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Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description

Evelyne Hubert*, Tobias Metzloff*[†], Cordian Riener[‡]

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Abstract

The Weyl group of a crystallographic root system has a nonlinear action on the compact torus. The orbit space of this action is a compact basic semi-algebraic set. We present a polynomial description of this set for the Weyl groups of type A, B, C, D and G. Our description is given through a polynomial matrix inequality. The novelty lies in an approach via Hermite quadratic forms and a closed formula for the matrix entries.

The orbit space of the nonlinear Weyl group action is the orthogonality region of generalized Chebyshev polynomials. In this polynomial basis, we show that the matrices obtained for the five types follow the same, surprisingly simple pattern. This is applied to the optimization of trigonometric polynomials with crystallographic symmetries.

Partial results of this article were presented as a poster at the ISSAC 2022 conference [[HMMR22](#)] and in the doctoral thesis [[Met22](#)].

Keywords: Multiplicative Invariants, Weyl Groups, Root Systems, Orbit Spaces, Semi-Algebraic Sets, Polynomial Matrix Inequalities, Chebyshev Polynomials

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1 Introduction

For a finite group acting on an affine variety, the set of all orbits is called orbit space. The coordinate ring contains the finitely generated subring of invariants. The image of the variety under fundamental invariants admits a parametrization of the orbit space. Being able to describe this image allows for numerous applications in differential geometry [Dub98], dynamical systems [Gat00], polynomial optimization [ALRT13] and quantum systems [AS83, GP01, SV05].

When the group acts linearly as, for instance, a matrix group on a real vector space, explicit descriptions of the orbit space are available for some groups. Specific examples are the symmetric group \mathfrak{S}_n [Pro78], the Weyl groups $\mathfrak{S}_n \times \{\pm 1\}^n$ and $\mathfrak{S}_n \times \{\pm 1\}_+^n$ [Tal20], some exceptional groups [Tal10] and low-dimensional groups [SV96]. Recent advances in the area of Fourier analysis and optimization have led to the study of Weyl groups acting nonlinearly on the compact torus. For all the infinite Weyl group families and one exceptional case, this article offers a unified and explicit description as a compact basic semi-algebraic set for the orbit space of this action.

In discrete Fourier analysis, the orbit space of the Weyl group acting on the compact torus is the region of orthogonality for several families of generalized Chebyshev polynomials [HW88]. These polynomials have been widely studied since the works of Koornwinder [Koo74], MacDonald [Mac90] and beyond [DL80, EL82, Bee91]. In some cases, one can show that the roots of these polynomials lie in the orbit space and provide nodes for Gaussian cubature formulae due to Xu et al. [LSX08, LX10, MP11, LSX12, Xu15, HHM16]. The roots are also suitable sampling points [MKNR13] and working in the Chebyshev basis brings advantages in interpolation [HS22].

Moreover, being able to describe the orbit space of a nonlinear Weyl group action allows for symmetry reductions in trigonometric optimization. The authors have recently applied the present result in [HMMR23], where an invariant trigonometric polynomial is rewritten as a linear combination of Chebyshev polynomials on the orbit space and then minimized as a classical polynomial on a semi-algebraic set. Such optimization problems arise, for example, in the computation of chromatic numbers [BDdV14].

These applications rely on the invariant-theoretic results of Bourbaki [Bou68], Steinberg [Ste75] and Farkas [Far84], which are presented in Lorenz' book [Lor05].

We briefly explain the setup (see also Section 2 for the details). A Weyl group can be seen as a finite Euclidean reflection group that leaves a full-dimensional lattice invariant [Kan01]. Such a group has a representation over the integers $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ by a change of basis stemming from the lattice generators. Thus, we have a nonlinear group action on the algebraic torus

$$\mathcal{G} \times (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n, \quad (B, x) \mapsto (x^{B_{\cdot 1}^{-1}}, \dots, x^{B_{\cdot n}^{-1}}). \quad (1.1)$$

The action leaves the compact torus $\mathbb{T}^n \subseteq (\mathbb{C}^*)^n$ invariant. The object of interest in the above applications is the orbit space \mathbb{T}^n/\mathcal{G} . This set can be embedded in \mathbb{R}^n as a compact basic semi-algebraic one, see for example Figure 1, and is parameterized as follows: The nonlinear action of \mathcal{G} on $(\mathbb{C}^*)^n$ induces an action on the ring of Laurent polynomials $\mathbb{Q}[x^\pm]$ in n variables. The invariant subring $\mathbb{Q}[x^\pm]^\mathcal{G}$ is generated by algebraically independent fundamental invariants $\theta_1, \dots, \theta_n$ [Bou68, Chapitre VI]. A canonical choice for the θ_i is to take orbit sums, where one may assume without loss of generality that the θ_i only take on real values on \mathbb{T}^n . Then the image of

$$\mathbb{T}^n \rightarrow \mathbb{R}^n, \quad x \mapsto (\theta_1(x), \dots, \theta_n(x)), \quad (1.2)$$

is a compact subset of \mathbb{R}^n and every image point corresponds in a 1 : 1 manner to an orbit in \mathbb{T}^n/\mathcal{G} . We call this image the real \mathbb{T} -orbit space of \mathcal{G} . The goal is to find polynomial inequalities that describe it as a basic semi-algebraic set.

This article gives such a polynomial description of the \mathbb{T} -orbit space for all four infinite Weyl group families A_{n-1} , B_n , C_n and D_n that leave the associated weight lattice invariant [Bou68, Planches I – IV]. Furthermore, there are five exceptional cases $E_{6,7,8}$, F_4 , G_2 and our result applies precisely to G_2 [Bou68, Planche IX]. In those cases, the Weyl group is $\mathcal{G} \cong \mathfrak{S}_n \times \{\pm 1\}^k$, where \mathfrak{S}_n is the symmetric group, \times denotes the semi-direct product and $0 \leq k \leq n$ is an integer.

We construct matrices H with entries in $\mathbb{Q}[z]$ that satisfy the following property: For $z \in \mathbb{R}^n$, the equation $z = (\theta_1(x), \dots, \theta_n(x))$ has a solution $x \in \mathbb{T}^n$ if and only if $H(z)$ is positive semi-definite. The matrices H for all five cases are given through formulae that only depend on the dimension n .

While closed expressions for the θ_i are known for certain groups and lattices, see [Lor05, Ham14], the concrete geometry of the \mathbb{T} -orbit space is mostly unexplored: To obtain a characterization, one could follow the general approach of Procesi and Schwarz for compact Lie groups acting on varieties [PS85, §4]. An orbit space is described there by the Gram matrix of the differentials that generate the \mathcal{G} -equivariant maps. This matrix is symmetric and subsequently has up to $n(n+1)/2$ distinct entries. To be of use, the entries need to be rewritten in the fundamental invariants, that is, in the coordinates of the orbit space. The effort to conduct this rewriting step grows with n and creates a bottleneck in computations. The matrices H we obtain in this article have the remarkable advantage to come directly in terms of the generating invariants without additional computations necessary. As they have Hankel structure, they only have up to $2n-1$ distinct entries. Furthermore, we obtain sharp bounds for the degree and the number of basis elements per entry in the Chebyshev basis.

In Theorems 4.3, 5.3, 6.4, 7.4 and 8.3, we give the entries for the matrices H in the coordinates z of the \mathbb{T} -orbit space, that is, in the standard monomial basis $\{z^\alpha \mid \alpha \in \mathbb{N}^n\}$ of $\mathbb{Q}[z]$. The definition of the generalized Chebyshev polynomials, which we denote by \widehat{T}_α , allows us to unify the formula for the five families. In the orthogonal basis $\{\widehat{T}_\alpha \mid \alpha \in \mathbb{N}^n\}$ of $\mathbb{Q}[z]$, we show that the matrices H are always identical and follow a remarkable pattern.

Theorem 1.1. *Let \mathcal{G} be a Weyl group of type A_{n-1} , B_n , C_n , D_n or G_2 . A real point z is contained in the real \mathbb{T} -orbit space of \mathcal{G} if and only if $H(z) \succeq 0$, where $H \in \mathbb{Q}[z]^{n \times n}$ is the matrix with Hankel structure¹*

$$H = \begin{bmatrix} \frac{\widehat{T}_0 - \widehat{T}_{2e_1}}{4} & \frac{\widehat{T}_{e_1} - \widehat{T}_{3e_1}}{8} & \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \dots \\ \frac{\widehat{T}_{e_1} - \widehat{T}_{3e_1}}{8} & \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \dots \\ \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \frac{5\widehat{T}_{e_1} - \widehat{T}_{3e_1} - 3\widehat{T}_{5e_1} - \widehat{T}_{7e_1}}{128} & \dots \\ \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \frac{5\widehat{T}_{e_1} - \widehat{T}_{3e_1} - 3\widehat{T}_{5e_1} - \widehat{T}_{7e_1}}{128} & \frac{5\widehat{T}_0 + 4\widehat{T}_{2e_1} - 4\widehat{T}_{4e_1} - 4\widehat{T}_{6e_1} - \widehat{T}_{8e_1}}{256} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(We give the proof and the closed formula for the entries in Theorem 9.1.)

The simplest instance is the univariate case $n = 1$. Then $\mathcal{G} = \{\pm I_1\} \subseteq \mathrm{GL}_1(\mathbb{Z})$ has \mathbb{T} -orbit space $[-1, 1] \subseteq \mathbb{R}$. Indeed, any element of $\mathbb{Q}[x, x^{-1}]^{\mathcal{G}}$ can be written uniquely as a polynomial in $(x + x^{-1})/2$ and the image of \mathbb{T} under $x \mapsto (x + x^{-1})/2$ is $[-1, 1]$. To obtain a polynomial description, we consider the matrix

$$H(z) = \left[\frac{\widehat{T}_0(z) - \widehat{T}_2(z)}{4} \right] = \left[\frac{1 - (2z^2 - 1)}{4} \right] = \left[\frac{1 - z^2}{2} \right] \in \mathbb{Q}[z]^{1 \times 1},$$

where \widehat{T}_ℓ is the usual univariate Chebyshev polynomial of the first kind with $\widehat{T}_\ell((x + x^{-1})/2) = (x^\ell + x^{-\ell})/2$ for $\ell \in \mathbb{N}$. Hence, we have $H(z) \succeq 0$ if and only if $1 \geq z^2$, that is, if $z \in [-1, 1]$. This is the domain on which the \widehat{T}_ℓ are orthogonal with respect to the weight function $(2 \mathrm{Det}(H(z)))^{-1/2} = 1/\sqrt{1 - z^2}$. For $n \geq 2$, some popular examples of \mathbb{T} -orbit spaces are depicted in Figure 1.

We explain the idea of the proof for C_n , the other cases are very similar: The Weyl group of C_n is the semi-direct product $\mathcal{G} \cong \mathfrak{S}_n \times \{\pm 1\}^n$. In particular, the ring of invariants can be written as $\mathbb{Q}[x^\pm]^{\mathcal{G}} \cong \mathbb{Q}[(y + y^{-1})/2]^{\mathfrak{S}_n}$. Here, the entries of the vectors y and y^{-1} (the inverse applies entry-wise) are the $2n$ monomials appearing in the first fundamental invariant θ_1 . The group \mathfrak{S}_n acts by permuting the $(y_i + y_i^{-1})/2$. Hence, the problem reduces to rewriting the \mathfrak{S}_n -invariants, that is, the elementary symmetric polynomials $\sigma_i((y + y^{-1})/2)$, in terms of the \mathcal{G} -invariants $\theta_i(x)$. The case C_n is the easiest, because we immediately have

$$\sigma_i \left(\frac{y(x) + y(x)^{-1}}{2} \right) = c_i \theta_i(x), \quad (1.3)$$

¹The entry (i, j) only depends on $i + j$.

If \mathbb{R} is of type A_{n-1} and $n \geq 3$, then $z \in \mathbb{R}^{n-1}$ and H is $n \times n$, because \mathbb{R} has rank $n - 1$ while $\mathcal{G} \cong \mathfrak{S}_n$ (similar for G_2).

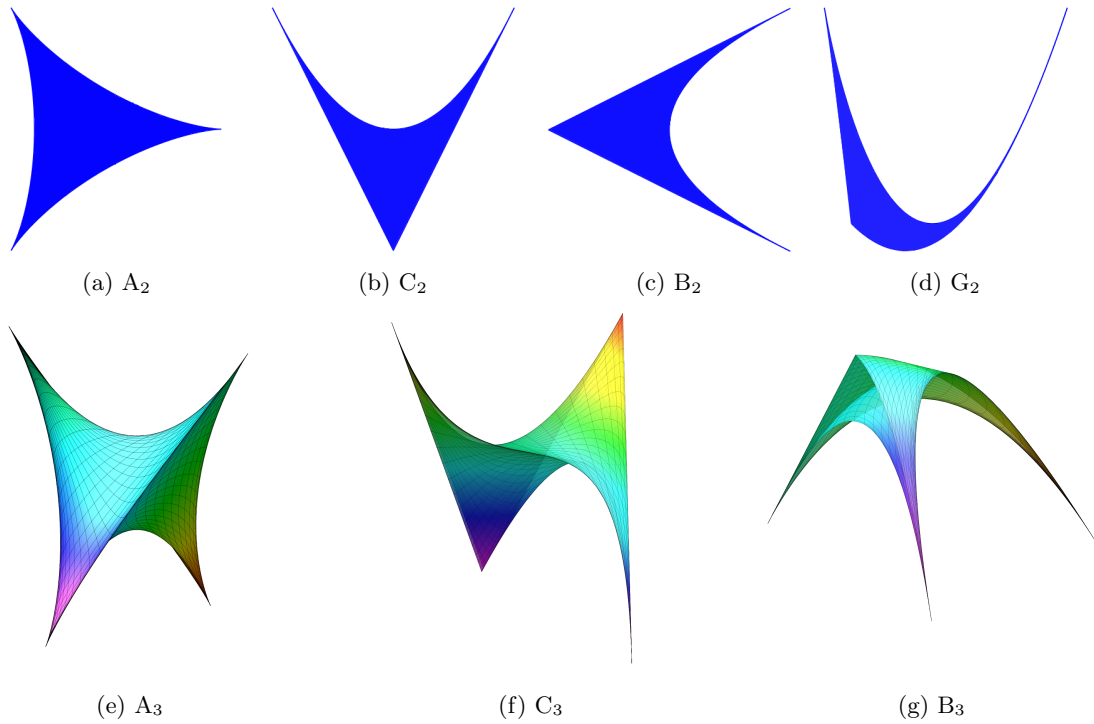


Figure 1: The real \mathbb{T} -orbit space for the irreducible root systems of rank 2 and 3.

where $c_i \in \mathbb{Q}$ is a constant. Hence, when $z \in \mathbb{C}^n$ with $z_i = \theta_i(x)$ for some $x \in (\mathbb{C}^*)^n$, then the $c_i z_i$ are the coefficients of a polynomial with root vector $(y + y^{-1})/2$. We show that z is contained in the \mathbb{T} -orbit space if and only if $y(x) \in \mathbb{T}^n$, that is, if and only if all the roots are contained in the interval $[-1, 1]$. A necessary and sufficient criterion for the latter is that the associated Hermite quadratic form is positive semi-definite. We compute the matrix H of this form and express its entries as polynomials in z_i .

The contents of this article are as follows. In [Section 2](#), we introduce the required definitions and notations. This section contains mostly standard material from the theory of root systems and lattices [[Bou68](#)]. The following [Section 3](#) is our toolbox for characterizing when all solutions of a symmetric system such as [Equation \(1.3\)](#) are contained in the compact torus. These tools are based on the material in [[CLO05](#), [CLO15](#)] and then used in [Sections 4](#) to [8](#), to give the formula from [Theorem 1.1](#) in the basis of standard monomials. Finally, we prove [Theorem 1.1](#) in [Section 9](#) by using the definition of the generalized Chebyshev polynomials.

