0$ be an equilibrium with non vanishing mean velocity. The following statements are equivalent*

1. *The function $\psi = \psi(v)$ belongs to $\ker \mathcal{L}_f^\star$.*
2. *The function $\psi = \psi(v)$ satisfies*

$$\sigma \frac{\mathrm{div}_v (M_u \nabla_v \psi)}{M_u(v)} + (v-u) \cdot W = 0 \tag{24}$$

for some vector $W \in \ker(\mathcal{M}_u - \sigma I_d)$.

Moreover, the linear map $W : \ker \mathcal{L}_f^\star \to \ker(\mathcal{M}_u - \sigma I_d)$, defined by $W[\psi] = \int_{\mathbb{R}^d} M_u(v) \nabla_v \psi\ \mathrm{d}v$ induces an isomorphism between the vector spaces $\ker \mathcal{L}_f^\star / \ker W$ and $\ker(\mathcal{M}_u - \sigma I_d)$, where $\ker W$ is the set of constant functions.

Proof.
1. $\Longrightarrow$ 2. Let ψ be an element of $\ker \mathcal{L}_f^\star$. By the last statement in Proposition 3.1 we deduce

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \mathcal{L}_f^\star \psi \; f(v-u)\, \mathrm{d}v = \int_{\mathbb{R}^d} \psi(v) \mathcal{L}_f(f(v-u))\, \mathrm{d}v \\
&= \int_{\mathbb{R}^d} \psi(v) \left[\sigma \nabla_v f - \mathrm{div}_v(f \mathcal{M}_u)\right]\, \mathrm{d}v \\
&= -\sigma \int_{\mathbb{R}^d} f(v) \nabla_v \psi \, \mathrm{d}v + \mathcal{M}_u \int_{\mathbb{R}^d} f(v) \nabla_v \psi \, \mathrm{d}v \\
&= \rho(\mathcal{M}_u - \sigma I_d) W[\psi].
\end{aligned}$$

As $\rho > 0$ we deduce that $W[\psi] \in \ker(\mathcal{M}_u - \sigma I_d)$ and by the second statement in Proposition 3.1 it comes that

$$\sigma \frac{\mathrm{div}_v(M_u \nabla_v \psi)}{M_u(v)} + (v-u)\cdot W = 0, \;\; W = W[\psi] \in \ker(\mathcal{M}_u - \sigma I_d).$$

2. $\Longrightarrow$ 1. Let ψ be a function satisfying (24) for some vector $W \in \ker(\mathcal{M}_u - \sigma I_d)$. Multiplying by $M_u(v)(v-u)$ and integrating with respect to v yields

$$-\sigma \int_{\mathbb{R}^d} M_u(v) \nabla_v \psi \, \mathrm{d}v + \mathcal{M}_u W = 0.$$

As we know that $W \in \ker(\mathcal{M}_u - \sigma I_d)$, we deduce that $W = W[\psi]$, implying that ψ belongs to $\ker \mathcal{L}_f^\star$

$$\mathcal{L}_f^\star \psi = \sigma \frac{\mathrm{div}_v(M_u \nabla_v \psi)}{M_u} + (v-u)\cdot W[\psi] = \sigma \frac{\mathrm{div}_v(M_u \nabla_v \psi)}{M_u} + (v-u)\cdot W = 0.$$

□

We focus on the eigenspace $\ker(\mathcal{M}_u - \sigma I_d)$.

Lemma 3.3 *Let M_u be an equilibrium with non vanishing mean velocity. Then we have*

$$\mathcal{M}_u - \sigma I_d = \sigma^2 \frac{\partial_{ll}^2 Z(\sigma, l(\sigma))}{Z(\sigma, l(\sigma))} \Omega \otimes \Omega \leq 0, \;\; \Omega = \frac{u}{|u|}.$$

In particular $(\mathbb{R}u)^\perp \subset \ker(\mathcal{M}_u - \sigma I_d)$ with equality iff $\partial_{ll}^2 Z(\sigma, l(\sigma)) \neq 0$.

Proof. Let us consider $\{E_1, \ldots, E_{d-1}\}$ an orthonormal basis of $(\mathbb{R}\Omega)^\perp$. By using the decomposition

$$v - u = (\Omega \otimes \Omega)(v-u) + \sum_{i=1}^{d-1} (E_i \otimes E_i)(v-u) = (\Omega \otimes \Omega)(v-u) + \sum_{i=1}^{d-1} (E_i \otimes E_i) v$$

we obtain

$$\begin{aligned}
\mathcal{M}_u &= \int_{\mathbb{R}^d} \left[\Omega \otimes \Omega (v-u) + \sum_{i=1}^{d-1} E_i \otimes E_i v\right] \otimes \left[\Omega \otimes \Omega (v-u) + \sum_{j=1}^{d-1} E_j \otimes E_j v\right] M_u(v)\, \mathrm{d}v \\
&= (\mathcal{M}_u \Omega \cdot \Omega) \Omega \otimes \Omega + \sum_{i=1}^{d-1} (\mathcal{M}_u E_i \cdot E_i) E_i \otimes E_i
\end{aligned}$$

since we have

$$\mathcal{M}_u\Omega\cdot E_j=0,\ \ 1\le j\le d-1 \tag{25}$$

and

$$\mathcal{M}_uE_i\cdot E_j=\delta_{ij}\int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u(v)\,\mathrm{d}v,\ \ 1\le i,j\le d-1. \tag{26}$$

The formula (25) comes by the change of variable $v=(I_d-2E_j\otimes E_j)v'$, by noticing that $I_d-2E_j\otimes E_j\in\mathcal{T}_u$, for any $1\le j\le d-1$

$$\begin{aligned}\mathcal{M}_u\Omega\cdot E_j&=\int_{\mathbb{R}^d}\Omega\cdot(v-u)(E_j\cdot v)M_u(v)\,\mathrm{d}v\\&=-\int_{\mathbb{R}^d}\Omega\cdot(v'-u)(E_j\cdot v')M_u(v')\,\mathrm{d}v'\\&=-\mathcal{M}_u\Omega\cdot E_j=0,\ \ 1\le j\le d-1.\end{aligned}$$

For the formula (26) with $i\neq j$ we use the rotation $\mathcal{O}_{ij}\in\mathcal{T}_u$

$$v=\mathcal{O}_{ij}v',\ \ \mathcal{O}_{ij}=\Omega\otimes\Omega+\sum_{k\notin\{i,j\}}E_k\otimes E_k+E_i\otimes E_j-E_j\otimes E_i.$$

Notice that

$$(E_i\cdot v)(E_j\cdot v)=-(E_j\cdot v')(E_i\cdot v'),\ \ (E_i\cdot v)^2=(E_j\cdot v')^2$$

and therefore,

$$\begin{aligned}\mathcal{M}_uE_i\cdot E_j&=\int_{\mathbb{R}^d}(E_i\cdot v)(E_j\cdot v)M_u(v)\,\mathrm{d}v\\&=-\int_{\mathbb{R}^d}(E_j\cdot v')(E_i\cdot v')M_u(v')\,\mathrm{d}v'\\&=-\mathcal{M}_uE_i\cdot E_j=0,\ \ 1\le i\neq j\le d-1\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_uE_i\cdot E_i&=\int_{\mathbb{R}^d}(E_i\cdot v)^2M_u(v)\,\mathrm{d}v\\&=\int_{\mathbb{R}^d}(E_j\cdot v')^2M_u(v')\,\mathrm{d}v'\\&=\mathcal{M}_uE_j\cdot E_j=0,\ \ 1\le i,j\le d-1.\end{aligned}$$

As $\sum_{i=1}^{d-1}(E_i\cdot v)^2=|v|^2-(v\cdot\Omega)^2$, we obtain

$$\int_{\mathbb{R}^d}(E_i\cdot v)^2M_u(v)\,\mathrm{d}v=\int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u(v)\,\mathrm{d}v,\ \ 1\le i\le d-1$$

and

$$\mathcal{M}_u=\int_{\mathbb{R}^d}(\,(v-u)\cdot\Omega)^2M_u(v)\,\mathrm{d}v\,\Omega\otimes\Omega+\int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u(v)\,\mathrm{d}v\,(I_d-\Omega\otimes\Omega).$$

We claim that $\int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u(v)\ \mathrm{d}v=\sigma$. Multiplying $\sigma\nabla_vM_u+M_u(v)\nabla_v\Phi_u=0$ by $(|v|^2I_d-v\otimes v)\Omega$ we obtain

$$\int_{\mathbb{R}^d}\sigma\nabla_vM_u\cdot(|v|^2I_d-v\otimes v)\Omega\,\mathrm{d}v+\int_{\mathbb{R}^d}M_u(v)\nabla_v\Phi_u\cdot(|v|^2I_d-v\otimes v)\Omega\,\mathrm{d}v=0.$$

But we have

$$\mathrm{div}_v[(|v|^2 I_d - v\otimes v)\Omega] = \mathrm{div}_v[|v|^2\Omega - (v\cdot\Omega)v] = -(d-1)(v\cdot\Omega)$$

and

$$\nabla_v\Phi_u\cdot(|v|^2 I_d - v\otimes v)\Omega = \left(v-u+V'(|v|)\frac{v}{|v|}\right)\cdot(|v|^2 I_d - v\otimes v)\Omega = -(|v|^2-(v\cdot\Omega)^2)|u|.$$

We deduce that

$$(d-1)\sigma\underbrace{\int_{\mathbb{R}^d}(v\cdot\Omega)M_u(v)\,\mathrm{d}v}_{=|u|} - |u|\int_{\mathbb{R}^d}[|v|^2-(v\cdot\Omega)^2]M_u(v)\,\mathrm{d}v = 0$$

and by taking into account that $|u| = \int_{\mathbb{R}^d}(v\cdot\Omega)M_u(v)\,\mathrm{d}v$, we obtain

$$\int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u(v)\,\mathrm{d}v = \sigma.$$

