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SOME REMARKS ON SHARPLY 2-TRANSITIVE GROUPS AND NEAR-DOMAINS

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ABSTRACT. A sharply 2-transitive permutation group of characteristic 0 whose point stabiliser has an abelian subgroup of finite index splits. More generally, a near-domain of characteristic 0 with a multiplicative subgroup of finite index avoiding all multipliers $d_{a,b}$ must be a near-field. In particular this answers question 12.48 b) of the Kourovka Notebook in characteristic 0.

1. INTRODUCTION

Recall that a permutation group G acting on a set X is sharply 2-transitive if for any two pairs (x, y) and (x', y') of distinct elements of X there is a unique $g \in G$ with $gx = x'$ and $gy = y'$. Then G has involutions, and either involutions have fixed points and G is of permutation characteristic 2, or the action on X is equivalent to the conjugation action on the set I of involutions. In that case, all translations, i.e. products of two distinct involutions, are also conjugate and have the same order p , which is either an odd prime or ∞ ; the number p (or 0 if $p = \infty$) is the permutation characteristic of G . We say that G splits if it has a regular normal subgroup N ; in that case $G = N \rtimes C_G(i)$ for any involution $i \in I$. Note that Tent, Rips and Segev have constructed non-split sharply 2-transitive permutation groups of characteristic 0 and 2.

V. D. Mazurov asked in the Kourovka Notebook (question 12.48):

Let G be a sharply 2-transitive permutation group.

- (1) Does G possess a regular normal subgroup if a point stabilizer is locally finite?
- (2) Does G possess a regular normal subgroup if a point stabilizer has an abelian subgroup of finite index?

We shall answer question (b) affirmatively in permutation characteristic 0. In fact, we shall show a more general result for near-domains.

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2. NEAR-DOMAINS AND NEAR-FIELDS.

Instead of working with sharply 2-transitive groups, we shall work in the equivalent setting of near-domains.

Definition 1. $(K, 0, 1, +, \cdot)$ is a *near-domain* if for all $a, b, c \in K$

- (1) $(K, 0, +)$ is a *loop*, i.e. $a + x = b$ and $y + a = b$ have unique solutions, with $a + 0 = 0 + a = a$;
- (2) $(K \setminus \{0\}, 1, \cdot)$ is a group, and $0 \cdot a = a \cdot 0 = 0$;
- (3) left distributivity holds: $a \cdot (b + c) = a \cdot b + a \cdot c$;
- (4) for all $a, b \in K$ there is $d_{a,b} \in K$ such that $a + (b + x) = (a + b) + d_{a,b} \cdot x$ for all x .

A near-domain is a *near-field* if addition is associative.

Hence a near-field is a skew field iff right distributivity holds.

Fact 2 (Tits, Karzel). *A sharply 2-transitive permutation group G is isomorphic to the group of affine transformations of some near-domain K , i.e. of the set of permutations $\{x \mapsto a + bx : a, b \in K, b \neq 0\}$; the centraliser of any involution is isomorphic to the multiplicative group K^\times . It is split iff K is a near-field.*

Let E be the set $\{d \in K : 1 + d = d + 1\}$. Since the additive loop of K is power-associative, it is easy to see that 1 generates a subfield of K contained in E , which is either \mathbb{Q} or \mathbb{F}_p . Thus K has a characteristic, which is easily seen to be equal to the permutation characteristic of G . Note that in characteristic > 2 there is a unique maximal sub-near-field, which is equal to E .

Fact 3 ([1]). *For all $a, b, c \in K$ we have:*

- (1) $d_{a,a} = 1$.
- (2) $d_{a,b}(b + a) = a + b$.
- (3) $cd_{a,b}c^{-1} = d_{ca,cb}$.
- (4) $d_{a,b} = d_{a,c}d_{c+a,-c+b}d_{-c,b}$.
- (5) *If $a, b \in E$ then $(a + b)2 \in E$.*
- (6) $|K^\times : C_{K^\times}(d_{a,b})| = \infty$ if $d_{a,b} \neq 1$.

Let now A be any subgroup of finite index in K^\times which avoids all non-trivial coefficients $d_{a,b}$ for $a, b \in K$. Kerby [1, Theorem 8.26] has shown that K must be a near-field in the following cases:

- (1) $\text{char}K = 0$ and $|K^\times : A| = 2$,
- (2) $\text{char}K = 2$, $|K^\times : A| = 2$ and $|E| > 2$,
- (3) $\text{char}K = p > 2$ and $|K^\times : A| < |E|$.

We shall adapt the proof of (3) to characteristic 0.

Lemma 4. *Suppose $d_{a,1/k} = 1$. Then $d_{a,n/k} = 1$ for all $n \in \mathbb{N}$.*

Proof. By induction on n . This is clear for $n = 0$ and $n = 1$. So suppose it holds for n , and consider

$$\begin{aligned} \frac{n+1}{k} + a &= \left(\frac{n}{k} + \frac{1}{k}\right) + a = \frac{n}{k} + \left(\frac{1}{k} + a\right) = \frac{n}{k} + \left(a + \frac{1}{k}\right) \\ &= \left(\frac{n}{k} + a\right) + \frac{1}{k} = \left(a + \frac{n}{k}\right) + \frac{1}{k} = a + \left(\frac{n}{k} + \frac{1}{k}\right) = a + \frac{n+1}{k}. \quad \square \end{aligned}$$

Proposition 5. *If $A \leq K^\times$ is a subgroup of finite index avoiding all nontrivial $d_{a,b}$ and $\text{char}(K) = 0$, then K is a near-field.*

Proof. Recall that $\mathbb{Q} \subseteq E$. If $K = E$, then $d_{a,b} = 1$ for all $a, b \in K$ and K is a near-field. So assume $E \subsetneq K^\times$, and take $a \in K \setminus E2^{-1}$. Let $n = |K^\times : A|$. Then there are distinct $i > j$ in $\{0, 1, 2, \dots, n\}/n!$ with $d_{a,i}A = d_{a,j}A$; since $d_{-j,i} = 1$ we obtain

$$d_{a,i} = d_{a,j}d_{j+a,-j+i}d_{-j,i} = d_{a,j}d_{j+a,-j+i}.$$

Hence $d_{j+a,-j+i} \in A$, and $d_{j+a,-j+i} = 1$ by assumption.

Now $d_{(i-j)^{-1}(j+a),1} = d_{j+a,-j+i} = 1$, so $(i-j)^{-1}(j+a) \in E$. Since $-(i-j)^{-1}j \in \mathbb{Q} \subseteq E$, we have

$$[-(i-j)^{-1}j + (i-j)^{-1}(j+a)]2 = (i-j)^{-1}a2 \in E,$$

and $d_{a2,i-j} = 1$. But $0 < (i-j)n! \leq n$ is integer, and there is an integer $k > 0$ with $i-j = \frac{1}{k}$. By Lemma 4 we obtain $d_{a2,1} = 1$ and $a2 \in E$, a contradiction. \square

Corollary 6. *Let G be a sharply doubly transitive permutation group of characteristic 0 whose point stabilizer is virtually abelian. Then G is split.*

Proof. If K is the associated near-domain, K^\times has an abelian subgroup A of finite index. Now any non-trivial $d_{a,b}$ has a centralizer of infinite index in K^\times , so $d_{a,b} \notin A$. We finish by Proposition 5. \square

REFERENCES

- [1] Kerby, William. *On infinite sharply multiply transitive groups*, Hamburger Mathematische Einzelschriften, Neue Folge, Heft 6. Vandenhoeck & Ruprecht, Göttingen 1974.

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