The description of \mathbb{T}^n/\mathcal{G} we offer is by no means unique as it depends on the choice of invariants θ_i . However, since the ring of invariants is a polynomial algebra, there exist unique polynomial mappings between different invariants, say $\theta_i = \pi_i(\tilde{\theta}_1, \dots, \tilde{\theta}_n)$. Hence, if $H(z) \succeq 0$ describes the \mathbb{T} -orbit space defined by the θ_i , then $H(\pi(z)) \succeq 0$ describes the one defined by the $\tilde{\theta}_i$. Alternatively, one can express the left-hand side of [Equation \(1.3\)](#) in terms of the $\tilde{\theta}_i$ to get the matrix immediately.

To easily access the matrices, we offer a Maple package `GENERALIZEDCHEBYSHEV`²: The output of

- `RHermiteMatrix(R, n)` is the matrix H in the standard monomials z^α ,
- `RTHermiteMatrix(R, n)` is the matrix H in the Chebyshev polynomials \widehat{T}_α ,

where R is the type A, B, C, D or G and $n \in \mathbb{N}$ is the dimension. All figures in this article were produced with this package and the plotting tools of Maple.

²The package is available under <https://github.com/TobiasMetzlaff/GeneralizedChebyshev>.

2 The \mathbb{T} -orbit space of a Weyl group

Based on [Bou68], we revisit the definition of a crystallographic root system and how it admits a Weyl group with an invariant, full-dimensional lattice. From [Lor05], we then recall actions of Weyl groups on tori and define the orbit space. A key theorem, proven in [Bou68], is that the ring of invariants for such actions is a polynomial algebra. This brings the definition of generalized Chebyshev polynomials. Finally, we prove that the orbit space is always as a real compact set.

2.1 Root systems and Weyl groups

We take the definitions from [Bou68, Ch. VI]. For a finite-dimensional real vector space V with inner product $\langle \cdot, \cdot \rangle$, a subset $R \subseteq V$ is called a **root system** in V , if the following conditions hold.

R1 R is finite, spans V and does not contain 0 .

R2 If $\rho, \tilde{\rho} \in R$, then $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$, where $\rho^\vee := \frac{2\rho}{\langle \rho, \rho \rangle}$.

R3 If $\rho, \tilde{\rho} \in R$, then $s_\rho(\tilde{\rho}) \in R$, where s_ρ is the reflection $s_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$ for $u \in V$.

The elements of R are called **roots** and the **rank** of R is $\text{Rank}(R) := \dim(V)$. The **Weyl group** \mathcal{W} of R is the group generated by the reflections s_ρ for $\rho \in R$. This is a finite subgroup of the orthogonal group on V with respect to the inner product $\langle \cdot, \cdot \rangle$. Furthermore, R is said to be **reduced**, if the following holds.

R4 For $\rho \in R$ and $c \in \mathbb{R}$, we have $c\rho \in R$ if and only if $c = \pm 1$.

We assume that the “reduced” property R4 always holds when we speak of a “root system”. An example is given in [Example 2.5](#).

A **weight** of R is an element $\mu \in V$, such that, for all $\rho \in R$, we have $\langle \mu, \rho^\vee \rangle \in \mathbb{Z}$. By the “crystallographic” property R2, every root is a weight. The set of all weights is called the **weight lattice** Ω . This lattice is full-dimensional and invariant under the action of \mathcal{W} on V , that is, $\Omega \otimes_{\mathbb{Z}} \mathbb{R} = V$ and $\mathcal{W}\Omega = \Omega$.

Let $\dim(V) = n$ and $B = \{\rho_1, \dots, \rho_n\} \subseteq R$ be a set of roots with the following properties.

B1 B is an \mathbb{R} -vector space basis for V .

B2 Every root $\rho \in R$ can be written as $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$ or $\rho = -\alpha_1 \rho_1 - \dots - \alpha_n \rho_n$ for some $\alpha \in \mathbb{N}^n$.

Such a set B always exists [Bou68, Ch. VI, §1, Thm. 3] and we define the **fundamental weights** as the linearly independent $\omega_1, \dots, \omega_n \in \Omega$, such that, for $1 \leq i, j \leq n$, we have $\langle \omega_i, \rho_j^\vee \rangle = \delta_{i,j}$.

Proposition 2.1. *Denote by \mathfrak{S}_n the symmetric group on n points. There exists a permutation $\sigma \in \mathfrak{S}_n$ of order at most 2, such that, for all $1 \leq i \leq n$, we have $-\omega_i \in \mathcal{W}\omega_{\sigma(i)}$.*

Proof. The statement follows from the existence of the longest Weyl group element $s_0 \in \mathcal{W}$, which takes $\{\rho_1, \dots, \rho_n\}$ to $\{-\rho_1, \dots, -\rho_n\}$, see [Bou68, Ch. VI, §1, Coro. 3 of Prop. 17]. Hence, there is a permutation $\sigma \in \mathfrak{S}_n$ with $s_0(\rho_i) = -\rho_{\sigma(i)}$. Since $s_0^2 = \text{Id}_V$ and the inner product is \mathcal{W} -invariant, σ is of order at most 2 and

$$-s_0(\omega_i) = \sum_{j=1}^n \langle -s_0(\omega_i), \rho_j^\vee \rangle \omega_j = \sum_{j=1}^n \langle \omega_i, -s_0(\rho_j^\vee) \rangle \omega_j = \sum_{j=1}^n \langle \omega_i, \rho_{\sigma(j)}^\vee \rangle \omega_j = \omega_{\sigma(i)}$$

is also a fundamental weight. □

Assume that $V = V^{(1)} \oplus \dots \oplus V^{(k)}$ is the direct sum of proper orthogonal subspaces and that, for each $1 \leq i \leq k$, $R^{(i)}$ is a root system in $V^{(i)}$. Then $R := R^{(1)} \cup \dots \cup R^{(k)}$ is a root system in V called the **direct sum** of the $R^{(i)}$. If a root system is not the direct sum of at least two root systems, then it is called **irreducible** [Bou68, Ch. VI, §1.2].

Every root system can be uniquely decomposed into irreducible components [Bou68, Ch. VI, §1, Prop. 6, 7]. There exist seven families of irreducible root systems, which are denoted by A_{n-1} , B_n , C_n , D_n , $E_{6,7,8}$, F_4 and G_2 , see [Bou68, Ch. VI, §4, Thm. 3] and [Bou68, Planches I – IX].

2.2 Multiplicative actions

To introduce the orbit space of a Weyl group, we require some multiplicative invariant theory from [Lor05]. Let \mathcal{W} be a finite subgroup of $\mathrm{GL}_n(\mathbb{R})$ and $\Omega \subseteq \mathbb{R}^n$ be an n -dimensional lattice, spanned by $\omega_1, \dots, \omega_n$. We assume that Ω is \mathcal{W} -invariant. Then the \mathbb{Z} -module isomorphism $\varphi : \mathbb{Z}^n \rightarrow \Omega$, $e_i \mapsto \omega_i$ yields a change of basis, so that

$$\mathcal{G} := \{\varphi^{-1} s \varphi \mid s \in \mathcal{W}\} \quad (2.1)$$

is a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$. We call \mathcal{G} the **integer representation** of \mathcal{W} with respect to $\omega_1, \dots, \omega_n$. The group \mathcal{G} acts on the standard basis vectors $e_i \in \mathbb{Z}^n$ as \mathcal{W} acts on the lattice generators $\omega_i \in \Omega$. Clearly, this group does not only depend on \mathcal{W} itself, but also on the lattice Ω .

Denote by $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$ the algebraic n -torus. For $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$ and a column vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we define $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}^*$. Then we have a nonlinear action

$$\begin{aligned} \star : \mathcal{G} \times (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ (B, x) &\mapsto B \star x := x^{B^{-1}} := (x^{B^{-1} \cdot 1}, \dots, x^{B^{-1} \cdot n}), \end{aligned} \quad (2.2)$$

where $B_{\cdot i}^{-1} \in \mathbb{Z}^n$ denotes the i -th column vector of $B^{-1} \in \mathcal{G}$. The orbit of an element $x \in (\mathbb{C}^*)^n$ is denoted by $\mathcal{G} \star x$.

The coordinate ring of $(\mathbb{C}^*)^n$ with rational coefficients is the ring of multivariate Laurent polynomials $\mathbb{Q}[x^\pm] := \mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. The monomials of $\mathbb{Q}[x^\pm]$ are the $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}^n$ and \star induces a linear action on $\mathbb{Q}[x^\pm]$, given by its values on the monomial basis

$$\begin{aligned} \cdot : \mathcal{G} \times \mathbb{Q}[x^\pm] &\rightarrow \mathbb{Q}[x^\pm], \\ (B, x^\alpha) &\mapsto B \cdot x^\alpha := x^{B\alpha}. \end{aligned} \quad (2.3)$$

For $f = \sum_\alpha f_\alpha x^\alpha \in \mathbb{Q}[x^\pm]$, $B \in \mathcal{G}$ and $x \in (\mathbb{C}^*)^n$, we have $(B \cdot f)(x) = \sum_\alpha f_\alpha x^{B\alpha} = f(B^{-1} \star x)$. If $B \cdot f = f$ whenever $B \in \mathcal{G}$, then f is called **\mathcal{G} -invariant**. The set of all \mathcal{G} -invariant Laurent polynomials is a subring denoted by $\mathbb{Q}[x^\pm]^\mathcal{G}$. The **orbit polynomial** associated to $\alpha \in \mathbb{Z}^n$ is

$$\Theta_\alpha := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B\alpha} \in \mathbb{Q}[x^\pm]^\mathcal{G}. \quad (2.4)$$

If $\tilde{\alpha} \in \mathcal{G}\alpha$, then $\Theta_{\tilde{\alpha}} = \Theta_\alpha$. The distinct Θ_α form a basis for $\mathbb{Q}[x^\pm]^\mathcal{G}$ as a \mathbb{Q} -vector space [Lor05, Eq. (3.4)]. We have $\Theta_0 = 1$ and, for $\alpha, \beta \in \mathbb{Z}^n$,

$$\Theta_\alpha \Theta_\beta = \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \Theta_{\alpha+B\beta}. \quad (2.5)$$

As a \mathbb{Q} -algebra, $\mathbb{Q}[x^\pm]^\mathcal{G}$ is finitely generated [Lor05, Coro. 3.3.2]. Hence, we may assume that $\mathbb{Q}[x^\pm]^\mathcal{G} = \mathbb{Q}[\theta_1, \dots, \theta_m]$ for some minimal $m \in \mathbb{N}$ and $\theta_1, \dots, \theta_m \in \mathbb{Q}[x^\pm]^\mathcal{G}$. The generators θ_i are called the **fundamental invariants** of \mathcal{G} and we define the map

$$\begin{aligned} \vartheta : (\mathbb{C}^*)^n &\rightarrow \mathbb{C}^m, \\ x &\mapsto (\theta_1(x), \dots, \theta_m(x)). \end{aligned} \quad (2.6)$$

This map is surjective. The question is what happens when we restrict to a maximal compact subgroup: We set $\mathbb{T} := \{x \in \mathbb{C} \mid x\bar{x} = 1\} \subseteq \mathbb{C}^*$, where \bar{x} is the complex conjugate. For $x \in \mathbb{T}$, we have $x^{-1} = \bar{x}$ and the compact n -torus \mathbb{T}^n is \mathcal{G} -invariant, that is, $\mathcal{G} \star \mathbb{T}^n = \mathbb{T}^n$.

Analogously to [CLO15, Ch. 7, §4, Thm. 10], one can show the following.

Proposition 2.2. *The map*

$$\begin{aligned} \mathbb{T}^n/\mathcal{G} &\rightarrow \mathbb{C}^m, \\ \mathcal{G} \star x &\mapsto \vartheta(x) = (\theta_1(x), \dots, \theta_m(x)) \end{aligned}$$

is well-defined and injective.

2.3 \mathbb{T} -orbit space and Chebyshev polynomials

Our goal is now to describe the image $\vartheta(\mathbb{T}^n)$ of the compact torus under the map ϑ , when the lattice Ω is the weight lattice of a root system with Weyl group \mathcal{W} . The integer representation is denoted by \mathcal{G} . For $1 \leq i \leq n$, we fix $\theta_i := \Theta_{e_i}$ as the orbit polynomial associated to e_i from Equation (2.4).

Theorem 2.3. [Bou68, Ch. VI, §3, Thm. 1] *The orbit polynomials $\theta_1, \dots, \theta_n$ are algebraically independent and $\mathbb{Q}[x^\pm]^\mathcal{G} = \mathbb{Q}[\theta_1, \dots, \theta_n]$ is a polynomial algebra.*

Conversely, Steinberg [Ste75] and Farkas [Far84] proved that $\mathbb{Q}[x^\pm]^\mathcal{G}$ is a polynomial algebra if and only if \mathcal{G} is the integer representation of a Weyl group defined via the weight lattice as in Equation (2.1).

According to Proposition 2.2, the map $x \mapsto \vartheta(x) := (\theta_1(x), \dots, \theta_n(x))$ is injective and thus the image $\vartheta(\mathbb{T}^n)$ is in bijection to \mathbb{T}^n/\mathcal{G} .

Definition 2.4. *We call $\mathcal{T} := \vartheta(\mathbb{T}^n)$ the \mathbb{T} -orbit space of \mathcal{G} .*

By fixing the fundamental invariants, we have chosen an embedding of the orbit space in the variety defined by the relations of the fundamental invariants θ_i . Since those are algebraically independent, it is an embedding in a real subspace isomorphic to \mathbb{R}^n . We shall see that the \mathbb{T} -orbit space \mathcal{T} is a compact basic semi-algebraic set.

To give the defining polynomial inequalities, it suffices to consider the irreducible root systems. Indeed, the Weyl group \mathcal{W} of a root system \mathbb{R} is the outer product of the Weyl groups corresponding to the irreducible components, see the discussion before [Bou68, Ch. VI, §1, Prop. 5]. If \mathbb{R} is not irreducible, that is, if $\mathbb{R} = \mathbb{R}^{(1)} \cup \dots \cup \mathbb{R}^{(k)}$, then the \mathbb{T} -orbit space \mathcal{T} can be written as a Cartesian product of \mathbb{T} -orbit spaces $\mathcal{T}^{(1)} \times \dots \times \mathcal{T}^{(k)}$.

Example 2.5. *The symmetric group \mathfrak{S}_3 acts on $\mathbb{R}^3/[1, 1, 1]^t$ by permutation of coordinates. It is the Weyl group \mathcal{W} of the root system A_2 . The weight lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is hexagonal with $-\omega_1 \in \mathcal{W}\omega_2$, see Figure 2. Over the integers, \mathcal{W} is represented as the group $\mathcal{G} \subseteq \mathrm{GL}_2(\mathbb{Z})$ generated by*

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The \mathbb{T} -orbit space \mathcal{T} of \mathcal{G} is a 2-dimensional real semi-algebraic set, namely the deltoid in Figure 2, whose polynomial description is given in Section 4. It is parametrized by the map

$$\vartheta : \mathbb{T}^2 \rightarrow \{z \in \mathbb{C}^2 \mid z_1 = \bar{z}_2\} \cong \mathbb{R}^2, \quad x \mapsto (\theta_1(x), \theta_2(x)) = \left(\frac{x_1 + x_2/x_1 + 1/x_2}{3}, \frac{x_2 + x_1/x_2 + 1/x_1}{3} \right).$$

The cusps of \mathcal{T} are the three points $\vartheta(\mathfrak{e}^u)$, where $u \in \{0, \omega_1, \omega_2\}$ and $\mathfrak{e}^u := (e^{-2\pi i \langle \omega_1, u \rangle}, e^{-2\pi i \langle \omega_2, u \rangle}) \in \mathbb{T}^2$.