By Remark 2.2, we know that

$$\sigma^2\frac{\partial^2_{ll}Z(\sigma,l(\sigma))}{Z(\sigma,l(\sigma))} = \int_{\mathbb{R}^d}M_u(v)\{((v-u)\cdot\Omega)^2-\sigma\}\,\mathrm{d}v = \int_{\mathbb{R}^d}M_u(v)((v-u)\cdot\Omega)^2\,\mathrm{d}v - \sigma$$

and finally we have

$$\mathcal{M}_u - \sigma I_d = \left(\int_{\mathbb{R}^d}((v-u)\cdot\Omega)^2 M_u(v)\,\mathrm{d}v - \sigma\right)\Omega\otimes\Omega = \sigma^2\frac{\partial^2_{ll}Z(\sigma,l(\sigma))}{Z(\sigma,l(\sigma))}\,\Omega\otimes\Omega.$$

As $l(\sigma)$ is a maximum point of $Z(\sigma,\cdot)$, we have $\partial^2_{ll}Z(\sigma,l(\sigma))\le 0$ and therefore $\mathcal{M}_u\le\sigma I_d$. □

4 The kernel of $\mathcal{L}_f^\star$

By Lemmas 3.2, 3.3, any solution of (24) with $W\in(\mathbb{R}u)^\perp$ belongs to the kernel of the formal adjoint $\mathcal{L}_f^\star$. Generally we will solve the elliptic problem

$$-\sigma\mathrm{div}_v(M_u\nabla_v\psi) = (v-u)\cdot WM_u(v),\quad v\in\mathbb{R}^d \tag{27}$$

for any $W\in\mathbb{R}^d$. We consider the continuous bilinear symmetric form $a_u : H^1_{M_u}\times H^1_{M_u}\to\mathbb{R}$ defined by

$$a_u(\varphi,\theta) = \sigma\int_{\mathbb{R}^d}\nabla_v\varphi\cdot\nabla_v\theta M_u(v)\,\mathrm{d}v,\quad \varphi,\theta\in H^1_{M_u}$$

and the linear form $L : H^1_{M_u}\to\mathbb{R}, L(\theta) = \int_{\mathbb{R}^d}\theta(v)(v-u)\cdot WM_u(v)\,\mathrm{d}v,\ \theta\in H^1_{M_u}$. Notice that under the hypothesis (12) L is bounded on $H^1_{M_u}$

$$\int_{\mathbb{R}^d}|\theta(v)(v-u)\cdot W|M_u\,\mathrm{d}v \le \left(\int_{\mathbb{R}^d}(\theta(v))^2M_u\,\mathrm{d}v\right)^{1/2}\left(\int_{\mathbb{R}^d}(|v|+|u|)^2M_u\,\mathrm{d}v\right)^{1/2}|W|.$$

We are looking for variational solutions of (27) *i.e.,*

$$\psi\in H^1_{M_u}\ \text{ and } a_u(\psi,\theta) = L(\theta)\quad\text{for any } \theta\in H^1_{M_u}. \tag{28}$$

When taking $\theta = 1 \in H^1_{M_u}$, we obtain the following necessary condition for the solvability of (27)

$$L(1) = \int_{\mathbb{R}^d} (v-u)\cdot W M_u(v)\,\mathrm{d}v = 0 \tag{29}$$

which is satisfied for any $W \in \mathbb{R}^d$, because M_u has mean velocity u. It happens that (29) also guarantees the solvability of (27). For that, it is enough to observe that the bilinear form a_u is coercive on the Hilbert space $\tilde{H}^1_{M_u} := \{\theta \in H^1_{M_u} \ : \ ((\theta,1))_{M_u} = 0\}$. Indeed, for any $\theta \in H^1_{M_u}$ such that $((\theta,1))_{M_u} = 0$, we have thanks to the Poincaré inequality (23)

$$\sigma \int_{\mathbb{R}^d} |\nabla_v \theta|^2 M_u(v)\,\mathrm{d}v \geq \lambda_u \int_{\mathbb{R}^d} (\theta(v))^2 M_u(v)\,\mathrm{d}v,$$

and therefore

$$a_u(\theta,\theta) \geq \frac{\lambda_u}{2}\int_{\mathbb{R}^d} (\theta(v))^2 M_u(v)\,\mathrm{d}v + \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla_v\theta|^2 M_u(v)\,\mathrm{d}v \geq \frac{\min\{\sigma,\lambda_u\}}{2}\|\theta\|^2_{M_u}.$$

Thanks to Lax-Milgram lemma on the Hilbert space $\tilde{H}^1_{M_u}$, there is a unique function $\psi \in \tilde{H}^1_{M_u}$ such that

$$a_u(\psi,\tilde{\theta}) = L(\tilde{\theta}) \quad \text{for any } \tilde{\theta} \in \tilde{H}^1_{M_u}. \tag{30}$$

The condition (29) allows us to extend (30) to $H^1_{M_u}$ (apply (30) with $\tilde{\theta} = \theta - ((\theta,1))_{M_u}$, for any $\theta \in H^1_{M_u}$). The uniqueness of the solution of (30) implies the uniqueness, up to a constant, for the solution of (28).

From now on, for any $W \in \mathbb{R}^d$, we denote by ψ_W the unique solution of (28), verifying $\int_{\mathbb{R}^d} \psi_W(v) M_u(v)\,\mathrm{d}v = 0$. Notice that $\psi_0 = 0$. The solution ψ_W depends linearly on $W \in \mathbb{R}^d$. Let us introduce the Hilbert spaces

$$\mathbf{L}^2_{M_u} = \{\xi : \mathbb{R}^d \to \mathbb{R}^d \ \text{ measurable }, \sum_{i=1}^d \int_{\mathbb{R}^d} (\xi_i(v))^2 M_u(v)\,\mathrm{d}v < +\infty\}$$

$$\mathbf{H}^1_{M_u} = \{\xi : \mathbb{R}^d \to \mathbb{R}^d \ \text{ measurable }, \sum_{i=1}^d \int_{\mathbb{R}^d} \{(\xi_i(v))^2 + |\nabla_v \xi_i|^2\} M_u(v)\,\mathrm{d}v < +\infty\}$$

endowed with the scalar product

$$(\xi,\eta)_{M_u} = \sum_{i=1}^d \int_{\mathbb{R}^d} \xi_i(v)\eta_i(v) M_u(v)\,\mathrm{d}v, \ \ \xi,\eta \in \mathbf{L}^2_{M_u}$$

$$((\xi,\eta))_{M_u} = \sum_{i=1}^d \int_{\mathbb{R}^d} \{\xi_i(v)\eta_i(v) + \nabla_v\xi_i\cdot\nabla_v\eta_i\} M_u(v)\,\mathrm{d}v, \ \ \xi,\eta \in \mathbf{H}^1_{M_u}.$$

We denote the induced norms by $|\xi|_{M_u} = (\xi,\xi)^{1/2}_{M_u}$, $\xi \in \mathbf{L}^2_{M_u}$ and $\|\xi\|_{M_u} = ((\xi,\xi))^{1/2}_{M_u}$, $\xi \in \mathbf{H}^1_{M_u}$. Obviously, a vector field $\xi = \xi(v)$ belongs to $\mathbf{H}^1_{M_u}$ iff $\xi_i \in H^1_{M_u}$ for any $i \in \{1,...,d\}$ and we have

$$\|\xi\|^2_{M_u} = \sum_{i=1}^d \|\xi_i\|^2_{M_u}.$$

Let us consider the closed subspace

$$\tilde{\mathbf{H}}^1_{M_u} = \{\xi \in \mathbf{H}^1_{M_u} \ : \ \int_{\mathbb{R}^d} \xi(v) M_u(v)\,\mathrm{d}v = 0\}.$$

Thanks to (23), for any $\xi \in \mathbf{H}^1_{M_u}$ we have the inequality

$$\sigma \sum_{i=1}^{d} \int_{\mathbb{R}^d} |\nabla_v \xi_i|^2 M_u(v)\, \mathrm{d}v \geq \lambda_u \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left| \xi_i(v) - \int_{\mathbb{R}^d} \xi_i(v') M_u(v')\, \mathrm{d}v' \right|^2 M_u(v)\, \mathrm{d}v$$

and therefore

$$\begin{aligned} \sigma \sum_{i=1}^{d} \int_{\mathbb{R}^d} |\nabla_v \xi_i|^2 M_u(v)\, \mathrm{d}v &\geq \frac{\min\{\sigma, \lambda_u\}}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^d} [(\xi_i(v))^2 + |\nabla_v \xi_i|^2] M_u(v)\, \mathrm{d}v \\ &= \frac{\min\{\sigma, \lambda_u\}}{2} \|\xi\|^2_{M_u}, \ \ \xi \in \tilde{\mathbf{H}}^1_{M_u}. \end{aligned} \tag{31}$$

We introduce the continuous bilinear symmetric form $\mathbf{a}_u : \mathbf{H}^1_{M_u} \times \mathbf{H}^1_{M_u} \to \mathbb{R}$ defined by

$$\mathbf{a}_u(\xi, \eta) = \sigma \int_{\mathbb{R}^d} \partial_v \xi : \partial_v \eta \ M_u(v)\, \mathrm{d}v = \sum_{i=1}^{d} a_u(\xi_i, \eta_i), \ \ \xi, \eta \in \mathbf{H}^1_{M_u}$$

and the linear form $\mathbf{L} : \mathbf{H}^1_{M_u} \to \mathbb{R}$, $\mathbf{L}(\eta) = \int_{\mathbb{R}^d} (v - u) \cdot \eta(v) M_u(v)\, \mathrm{d}v, \eta \in \mathbf{H}^1_{M_u}$. Under the hypothesis (12), it is easily seen that $\mathbf{L}$ is bounded on $\mathbf{H}^1_{M_u}$

$$\int_{\mathbb{R}^d} |(v - u) \cdot \eta(v)| M_u(v)\, \mathrm{d}v \leq \left(\int_{\mathbb{R}^d} (|v| + |u|)^2 M_u(v)\, \mathrm{d}v \right)^{1/2} \|\eta\|_{M_u}, \ \ \eta \in \mathbf{H}^1_{M_u}.$$

Proposition 4.1 *There is a unique solution F of the variational problem*

$$F \in \tilde{\mathbf{H}}^1_{M_u} \ \ \textit{and} \ \ \mathbf{a}_u(F, \eta) = \mathbf{L}(\eta), \ \ \textit{for any } \eta \in \mathbf{H}^1_{M_u}.$$

For any $W \in \mathbb{R}^d$ we have $\psi_W(v) = F(v) \cdot W, v \in \mathbb{R}^d$. The vector field F is left invariant by the family $\mathcal{T}_u$.