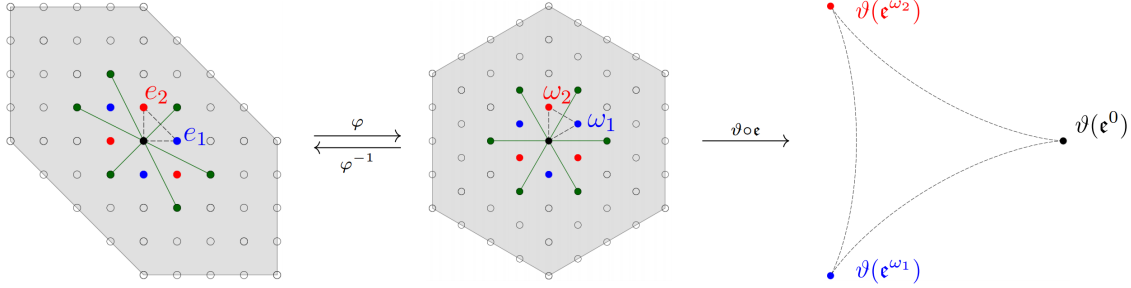


Figure 2: The root system A_2 admits a hexagonal weight lattice Ω (circles \circ) with fundamental weights ω_1, ω_2 and their orbits. Here are the usual orthogonal representation with \mathcal{W} -symmetry (middle) and the integer representation with \mathcal{G} -symmetry (left). The isomorphism $\varphi : \mathbb{Z}^2 \rightarrow \Omega$ yields the change of basis from $\{e_1, e_2\}$ to $\{\omega_1, \omega_2\}$. The \mathbb{T} -orbit space of \mathcal{G} (right) is the “deltoid”, also known as “Steiner’s hypocycloid”.

Thanks to [Theorem 2.3](#), we can define a family of multivariate polynomials associated to the weight lattice $\Omega \cong \mathbb{Z}^n$, which generalize the univariate Chebyshev polynomials. Denote by $\mathbb{Q}[z] := \mathbb{Q}[z_1, \dots, z_n]$ the multivariate polynomial ring.

Definition 2.6. *The generalized Chebyshev polynomials of the first kind associated to $\alpha \in \mathbb{Z}^n$ is the unique $T_\alpha \in \mathbb{Q}[z]$ satisfying $T_\alpha(\vartheta(x)) = \Theta_\alpha(x)$.*

For every $\tilde{\alpha} \in \mathbb{Z}^n$, there exists a unique $\alpha \in \mathbb{N}^n$, such that $\tilde{\alpha} \in \mathcal{G}_\alpha$ [[Bou68](#), Ch. VI, §1, Thm. 2]. Hence, we have $T_\alpha = T_{\tilde{\alpha}}$ and $\{T_\alpha \mid \alpha \in \mathbb{N}^n\}$ is a basis for the \mathbb{R} -vector space $\mathbb{Q}[z]$. We have $T_0 = 1$ and $T_{e_i} = z_i$ for $1 \leq i \leq n$. The T_α satisfy the same recurrence formula as the Θ_α , see [Equation \(2.5\)](#).

Beyond those properties, one can show that the T_α are orthogonal on the \mathbb{T} -orbit space and the weight function is closely related to the determinant of the Jacobian of the map ϑ . For a proof, we refer to [[HW88](#)].

Example 2.7. *The generalized Chebyshev polynomials of the first kind associated to A_2 are*

$$\begin{aligned} T_{02} &= 3z_2^2 - 2z_1, T_{11} = 3/2 z_1 z_2 - 1/2, T_{20} = 3z_1^2 - 2z_2, T_{03} = 9z_2^3 - 9z_1 z_2 + 1, \\ T_{12} &= 9/2 z_1 z_2^2 - 3z_1^2 - 1/2 z_2, T_{21} = -3z_2^2 - 1/2 z_1 + 9/2 z_1^2 z_2, T_{30} = 9z_1^3 - 9z_1 z_2 + 1, \dots \end{aligned}$$

The generalized Chebyshev polynomials of the first kind associated to B_2 are

$$\begin{aligned} T_{02} &= 4z_2^2 - 2z_1 - 1, T_{11} = 2z_1 z_2 - z_2, T_{20} = 4z_1^2 - 8z_2^2 + 4z_1 + 1, T_{03} = 16z_2^3 - 12z_1 z_2 - 3z_2, \\ T_{12} &= 8z_1 z_2^2 - 4z_1^2 - 3z_1, T_{21} = 8z_1^2 z_2 - 16z_2^3 + 6z_1 z_2 + 3z_2, T_{30} = 16z_1^3 - 48z_1 z_2^2 + 24z_1^2 + 9z_1, \dots \end{aligned}$$

The following is only relevant if the permutation $\sigma \in \mathfrak{S}_n$ from [Proposition 2.1](#) is not the identity. In this case, $-I_n \notin \mathcal{G}$ and \mathcal{T} is not necessarily a subset of \mathbb{R}^n . The irreducible root systems, for which this is the case, are both A_{n-1} and D_{2n-1} whenever $n \geq 3$ as well as E_6 . For $x \in \mathbb{T}^n$ and $1 \leq i \leq n$, we have

$$\overline{\theta_i(x)} = \theta_i(-I_n \star x) = (-I_n \cdot \theta_i)(x) = \theta_{\sigma(i)}(x).$$

When $j = \sigma(j)$, we leave the j -th coordinate of ϑ as it is and set $\theta_{j,\mathbb{R}} := \theta_j$. When $j < \sigma(j)$, we replace the j -th, respectively $\sigma(j)$ -th, coordinate of ϑ by $\theta_{j,\mathbb{R}} := \Re(\theta_j) = (\theta_j + \theta_{\sigma(j)})/2$, respectively $\theta_{\sigma(j),\mathbb{R}} := \Im(\theta_j) = (\theta_j - \theta_{\sigma(j)})/(2i)$. The so obtained map

$$\begin{aligned} \vartheta_{\mathbb{R}} : \mathbb{T}^n &\rightarrow \mathbb{R}^n, \\ x &\mapsto (\theta_{1,\mathbb{R}}(x), \dots, \theta_{n,\mathbb{R}}(x)) \end{aligned} \tag{2.7}$$

has image $\mathcal{T}_{\mathbb{R}} \subseteq [-1, 1]^n \subseteq \mathbb{R}^n$. This image is called the **real \mathbb{T} -orbit space** of \mathcal{G} . On $\mathcal{T}_{\mathbb{R}}$, we now define the real generalized Chebyshev polynomials.

Proposition 2.8. *Let $\alpha, \hat{\alpha} \in \mathbb{N}^n$ with $-\alpha \in \mathcal{G}\hat{\alpha}$. Then there exist unique polynomials $\hat{T}_\alpha, \hat{T}_{\hat{\alpha}} \in \mathbb{Q}[z]$, such that*

$$T_\alpha(\vartheta(x)) = \hat{T}_\alpha(\vartheta_{\mathbb{R}}(x)) + i\hat{T}_{\hat{\alpha}}(\vartheta_{\mathbb{R}}(x)) \quad \text{and} \quad T_{\hat{\alpha}}(\vartheta(x)) = \hat{T}_\alpha(\vartheta_{\mathbb{R}}(x)) - i\hat{T}_{\hat{\alpha}}(\vartheta_{\mathbb{R}}(x)).$$

Proof. Assume that $T_\alpha = \sum_{\beta} c_\beta z^\beta \in \mathbb{Q}[z]$ for some $c_\beta \in \mathbb{Q}$ and $\beta \in \mathbb{N}^n$. Then we have

$$T_\alpha(\vartheta(x)) = \sum_{\beta} c_\beta \prod_{j=1}^n (\Re(\theta_j(x)) + i\Im(\theta_j(x)))^{\beta_j} \quad \text{and} \quad T_{\hat{\alpha}}(\vartheta(x)) = \sum_{\beta} c_\beta \prod_{j=1}^n (\Re(\theta_{\sigma(j)}(x)) + i\Im(\theta_{\sigma(j)}(x)))^{\beta_j}.$$

If $j = \sigma(j)$, then $\Re(\theta_j) = \theta_{j,\mathbb{R}}$ and $\Im(\theta_j) = 0$. Otherwise, the definition of $\theta_{j,\mathbb{R}}$ implies

$$\Re(\theta_j(x)) = \begin{cases} \theta_{j,\mathbb{R}}(x), & \text{if } j < \sigma(j) \\ \theta_{\sigma(j),\mathbb{R}}(x), & \text{if } j > \sigma(j) \end{cases} \quad \text{and} \quad \Im(\theta_j(x)) = \begin{cases} \theta_{\sigma(j),\mathbb{R}}(x), & \text{if } j < \sigma(j) \\ -\theta_{j,\mathbb{R}}(x), & \text{if } j > \sigma(j) \end{cases}.$$

Altogether, we obtain

$$\begin{aligned} \frac{T_\alpha(\vartheta(x)) + T_{\hat{\alpha}}(\vartheta(x))}{2} &= \sum_{\beta} \frac{c_\beta}{2} \prod_{j=\sigma(j)} \theta_{j,\mathbb{R}}(x)^{\beta_j} \\ &\quad \left(\prod_{j < \sigma(j)} (\theta_{j,\mathbb{R}}(x) + i\theta_{\sigma(j),\mathbb{R}}(x))^{\beta_j} (\theta_{j,\mathbb{R}}(x) - i\theta_{\sigma(j),\mathbb{R}}(x))^{\beta_{\sigma(j)}} \right. \\ &\quad \left. + \prod_{j < \sigma(j)} (\theta_{j,\mathbb{R}}(x) - i\theta_{\sigma(j),\mathbb{R}}(x))^{\beta_j} (\theta_{j,\mathbb{R}}(x) + i\theta_{\sigma(j),\mathbb{R}}(x))^{\beta_{\sigma(j)}} \right). \end{aligned}$$

This is a polynomial in $\vartheta_{\mathbb{R}}(x) = (\theta_{1,\mathbb{R}}(x), \dots, \theta_{n,\mathbb{R}}(x))$, denoted by \hat{T}_α . Since the left hand side is real for every x , the coefficient in front of i must be 0. Hence, we have $\hat{T}_\alpha \in \mathbb{Q}[z]$. Similarly, by computing $(T_\alpha - T_{\hat{\alpha}})/(2i)$, we obtain $\hat{T}_{\hat{\alpha}} \in \mathbb{Q}[z]$. Since this is an analytical identity on \mathbb{C}^n , the polynomials are unique. \square

Example 2.9. *Consider the root system A_2 with Weyl group \mathfrak{S}_3 . The generators of the integer representation \mathcal{G} are given in [Example 2.5](#) and we have $\widehat{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{Z}^2$, that is, $-e_1 \in \mathcal{G}e_2$. The generalized Chebyshev polynomials of degree 2 are*

$$T_{02} = 3z_2^2 - 2z_1, \quad T_{11} = 3/2 z_1 z_2 - 1/2, \quad T_{20} = 3z_1^2 - 2z_2 \in \mathbb{Q}[z].$$

After substitution $z_1 \mapsto \hat{z}_1 + i\hat{z}_2$, $z_2 \mapsto \hat{z}_1 - i\hat{z}_2$, we have

$$T_{20}, T_{02} = (3\hat{z}_1^2 - 3\hat{z}_2^2 - 2\hat{z}_1) \pm i(6\hat{z}_1\hat{z}_2 + 2\hat{z}_2), \quad T_{11} = (3/2\hat{z}_1^2 + 3/2\hat{z}_2^2 - 1/2),$$

and so the new polynomials from [Proposition 2.8](#) are

$$\hat{T}_{02} = 6\hat{z}_1\hat{z}_2 + 2\hat{z}_2, \quad \hat{T}_{11} = 3/2\hat{z}_1^2 + 3/2\hat{z}_2^2 - 1/2, \quad \hat{T}_{20} = 3\hat{z}_1^2 - 3\hat{z}_2^2 - 2\hat{z}_1.$$

Those are defined on the embedding of the “deltoid” in \mathbb{R}^2 from [Example 2.5](#), which is the real \mathbb{T} -orbit space $\mathcal{T}_{\mathbb{R}}$. At higher degrees, the real generalized Chebyshev polynomials of the first kind associated to A_2 are

$$\begin{aligned} \hat{T}_{03} &= 27\hat{z}_1^2\hat{z}_2 - 9\hat{z}_2^3, & \hat{T}_{12} &= 6\hat{z}_1\hat{z}_2 - 1/2\hat{z}_2 + 9/2\hat{z}_1^2\hat{z}_2 + 9/2\hat{z}_2^3, \\ \hat{T}_{21} &= -3\hat{z}_1^2 + 3\hat{z}_2^2 - 1/2\hat{z}_1 + 9/2\hat{z}_1^3 + 9/2\hat{z}_1\hat{z}_2^2, & \hat{T}_{30} &= 9\hat{z}_1^3 - 27\hat{z}_1\hat{z}_2^2 - 9\hat{z}_1^2 - 9\hat{z}_2^2 + 1, \\ \hat{T}_{04} &= 108\hat{z}_1^3\hat{z}_2 - 108\hat{z}_1\hat{z}_2^3 - 36\hat{z}_1^2\hat{z}_2 - 36\hat{z}_2^3 - 12\hat{z}_1\hat{z}_2 + 4\hat{z}_2, \\ \hat{T}_{13} &= 27/2\hat{z}_2^3 - 5/2\hat{z}_2 + 27\hat{z}_1\hat{z}_2^3 + 27/2\hat{z}_1^2\hat{z}_2 + 27\hat{z}_1^3\hat{z}_2 - 3\hat{z}_1\hat{z}_2, \\ \hat{T}_{22} &= -18\hat{z}_1^3 - 1/2 + 54\hat{z}_1\hat{z}_2^2 + 6\hat{z}_1^2 + 6\hat{z}_2^2 + 27/2\hat{z}_1^4 + 27\hat{z}_1^2\hat{z}_2^2 + 27/2\hat{z}_2^4, \\ \hat{T}_{31} &= -27/2\hat{z}_1^3 - 27/2\hat{z}_1\hat{z}_2^2 + 5/2\hat{z}_1 - 3/2\hat{z}_1^2 + 3/2\hat{z}_2^2 + 27/2\hat{z}_1^4 - 27/2\hat{z}_2^4, \\ \hat{T}_{40} &= 27\hat{z}_1^4 - 162\hat{z}_1^2\hat{z}_2^2 + 27\hat{z}_2^4 - 36\hat{z}_1^3 - 36\hat{z}_1\hat{z}_2^2 + 6\hat{z}_1^2 - 6\hat{z}_2^2 + 4\hat{z}_1, \dots \end{aligned}$$

3 Symmetric polynomial systems

In order to give a polynomial description for the \mathbb{T} -orbit space \mathcal{T} of a Weyl group \mathcal{G} , our first step is to characterize when a symmetric polynomial system has a solution in the compact torus. The present section should be regarded as a toolbox for the proofs of the main results.

For $1 \leq i \leq n$, we denote by $\sigma_i(y_1, \dots, y_n) = \sum_{|K|=i} \prod_{k \in K} y_k$ the i -th elementary symmetric function in n indeterminates y_i , where K ranges over all subsets of $\{1, \dots, n\}$ with cardinality i . We shall be confronted with the following two types of polynomial systems.

$$\begin{aligned} \text{(I)} \quad \sigma_i(y_1, \dots, y_n) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{C} \\ \text{(II)} \quad \sigma_i\left(\frac{y_1+y_1^{-1}}{2}, \dots, \frac{y_n+y_n^{-1}}{2}\right) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{R} \end{aligned}$$

The goal is to determine, whether all complex solutions $y = (y_1, \dots, y_n)$ of system (I), respectively (II), are contained in the compact torus \mathbb{T}^n .

We pursue this goal, because the above situation arises in [Sections 4 to 9](#). Later, the y_i will assume the role of multivariate monomials in $\mathbb{Q}[x^\pm]$, such that $y(x) \in \mathbb{T}^n$ if and only if $z = \vartheta(x) \in \mathcal{T}$ and the left hand side of system (I) or (II) is an invariant in $\mathbb{Q}[x^\pm]^\mathcal{G}$. This section will provide an indirect criterion for $z \in \mathcal{T}$ in terms of the coefficients c_i , which will later be substituted by polynomials in z .

3.1 Solutions in the compact torus

Recall Vieta's formula $\prod_{k=1}^n (x - y_k) = x^n + \sum_{i=1}^n (-1)^i \sigma_i(y_1, \dots, y_n) x^{n-i}$. Here, x is a univariate indeterminate.

Lemma 3.1. 1. *The map*

$$\begin{aligned} \mathbb{C}^* &\rightarrow \mathbb{C}, \\ x &\mapsto \frac{x+x^{-1}}{2} \end{aligned}$$

is surjective and the preimage of the real interval $[-1, 1] \subseteq \mathbb{R}$ is \mathbb{T} .

2. *System (I) has a unique solution $y \in \mathbb{C}^n$ up to permutation of its coordinates y_i .*

3. *System (II) has a unique solution $y \in (\mathbb{C}^*)^n$ up to permutation and inversion of its coordinates y_i .*

Proof. 1. For $r \in \mathbb{C}$, consider the univariate polynomial $p := x^2 - 2rx + 1$. Then 0 is not a root of p and we have $p(x) = 0$ if and only if $r = (x + x^{-1})/2$, that is, r is in the image of the map. If $r \in [-1, 1]$, then the discriminant of p is $r^2 - 1 \leq 0$ and the two roots are $x, \bar{x} = x, x^{-1} = r \pm i\sqrt{1-r^2} \in \mathbb{T}$. On the other hand, for $x \in \mathbb{T}$, we have $(x + x^{-1})/2 = (x + \bar{x})/2 = \Re(x) \in [-1, 1]$.

2. By Vieta's formula, a solution y of system (I) is the vector of roots of the polynomial $x^n + c_1 x^{n-1} + \dots + c_n$. Since \mathbb{C} is algebraically closed, the statement follows.

3. By 1., we can write the roots r_1, \dots, r_n of the polynomial $x^n + c_1 x^{n-1} + \dots + c_n$ as $r_i = (y_i + y_i^{-1})/2$ for some $y \in (\mathbb{C}^*)^n$. Then y is a unique solution of (II) up to permutation and inversion. \square

From now on, we speak of “the” solution of system (I), respectively (II).

Proposition 3.2. *For $c_1, \dots, c_n \in \mathbb{R}$, the corresponding solution of system (II) is contained in \mathbb{T}^n if and only if all the roots of the univariate polynomial*

$$x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$$

are contained in $[-1, 1]$.

Proof. If $y \in (\mathbb{C}^*)^n$ is the solution of system (II), then the roots of $x^n + c_1 x^{n-1} + \dots + c_n$ are $r_i := (y_i + y_i^{-1})/2$. According to [Lemma 3.1](#), we have $y \in \mathbb{T}^n$ if and only if $r_1, \dots, r_n \in [-1, 1]$. \square

Similarly, we can characterize solutions of system (I). Recall that the univariate Chebyshev polynomial of the first kind associated to $\ell \in \mathbb{N}$ is the unique T_ℓ with $T_\ell((x + x^{-1})/2) = (x^\ell + x^{-\ell})/2$. For $p \neq 0$ a polynomial in x , we denote by $\text{Coeff}(x^\ell, p)$ the coefficient of the monomial x^ℓ in p for $0 \leq \ell \leq \deg(p)$.

Proposition 3.3. *For $c_1, \dots, c_{n-1} \in \mathbb{C}$ with $\bar{c}_i = (-1)^n c_{n-i}$ and $c_0 := (-1)^n c_n := 1$, the corresponding solution of system (I) is contained in \mathbb{T}^n if and only if all the roots of the univariate polynomial*

$$T_n(x) + d_1 T_{n-1}(x) + \dots + d_{n-1} T_1(x) + \frac{d_n}{2} T_0(x) \quad \text{with} \quad d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

are contained in $[-1, 1]$.

Proof. The solution of system (I) is contained in \mathbb{T}^n if and only if all the roots of the univariate polynomial $p := x^n + c_1 x^{n-1} + \dots + c_n$ are contained in \mathbb{T} .

The roots of p are nonzero, because $p(0) = c_n = (-1)^n \neq 0$. We fix another univariate polynomial $\tilde{p} := x^n + \bar{c}_1 x^{n-1} + \dots + \bar{c}_n$. Since $\tilde{p}(x) = (-x)^n p(x^{-1})$, the set of roots of $p\tilde{p}$ consists of the roots of p and their inverses. In particular, all the roots of $p\tilde{p}$ are contained in \mathbb{T} if and only if all the roots of p are. For $0 \leq \ell \leq n$, the coefficients of $p\tilde{p}$ satisfy

$$\text{Coeff}(x^\ell, p\tilde{p}) = \sum_{i=0}^{\ell} c_{n-i} \bar{c}_{n-\ell+i} = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} = \sum_{i=0}^{\ell} \text{Coeff}(x^{n-i}, \tilde{p}) \text{Coeff}(x^{n-\ell+i}, p) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}).$$

The coefficients are real, because

$$\sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} = \sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \underbrace{(\bar{c}_i c_{\ell-i} + \bar{c}_{\ell-i} c_i)}_{\in \mathbb{R}} + \begin{cases} c_{\ell/2} \bar{c}_{\ell/2}, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases} \in \mathbb{R}.$$

Thus, $\text{Coeff}(x^\ell, p\tilde{p}) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}) = d_\ell \in \mathbb{R}$ and we can write

$$p\tilde{p} = \sum_{\ell=1}^n d_{n-\ell} (x^{n+\ell} + x^{n-\ell}) + d_n = 2x^n \left(\sum_{\ell=1}^n d_{n-\ell} T_\ell \left(\frac{x+x^{-1}}{2} \right) + \frac{d_n}{2} T_0 \left(\frac{x+x^{-1}}{2} \right) \right) =: 2x^n g \left(\frac{x+x^{-1}}{2} \right).$$

With [Lemma 3.1](#), we see that $x \in \mathbb{T}$ is a root of $p\tilde{p}$ if and only if $(x + x^{-1})/2 \in [-1, 1]$ is a root of g . \square

The following example illustrates the proof.

Example 3.4. *For $c \in \mathbb{C}$, set $p = x^3 - cx^2 + \bar{c}x - 1$ and consider the palindromic polynomial $p\tilde{p} \in \mathbb{R}[x]$ with*

$$\begin{aligned} & \frac{1}{2x^3} p(x)\tilde{p}(x) \\ &= \frac{1}{2x^3} \left((x^6 + 1) - (c + \bar{c})(x^5 + x) + (c\bar{c} + c + \bar{c})(x^4 + x^2) - \frac{c^2 + \bar{c}^2 + 2}{2} x^3 \right) \\ &= \frac{x^3 + x^{-3}}{2} - (c + \bar{c}) \frac{x^2 + x^{-2}}{2} + (c\bar{c} + c + \bar{c}) \frac{x + x^{-1}}{2} - \frac{c^2 + \bar{c}^2 + 2}{2} \\ &= T_3 \left(\frac{x + x^{-1}}{2} \right) - (c + \bar{c}) T_2 \left(\frac{x + x^{-1}}{2} \right) + (c\bar{c} + c + \bar{c}) T_1 \left(\frac{x + x^{-1}}{2} \right) - \frac{c^2 + \bar{c}^2 + 2}{2} T_0 \left(\frac{x + x^{-1}}{2} \right) \\ &= g \left(\frac{x + x^{-1}}{2} \right). \end{aligned}$$

All the roots of p are contained in \mathbb{T} if and only if all the roots of g are contained in $[-1, 1]$, that is, if and only if the solution of system (I) with right hand side $c_1 = -c, c_2 = \bar{c}, c_3 = -1$ is in \mathbb{T}^3 .

3.2 Characterization via Hermite quadratic forms

Let $p, q \in \mathbb{R}[x]$ be univariate polynomials with $p \neq 0$. The multiplication by q in the algebra $\mathbb{R}[x]/\langle p \rangle$ is

$$\begin{aligned} m_q : \mathbb{R}[x]/\langle p \rangle &\rightarrow \mathbb{R}[x]/\langle p \rangle, \\ f + \langle p \rangle &\mapsto qf + \langle p \rangle. \end{aligned}$$

A classical result, usually referred to as Sylvester's version of Sturm's theorem, allows us to effectively characterize when a univariate polynomial has all of its roots in $[-1, 1]$.

Theorem 3.5. *All the roots of p are contained in $S(q) := \{x \in \mathbb{R} \mid q(x) \geq 0\}$ if and only if the Hermite quadratic form*

$$\begin{aligned} H(p, q) : \mathbb{R}[x]/\langle p \rangle &\rightarrow \mathbb{R}, \\ f + \langle p \rangle &\mapsto \text{Tr}(m_q f^2) \end{aligned}$$

is positive semi-definite.

Proof. Let $H \in \mathbb{R}^{n \times n}$ be the symmetric matrix associated to $H(p, q)$ for a fixed basis of $\mathbb{R}[x]/\langle p \rangle$, where $n = \deg(p)$. Denote by N_+ , respectively N_- , the number of strictly positive, respectively negative, eigenvalues of H , including their multiplicities. By [CLO05, Ch. 2, Thm. 5.2], the rank and the signature of $H(p, q)$ are

$$\begin{aligned} N_+ + N_- &= \text{Rank}(H(p, q)) = |\{x \in \mathbb{C} \mid p(x) = 0, q(x) \neq 0\}|, \\ N_+ - N_- &= \text{Sign}(H(p, q)) = \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) > 0\}|}_{=:n_+} - \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) < 0\}|}_{=:n_-}. \end{aligned}$$

If all the roots of p are contained in $S(q)$, then $n_- = 0$, and thus

$$N_+ + N_- = \text{Rank}(H(p, q)) = \text{Sign}(H(p, q)) = n_+ = N_+ - N_-.$$

Hence, $N_- = 0$ and all eigenvalues of H are nonnegative, that is, $H(p, q)$ is positive semi-definite.

Conversely, assume that $H(p, q)$ is positive semi-definite. Then $N_- = 0$ and

$$N_+ = \text{Sign}(H(p, q)) = n_+ - n_- \leq \text{Rank}(H(p, q)) = N_+,$$

that is, $n_+ - n_- = \text{Rank}(H(p, q))$. On the other hand, $\text{Rank}(H(p, q)) \geq n_+ + n_-$. Hence, $n_- = 0$ and $\text{Rank}(H(p, q)) = n_+$ implies that all the roots of p are real and contained in $S(q)$. \square

Finally, we compute the matrix of the Hermite quadratic form. If $p = x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$, then the matrix of the multiplication m_x in $\mathbb{R}[x]/\langle p \rangle$ in the basis $\{1, x, \dots, x^{n-1}\}$ is the companion matrix

$$\begin{bmatrix} 0 & 0 & -c_n \\ 1 & \ddots & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & 1 & -c_1 \end{bmatrix} \quad (3.1)$$

of p , because $x x^i = x^{i+1}$ whenever $0 \leq i \leq n-2$ and $x x^{n-1} = x^n \equiv -c_1 x^{n-1} - \dots - c_{n-1} x - c_n \pmod{\langle p \rangle}$.

Corollary 3.6. *For $c_1, \dots, c_n \in \mathbb{R}$, denote by $C \in \mathbb{R}^{n \times n}$ the matrix from Equation (3.1). Then the solution of system (II) is contained in \mathbb{T}^n if and only if $H \succeq 0$, where $H \in \mathbb{R}^{n \times n}$ has entries $H_{ij} = \text{Tr}(C^{i+j-2} - C^{i+j})$.*

Proof. H is the matrix associated to the Hermite quadratic form $H(p, q)$ from Theorem 3.5 with $q = 1 - x^2$ in the basis $\{1, x, x^2, \dots, x^{n-1}\}$. Indeed, by [CLO05, Ch. 2, Prop. 4.2], the matrix entries are

$$H_{ij} = \text{Tr}(m_q x^{i-1} x^{j-1}) = \text{Tr}(m_x^{i+j-2} - m_x^{i+j}) = \text{Tr}(m_x^{i+j-2} - m_x^{i+j})$$

for $1 \leq i, j \leq n$. Since the trace is basis independent, we can replace m_x with its matrix C given by Equation (3.1). The statement now follows from Proposition 3.2, because $\{x \in \mathbb{R} \mid q(x) \geq 0\} = [-1, 1]$. \square

On the other hand, the univariate Chebyshev polynomials also form a basis $\{T_0, T_1, \dots, T_{n-1}\}$ of $\mathbb{R}[x]/\langle p \rangle$. If $n \geq 3$ and $p = T_n + d_1 T_{n-1} + \dots + d_{n-1} T_1 + d_n/2 T_0 \in \mathbb{R}[x]$, then the matrix of m_x is

$$\begin{bmatrix} 0 & 1/2 & & & 0 & -d_n/4 \\ 1 & 0 & \ddots & & & -d_{n-1}/2 \\ & 1/2 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3/2 \\ & & & \ddots & 0 & (1-d_2)/2 \\ 0 & & & & 1/2 & -d_1/2 \end{bmatrix}. \quad (3.2)$$

The entries originate from the recurrence formula $2xT_j = T_{j+1} + T_{j-1}$. In particular, the last column is obtained through

$$2xT_{n-1} = T_n + T_{n-2} \equiv -d_1 T_{n-1} + (1-d_2)T_{n-2} - d_3 T_{n-3} - \dots - d_{n-1} T_1 - d_n/2 T_0 \pmod{\langle p \rangle}.$$

Corollary 3.7. *Let $n \geq 3$. For $c_1, \dots, c_{n-1} \in \mathbb{C}$ with $\bar{c}_i = (-1)^n c_{n-i}$ and $c_0 := (-1)^n c_n := 1$, set*

$$d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

for $1 \leq \ell \leq n$ and denote by $C \in \mathbb{R}^{n \times n}$ the matrix from Equation (3.2). Then the solution of system (I) is contained in \mathbb{T}^n if and only if $H \geq 0$, where $H \in \mathbb{R}^{n \times n}$ has entries $H_{ij} = \text{Tr}(C^{i+j-2} - C^{i+j})$.