Proof. The bilinear for $\mathbf{a}_u$ is coercive on $\tilde{\mathbf{H}}^1_{M_u}$, thanks to (31)

$$\mathbf{a}_u(\xi, \xi) \geq \frac{\min\{\sigma, \lambda_u\}}{2} \|\xi\|^2_{M_u}, \quad \text{for any } \xi \in \tilde{\mathbf{H}}^1_{M_u}.$$

By Lax-Milgram lemma, applied on the Hilbert space $\tilde{\mathbf{H}}^1_{M_u}$, there is a unique vector field $F \in \tilde{\mathbf{H}}^1_{M_u}$ such that

$$\mathbf{a}_u(F, \eta) = \mathbf{L}(\eta), \ \ \text{for any } \eta \in \tilde{\mathbf{H}}^1_{M_u}.$$

Actually, the above equality holds true for any $\eta \in \mathbf{H}^1_{M_u}$

$$\begin{aligned} \mathbf{a}_u(F, \eta) &= \sum_{i=1}^{d} a_u(F_i, \eta_i) = \sum_{i=1}^{d} a_u(F_i, \eta_i - (\eta_i, 1)_{M_u}) \\ &= \mathbf{L}(\eta_1 - (\eta_1, 1)_{M_u}, ..., \eta_d - (\eta_d, 1)_{M_u}) \\ &= \sum_{i=1}^{d} \int_{\mathbb{R}^d} (v_i - u_i)[\eta_i(v) - (\eta_i, 1)_{M_u}] M_u(v)\, \mathrm{d}v \\ &= \sum_{i=1}^{d} \int_{\mathbb{R}^d} (v_i - u_i) \eta_i(v) M_u(v)\, \mathrm{d}v \\ &= \mathbf{L}(\eta). \end{aligned}$$

It remains to check that for any $W \in \mathbb{R}^d$, $v \mapsto F(v) \cdot W$ solves (30), on $H^1_{M_u}$. Observe that $F \cdot W \in \tilde{H}^1_{M_u}$. Notice also that for any $\theta \in H^1_{M_u}$ we have $\theta W \in \mathbf{H}^1_{M_u}$ and

$$\begin{aligned} a_u(F \cdot W, \theta) &= \sigma \int_{\mathbb{R}^d} {}^t\partial_v F W \cdot \nabla_v \theta M_u(v)\, \mathrm{d}v \\ &= \sigma \int_{\mathbb{R}^d} \partial_v F : \partial_v(\theta W) M_u(v)\, \mathrm{d}v \\ &= \mathbf{a}_u(F, \theta W) = \mathbf{L}(\theta W) \\ &= \int_{\mathbb{R}^d} (v-u) \cdot W \theta(v) M_u(v)\, \mathrm{d}v \\ &= L(\theta). \end{aligned}$$

Thank to the uniqueness we obtain $\psi_W(v) = F(v) \cdot W, v \in \mathbb{R}^d, W \in \mathbb{R}^d$. Consider now $\mathcal{O} \in \mathcal{T}_u$. We are done if we prove that $v \to \mathcal{O}F({}^t\mathcal{O}v)$ solves the same problem as F. Clearly we have

$$\int_{\mathbb{R}^d} |\mathcal{O}F({}^t\mathcal{O}v)|^2 M_u(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} |F(v')|^2 M_u(v')\, \mathrm{d}v' < +\infty$$

$$\begin{aligned} \int_{\mathbb{R}^d} \partial[\mathcal{O}F({}^t\mathcal{O}\,\cdot)] : \partial[\mathcal{O}F({}^t\mathcal{O}\,\cdot)] M_u(v)\, \mathrm{d}v &= \int_{\mathbb{R}^d} \partial F({}^t\mathcal{O}\,\cdot) : \partial F({}^t\mathcal{O}\,\cdot) M_u(v)\, \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \partial F({}^t\mathcal{O}v)\, {}^t\mathcal{O} : \partial F({}^t\mathcal{O}v)\, {}^t\mathcal{O} M_u(v)\, \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \partial F(v') : \partial F(v') M_u(v')\, \mathrm{d}v' < +\infty. \end{aligned}$$

and

$$\int_{\mathbb{R}^d} \mathcal{O}F({}^t\mathcal{O}v) M_u(v)\, \mathrm{d}v = \mathcal{O} \int_{\mathbb{R}^d} F(v') M_u(\mathcal{O}v')\, \mathrm{d}v' = 0$$

saying that $v \mapsto \mathcal{O}F({}^t\mathcal{O}v)$ belongs to $\tilde{\mathbf{H}}^1_{M_u}$. For any $\eta \in \mathbf{H}^1_{M_u}$ we have ${}^t\mathcal{O}\eta(\mathcal{O}\,\cdot) \in \mathbf{H}^1_{M_u}$ and

$$\begin{aligned} \mathbf{a}_u(\mathcal{O}F({}^t\mathcal{O}\,\cdot), \eta) &= \sigma \int_{\mathbb{R}^d} \partial(\mathcal{O}F({}^t\mathcal{O}\,\cdot)) : \partial\eta M_u(v)\, \mathrm{d}v \\ &= \sigma \int_{\mathbb{R}^d} \mathcal{O}\partial F({}^t\mathcal{O}v)\, {}^t\mathcal{O} : \partial\eta M_u(v)\, \mathrm{d}v \\ &= \sigma \int_{\mathbb{R}^d} \partial F(v') :\ {}^t\mathcal{O}(\partial\eta)(\mathcal{O}v')\mathcal{O} M_u(\mathcal{O}v')\, \mathrm{d}v' \\ &= \sigma \int_{\mathbb{R}^d} \partial F(v') : \partial({}^tO\eta(\mathcal{O}\,\cdot))(v') M_u(v')\, \mathrm{d}v' \\ &= \int_{\mathbb{R}^d} (v'-u) \cdot\ {}^t\mathcal{O}\eta(\mathcal{O}v') M_u(v')\, \mathrm{d}v' \\ &= \int_{\mathbb{R}^d} (v-u) \cdot \eta(v) M_u(v)\, \mathrm{d}v = \mathbf{L}(\eta). \end{aligned}$$

□

The vector field F expresses in terms of two functions which are left invariant by the family $\mathcal{T}_u$.

Proposition 4.2 *There is a function ψ, which is left invariant by the family $\mathcal{T}_u$, such that*

$$F(v) = \psi(v) \frac{v - (v \cdot \Omega)\Omega}{\sqrt{|v|^2 - (v \cdot \Omega)^2}} + \psi_\Omega(v)\Omega, \quad v \in \mathbb{R}^d \setminus (\mathbb{R}\Omega).$$

Proof. Obviously we have $F = (F\cdot\Omega)\Omega + F' = \psi_\Omega\Omega + F'$, with $F' = (I_d - \Omega\otimes\Omega)F$. The vector field F' is orthogonal to Ω and is left invariant by the family $\mathcal{T}_u$

$$F'({}^t\mathcal{O}v) = F({}^t\mathcal{O}v) - (F({}^t\mathcal{O}v)\cdot\Omega)\Omega = {}^t\mathcal{O}F(v) - ({}^t\mathcal{O}F(v)\cdot\Omega)\Omega$$
$$= {}^t\mathcal{O}(F(v) - (F(v)\cdot\Omega)\Omega) = {}^t\mathcal{O}F'(v),\ \ v\in\mathbb{R}^d.$$

We claim that $F'(v)$ is parallel to the orthogonal projection of v over $(\mathbb{R}\Omega)^\perp$. Indeed, for any $v\in\mathbb{R}^d\setminus(\mathbb{R}\Omega)$, let us consider

$$E(v) = \frac{(I_d - \Omega\otimes\Omega)v}{\sqrt{|v|^2 - (v\cdot\Omega)^2}}.$$

When $d=2$, since $E(v)$ and $F'(v)$ are both orthogonal to Ω, there exists a function $\psi = \psi(v)$ such that

$$F'(v) = \psi(v)E(v) = \psi(v)\frac{(I_2 - \Omega\otimes\Omega)v}{\sqrt{|v|^2 - (\Omega\cdot v)^2}},\ \ v\in\mathbb{R}^2\setminus(\mathbb{R}\Omega)\,.$$

If $d\geq 3$, let us denote by ${}^\perp E$, any unitary vector orthogonal to E and Ω. Introducing the orthogonal matrix $\mathcal{O} = I_d - 2\,{}^\perp E\otimes{}^\perp E\in\mathcal{T}_u$, we obtain $F'({}^t\mathcal{O}\,\cdot) = {}^t\mathcal{O}F'$. Observe that

$$0 = {}^\perp E\cdot E(v) = {}^\perp E\cdot\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}} = \frac{{}^\perp E\cdot v}{\sqrt{|v|^2-(v\cdot\Omega)^2}},\ \ \mathcal{O}v = v$$

and thus

$$F'(v) = F'(\mathcal{O}v) = \mathcal{O}F'(v) = (I_d - 2\,{}^\perp E\otimes{}^\perp E)F'(v) = F'(v) - 2({}^\perp E\cdot F'(v))\,{}^\perp E$$

from which it follows that ${}^\perp E\cdot F'(v) = 0$, for any vector ${}^\perp E$ orthogonal to E and Ω. Hence, there exists a function $\psi(v)$ such that

$$F'(v) = \psi(v)E(v) = \psi(v)\frac{(I_d - \Omega\otimes\Omega)v}{\sqrt{|v|^2 - (v\cdot\Omega)^2}},\ \ v\in\mathbb{R}^d\setminus(\mathbb{R}\Omega).$$

It is easily seen that the function ψ is left invariant by the family $\mathcal{T}_u$. Indeed, for any $\mathcal{O}\in\mathcal{T}_u$ we have

$$\psi({}^t\mathcal{O}v) = F'({}^t\mathcal{O}v)\cdot E({}^t\mathcal{O}v) = {}^t\mathcal{O}F'(v)\cdot\,{}^t\mathcal{O}E(v) = F'(v)\cdot E(v) = \psi(v),\ v\in\mathbb{R}^d.$$

Similarly, ψ_Ω is left invariant by the family $\mathcal{T}_u$

$$\psi_\Omega({}^t\mathcal{O}v) = F({}^t\mathcal{O}v)\cdot\Omega = {}^t\mathcal{O}F(v)\cdot\Omega = F(v)\cdot\mathcal{O}\Omega = F(v)\cdot\Omega = \psi_\Omega(v),\ v\in\mathbb{R}^d,\ \mathcal{O}\in\mathcal{T}_u.$$

□

The functions ψ, ψ_Ω will enter the fluid model satisfied by the macroscopic quantities $\rho, \Omega, |u|$. It is convenient to determine the elliptic partial differential equations satisfied by them.

Proposition 4.3 *There are two functions $\chi = \chi(c,r) :]-1,1[\times]0,+\infty[\to\mathbb{R}$, $\chi_\Omega = \chi_\Omega(c,r) :]-1,1[\times]0,+\infty[\to\mathbb{R}$ such that $\psi(v) = \chi(v\cdot\Omega/|v|,|v|)\,,\psi_\Omega(v) = \chi_\Omega(v\cdot\Omega/|v|,|v|)$, $v\in\mathbb{R}^d\setminus(\mathbb{R}\Omega)$. The above functions satisfy*

$$-\sigma\partial_c\{r^{d-3}(1-c^2)^{\frac{d-1}{2}}e(c,r,|u|)\partial_c\chi\} - \sigma\partial_r\{r^{d-1}(1-c^2)^{\frac{d-3}{2}}e(c,r,|u|)\partial_r\chi\} \tag{32}$$
$$+\sigma(d-2)r^{d-3}(1-c^2)^{\frac{d-5}{2}}e(c,r,|u|)\chi = r^d(1-c^2)^{\frac{d-2}{2}}e(c,r,|u|),\ (c,r)\in]-1,1[\times]0,+\infty[$$

and

$$-\sigma\partial_c\{r^{d-3}(1-c^2)^{\frac{d-1}{2}}e(c,r,|u|)\partial_c\chi_\Omega\} - \sigma\partial_r\{r^{d-1}(1-c^2)^{\frac{d-3}{2}}e(c,r,|u|)\partial_r\chi_\Omega\} \tag{33}$$
$$= r^{d-1}(rc-|u|)(1-c^2)^{\frac{d-3}{2}}e(c,r,|u|),\ (c,r)\in]-1,1[\times]0,+\infty[$$

where $e(c,r,l) = \exp\left(-\frac{r^2}{2\sigma} + \frac{rcl}{\sigma} - \frac{V(r)}{\sigma}\right)$.

Proof. The function $\psi_\Omega = F \cdot \Omega$ satisfies

$$\psi_\Omega \in \tilde{H}^1_{M_u} \text{ and } \sigma \int_{\mathbb{R}^d} \nabla_v \psi_\Omega \cdot \nabla_v \theta \, M_u(v) \, \mathrm{d}v = \int_{\mathbb{R}^d} (v-u) \cdot \Omega \, \theta(v) M_u(v) \, \mathrm{d}v, \; \theta \in \tilde{H}^1_{M_u}. \quad (34)$$

By Remark 2.1 we know that there is $\chi_\Omega = \chi_\Omega(c,r)$ such that $\psi_\Omega(v) = \chi_\Omega(v \cdot \Omega/|v|, |v|), v \in \mathbb{R}^d \setminus (\mathbb{R}\Omega)$. As ψ_Ω belongs to $\tilde{H}^1_{M_u}$, which is equivalent to

$$\int_{\mathbb{R}^d} \psi_\Omega(v) M_u(v) \, \mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} |\nabla_v \psi_\Omega|^2 M_u(v) \, \mathrm{d}v < +\infty$$

we are led to the Hilbert space

$$H_{\|,|u|} = \{h :]-1,1[\times]0,+\infty[\to \mathbb{R}, \; \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} h(c,r) e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}} \, \mathrm{d}c\mathrm{d}r = 0,$$
$$\int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} \left[(\partial_c h)^2 \frac{1-c^2}{r^2} + (\partial_r h)^2 \right] e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}} \, \mathrm{d}c\mathrm{d}r < +\infty\}$$

endowed with the scalar product

$$(h,g)_{\|,|u|} = \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} \left[\partial_c h \partial_c g \frac{1-c^2}{r^2} + \partial_r h \partial_r g \right] e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}} \, \mathrm{d}c\mathrm{d}r, \; h,g \in H_{\|,|u|}.$$

Taking in (34) $\theta(v) = h(v \cdot \Omega/|v|, |v|)$, with $h \in H_{\|,|u|}$ (which means $\theta \in \tilde{H}^1_{M_u}$), we obtain

$$\sigma \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} \left[\partial_c \chi_\Omega \partial_c h \frac{1-c^2}{r^2} + \partial_r \chi_\Omega \partial_r h \right] e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}} \, \mathrm{d}c\mathrm{d}r$$
$$= \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} (rc - |u|) h(c,r) e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}} \, \mathrm{d}c\mathrm{d}r$$

which implies (33). We focus now on the equation satisfied by ψ. Let us consider an orthonormal basis $\{E_1, ..., E_{d-1}\}$ of $(\mathbb{R}\Omega)^\perp$. By Remark 2.1 we know that there is $\chi = \chi(c,r)$ such that $\psi(v) = \chi(v \cdot \Omega/|v|, |v|)$ and

$$\psi_{E_i}(v) = F(v) \cdot E_i = \psi(v) \frac{v \cdot E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}} = \chi(v \cdot \Omega/|v|, |v|) \frac{v \cdot E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}}$$

for $v \in \mathbb{R}^d \setminus (\mathbb{R}\Omega), i \in \{1, ..., d-1\}$. Let us consider $\psi_{E_i,h}(v) = h(v \cdot \Omega/|v|, |v|) \frac{v \cdot E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}}$, where $h = h(c,r)$ is a function such that $\psi_{E_i,h} \in H^1_{M_u}$. Actually, once that $\psi_{E_i,h} \in H^1_{M_u}$, then $((\psi_{E_i,h}, 1))_{M_u} = \int_{\mathbb{R}^d} h(v \cdot \Omega/|v|, |v|) \frac{v \cdot E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}} M_u(v) \, \mathrm{d}v = 0$, saying that $\psi_{E_i,h} \in \tilde{H}^1_{M_u}$. A straightforward computation shows that

$$\nabla_v \psi_{E_i,h} = \frac{v \cdot E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}} \left[\partial_c h \frac{I_d - \frac{v \otimes v}{|v|^2}}{|v|} \Omega + \partial_r h \frac{v}{|v|} \right]$$
$$+ h \left(\frac{v \cdot \Omega}{|v|}, |v| \right) \left[I_d - \frac{(v - (v \cdot \Omega)\Omega) \otimes v}{|v|^2 - (v \cdot \Omega)^2} \right] \frac{E_i}{\sqrt{|v|^2 - (v \cdot \Omega)^2}}$$

and

$$|\nabla_v \psi_{E_i,h}|^2 = \frac{(v \cdot E_i)^2}{|v|^4} (\partial_c h)^2 + \frac{(v \cdot E_i)^2}{|v|^2 - (v \cdot \Omega)^2} (\partial_r h)^2$$
$$+ \frac{|v|^2 - (v \cdot \Omega)^2 - (v \cdot E_i)^2}{(|v|^2 - (v \cdot \Omega)^2)^2} h^2 \left(\frac{v \cdot \Omega}{|v|}, |v| \right).$$

The condition $\psi_{E_i,h} \in H^1_{M_u}$ writes

$$\int_{\mathbb{R}^d} (\psi_{E_i,h})^2 M_u(v)\, \mathrm{d}v < +\infty, \quad \int_{\mathbb{R}^d} |\nabla_v \psi_{E_i,h}|^2 M_u(v)\, \mathrm{d}v < +\infty$$

which is equivalent, thanks to the Poincaré inequality (23) to

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla_v \psi_{E_i,h}|^2 M_u(v)\, \mathrm{d}v \\
&= \int_{\mathbb{R}^d} \left[\frac{(v\cdot E_i)^2}{|v|^4} (\partial_c h)^2 + \frac{(v\cdot E_i)^2 (\partial_r h)^2}{|v|^2 - (v\cdot\Omega)^2} + \frac{|v|^2 - (v\cdot\Omega)^2 - (v\cdot E_i)^2}{(|v|^2-(v\cdot\Omega)^2)^2} h^2 \right] M_u(v)\, \mathrm{d}v < +\infty
\end{aligned}$$

and therefore to $h \in H_{\perp,|u|}$, where we consider he Hilbert space

$$H_{\perp,|u|} = \{h :]-1,1[\times]0,+\infty[\to \mathbb{R}, \ \ \|h\|^2_{\perp,|u|} = (h,h)_{\perp,|u|} < +\infty\}$$

endowed with the scalar product

$$\begin{aligned}
(g,h)_{\perp,|u|} = \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} \left[\frac{1-c^2}{r^2} \partial_c g \partial_c h + \partial_r g \partial_r h + \frac{(d-2)gh}{r^2(1-c^2)} \right] \\
e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}}\, \mathrm{d}c\mathrm{d}r, g, h, \in H_{\perp,|u|}.
\end{aligned}$$

Taking $\theta = \psi_{E_i,h} \in \tilde{H}^1_{M_u}$ in (30) leads to

$$\sigma \int_{\mathbb{R}^d} \nabla_v \psi_{E_i} \cdot \nabla_v \psi_{E_i,h} M_u(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} \psi_{E_i,h}\, (v\cdot E_i) M_u(v)\, \mathrm{d}v, \ \ h \in H_{\perp,|u|}$$

or equivalently

$$\begin{aligned}
&\sigma \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} \left[\frac{1-c^2}{r^2} \partial_c \chi \partial_c h + \partial_r \chi \partial_r h + \frac{(d-2)\chi h}{r^2(1-c^2)} \right] e(c,r,|u|)(1-c^2)^{\frac{d-3}{2}}\, \mathrm{d}c\mathrm{d}r \\
&= \int_{\mathbb{R}_+} r^{d-1} \int_{-1}^{+1} r h(c,r) e(c,r,|u|)(1-c^2)^{\frac{d-2}{2}}\, \mathrm{d}c\mathrm{d}r, \ h \in H_{\perp,|u|}
\end{aligned}$$

which implies (32). □

5 The fluid model

The balances for the macroscopic quantities ρ, u follow by using the elements in the kernel of $\mathcal{L}^\star_f$.