Proof. The proof is analogous to the one of Corollary 3.6 with Proposition 3.3. \square

4 Type A_{n-1}

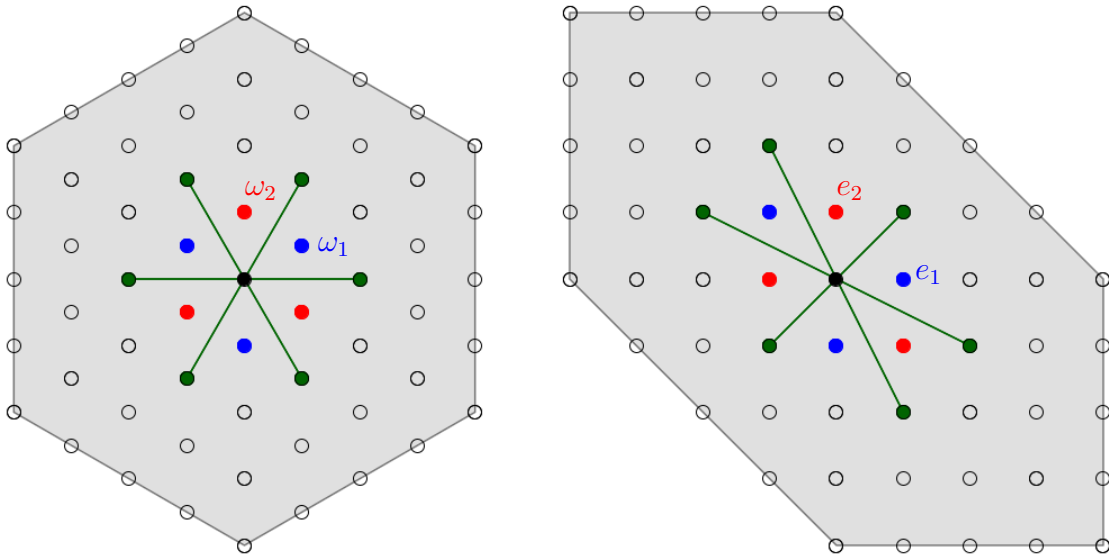


Figure 3: The root system A_2 and its weight lattice in the usual orthogonal representation and the integer representation. The orbits of the fundamental weights are the blue and red lattice elements.

The group \mathfrak{S}_n acts on \mathbb{R}^n by permutation of coordinates and leaves the subspace $V := \mathbb{R}^n / \langle [1, \dots, 1]^t \rangle = \{u \in \mathbb{R}^n \mid u_1 + \dots + u_n = 0\}$ invariant. The root system A_{n-1} given in [Bou68, Planche I] is a root system of rank $n - 1$ in V with fundamental weights

$$\omega_i = \sum_{j=1}^i e_j - \frac{i}{n} \sum_{j=1}^n e_j = \frac{1}{n} \underbrace{(n-i, \dots, n-i)}_{i \text{ times}}, \underbrace{(-i, \dots, -i)}_{n-i \text{ times}} \quad (1 \leq i \leq n-1). \quad (4.1)$$

Here, the $e_i \in \mathbb{R}^n$ are the Euclidean standard basis vectors. In particular, the weight lattice is $\Omega := \{u/n \mid u \in V \cap \mathbb{Z}^n\}$. The Weyl group $\mathcal{W} \cong \mathfrak{S}_n$ of A_{n-1} is orthogonal with respect to the Euclidean scalar product and we have $-\omega_i \in \mathcal{W}\omega_{n-i}$. The orbit $\mathcal{W}\omega_i$ has cardinality $\binom{n}{i}$ for $1 \leq i \leq n-1$, see Figure 3.

In this section, we give a closed formula for the matrix polynomial from Theorem 1.1 in the standard monomial basis for the root system A_{n-1} . The ring of Laurent polynomials is $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}]$ and the polynomial ring is $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_{n-1}]$.

4.1 Orbit polynomials

With $\mathcal{W} \cong \mathfrak{S}_n$ and Equation (4.1), one observes that a weight $\mu \in \Omega$ is contained in the \mathcal{W} -orbit of the first fundamental weight ω_1 if and only if

$$\mu = \begin{cases} \omega_1, \\ -\omega_{n-1}, & \text{or} \\ \omega_{i+1} - \omega_i, & 1 \leq i \leq n-2. \end{cases}$$

We denote by $\mathcal{G} \in \text{GL}_{n-1}(\mathbb{Z})$ the integer representation of \mathcal{W} defined by Equation (2.1). Then the corresponding orbit of monomials $\mathcal{G} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{G}\} \subseteq \mathbb{Q}[x^\pm]$ consists of n distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-1} := x_{n-1} x_{n-2}^{-1}, \quad y_n := x_{n-1}^{-1}. \quad (4.2)$$

For $1 \leq i \leq n$, let σ_i be the i -th elementary symmetric function in n indeterminates. Recall that, for $1 \leq i \leq n-1$, the \mathcal{G} -invariant orbit polynomial associated to $e_i \in \mathbb{Z}^{n-1}$ from Equation (2.4) is denoted by $\theta_i = \Theta_{e_i}$.

One can now show that $\sigma_i(y(x)) \in \mathbb{Q}[x^\pm]^\mathcal{G}$ is invariant and thus express it explicitly in terms of the fundamental invariants.

Proposition 4.1. *In $\mathbb{Q}[x^\pm]$, we have*

$$\sigma_i(y_1(x), \dots, y_n(x)) = \binom{n}{i} \theta_i(x) \quad (1 \leq i \leq n-1) \quad \text{and} \quad \sigma_n(y_1(x), \dots, y_n(x)) = 1.$$

Proof. It follows from Equation (4.2) that $x_i = y_1(x) \dots y_i(x)$ and \mathcal{G} acts on the $y_i(x)$ by permutation. Hence,

$$\theta_i(x) = \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B e_i} = \frac{|\text{Stab}_\mathcal{G}(x_i)|}{|\mathcal{G}|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} y_j(x) = \frac{1}{|\mathcal{G} \cdot x_i|} \sigma_i(y_1(x), \dots, y_n(x)).$$

With $|\mathcal{G} \cdot x_i| = \binom{n}{i}$ and $1 = y_1(x) \dots y_n(x)$, we obtain the statement. \square

We denote $\mathbb{T}_1^n := \{y \in \mathbb{T}^n \mid y_1 \dots y_n = 1\}$. The next statement follows immediately from Equation (4.2).

Lemma 4.2. *The map*

$$\Upsilon : \mathbb{T}^{n-1} \rightarrow \mathbb{T}_1^n, \\ x \mapsto (y_1(x), \dots, y_n(x)),$$

is bijective.

4.2 Hermite characterization with standard monomials

We now characterize, whether a given point z is contained in the \mathbb{T} -orbit space \mathcal{T} of \mathcal{G} , that is, if the equation $\theta_i(x) = z_i$ has a solution $x \in \mathbb{T}^{n-1}$. As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (I) has its solution in \mathbb{T}^n . We state our main result for A_{n-1} in the standard monomial basis.

Theorem 4.3. *Let $n \geq 3$. Define the $(n-1)$ -dimensional \mathbb{R} -vector space $\mathcal{Z} := \{z \in \mathbb{C}^{n-1} \mid \forall 1 \leq i \leq n-1 : \bar{z}_i = z_{n-i}\}$ and the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by*

$$H(z)_{ij} = \text{Tr}((C(z))^{i+j-2} - (C(z))^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & 1/2 & & 0 & -d_n(z)/4 \\ 1 & 0 & \ddots & & -d_{n-1}(z)/2 \\ & 1/2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3(z)/2 \\ & & & \ddots & 0 & (1-d_2(z))/2 \\ 0 & & & & 1/2 & -d_1(z)/2 \end{bmatrix},$$

$$d_\ell(z) = (-1)^\ell \sum_{i=0}^{\ell} \binom{n}{i} \binom{n}{\ell-i} z_i z_{n-\ell+i} \Big|_{z_0=z_n=1} \quad \text{for } 1 \leq \ell \leq n.$$

For $z \in \mathcal{Z}$, we have $H(z) \in \mathbb{R}^{n \times n}$ and $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$.

Proof. For $z \in \mathbb{C}^{n-1}$, define $c_0 := 1$, $c_n := (-1)^n$, $c_i := (-1)^i \binom{n}{i} z_i \in \mathbb{C}$ for $1 \leq i \leq n-1$ as well as $d_\ell := d_\ell(z)$ for $1 \leq \ell \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$ and fix $x \in \mathbb{T}^{n-1}$, such that $\theta_i(x) = z_i$. By [Proposition 4.1](#) and [Lemma 4.2](#), the unique solution of

$$(I) \quad \sigma_i(y_1, \dots, y_n) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is $y = \Upsilon(x) \in \mathbb{T}_1^n$. Note that $\theta_j(x)$ and $\theta_{n-j}(x)$ are complex conjugates, because $-\omega_j \in \mathcal{W}_{\omega_{n-j}}$. Therefore, $z \in \mathcal{Z}$ and $d_\ell = \sum_{i=0}^{\ell} c_i \bar{c}_{\ell-i} \in \mathbb{R}$ yields the last column of $C(z)$. [Corollary 3.7](#) gives us $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $z \in \mathcal{Z}$ with $H(z) \succeq 0$. By [Corollary 3.7](#), the solution y of system (I) is contained in \mathbb{T}_1^n . Let $x \in \mathbb{T}^{n-1}$ be the unique preimage of y under Υ . Then by [Proposition 4.1](#),

$$z_i = (-1)^i \binom{n}{i}^{-1} c_i = \binom{n}{i}^{-1} \sigma_i(y_1, \dots, y_n) = \theta_i(x)$$

for $1 \leq i \leq n-1$ and so $z = \vartheta(x) \in \mathcal{T}$. □

For $n = 2$, we are in the univariate case with $\mathcal{T} = [-1, 1]$.

Example: A_2

We study the root system from [Figure 3](#) with Weyl group $\mathcal{W} \cong \mathfrak{S}_3$. Since $\mathcal{W}\omega_1 = \{\omega_1, \omega_2 - \omega_1, -\omega_2\}$ and $\mathcal{W}\omega_2 = \{\omega_2, \omega_1 - \omega_2, -\omega_1\}$, the integer representation of \mathcal{W} is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Let $z_1, z_2 \in \mathbb{R}$ and $z = (z_1 + iz_2, z_1 - iz_2)$. The matrix $C \in \mathbb{Q}[z]^{3 \times 3}$ from [Theorem 4.3](#) is

$$C(z) = \begin{bmatrix} 0 & 1/2 & (1 + 9z_1^2 - 9z_2^2)/2 \\ 1 & 0 & (1 - 9z_1^2 - 9z_2^2 - 6z_1)/2 \\ 0 & 1/2 & 3z_1 \end{bmatrix}.$$

Define the matrix $H \in \mathbb{Q}[z]^{3 \times 3}$ with entries $(H(z))_{ij} = \text{Tr}(C(z)^{i+j-2}) - \text{Tr}(C(z)^{i+j})$. Then $z \in \mathcal{T}$, or equivalently $(z_1, z_2) \in \mathcal{T}_{\mathbb{R}}$, if and only if $H(z)$ is positive semi-definite. Assume that

$$\text{Det}(x I_3 - H(z)) = x^3 - h_1(z) x^2 + h_2(z) x - h_3(z)$$

is the characteristic polynomial of $H(z)$, where

$$\begin{aligned} h_3(z) &= -\text{Coeff}(x^0, \text{Det}(x I_3 - H(z))) \\ &= 2187/64 z_2^4 (3 z_1 + 1)^2 (-3 z_1^4 - 6 z_1^2 z_2^2 - 3 z_2^4 + 8 z_1^3 - 24 z_1 z_2^2 - 6 z_1^2 - 6 z_2^2 + 1) \\ h_2(z) &= \text{Coeff}(x^1, \text{Det}(x I_3 - H(z))) \\ &= 243/256 z_2^2 (-243 z_1^8 - 972 z_1^6 z_2^2 - 1458 z_1^4 z_2^4 - 972 z_1^2 z_2^6 - 243 z_2^8 + 324 z_1^7 - 1620 z_1^5 z_2^2 \\ &\quad - 4212 z_1^3 z_2^4 - 2268 z_1 z_2^6 - 432 z_1^6 - 2052 z_1^4 z_2^2 - 5400 z_1^2 z_2^4 - 324 z_2^6 + 180 z_1^5 - 3384 z_1^3 z_2^2 \\ &\quad - 684 z_1 z_2^4 + 18 z_1^4 - 804 z_1^2 z_2^2 + 42 z_2^4 + 76 z_1^3 + 404 z_1 z_2^2 - 8 z_1^2 - 108 z_2^2 + 60 z_1 + 25), \\ h_1(z) &= -\text{Coeff}(x^2, \text{Det}(x I_3 - H(z))) \\ &= 1/32 (-729 z_1^6 + 1458 z_1^5 + (10935 z_2^2 - 1215) z_1^4 + (-2916 z_2^2 + 540) z_1^3 + 351 z_2^2 + 63 \\ &\quad + (-10935 z_2^4 + 1458 z_2^2 - 135) z_1^2 + (-4374 z_2^4 + 972 z_2^2 + 18) z_1 + 729 z_2^6 - 1215 z_2^4). \end{aligned}$$

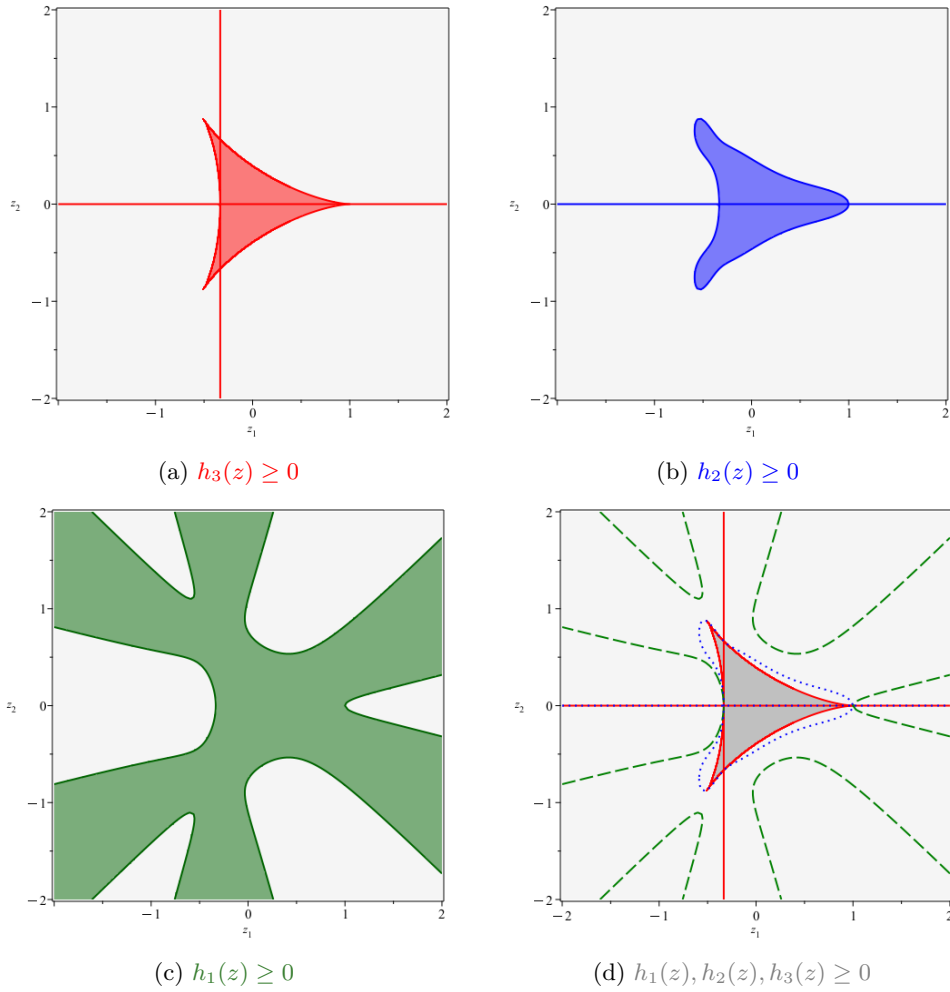


Figure 4: The intersection of the semi-algebraic sets defined by the coefficients of the characteristic polynomial of $H(z)$ is the \mathbb{T} -orbit space associated to A_2 .