Proof. (of Theorem 1.1)
The use of $\psi = 1 \in \ker \mathcal{L}^\star_f$ leads to (7). By Lemma 3.3, we know that $(\mathbb{R}u)^\perp \subset \ker(\mathcal{M}_u - \sigma I_d)$ and thus, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the vector field

$$v \mapsto F'(t,x,v) = \chi\left(\frac{v\cdot\Omega(t,x)}{|v|}, |v| \right) \frac{(I_d - \Omega(t,x)\otimes\Omega(t,x))v}{\sqrt{|v|^2 - (v\cdot\Omega(t,x))^2}} = \sum_{i=1}^{d-1} \psi_{E_i}(v) E_i$$

belongs to the kernel of $\mathcal{L}^\star_f$, implying that

$$\int_{\mathbb{R}^d} \partial_t f\ F'(t,x,v)\, \mathrm{d}v + \int_{\mathbb{R}^d} v\cdot\nabla_x f\ F'(t,x,v)\, \mathrm{d}v = 0, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

We have $\partial_t f = \partial_t \rho M_u + \rho \frac{M_u}{\sigma}(v-u)\cdot \partial_t u$ and we obtain

$$\int_{\mathbb{R}^d} \partial_t f\ F'\ \mathrm{d}v = \int_{\mathbb{R}^d} \Big(\partial_t \rho + \frac{\rho}{\sigma}(v-u)\cdot\partial_t u\Big)\,\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}} M_u(v)\ \mathrm{d}v$$
$$= \partial_t\rho \int_{\mathbb{R}^d} \chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}} M_u(v)\ \mathrm{d}v$$
$$+ \frac{\rho}{\sigma}\int_{\mathbb{R}^d} \chi\left(\frac{v\cdot\Omega}{|v|},|v|\right) M_u(v)\frac{[v-(v\cdot\Omega)\Omega]\otimes[v-(v\cdot\Omega)\Omega+(v\cdot\Omega)\Omega-u]}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v\ \partial_t u.$$

It is easily seen (use the change of variable $v=(I_d-2E_i\otimes E_i)v', 1\le i\le d-1$) that

$$\int_{\mathbb{R}^d}\chi\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}}M_u(v)\ \mathrm{d}v = \sum_{i=1}^{d-1}\int_{\mathbb{R}^d}\chi\frac{(v\cdot E_i)E_i}{\sqrt{|v|^2-(v\cdot\Omega)^2}}M_u(v)\ \mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^d}\chi M_u\frac{[v-(v\cdot\Omega)\Omega]\otimes[(v\cdot\Omega)\Omega-u]}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v = \sum_{i=1}^{d-1}\int_{\mathbb{R}^d}\chi M_u\frac{(v\cdot E_i)E_i\otimes[(v\cdot\Omega)\Omega-u]}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v = 0,$$

and

$$\int_{\mathbb{R}^d}\chi M_u\frac{[v-(v\cdot\Omega)\Omega]\otimes[v-(v\cdot\Omega)\Omega]}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v = \sum_{1\le i,j\le d-1}\int_{\mathbb{R}^d}\chi M_u\frac{(v\cdot E_i)(v\cdot E_j)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v E_i\otimes E_j$$
$$= \sum_{i=1}^{d-1}\int_{\mathbb{R}^d}\chi\frac{(v\cdot E_i)^2}{\sqrt{|v|^2-(v\cdot\Omega)^2}}M_u(v)\ \mathrm{d}v E_i\otimes E_i$$
$$= \int_{\mathbb{R}^d}\chi\frac{\sqrt{|v|^2-(v\cdot\Omega)^2}}{d-1}M_u(v)\ \mathrm{d}v(I_d-\Omega\otimes\Omega).$$

Therefore one gets

$$\int_{\mathbb{R}^d}\partial_t f F'(t,x,v)\ \mathrm{d}v = c_{\perp,1}\frac{\rho}{\sigma}(I_d-\Omega\otimes\Omega)\partial_t u \tag{35}$$

with

$$c_{\perp,1} = \int_{\mathbb{R}^d}\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)\frac{\sqrt{|v|^2-(v\cdot\Omega)^2}}{d-1}M_u(v)\ \mathrm{d}v.$$

Observe also that

$$v\cdot\nabla_x f = (v\cdot\nabla_x\rho)M_u + \frac{\rho}{\sigma}\partial_x u v\cdot(v-u)M_u$$
$$= (v\cdot\nabla_x\rho)M_u + \frac{\rho}{\sigma}\partial_x u v\cdot(v-(v\cdot\Omega)\Omega+(v\cdot\Omega)\Omega-u)M_u,$$

and therefore

$$\int_{\mathbb{R}^d}(v\cdot\nabla_x f)\ F'\ \mathrm{d}v = \int_{\mathbb{R}^d}\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)M_u(v)\frac{(v-(v\cdot\Omega)\Omega)\otimes v}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v\ \nabla_x\rho \tag{36}$$
$$+\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)M_u(v)\frac{(v-(v\cdot\Omega)\Omega)\otimes(v-(v\cdot\Omega)\Omega+(v\cdot\Omega)\Omega-u)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \partial_x u v\ \mathrm{d}v.$$

As before, using the change of variable $v=(I_d-2E_i\otimes E_i)v', 1\le i\le d-1$, we have

$$\int_{\mathbb{R}^d}\chi M_u(v)\frac{(v-(v\cdot\Omega)\Omega)\otimes v}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v = \int_{\mathbb{R}^d}\chi M_u\frac{(v-(v\cdot\Omega)\Omega)\otimes(v-(v\cdot\Omega)\Omega)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v$$
$$+\int_{\mathbb{R}^d}\chi M_u\frac{(v-(v\cdot\Omega)\Omega)\otimes(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\ \mathrm{d}v$$
$$= c_{\perp,1}(I_d-\Omega\otimes\Omega).$$

For the second integral in the right hand side of (36), by noticing that

$$\int_{\mathbb{R}^d}(v\cdot E_i)(v\cdot E_j)(v\cdot E_k)\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)M_u(v)\,\mathrm{d}v=0,\quad i,j,k\in\{1,...,d-1\},$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\chi M_u(v)\frac{(v-(v\cdot\Omega)\Omega)\otimes(v-(v\cdot\Omega)\Omega+(v\cdot\Omega)\Omega-u)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\partial_x uv\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}\chi M_u\frac{(v-(v\cdot\Omega)\Omega)\otimes(v-(v\cdot\Omega)\Omega)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\partial_x u\Omega\,(v\cdot\Omega)\,\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^d}\chi M_u\frac{(v-(v\cdot\Omega)\Omega)\otimes((v\cdot\Omega)\Omega-u)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\partial_x u(v-(v\cdot\Omega)\Omega)\,\mathrm{d}v\\
&=c_{\perp,2}(I_d-\Omega\otimes\Omega)\partial_x u\Omega+\int_{\mathbb{R}^d}\chi M_u\frac{(v-(v\cdot\Omega)\Omega)\otimes(v-(v\cdot\Omega)\Omega)}{\sqrt{|v|^2-(v\cdot\Omega)^2}}\,{}^t\partial_x u[(v\cdot\Omega)\Omega-u]\,\mathrm{d}v\\
&=c_{\perp,2}(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)\Omega-c_{\perp,1}(I_d-\Omega\otimes\Omega)\,(u\cdot\partial_x)u,
\end{aligned}$$

where

$$c_{\perp,2}=\int_{\mathbb{R}^d}(v\cdot\Omega)\chi\frac{\sqrt{|v|^2-(v\cdot\Omega)^2}}{d-1}M_u(v)\,\mathrm{d}v.$$

Therefore we deduce

$$\begin{aligned}
\int_{\mathbb{R}^d}(v\cdot\nabla_x f)F'(t,x,v)\,\mathrm{d}v=c_{\perp,1}(I_d-\Omega\otimes\Omega)\nabla_x\rho+\frac{\rho}{\sigma}c_{\perp,2}(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)\Omega\\
-\frac{\rho}{\sigma}c_{\perp,1}(I_d-\Omega\otimes\Omega)\,(u\cdot\partial_x)u
\end{aligned}\tag{37}$$

and finally (35), (37) yield

$$(I_d-\Omega\otimes\Omega)\partial_t u+\sigma(I_d-\Omega\otimes\Omega)\frac{\nabla_x\rho}{\rho}+c_\perp(I_d-\Omega\otimes\Omega)(u\cdot\partial_x)u+(c_\perp-1)(I_d-\Omega\otimes\Omega)\nabla_x\frac{|u|^2}{2}=0\tag{38}$$

where

$$\begin{aligned}
c_\perp=\frac{c_{\perp,2}}{|u|\;c_{\perp,1}}&=\frac{\int_{\mathbb{R}^d}(v\cdot\Omega)\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)\sqrt{|v|^2-(v\cdot\Omega)^2}M_u(v)\,\mathrm{d}v}{|u|\;\int_{\mathbb{R}^d}\chi\left(\frac{v\cdot\Omega}{|v|},|v|\right)\sqrt{|v|^2-(v\cdot\Omega)^2}M_u(v)\,\mathrm{d}v}\\
&=\frac{\int_{\mathbb{R}_+}r^{d+1}\int_0^\pi\cos\theta\chi(\cos\theta,r)e(\cos\theta,r,l(\sigma))\sin^{d-1}\theta\,\mathrm{d}\theta\mathrm{d}r}{l(\sigma)\int_{\mathbb{R}_+}r^d\int_0^\pi\chi(\cos\theta,r)e(\cos\theta,r,l(\sigma))\sin^{d-1}\theta\,\mathrm{d}\theta\mathrm{d}r}.
\end{aligned}$$

Recall that $|u|=l(\sigma)$ and therefore we have $u\cdot\partial_t u=\frac{1}{2}\partial_t|u|^2=0$, $(u\cdot\partial_x)u=\frac{1}{2}\nabla_x|u|^2=0$, implying that

$$\Omega\cdot\partial_t u=0,\quad {}^t\partial_x u\Omega=0,\quad \Omega\cdot\partial_x u\Omega=0.$$

The equation (38) becomes

$$\partial_t\Omega+l(\sigma)c_\perp(\Omega\cdot\nabla_x)\Omega+\frac{\sigma}{l(\sigma)}(I_d-\Omega\otimes\Omega)\frac{\nabla_x\rho}{\rho}=0.$$

We have to check that $c_{\perp,1}\neq 0$. This comes by using the elliptic equations satisfied by ψ_{E_i}, that is

$$-\sigma\mathrm{div}_v(M_u\nabla_v\psi_{E_i})=(v\cdot E_i)M_u(v),\quad v\in\mathbb{R}^d,\quad i\in\{1,...,d-1\}.$$

Indeed, we have

$$c_{\perp,1} = \int_{\mathbb{R}^d} \chi \frac{(v\cdot E_i)^2}{\sqrt{|v|^2-(v\cdot\Omega)^2}} M_u(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (F(v)\cdot E_i)(v\cdot E_i) M_u(v)\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^d} \psi_{E_i}(v)(v\cdot E_i) M_u(v)\,\mathrm{d}v = \sigma \int_{\mathbb{R}^d} |\nabla_v \psi_{E_i}|^2 M_u(v)\,\mathrm{d}v > 0.$$

□

Other potentials $v \mapsto V(|v|)$ can be handled as well. For example, let us assume that there is $\sigma > 0$, $0 \le l_1(\sigma) < l_2(\sigma) \le +\infty$ such that the function $l \mapsto Z(\sigma, l)$ is stricly increasing on $[0, l_1(\sigma)]$, constant on $[l_1(\sigma), l_2(\sigma)[$, and strictly decreasing on $[l_2(\sigma), +\infty[$. In that case, for any $l \in [l_1(\sigma), l_2(\sigma)[$ we have $\partial^2_{ll} Z(\sigma, l) = 0$ and by Lemma 3.3 we deduce that $\mathcal{M}_u = \sigma I_d$, saying that $\ker(\mathcal{M}_u - \sigma I_d) = \mathbb{R}^d$. Using the function ψ_Ω, we obtain a balance for $|u|$ as well.

Proof. (of Theorem 1.2)
In this case ψ_Ω belongs to $\ker \mathcal{L}^\star_f$, and therefore we also have the balance

$$\int_{\mathbb{R}^d} \partial_t f \psi_{\Omega(t,x)}(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} (v\cdot\nabla_x f)\psi_{\Omega(t,x)}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} \mathcal{L}_{f(t,x,\cdot)}(f^1)\psi_{\Omega(t,x)}\,\mathrm{d}v = 0.$$

As before, using also $\int_{\mathbb{R}^d} \psi_\Omega(v) M_u(v)\,\mathrm{d}v = 0$, we write

$$\int_{\mathbb{R}^d} \partial_t f \psi_\Omega\,\mathrm{d}v = \int_{\mathbb{R}^d} \Big[\partial_t \rho M_u(v) + \frac{\rho}{\sigma} M_u(v)(v-u)\cdot\partial_t u\Big]\psi_\Omega(v)\,\mathrm{d}v \tag{39}$$
$$= \Big(\partial_t\rho - \frac{\rho}{\sigma} u\cdot\partial_t u\Big)\int_{\mathbb{R}^d}\psi_\Omega(v) M_u(v)\,\mathrm{d}v$$
$$+ \frac{\rho}{\sigma}\int_{\mathbb{R}^d} \chi_\Omega M_u(v)[v-(v\cdot\Omega)\Omega + (v\cdot\Omega)\Omega]\,\mathrm{d}v\cdot\partial_t u$$
$$= \frac{\rho}{\sigma} c_{\parallel,1}\Omega\cdot\partial_t u$$

where

$$c_{\parallel,1} = \int_{\mathbb{R}^d} (v\cdot\Omega)\psi_\Omega(v) M_u(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (v-u)\cdot\Omega\,\psi_\Omega M_u\,\mathrm{d}v = \sigma\int_{\mathbb{R}^d} |\nabla_v\psi_\Omega|^2 M_u(v)\,\mathrm{d}v > 0.$$

Similarly, observe that

$$\int_{\mathbb{R}^d} (v\cdot\nabla_x f)\psi_\Omega\,\mathrm{d}v = \int_{\mathbb{R}^d}\Big[v\cdot\nabla_x\rho + \frac{\rho}{\sigma}\partial_x u v\cdot(v-u)\Big] M_u(v)\psi_\Omega(v)\,\mathrm{d}v \tag{40}$$
$$\int_{\mathbb{R}^d} (v\cdot\Omega)\psi_\Omega(v) M_u(v)\,\mathrm{d}v\,(\Omega\cdot\nabla_x\rho) + \frac{\rho}{\sigma}\int_{\mathbb{R}^d}\psi_\Omega(v) M_u(v)(v-u)\otimes v\,\mathrm{d}v : \partial_x u$$
$$= c_{\parallel,1}\Omega\cdot\nabla_x\rho + \frac{\rho}{\sigma}\int_{\mathbb{R}^d}\psi_\Omega M_u\{[v-(v\cdot\Omega)\Omega]\otimes[v-(v\cdot\Omega)\Omega] + (v\cdot\Omega)^2\Omega\otimes\Omega\}\,\mathrm{d}v : \partial_x u$$
$$- \frac{\rho}{\sigma}\int_{\mathbb{R}^d}\psi_\Omega(v) M_u(v) v\,\mathrm{d}v\cdot\ {}^t\partial_x u u = c_{\parallel,1}\Omega\cdot\nabla_x\rho$$
$$+ \frac{\rho}{\sigma}\left\{\int_{\mathbb{R}^d}\psi_\Omega M_u\frac{|v|^2-(v\cdot\Omega)^2}{d-1}\,\mathrm{d}v\,(I_d-\Omega\otimes\Omega) + \int_{\mathbb{R}^d}(v\cdot\Omega)^2\psi_\Omega M_u\,\mathrm{d}v\,\Omega\otimes\Omega\right\} : \partial_x u$$
$$- \frac{\rho}{\sigma} c_{\parallel,1}\Omega\cdot\ {}^t\partial_x u u$$
$$= c_{\parallel,1}\Omega\cdot\nabla_x\rho + \frac{\rho}{\sigma}(2c_{\parallel,2} - |u|c_{\parallel,1})\Omega\otimes\Omega : \partial_x u + \frac{\rho}{\sigma} c_{\parallel,3}(I_d - \Omega\otimes\Omega) : \partial_x u$$
$$= c_{\parallel,1}\Omega\cdot\nabla_x\rho + \frac{\rho}{\sigma}\frac{c_{\parallel,2}}{|u|}(\Omega\cdot\partial_x u u) + \frac{\rho}{\sigma}\left(\frac{c_{\parallel,2}}{|u|} - c_{\parallel,1}\right)\left(\Omega\cdot\nabla_x\frac{|u|^2}{2}\right) + \frac{\rho}{\sigma} c_{\parallel,3}|u|\mathrm{div}_x\Omega$$

where

$$c_{\|,2} = \int_{\mathbb{R}^d} \frac{(v\cdot\Omega)^2}{2}\psi_\Omega(v)M_u(v)\,\mathrm{d}v,\;\; c_{\|,3} = \int_{\mathbb{R}^d} \frac{|v|^2-(v\cdot\Omega)^2}{d-1}\psi_\Omega(v)M_u(v)\,\mathrm{d}v.$$

In the above computations we have used the identity $(I_d - \Omega\otimes\Omega) : \partial_x u = |u|\mathrm{div}_x\Omega$. Combining (39), (40) leads to

$$\Omega\cdot\partial_t u + \sigma\Omega\cdot\frac{\nabla_x\rho}{\rho} + c_\|(\Omega\cdot(u\cdot\partial_x)u) + (c_\|-1)\left(\Omega\cdot\nabla_x\frac{|u|^2}{2}\right) + c'_\||u|^2\mathrm{div}_x\Omega = 0 \qquad (41)$$

where $c_\| = \frac{c_{\|,2}}{|u|c_{\|,1}}$, $c'_\| = \frac{c_{\|,3}}{|u|c_{\|,1}}$. Finally we deduce from (38), (41) the balance for the mean velocity u

$$\begin{aligned}\partial_t u + c_\perp(I_d-\Omega\otimes\Omega)\partial_x uu + c_\|(\Omega\otimes\Omega)(u\cdot\partial_x)u + (c_\perp-1)(I_d-\Omega\otimes\Omega)\nabla_x\frac{|u|^2}{2}\\ + (c_\|-1)(\Omega\otimes\Omega)\nabla_x\frac{|u|^2}{2} + \sigma\frac{\nabla_x\rho}{\rho} + c'_\|\mathrm{div}_x\Omega|u|u = 0.\end{aligned}$$

□

Remark 5.1 *When $V = 0$, the equilibria are Maxwellians parametrized by $\rho\in\mathbb{R}_+$ and $u\in\mathbb{R}^d$*

$$M_u(v) = \frac{\rho}{(2\pi\sigma)^{d/2}}\exp\left(-\frac{|v-u|^2}{2\sigma}\right),\;\; v\in\mathbb{R}^d.$$

In that case the function $l\to Z(\sigma,l)$ is constant

$$Z(\sigma,l) = \int_{\mathbb{R}^d}\exp\left(-\frac{|v-u|^2}{2\sigma}\right)\,\mathrm{d}v = (2\pi\sigma)^{d/2},\;\; l\in\mathbb{R}_+^\star.$$

It is easily seen that the solution of

$$-\sigma\mathrm{div}_v\{M_u\partial_v F\} = (v-u)M_u(v),\;\; v\in\mathbb{R}^d,\;\; \int_{\mathbb{R}^d}M_u(v)F(v)\,\mathrm{d}v = 0$$

is $F(v) = v-u, v\in\mathbb{R}^d$, which belongs to $\tilde{\mathbf{H}}^1_{M_u}$, and therefore the functions ψ,ψ_Ω such that

$$F(v) = \psi(v)\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}} + \psi_\Omega(v)\Omega,\;\; v\in\mathbb{R}^d\setminus(\mathbb{R}\Omega)$$

are given by

$$\psi(v) = (v-u)\cdot\frac{v-(v\cdot\Omega)\Omega}{\sqrt{|v|^2-(v\cdot\Omega)^2}} = \sqrt{|v|^2-(v\cdot\Omega)^2},\;\; \psi_\Omega(v) = (v-u)\cdot\Omega,\;\; v\in\mathbb{R}^d$$

and $\psi_{E_i}(v) = F(v)\cdot E_i = (v\cdot E_i), v\in\mathbb{R}^d, 1\le i\le d-1$.