The matrix $H(z)$ is positive semi-definite if and only if $h_i(z) \geq 0$ for $1 \leq i \leq 3$. In Figure 4, a solid red line, blue dots and green dashes indicate the varieties of these three polynomials. From those plots, we suspect that the real \mathbb{T} -orbit space is invariant under rotation by $2\pi/3$ and horizontal reflection, that is, under the dihedral group \mathfrak{D}_3 of order 6. The \mathfrak{D}_3 -invariants are $\mathbb{Q}[z]^{\mathfrak{D}_3} = \mathbb{R}[g_1, g_2]$ with $g_1(z) := z_1^2 + z_2^2$ and $g_2(z) := z_1(z_1^2 - 3z_2^2)$. Hence, we have

$$h_3(z) = \underbrace{2187/64 z_2^4 (3z_1 + 1)^2}_{\geq 0} \underbrace{(-6g_1(z) - 3g_1(z)^2 + 8g_2(z) + 1)}_{\mathfrak{D}_3\text{-invariant}}.$$

and the variety of the \mathfrak{D}_3 -invariant factor is the boundary of $\mathcal{T}_{\mathbb{R}}$. We now identify the vertices of $\mathcal{T}_{\mathbb{R}}$, which correspond to the fundamental weights and the origin. With $\vartheta_{\mathbb{R}} = (\frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 - \theta_2}{2i})$, those are

$$\text{Vertex}_1 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_1 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{4}{3}\pi i\right), \exp\left(-\frac{2}{3}\pi i\right)\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_2 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{2}{3}\pi i\right), \exp\left(-\frac{4}{3}\pi i\right)\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_3 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta_{\mathbb{R}}(1, 1) = (1, 0).$$

For a generic point $(z_1, z_2) \in \mathbb{R}^2$, $H(z)$ has rank 3. The variety “ $h_1 = h_2 = h_3 = 0$ ” corresponds to rank 0 Hermite matrices $H(z)$ and contains Vertex_3 as well as

$$\vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, (\omega_1 + \omega_2)/2 \rangle), \exp(-2\pi i \langle \omega_2, (\omega_1 + \omega_2)/2 \rangle)) = \vartheta_{\mathbb{R}}(\exp(-\pi i), \exp(\pi i)) = \left(-\frac{1}{3}, 0\right).$$

The varieties “ $h_2 = h_3 = 0$ ”, containing Vertex_1 and Vertex_2 , and “ $h_1 = h_3 = 0$ ”, containing $(-1/3, \pm 2/3)$, determine rank 1 Hermite matrices. Every other point on the boundary of $\mathcal{T}_{\mathbb{R}}$ corresponds to a rank 2 Hermite matrix.

5 Type C_n

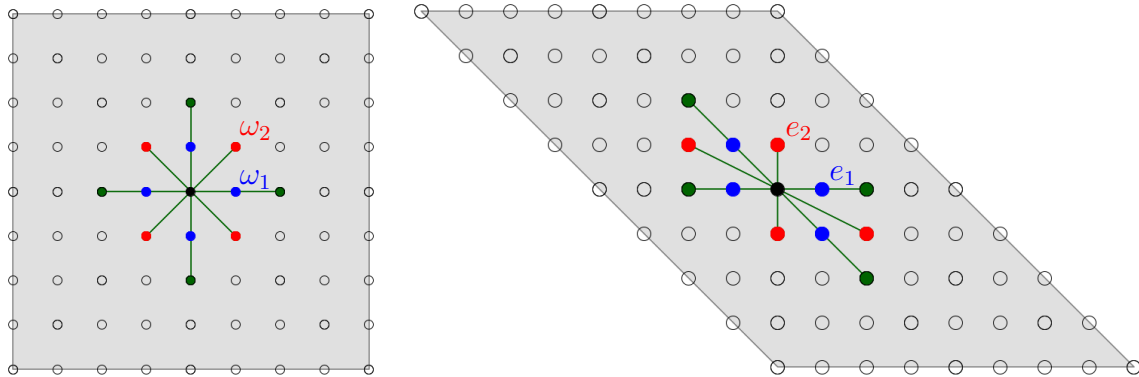


Figure 5: The root system C_2 and its weight lattice in the usual orthogonal representation and the integer representation. The orbits of the fundamental weights are the blue and red lattice elements.

The groups \mathfrak{S}_n and $\{\pm 1\}^n$ act on \mathbb{R}^n by permutation of coordinates and multiplication of coordinates by ± 1 . The root system C_n given in [Bou68, Planche III] is a root system in \mathbb{R}^n with fundamental weights

$$\omega_i = e_1 + \dots + e_i \quad (1 \leq i \leq n). \quad (5.1)$$

The weight lattice is $\Omega = \mathbb{Z}^n$. The Weyl group $\mathcal{W} \cong \mathfrak{S}_n \ltimes \{\pm 1\}^n$ of C_n is orthogonal with respect to the Euclidean scalar product. We have $-I_n \in \mathcal{W}$ and thus $-\omega_i \in \mathcal{W}\omega_i$. Furthermore, the orbit $\mathcal{W}\omega_i$ has cardinality $2^i \binom{n}{i}$ for $1 \leq i \leq n$, see [Figure 5](#).

In this section, we give a closed formula for the matrix polynomial from [Theorem 1.1](#) in the standard monomial basis for C_n . This is a root system of rank n . Hence, the ring of Laurent polynomials is $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ and the polynomial ring is $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$.

5.1 Orbit polynomials

We denote by $\mathcal{G} \in \text{GL}_n(\mathbb{Z})$ the integer representation of \mathcal{W} with respect to the fundamental weights in [Equation \(5.1\)](#). Then the orbit $\mathcal{G} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{G}\} \subseteq \mathbb{Q}[x^\pm]$ consists of $2n$ distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_n := x_n x_{n-1}^{-1} \quad (5.2)$$

and their inverses. For $1 \leq i \leq n$, let σ_i be the i -th elementary symmetric function in n indeterminates and recall that $\theta_i = \Theta_{e_i}$ is the \mathcal{G} -invariant orbit polynomial associated to $e_i \in \mathbb{Z}^n$ from [Equation \(2.4\)](#).

Proposition 5.1. *In $\mathbb{Q}[x^\pm]$, we have*

$$\sigma_i \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) = \binom{n}{i} \theta_i(x) \quad (1 \leq i \leq n).$$

Proof. It follows from [Equation \(5.2\)](#) that $x_i = y_1(x) \dots y_i(x)$ and \mathcal{G} acts on the $y_i^{\pm 1}(x)$ by permutation. Hence,

$$\theta_i(x) = \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B e_i} = \frac{|\text{Stab}_{\mathcal{G}}(x_i)|}{|\mathcal{G}|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \sum_{\delta \in \{\pm 1\}^J} \prod_{j \in J} y_j(x)^{\delta_j} = \frac{1}{|\mathcal{G} \cdot x_i|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} (y_j(x) + y_j(x)^{-1})$$

With $|\mathcal{G} \cdot x_i| = 2^i \binom{n}{i}$, we obtain the statement. \square

The next statement follows immediately from [Equation \(5.2\)](#).

Lemma 5.2. *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)), \end{aligned}$$

is bijective.

5.2 Hermite characterization with standard monomials

We now characterize, whether a given point z is contained in the \mathbb{T} -orbit space \mathcal{T} of \mathcal{G} , that is, if the equation $\theta_i(x) = z_i$ has a solution $x \in \mathbb{T}^n$. As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in \mathbb{T}^n . We state our main result for C_n in the standard monomial basis.

Theorem 5.3. *Define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by*

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \dots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad \text{for } 1 \leq i \leq n.$$

Then $\mathcal{T} = \mathcal{T}_{\mathbb{R}} = \{z \in \mathbb{R}^n \mid H(z) \geq 0\}$.

Proof. Let $z \in \mathbb{R}^n$ and set $c_i := c_i(z) \in \mathbb{R}$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. By [Proposition 5.1](#) and [Lemma 5.2](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left(\frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i c_i \quad (1 \leq i \leq n)$$

is $y = \Upsilon(x) \in \mathbb{T}^n$. Applying [Corollary 3.6](#) yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $H(z) \succeq 0$. By [Corollary 3.6](#), the solution y of the above system (II) is contained in \mathbb{T}^n . Let $x \in \mathbb{T}^n$ be the unique preimage of y under Υ . Then $z_i = \theta_i(x)$ and so $z = \vartheta(x)$ is contained in \mathcal{T} . \square

Example: C_2

We study the root system from [Figure 5](#) with Weyl group $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$. Since $\mathcal{W}\omega_1 = \{\omega_1, -\omega_1, \omega_1 - \omega_2, \omega_2 - \omega_1\}$ and $\mathcal{W}\omega_2 = \{\omega_2, -\omega_2, 2\omega_1 - \omega_1, \omega_2 - 2\omega_1\}$, the integer representation of \mathcal{W} is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let $z = (z_1, z_2) \in \mathbb{R}^2$. The matrix $C \in \mathbb{Q}[z]^{2 \times 2}$ from [Theorem 5.3](#) is

$$C(z) = \begin{bmatrix} 0 & -z_2 \\ 1 & 2z_1 \end{bmatrix}.$$

Then z is contained in \mathcal{T} if and only if the resulting Hermite matrix

$$H(z) = \begin{bmatrix} -4z_1^2 + 2z_2 + 2 & -8z_1^3 + 6z_1z_2 + 2z_1 \\ -8z_1^3 + 6z_1z_2 + 2z_1 & -16z_1^4 + 16z_1^2z_2 + 4z_1^2 - 2z_2^2 - 2z_2 \end{bmatrix}$$

is positive semi-definite, which is equivalent to its determinant and trace

$$\begin{aligned} \text{Det}(H(z)) &= -4(z_1^2 - z_2)(2z_1 + 1 + z_2)(2z_1 - 1 - z_2) \\ \text{Tr}(H(z)) &= -16z_1^4 + 16z_1^2z_2 - 2z_2^2 + 2 \end{aligned}$$

being nonnegative. The varieties of these two polynomials in z_1, z_2 are depicted in [Figure 6](#).

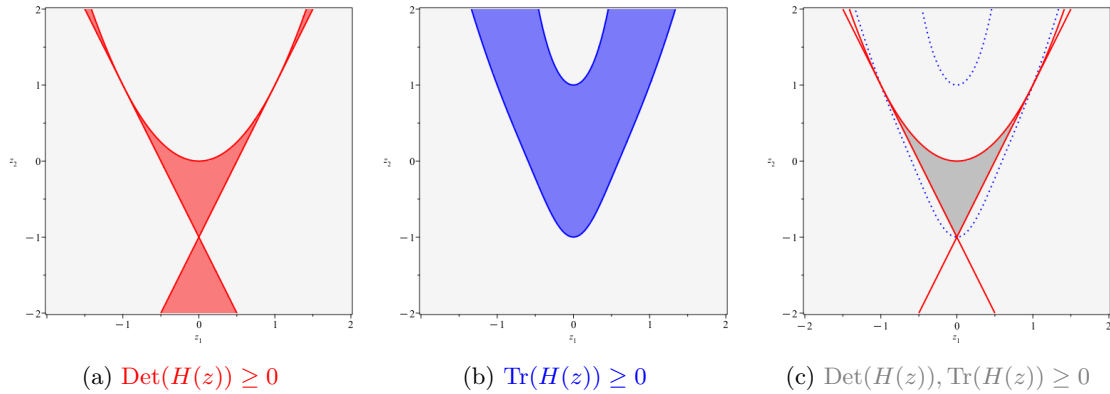


Figure 6: The semi-algebraic sets defined by the determinant and trace of $H(z)$.

We observe three intersections of “ $\text{Det}(H(z)) = 0$ ” (red solid line) and “ $\text{Tr}(H(z)) = 0$ ” (blue dots) in the the vertices

$$\text{Vertex}_1 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_1/2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1/2 \rangle)) = \vartheta(-1, -1) = (0, -1),$$

$$\text{Vertex}_2 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_2/2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2/2 \rangle)) = \vartheta(-1, 1) = (-1, 1),$$

$$\text{Vertex}_3 := \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1).$$

The shape of this domain is dictated by the determinant, but from the positivity condition one can observe that the trace is also required. Alternatively, the inequality given by the trace could be replaced by the constraint that the orbit space is contained in the square $[-1, 1]^2$.

6 Type B_n

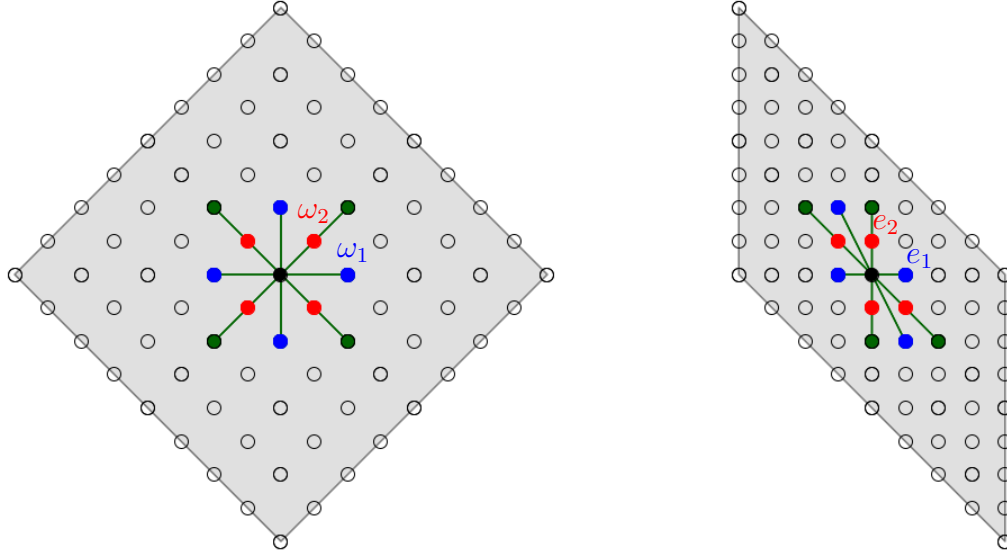


Figure 7: The root system B_2 and its weight lattice in the usual orthogonal representation and the integer representation. The orbits of the fundamental weights are the blue and red lattice elements.

The root system B_n given in [Bou68, Planche II] is a root system in \mathbb{R}^n . Its Weyl group is isomorphic to that of C_n , that is $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}^n$. However, the fundamental weights are now

$$\omega_i = e_1 + \dots + e_i \quad (1 \leq i \leq n-1) \quad \text{and} \quad \omega_n = (e_1 + \dots + e_n)/2. \quad (6.1)$$

In particular, the \mathbb{T} -orbit space can be very different from C_n , see Figure 1. We have $-I_n \in \mathcal{W}$ and thus $-\omega_i \in \mathcal{W}\omega_i$. Furthermore, the orbit $\mathcal{W}\omega_i$ has cardinality $2^i \binom{n}{i}$ for $1 \leq i \leq n$, see Figure 7.

In this section, we give a closed formula for the matrix polynomial from Theorem 1.1 in the standard monomial basis for B_n . This is a root system of rank n . Hence, the ring of Laurent polynomials is $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ and the polynomial ring is $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$.

6.1 Orbit polynomials

We denote by $\mathcal{G} \in \mathrm{GL}_n(\mathbb{Z})$ the integer representation of \mathcal{W} with respect to the fundamental weights in [Equation \(6.1\)](#). Then the orbit $\mathcal{G} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{G}\} \subseteq \mathbb{Q}[x^\pm]$ consists of $2n$ distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-1} := x_{n-1} x_{n-2}^{-1}, \quad y_n := x_n^2 x_{n-1}^{-1} \quad (6.2)$$

and their inverses. For $1 \leq i \leq n$, let σ_i be the i -th elementary symmetric function in n indeterminates and recall that $\theta_i = \Theta_{e_i}$ is the \mathcal{G} -invariant orbit polynomial associated to $e_i \in \mathbb{Z}^n$ from [Equation \(2.4\)](#).

Proposition 6.1. *In $\mathbb{Q}[x^\pm]$, we have*

$$\sigma_i \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) = \binom{n}{i} \theta_i(x) \quad (1 \leq i \leq n-1)$$

and $\sigma_n \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) = \Theta_{2e_n}(x).$

Proof. It follows from [Equation \(6.2\)](#) that $x_i = y_1(x) \dots y_i(x)$ and $x_n^2 = y_1(x) \dots y_n(x)$. Then the proof is analogous to [Proposition 5.1](#). \square

We compute the explicit expression for the right hand side of [Proposition 6.1](#) in terms of the fundamental invariants θ_i .

Lemma 6.2. *In $\mathbb{Q}[x^\pm]$, we have*

$$\Theta_{2e_n} = 2^n \theta_n^2 - \sum_{j=1}^{n-1} \binom{n}{j} \theta_j - 1.$$

Proof. The cardinality of the orbit $\mathcal{G} e_n$ is 2^n . Let $\alpha \in \mathcal{G} e_n$ and distinguish between the following cases.

1. If $\alpha = e_n$, then $\Theta_{e_n + \alpha} = \Theta_{2e_n}$ is the term for which we seek an explicit formula.
2. If $\alpha = -e_n$, then $\Theta_{e_n + \alpha} = \Theta_0 = 1$.
3. Otherwise, it follows from [Equation \(6.1\)](#) that there exists $1 \leq j \leq n-1$, such that $e_n + \alpha \in \mathcal{G} e_j$. The number of α , for which this is the case, is $\binom{n}{j}$.

With the recurrence formula from [Equation \(2.5\)](#), we conclude

$$2^n \theta_n^2 = |\mathcal{G} e_n| \Theta_{e_n} \Theta_{e_n} = \sum_{\alpha \in \mathcal{G} e_n} \Theta_{\alpha + e_n} = \Theta_{2e_n} + \sum_{j=1}^{n-1} \binom{n}{j} \Theta_{e_j} + 1 = \Theta_{2e_n} + \sum_{j=1}^{n-1} \binom{n}{j} \theta_j + 1$$

and obtain the formula for Θ_{2e_n} . \square

Lemma 6.3. *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)), \end{aligned}$$

is surjective. Furthermore, every $y \in \mathbb{T}^n$ has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ with

$$\theta_i(x) = \theta_i(x') \quad (1 \leq i \leq n-1) \quad \text{and} \quad \theta_n(x) = -\theta_n(x').$$

Proof. For $y \in \mathbb{T}^n$, choose $x \in \mathbb{T}^n$ with $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-1} = y_{n-1} x_{n-2}$ and $x_n^2 = y_n x_{n-1}$. Then x is a preimage of y under Υ and unique up to a sign in the last coordinate.

For $1 \leq i \leq n-1$, we have $\theta_i(x) = \theta_i(x')$, because $y_k(x) = y_k(x')$ for all $1 \leq k \leq n$.

It follows from Equation (6.1) that the stabilizer of ω_n in \mathcal{W} is \mathfrak{S}_n . Hence, we have $\mathcal{W}\omega_n = \{\pm 1\}^n \omega_n$ and, for all $\mu \in \mathcal{W}\omega_n$, there exist $\epsilon_i = \pm 1$ and $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-1}$, such that

$$\mu = \frac{\epsilon_1}{2} e_1 + \dots + \frac{\epsilon_n}{2} e_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-1} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_n}{2} (2\omega_n - \omega_{n-1}) = \epsilon_n \omega_n + \nu.$$

Now let $\alpha, \beta \in \mathbb{Z}^n$, such that $\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n$ and $\nu = \beta_1 \omega_1 + \dots + \beta_n \omega_n$. Then $\beta_n = 0$ and the monomial in θ_n corresponding to μ is $x^\alpha = x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}} x_n^{\epsilon_n}$. Thus, x^α is linear in x_n . Since every monomial in θ_n can be written in terms of such β and ϵ_i , θ_n is linear in x_n and with x, x' as above we have $\theta_n(x) = -\theta_n(x')$. \square

6.2 Hermite characterization with standard monomials

We now characterize, whether a given point z is contained in the \mathbb{T} -orbit space \mathcal{T} of \mathcal{G} , that is, if the equation $\theta_i(x) = z_i$ has a solution $x \in \mathbb{T}^n$. As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in \mathbb{T}^n . We state our main result for B_n in the standard monomial basis.

Theorem 6.4. *Define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by*

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad (1 \leq i \leq n-1) \quad \text{and} \quad c_n(z) = (-1)^n \left(2^n z_n^2 - \sum_{i=1}^{n-1} \binom{n}{i} z_i - 1 \right).$$

Then $\mathcal{T} = \mathcal{T}_{\mathbb{R}} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{R}^n$ and set $c_i := c_i(z) \in \mathbb{R}$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. By Proposition 6.1 and Lemmas 6.2 and 6.3, the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left(\frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is $y = \Upsilon(x) \in \mathbb{T}^n$. Applying Corollary 3.6 yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $H(z) \succeq 0$. By Corollary 3.6, the solution y of system (II) with coefficients c_i is contained in \mathbb{T}^n . According to Lemma 6.3, y has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ under Υ with $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$ and $x_n = -x'_n$. We have $z_i = \theta_i(x) = \theta_i(x')$ for $1 \leq i \leq n-1$ and $z_n^2 = \theta_n(x)^2 = \theta_n(x')^2$ with $\theta_n(x) = -\theta_n(x')$. Therefore, $z_n = \theta_n(x)$ or $z_n = -\theta_n(x) = \theta_n(x')$ and thus, z is contained in \mathcal{T} . \square

Example: B_2

We study the root system from [Figure 7](#) with Weyl group $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$. Since $\mathcal{W}\omega_1 = \{\omega_1, -\omega_1, \omega_1 - 2\omega_2, 2\omega_2 - \omega_1\}$ and $\mathcal{W}\omega_2 = \{\omega_2, -\omega_2, \omega_1 - \omega_1, \omega_2 - \omega_1\}$, the integer representation of \mathcal{W} is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let $z = (z_1, z_2) \in \mathbb{R}^2$. The matrix $C \in \mathbb{Q}[z]^{2 \times 2}$ from [Theorem 6.4](#) is

$$C(z) = \begin{bmatrix} 0 & -4z_2^2 + 2z_1 + 1 \\ 1 & 2z_1 \end{bmatrix}.$$

Then z is contained in \mathcal{T} if and only if the resulting Hermite matrix

$$H(z) = 16 \begin{bmatrix} -4z_1^2 + 8z_2^2 - 4z_1 & -8z_1^3 + 24z_1z_2^2 - 12z_1^2 - 4z_1 \\ -8z_1^3 + 24z_1z_2^2 - 12z_1^2 - 4z_1 & -16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 20z_1^2 + 8z_2^2 - 4z_1 \end{bmatrix}$$

is positive semi-definite, which is equivalent to its determinant and trace

$$\begin{aligned} \text{Det}(H(z)) &= -64z_2^2(-z_2^2 + z_1)(z_1 + 1 + 2z_2)(z_1 + 1 - 2z_2) \\ \text{Tr}(H(z)) &= -16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 24z_1^2 + 16z_2^2 - 8z_1 \end{aligned}$$

being nonnegative. The varieties of these two polynomials in z_1, z_2 are depicted below.

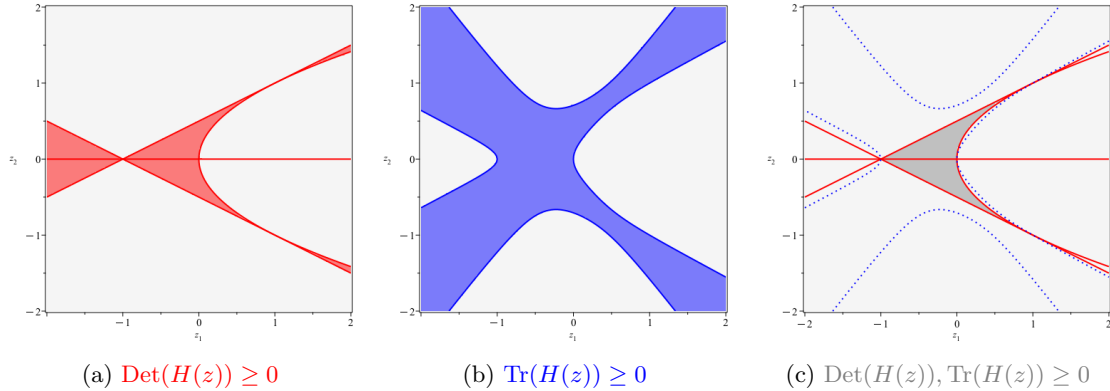


Figure 8: The semi-algebraic sets defined by the determinant and trace of $H(z)$.

The \mathbb{T} -orbit space in the B_2 -case is obtained from the C_2 -case in [Figure 6](#) by permuting z_1 and z_2 . In fact, this follows immediately from their Dynking graphs, which are the same up to permutation. Let us confirm that it follows also from our semi-algebraic description: Indeed, we have $\text{Det}(H(z))/\text{Det}(H^{C_2}(z_2, z_1)) = 16z_2^2 \geq 0$ on \mathbb{R}^2 . Furthermore, the determinant of $H(z)$ is 0 on the line “ $z_2 = 0$ ” and the rank is 0 in $z = (0, 0)$ (although this is not a vertex).

7 Type D_n

The group $\{\pm 1\}_+^n := \{\epsilon \in \{\pm 1\}^n \mid \epsilon_1 \dots \epsilon_n = 1\}$ acts on \mathbb{R}^n by multiplication of coordinates with ± 1 , where only an even amount of sign changes is admissible. The root system D_n given in [\[Bou68, Planche IV\]](#) is a root system in \mathbb{R}^n with fundamental weights

$$\omega_i = e_1 + \dots + e_i \quad (1 \leq i \leq n-2) \quad \text{and} \quad \omega_{n-1} = (e_1 + \dots + e_{n-1} - e_n)/2, \quad \omega_n = (e_1 + \dots + e_n)/2. \quad (7.1)$$

The Weyl group of D_n is $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}_+^n$. For all $1 \leq i \leq n$, we have $-\omega_i \in \mathcal{W}\omega_i$, except when n is odd, where $-\omega_{n-1} \in \mathcal{W}\omega_n$. Furthermore, we have $|\mathcal{W}\omega_i| = 2^i \binom{n}{i}$ for $1 \leq i \leq n-2$ and $|\mathcal{W}\omega_{n-1}| = |\mathcal{W}\omega_n| = 2^{n-1}$.

In this section, we give a closed formula for the matrix polynomial from [Theorem 1.1](#) in the standard monomial basis for D_n . This is a root system of rank n . Hence, the ring of Laurent polynomials is $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ and the polynomial ring is $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$.

7.1 Orbit polynomials

We denote by $\mathcal{G} \in \text{GL}_n(\mathbb{Z})$ the integer representation of \mathcal{W} with respect to the fundamental weights in [Equation \(7.1\)](#). Then the orbit $\mathcal{G} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{G}\} \subseteq \mathbb{Q}[x^\pm]$ consists of $2n$ distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-2} := x_{n-2} x_{n-3}^{-1}, \quad y_{n-1} := x_n x_{n-1} x_{n-2}^{-1}, \quad y_n := x_n x_{n-1}^{-1} \quad (7.2)$$

and their inverses. For $1 \leq i \leq n$, let σ_i be the i -th elementary symmetric function in n indeterminates and recall that $\theta_i = \Theta_{e_i}$ is the \mathcal{G} -invariant orbit polynomial associated to $e_i \in \mathbb{Z}^n$ from [Equation \(2.4\)](#).

Proposition 7.1. *In $\mathbb{Q}[x^\pm]$, we have*

$$\begin{aligned} \sigma_i \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) &= \binom{n}{i} \theta_i(x) \quad (1 \leq i \leq n-2) \\ \text{and } \sigma_{n-1} \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) &= n \Theta_{e_{n-1} + e_n}(x), \\ \sigma_n \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \dots, \frac{y_n(x) + y_n(x)^{-1}}{2} \right) &= \frac{\Theta_{2e_{n-1}}(x) + \Theta_{2e_n}(x)}{2}. \end{aligned}$$

Proof. Let $1 \leq i \leq n-2$. It follows from [Equation \(7.2\)](#) that $x_i = y_1(x) \dots y_i(x)$. Then the statement for θ_i is proven analogously to [Proposition 5.1](#).

With $x_n x_{n-1} = y_1(x) \dots y_{n-1}(x)$, we obtain the equation for $\Theta_{e_{n-1} + e_n}(x)$ as well.

Finally, we have $x_{n-1}^2 = y_1(x) \dots y_{n-1}(x) y_n(x)^{-1}$ and $x_n^2 = y_1(x) \dots y_n(x)$. Thus,

$$\frac{\Theta_{2e_{n-1}}(x) + \Theta_{2e_n}(x)}{2} = \frac{1}{2^n} \left(\sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = -1}} y(x)^\epsilon + \sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = 1}} y(x)^\epsilon \right) = \frac{1}{2^n} \sum_{\epsilon \in \{\pm 1\}^n} y(x)^\epsilon = \frac{1}{2^n} \prod_{i=1}^n (y_i(x) + y_i(x)^{-1}),$$

where $y^\epsilon := y_1^{\epsilon_1} \dots y_n^{\epsilon_n}$. This proves the last equation. \square

We give the explicit expression for the right hand side of [Proposition 7.1](#) in terms of the fundamental invariants.

Lemma 7.2. *1. If n is even, then*

$$\begin{aligned} \Theta_{e_{n-1} + e_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1}, \\ \Theta_{2e_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1, \\ \Theta_{2e_n} &= 2^{n-1} \theta_n^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1. \end{aligned}$$

2. If n is odd, then

$$\begin{aligned}\Theta_{e_{n-1}+e_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j} - \frac{1}{n}, \\ \Theta_{2e_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}, \\ \Theta_{2e_n} &= 2^{n-1} \theta_n^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}.\end{aligned}$$

Proof. With Equation (7.1) and the recurrence formula from Equation (2.5), the proof is analogous to that for the case of B_n in Lemma 6.2. The full computation can be found in [Met22, Lemma 1.43] \square

Lemma 7.3. *The map*

$$\begin{aligned}\Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)),\end{aligned}$$

is surjective. Furthermore, every $y \in \mathbb{T}^n$ has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ under Υ with

$$\theta_i(x) = \begin{cases} \theta_i(x'), & \text{if } 1 \leq i \leq n-2 \\ -\theta_i(x'), & \text{if } i \in \{n-1, n\} \end{cases}.$$

Proof. For $y \in \mathbb{T}^n$, choose $x \in \mathbb{T}^n$ with $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-2} = y_{n-2} x_{n-3}$ and $x_{n-1}^2 = y_{n-1}^{-1} y_{n-1} x_{n-2}$, $x_n = y_n x_{n-1}$. Then x is a preimage of y under Υ and unique up to a common sign in x_{n-1} and x_n . This second preimage is denoted by x' . By Proposition 7.1, we have $\theta_i(x) = \theta_i(x')$ whenever $1 \leq i \leq n-2$.