By straightforward computations we obtain

$$c_{\perp,1} = \int_{\mathbb{R}^d}\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u\,\mathrm{d}v = \int_{\mathbb{R}^d}(v\cdot E_1)^2M_u\,\mathrm{d}v = -\sigma\int_{\mathbb{R}^d}(v\cdot E_1)(\nabla_v M_u\cdot E_1)\,\mathrm{d}v = \sigma$$

$$\begin{aligned}c_{\perp,2} &= \int_{\mathbb{R}^d}(v\cdot\Omega)\frac{|v|^2-(v\cdot\Omega)^2}{d-1}M_u\,\mathrm{d}v = \int_{\mathbb{R}^d}(v\cdot\Omega)(v\cdot E_1)^2M_u\,\mathrm{d}v\\ &= -\sigma\int_{\mathbb{R}^d}(v\cdot\Omega)(v\cdot E_1)\mathrm{div}_v(M_uE_1)\,\mathrm{d}v = \sigma\int_{\mathbb{R}^d}M_u(v)E_1\cdot[(v\cdot E_1)\Omega + (v\cdot\Omega)E_1]\,\mathrm{d}v\\ &= \sigma|u|.\end{aligned}$$

$$c_\perp = \frac{c_{\perp,2}}{|u|c_{\perp,1}} = 1$$

$$c_{\parallel,1} = \sigma \int_{\mathbb{R}^d} |\nabla_v \psi_\Omega|^2 M_u(v)\, \mathrm{d}v = \sigma$$

$$c_{\parallel,2} = \int_{\mathbb{R}^d} \frac{(v\cdot\Omega)^2}{2} \psi_\Omega M_u \, \mathrm{d}v = -\sigma \int_{\mathbb{R}^d} \frac{(v\cdot\Omega)^2}{2} \mathrm{div}_v(M_u\Omega)\, \mathrm{d}v = \sigma \int_{\mathbb{R}^d} (v\cdot\Omega) M_u \, \mathrm{d}v = \sigma|u|$$

$$c_\parallel = \frac{c_{\parallel,2}}{|u|c_{\parallel,1}} = 1$$

$$c_{\parallel,3} = \int_{\mathbb{R}^d} \frac{|v|^2 - (v\cdot\Omega)^2}{d-1} \psi_\Omega M_u \, \mathrm{d}v = \int_{\mathbb{R}^d} (v\cdot E_1)^2 \psi_\Omega M_u \, \mathrm{d}v = -\sigma \int_{\mathbb{R}^d} (v\cdot E_1)^2 \mathrm{div}_v(M_u\Omega)\, \mathrm{d}v$$
$$= 2\sigma \int_{\mathbb{R}^d} (v\cdot E_1)(E_1\cdot\Omega) M_u \, \mathrm{d}v = 0.$$

In this case (10), (11) are the Euler equations, as expected when taking the limit $\varepsilon \searrow 0$ in the Fokker-Planck equations

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\mathrm{div}_v\{\sigma\nabla_v f^\varepsilon + f^\varepsilon(v-u[f^\varepsilon])\}, \;\; (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$$

that is

$$\partial_t\rho + \mathrm{div}_x(\rho u) = 0, \;\; \partial_t u + \partial_x u u + \sigma\frac{\nabla_x\rho}{\rho} = 0, \;\; (t,x)\in\mathbb{R}_+\times\mathbb{R}^d.$$

6 Examples

We analyze now the potentials $v\mapsto V_{\alpha,\beta}(|v|) = \beta\frac{|v|^4}{4} - \alpha\frac{|v|^2}{2}$. Clearly the hypothesis (12) is satisfied, and thus the function $Z(\sigma,|u|) = \int_{\mathbb{R}^d}\exp\left(-\frac{|v-u|^2}{2\sigma} - \frac{V_{\alpha,\beta}(|v|)}{\sigma}\right)\,\mathrm{d}v$ is well defined. As seen in Section 2, the sign of $\partial_l Z(\sigma,l)$, for small $\sigma>0$, depends on the sign of $V'_{\alpha,\beta}$. The potential $V_{\alpha,\beta}$ satisfy (17) with $r_0 = \sqrt{\alpha/\beta}$

$$V'_{\alpha,\beta}(r) = r(\beta r^2-\alpha) < 0 \;\text{ for }\; 0<r<\sqrt{\alpha/\beta} \;\text{ and }\; V'_{\alpha,\beta}(r)>0 \;\text{ for any }\; r>\sqrt{\alpha/\beta}.$$

One can check that these potentials belong to the family $\mathcal{V}$, see [45]. We include an example $V_{1,1}(|v|) = \frac{|v|^4}{4} - \frac{|v|^2}{2}$ for the sake of completeness. In this case the critical diffusion can be computed explicitly.

Proposition 6.1 *Consider the potential $v\mapsto V_{1,1}(|v|) = \frac{|v|^4}{4} - \frac{|v|^2}{2}$. The critical diffusion σ_0 writes*

$$\sigma_0^{1/2} = \frac{1}{d}\frac{\int_{\mathbb{R}_+}\exp(-z^4/4)z^{d+1}\,\mathrm{d}z}{\int_{\mathbb{R}_+}\exp(-z^4/4)z^{d-1}\,\mathrm{d}z}, \;\; d\geq 2.$$

In particular, for $d=2$ we have $\sigma_0 = 1/\pi$.

Proof. We have

$$\Phi_u(v) = \frac{|v-u|^2}{2} + V_{1,1}(|v|) = \frac{|v|^4}{4} - v\cdot u + \frac{|u|^2}{2}$$

and therefore

$$Z(\sigma,l) = \int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_u(v)}{\sigma}\right)\,\mathrm{d}v = |\mathbb{S}^{d-2}|\exp\left(-\frac{l^2}{2\sigma}\right)$$
$$\int_{\mathbb{R}_+}\exp\left(-\frac{r^4}{4\sigma}\right)r^{d-1}\int_0^\pi \exp\left(\frac{rl\cos\theta}{\sigma}\right)\sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r.$$

Taking the second derivative with respect to l one gets cf. Remark 2.2

$$\partial_{ll}^2 Z(\sigma,l) = \int_{\mathbb{R}^d} \exp\left(-\frac{|v|^4}{4\sigma} + \frac{v\cdot u}{\sigma} - \frac{l^2}{2\sigma}\right) \frac{(v\cdot\Omega - l)^2 - \sigma}{\sigma^2}\,\mathrm{d}v = |\mathbb{S}^{d-2}| \exp\left(-\frac{l^2}{2\sigma}\right)$$
$$\int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d-1} \int_0^\pi \exp\left(\frac{rl\cos\theta}{\sigma}\right) \frac{(r\cos\theta - l)^2 - \sigma}{\sigma^2} \sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r$$

and therefore

$$\partial_{ll}^2 Z(\sigma,0) = \frac{|\mathbb{S}^{d-2}|}{\sigma^2} \int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d+1}\,\mathrm{d}r \int_0^\pi \cos^2\theta \sin^{d-2}\theta\,\mathrm{d}\theta$$
$$- \frac{|\mathbb{S}^{d-2}|}{\sigma} \int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d-1}\,\mathrm{d}r \int_0^\pi \theta \sin^{d-2}\theta\,\mathrm{d}\theta.$$

It is easily seen that

$$\int_0^\pi \cos^2\theta \sin^{d-2}\theta\,\mathrm{d}\theta = \int_0^\pi \sin^{d-2}\theta\,\mathrm{d}\theta + \int_0^\pi \cos'\theta \sin^{d-1}\theta\,\mathrm{d}\theta$$
$$= \int_0^\pi \sin^{d-2}\theta\,\mathrm{d}\theta - (d-1)\int_0^\pi \cos^2\theta \sin^{d-2}\theta\,\mathrm{d}\theta$$

and thus

$$\int_0^\pi \cos^2\theta \sin^{d-2}\theta\,\mathrm{d}\theta = \frac{1}{d}\int_0^\pi \sin^{d-2}\theta\,\mathrm{d}\theta, \quad d \geq 2.$$

We obtain the following expression for the second derivative $\partial_{ll}^2 Z(\sigma,0)$

$$\frac{\partial_{ll}^2 Z(\sigma,0)}{|\mathbb{S}^{d-2}|} = \frac{\int_0^\pi \sin^{d-2}\theta\,\mathrm{d}\theta}{\sigma^2}\left\{\frac{1}{d}\int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d+1}\,\mathrm{d}r - \sigma\int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d-1}\,\mathrm{d}r\right\}.$$

Using the change of variable $r = \sigma^{1/4} z$, we have

$$\int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d+1}\,\mathrm{d}r = \int_{\mathbb{R}_+} \exp\left(-\frac{z^4}{4}\right) z^{d+1}\,\mathrm{d}z\,\sigma^{\frac{d+2}{4}}$$
$$\int_{\mathbb{R}_+} \exp\left(-\frac{r^4}{4\sigma}\right) r^{d-1}\,\mathrm{d}r = \int_{\mathbb{R}_+} \exp\left(-\frac{z^4}{4}\right) z^{d-1}\,\mathrm{d}z\,\sigma^{\frac{d}{4}}$$

and thus $\partial_{ll}^2 Z(\sigma,0) > 0$ iff

$$\sigma^{1/2} < \frac{1}{d}\frac{\int_{\mathbb{R}_+} \exp\left(-\frac{z^4}{4}\right) z^{d+1}\,\mathrm{d}z}{\int_{\mathbb{R}_+} \exp\left(-\frac{z^4}{4}\right) z^{d-1}\,\mathrm{d}z}.$$

The critical diffusion σ_0 is, cf. Proposition 2.3

$$\sigma_0^{1/2} = \frac{1}{d}\frac{\int_{\mathbb{R}_+} \exp(-z^4/4) z^{d+1}\,\mathrm{d}z}{\int_{\mathbb{R}_+} \exp(-z^4/4) z^{d-1}\,\mathrm{d}z}, \quad d \geq 2.$$

In particular, when $d = 2$, we obtain

$$\int_{\mathbb{R}_+} \exp(-z^4/4) z^3\,\mathrm{d}z = \int_{\mathbb{R}_+} \exp(-z^4/4)\,\mathrm{d}\frac{z^4}{4} = \int_{\mathbb{R}_+} \exp(-s)\,\mathrm{d}s = 1$$

and

$$\int_{\mathbb{R}_+} \exp(-z^4/4) z\,\mathrm{d}z = \int_{\mathbb{R}_+} \exp(-z^4/4)\,\mathrm{d}\frac{z^2}{2} = \int_{\mathbb{R}_+} \exp(-s^2)\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}$$

implying that $\sigma_0 = 1/\pi$. □

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References

[1] P. Aceves-Sánchez, M. Bostan, J.A. Carrillo, P. Degond, Hydrodynamic limits for kinetic flocking models of Cucker-Smale type, Mathematical Biosciences and Engineering, 16 (2019), pp. 7883-7910.

[2] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato, J. Soler, Vehicular traffic, crowds, and swarms: from kinetic theory and multiscale methods to applications and research perspectives, Math. Models Methods Appl. Sci. 29 (2019) 1901–2005.

[3] A. B. T. Barbaro and P. Degond, Phase transition and diffusion among socially interacting self-propelled agents, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), pp. 1249-1278.

[4] A. B. T. Barbaro, J.A. Cañizo, J.A. Carrillo, P. Degond, Phase transitions in a kinetic flocking model of Cucker-Smale type, Multiscale Model. Simul. 14 (2016) 1063-1088.

[5] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations. I. Formal derivations. J. Stat. Phys. 63(1991) 323-344.

[6] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. Comm. Pure Appl. Math. 46(1993) 667-753.

[7] N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, On the asymptotic theory from microscopic to macroscopic tissue models: An overview with perspectives, Math. Models Methods Appl. Sci. 22 (2012) 1130001.

[8] N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, On the multiscale modeling of vehicular traffic: From kinetic to hydrodynamics, Discr. Cont. Dyn. Syst. Ser. B 19 (2014) 18691888.

[9] N. Bellomo, P. Degond and E. Tadmor, eds. Active Particles, Vol. 1 (Birkhuser-Springer, 2017).

[10] F. Bolley, J.A. Cañizo, J.A. Carrillo, Stochastic Mean-Field Limit: Non-Lipschitz Forces & Swarming, Math. Models Methods Appl. Sci. 21(2011) 2179-2210.

[11] F. Bolley, J.A. Cañizo, J.A. Carrillo, Mean-field limit for the stochastic Vicsek model, Appl. Math. Lett. 25(2012) 339-343.

[12] V. Bonnaillie-Noël, J.A. Carrillo, T. Goudon, G.A. Pavliotis, Efficient numerical calculation of drift and diffusion coefficients in the diffusion approximation of kinetic equations, IMA J. Numer. Anal. (2016) 1-34.

[13] M. Bostan, The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime, Asymptot. Anal. 61(2009) 91-123.

[14] M. Bostan, Transport equations with disparate advection fields. Application to the gyrokinetic models in plasma physics, J. Differential Equations 249(2010) 1620-1663.

[15] M. Bostan, Gyro-kinetic Vlasov equation in three dimensional setting. Second order approximation, SIAM J. Multiscale Model. Simul. 8(2010) 1923-1957.

[16] M. Bostan, J.-A. Carrillo, Asymptotic fixed-speed reduced dynamics for kinetic equations in swarming, Math. Models Methods Appl. Sci. 23(2013) 2353-2393.

[17] M. Bostan, J.-A. Carrillo, Reduced fluid models for self-propelled populations, interacting through alignment, Math. Models Methods Appl. Sci. 27(2017) 1255-1299.

[18] M. Bostan, On the Boltzmann equation for charged particle beams under the effect of strong magnetic fields, Discrete Contin. Dyn. Syst. Ser. B 20(2015) 339-371.

[19] M. Bostan, High magnetic field equilibria for the Fokker-Planck-Landau equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 33(2016) 899-931.

[20] W. Braun, K. Hepp, The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles, Comm. Math. Phys. 56(1977) 101-113.

[21] D. Burini, S. De Lillo and L. Gibelli, Collective learning dynamics modeling based on the kinetic theory of active particles, Phys. Life Rev. 16 (2016) 123139.

[22] J.A. Cañizo, J.A. Carrillo, J. Rosado, A well-posedness theory in measures for some kinetic models of collective motion, Math. Models Methods Appl. Sci. 21(2011) 515-539.

[23] J.A. Carrillo, M.R. D'Orsogna, V. Panferov, Double milling in a self-propelled swarms from kinetic theory, Kinet. Relat. Models 2(2009) 363-378.

[24] J.A. Carrillo, M. Fornasier, J. Rosado, G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, SIAM J. Math. Anal. 42(2010) 218-236.

[25] J. A. Carrillo, M. Fornasier, G. Toscani, F. Vecil, Particle, Kinetic, and Hydrodynamic Models of Swarming. Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhauser (2010), 297-336.

[26] J. A. Carrillo, Y.-P. Choi, M. Hauray, The derivation of swarming models: mean-field limit and Wasserstein distances, in Collective dynamics from bacteria to crowds, volume 553 CISM Courses and Lect., pages 1-46. Springer, Vienna (2014).

[27] J. A. Carrillo, Y.-P. Choi, M. Hauray, S. Salem, Mean-field limit for collective behavior models with sharp sensitivity regions, J. Eur. Math. Soc. 21(2019) 121-161.

[28] J.A. Carrillo, A. Klar, S. Martin, S. Tiwari, Self-propelled interacting particle systems with roosting force, Math. Models Methods Appl. Sci 20 (2010) 1533-1552.

[29] J. A. Carrillo, R. S. Gvalani, G. A. Pavliotis, A. Schlichting, Long-time behaviour and phase transitions for the Mckean-Vlasov equation on the torus, Arch. Ration. Mech. Anal. 235 (2020) 635–690.

[30] L. Chayes and V. Panferov. The McKean-Vlasov equation in finite volume. *J. Stat. Phys.*, 138(1-3):351–380, 2010.

[31] Y.-L. Chuang, M.R. D'Orsogna, D. Marthaler, A.L. Bertozzi, L. Chayes, State transitions and the continuum limit for a 2D interacting, self-propelled particle system, Phys. D 232(2007) 33-47.

[32] F. Cucker and S. Smale. Emergent behavior in flocks. IEEE Trans. Automat. Control 52(2007) 852-862.

[33] P. Degond, A. Frouvelle, J.-G. Liu, Macroscopic limits and phase transition in a system of self-propelled particles, J. Nonlinear Sci. 23 (2013) 427-456.

[34] P. Degond, A. Frouvelle, J.-G. Liu, Phase transitions, hysteresis, and hyperbolicity for self-organized alignment dynamics, Arch. Ration. Mech. Anal. 216 (2015) 63115.

[35] P. Degond, S. Motsch, Continuum limit of self-driven particles with orientation interaction, Math. Models Methods Appl. Sci. 18(2008) 1193-1215.

[36] R. Dobrushin, Vlasov equations, Funct. Anal. Appl. 13(1979) 115-123.

[37] M.R. D'Orsogna, Y.-L. Chuang, A.L. Bertozzi, L. Chayes, Self-propelled particles with soft-core interactions : Patterns, stability and collapse, Phys. Rev. Lett. 96(2006) 104302-1/4.

[38] A. Frouvelle, J.-G. Liu, Dynamics in a kinetic model of oriented particles with phase transition, SIAM J. Math. Anal. 44 (2012) 791-826.

[39] S. N. Gomes, G. A. Pavliotis, Mean field limits for interacting diffusions in a two-scale potential, J. Nonlinear Sci. 28 (2018) 905–941.

[40] S.-Y. Ha, J.-G. Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, Commun. Math. Sci. 7(2009) 297-325.

[41] S.-Y. Ha, K. Lee, D. Levy, Emergence of Time-Asymptotic Flocking in a Stochastic Cucker-Smale System, Commun. Math. Sci. 7(2009) 453-469.

[42] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, Kinet. Relat. Models 1 (2008) 415-435.

[43] D. Levermore, Entropic convergence and the linearized limit for the Boltzmann equation. Comm. Partial Differential Equations 18(1993) 1231-1248.

[44] D. Levermore, Moment closure hierarchies for kinetic theories. J. Stat. Phys. 83 (1996) 1021-1065.

[45] X. Li, Flocking: Phase transition and asymptotic behaviour, preprint arXiv:1906.07517.

[46] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.

[47] S. Motsch, E. Tadmor, A new model for self-organized dynamics and its flocking behavior, J. Stat. Phys. 144(2011) 923-947.

[48] S. Motsch, E. Tadmor, Heterophilious dynamics enhances consensus, SIAM Rev. 56 (2014) 577-621.

[49] H. Neunzert, The Vlasov equation as a limit of Hamiltonian classical mechanical systems of interacting particles, Trans. Fluid Dynamics, 18(1977) 663-678.

[50] M. Reed, B. Simon, Methods of modern mathematical physics, Vol. IV Analysis of Operators, Academic Press, 1980.

[51] Y. Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):195–221, 1984.

[52] J Tugaut, Phase transitions of McKean-Vlasov processes in double-wells landscape, Stochastics 86 (2013) 257-284.

[53] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75 (1995) 1226-1229.