It follows from Equation (7.1) that the stabilizers of ω_{n-1} and ω_n in \mathcal{W} are both \mathfrak{S}_n . Hence, we have $\mathcal{W}\omega_{n-1} = \{\pm 1\}_+^n \omega_{n-1}$ and $\mathcal{W}\omega_n = \{\pm 1\}_+^n \omega_n$. Thus, for $\mu \in \mathcal{W}\omega_n$, there exist $\epsilon_i = \pm 1$ with $\epsilon_1 \dots \epsilon_n = 1$ and $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-2}$, such that

$$\begin{aligned}\mu &= \frac{\epsilon_1}{2} e_1 + \dots + \frac{\epsilon_n}{2} e_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-2} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_{n-1}}{2} (\omega_n + \omega_{n-1} - \omega_{n-2}) + \frac{\epsilon_n}{2} (\omega_n - \omega_{n-1}) \\ &= \frac{\epsilon_{n-1} + \epsilon_n}{2} \omega_n + \frac{\epsilon_{n-1} - \epsilon_n}{2} \omega_{n-1} + \nu \in \Omega.\end{aligned}$$

Now let $\alpha, \beta \in \mathbb{Z}^n$, such that $\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n$ and $\nu = \beta_1 \omega_1 + \dots + \beta_n \omega_n$. Then $\beta_{n-1} = \beta_n = 0$ and the monomial in θ_n corresponding to μ is

$$x^\alpha = x_1^{\beta_1} \dots x_{n-2}^{\beta_{n-2}} x_{n-1}^{(\epsilon_{n-1} - \epsilon_n)/2} x_n^{(\epsilon_{n-1} + \epsilon_n)/2}$$

with $(\epsilon_{n-1} \pm \epsilon_n)/2 \in \{-1, 0, 1\}$. Therefore, x^α is linear in x_{n-1} and independent of x_n or vice versa. With x, x' as above we have $x^\alpha = -(x')^\alpha$. Since every monomial in θ_n can be written in terms of such β and ϵ_i , we obtain $\theta_n(x) = -\theta_n(x')$. Analogously, for $\mu \in \mathcal{W}\omega_{n-1}$, we have $\epsilon_1 \dots \epsilon_n = -1$ and obtain the statement for θ_{n-1} . \square

7.2 Hermite characterization with standard monomials

We now characterize, whether a given point z is contained in the \mathbb{T} -orbit space \mathcal{T} of \mathcal{G} , that is, if the equation $\theta_i(x) = z_i$ has a solution $x \in \mathbb{T}^n$. As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in \mathbb{T}^n . We state our main result for D_n in the standard monomial basis.

Theorem 7.4. Define the n -dimensional \mathbb{R} -vector space

$$\mathcal{Z} := \begin{cases} \mathbb{R}^n, & \text{if } n \text{ is even} \\ \{z \in \mathbb{C}^n \mid z_1, \dots, z_{n-2} \in \mathbb{R}, \bar{z}_n = z_{n-1}\}, & \text{if } n \text{ is odd} \end{cases}$$

and the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad (1 \leq i \leq n-2) \quad \text{and}$$

$$c_{n-1}(z) = (-1)^{n-1} \begin{cases} 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} z_{2j-1}, & \text{if } n \text{ is even} \\ 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-3)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is odd} \end{cases},$$

$$c_n(z) = (-1)^n \begin{cases} 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is even} \\ 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} z_{2j+1}, & \text{if } n \text{ is odd} \end{cases}.$$

For all $z \in \mathcal{Z}$, we have $H(z) \in \mathbb{R}^{n \times n}$ and $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{C}^n$ and set $c_i := c_i(z)$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. Furthermore, we have $z \in \mathcal{Z}$ and $c_i \in \mathbb{R}$. By [Proposition 7.1](#) and [Lemmas 7.2](#) and [7.3](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left(\frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i \tilde{c}_i \quad (1 \leq i \leq n)$$

is $y = \Upsilon(x) \in \mathbb{T}^n$. Applying [Corollary 3.6](#) yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $z \in \mathcal{Z}$ with $H(z) \succeq 0$. Hence, $c_i \in \mathbb{R}$ and by [Corollary 3.6](#), the solution y of the above system (II) is contained in \mathbb{T}^n . According to [Lemma 7.3](#), y has two distinct preimages $x, x' \in \mathbb{T}^n$. We have $z_i = \theta_i(x) = \theta_i(x')$ for $1 \leq i \leq n-2$ and $\theta_{n-1}(x) = -\theta_{n-1}(x')$, $\theta_n(x) = -\theta_n(x')$. Furthermore, $z_{n-1}^2 + z_n^2 = \theta_{n-1}(x)^2 + \theta_n(x)^2 = \theta_{n-1}(x')^2 + \theta_n(x')^2$ and $z_{n-1} z_n = \theta_{n-1}(x) \theta_n(x) = \theta_{n-1}(x') \theta_n(x')$. Therefore, $\{z_{n-1}, z_n\} \in \{\{\theta_{n-1}(x), \theta_n(x)\}, \{\theta_{n-1}(x'), \theta_n(x')\}\}$. If $z_{n-1} = \theta_{n-1}(x)$, $z_n = \theta_n(x)$, then $z = \vartheta(x)$. Otherwise, set $\tilde{x} = (x_1, \dots, x_{n-2}, x_n, x_{n-1})$ to obtain $z_{n-1} = \theta_{n-1}(\tilde{x})$, $z_n = \theta_n(\tilde{x})$ and $z = \vartheta(\tilde{x})$. An analogous argument applies to x' and thus, z is contained in \mathcal{T} . \square

8 Type G_2

The group $\mathfrak{S}_3 \times \{\pm 1\}$ acts on \mathbb{R}^3 by permutation of coordinates and scalar multiplication with ± 1 and leaves the subspace $V = \mathbb{R}^3 / \langle [1, 1, 1]^t \rangle = \{u \in \mathbb{R}^n \mid u_1 + u_2 + u_3 = 0\}$ invariant. The root system G_2 from [[Bou68](#), Planche IX] is a root system in V with fundamental weights

$$\omega_1 = (1, -1, 0) \quad \text{and} \quad \omega_2 = (-2, 1, 1). \quad (8.1)$$

The Weyl group of G_2 is $\mathcal{W} \cong \mathfrak{S}_3 \times \{\pm 1\}$. We have $-I_3 \in \mathcal{W}$ and thus, $-\omega_1 \in \mathcal{W}\omega_1$ as well as $-\omega_2 \in \mathcal{W}\omega_2$. Furthermore, we have $|\mathcal{W}\omega_1| = |\mathcal{W}\omega_2| = 6$, see [Figure 9](#).

In this section, we give a closed formula for the matrix polynomial from [Theorem 1.1](#) in the standard monomial basis for G_2 . This is a root system of rank 2. Hence, the ring of Laurent polynomials is $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ and the polynomial ring is $\mathbb{Q}[z] = \mathbb{Q}[z_1, z_2]$.

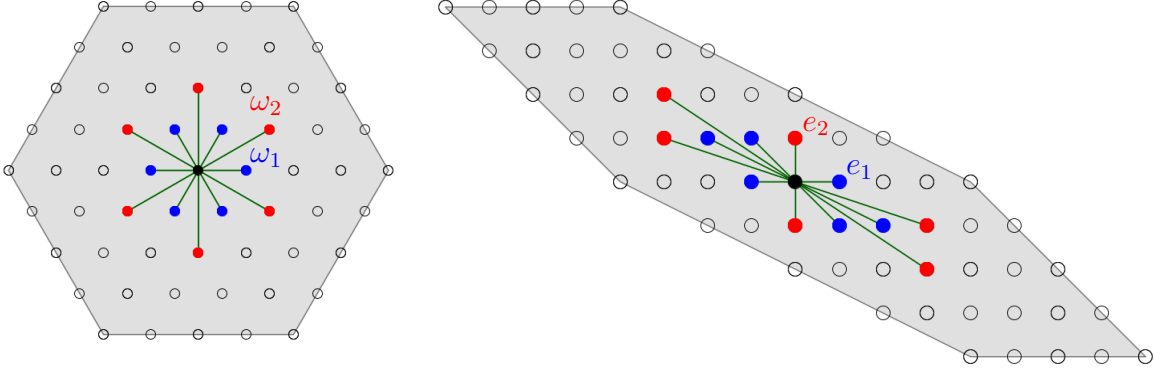


Figure 9: The root system G_2 and the weight lattice in the usual orthogonal representation and the integer representation. The orbits of the fundamental weights are the blue and red lattice elements.

8.1 Orbit polynomials

The integer representation of \mathcal{W} with respect to the fundamental weights in [Equation \(8.1\)](#) is

$$\mathcal{G} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \right\}.$$

Then the orbit $\mathcal{G} \cdot x_1 = \{x^{Be_1} \mid B \in \mathcal{G}\} \subseteq \mathbb{Q}[x^\pm]$ consists of 6 distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_1 x_2^{-1}, \quad y_3 := x_1^{-2} x_2 \quad (8.2)$$

and their inverses. For $1 \leq i \leq 3$, let σ_i be the i -th elementary symmetric function in 3 indeterminates and recall that, for $i \leq 2$, $\theta_i = \Theta_{e_i}$ is the \mathcal{G} -invariant orbit polynomial associated to $e_i \in \mathbb{Z}^2$ from [Equation \(2.4\)](#).

Proposition 8.1. *In $\mathbb{Q}[x^\pm]$, we have*

$$\begin{aligned} \sigma_1 \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \frac{y_2(x) + y_2(x)^{-1}}{2}, \frac{y_3(x) + y_3(x)^{-1}}{2} \right) &= 3\theta_1(x), \\ \sigma_2 \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \frac{y_2(x) + y_2(x)^{-1}}{2}, \frac{y_3(x) + y_3(x)^{-1}}{2} \right) &= \frac{3(\theta_1(x) + \theta_2(x))}{2} \\ \text{and } \sigma_3 \left(\frac{y_1(x) + y_1(x)^{-1}}{2}, \frac{y_2(x) + y_2(x)^{-1}}{2}, \frac{y_3(x) + y_3(x)^{-1}}{2} \right) &= \frac{9\theta_1(x)^2 - 3\theta_1(x) - 3\theta_2(x) - 1}{2}. \end{aligned}$$

Proof. In principle, this can be checked by hand, but we give a constructive proof to show how to obtain the equations. It follows from [Equation \(8.2\)](#) that $y_1(x)y_2(x)y_3(x) = 1$ and $x_1 = y_1(x)$, $x_1^{-1} = y_2(x)y_3(x)$ and $x_2 = y_1(x)^2 y_3(x)$, $x_2^{-1} = y_2(x)^2 y_3(x)$. Thus, after computing the orbits $\mathcal{G} \cdot x_1$ and $\mathcal{G} \cdot x_2$, we find polynomials $g_1, g_2 \in \mathbb{R}[y_1, y_2, y_3]$, such that $\theta_i(x) = g_i(y(x))$. Consider the ideal

$$\mathcal{I} := \langle \theta_1 - g_1(y), \theta_2 - g_2(y), 1 - y_1 y_2 y_3 \rangle \subseteq \mathbb{R}[y_1, y_2, y_3, \theta_1, \theta_2].$$

Choose an elimination ordering $\{y_1, y_2, y_3\} \succeq \{\theta_1, \theta_2\}$ and let G be a Gröbner basis of \mathcal{I} . Note that the expression on the left-hand side of the statement is \mathcal{G} -invariant. We can write it as $\sigma_i(y_1 + y_2 y_3, y_2 + y_1 y_3, y_3 + y_1 y_2)/2^i$, that is, a polynomial in $\mathbb{R}[y_1, y_2, y_3, \theta_1, \theta_2]$. Because of the \mathcal{G} -invariance, the normal form of this polynomial with respect to G only depends on θ_1 and θ_2 . One obtains the right-hand side. \square

The next statement follows immediately from [Equation \(8.2\)](#).

Lemma 8.2. *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^2 &\rightarrow \mathbb{T}_1^3, \\ x &\mapsto (y_1(x), y_2(x), y_3(x)) \end{aligned}$$

is bijective.

8.2 Hermite characterization with standard monomials

With [Proposition 8.1](#) and [Lemma 8.2](#), the proof of the following is analogous to [Theorem 5.3](#).

Theorem 8.3. *Define the matrix $H \in \mathbb{Q}[z]^{3 \times 3}$ by*

$$H(z)_{ij} = \text{Tr}((C(z))^{i+j-2} - (C(z))^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & 0 & (9z_1^2 - 3z_1 - 3z_2 - 1)/2 \\ 1 & 0 & -3(z_1 + z_2)/2 \\ 0 & 1 & 3z_1 \end{bmatrix}.$$

Then $\mathcal{T} = \mathcal{T}_{\mathbb{R}} = \{z \in \mathbb{R}^2 \mid H(z) \succeq 0\}$.

Example

Our result applies to the root system from [Figure 9](#) as follows. Let $z = (z_1, z_2) \in \mathbb{R}^2$. The Hermite matrix polynomial from [Theorem 8.3](#) is $4/3 H(z) =$

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -12z_1^2 + 4z_1 + 4z_2 + 4 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -36z_1^3 + 18z_1z_2 + 10z_1 + 6z_2 + 2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -36z_1^3 + 18z_1z_2 + 10z_1 + 6z_2 + 2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -324z_1^5 + 270z_1^3z_2 + 126z_1^3 + 45z_1^2z_2 - 45z_1z_2^2 + 15z_1^2 - 48z_1z_2 - 15z_2^2 - 11z_1 - 11z_2 - 2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -324z_1^5 + 270z_1^3z_2 + 126z_1^3 + 45z_1^2z_2 - 45z_1z_2^2 + 15z_1^2 - 48z_1z_2 - 15z_2^2 - 11z_1 - 11z_2 - 2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -972z_1^6 + 972z_1^4z_2 + 432z_1^4 + 162z_1^3z_2 - 243z_1^3z_2^2 + 63z_1^3 - 207z_1^2z_2 - 81z_1z_2^2 + 9z_2^3 - 45z_1^2 - 66z_1z_2 - 3z_2^2 - 14z_1 - 6z_2 - 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and shall have characteristic polynomial

$$\text{Det}(x I_3 - H(z)) = x^3 - h_1(z)x^2 + h_2(z)x - h_3(z).$$

Again, we have $z \in \mathcal{T}$ if and only if $h_i(z) \geq 0$ for $1 \leq i \leq 3$ with

$$\begin{aligned} h_3(z) &= -\text{Coeff}(x^0, \text{Det}(x I_3 - H(z))) \\ &= 81/16 (24z_1^3 - 12z_1z_2 - z_2^2 - 6z_1 - 4z_2 - 1) (3z_1^2 - 2z_2 - 1)^2 (3z_1 + 1)^2, \\ h_2(z) &= \text{Coeff}(x^1, \text{Det}(x I_3 - H(z))) \\ &= 27/16 (3z_1^2 - 2z_2 - 1) (648z_1^6 - 540z_1^5 - 972z_1^4z_2 - 216z_1^3z_2^2 - 648z_1^4 + 36z_1^3z_2 \\ &\quad + 369z_1^2z_2^2 + 126z_1z_2^3 + 9z_2^4 - 18z_1^3 + 480z_1^2z_2 + 246z_1z_2^2 + 36z_2^3 + 129z_1^2 + 202z_1z_2 \\ &\quad + 37z_2^2 + 44z_1 + 28z_2 + 4), \\ h_1(z) &= -\text{Coeff}(x^2, \text{Det}(x I_3 - H(z))) \\ &= 9/4 - 729z_1^6 + 243z_1^4 + 729z_1^4z_2 - 729/4z_1^2z_2^2 + 243/2z_1^3z_2 + 189/4z_1^3 - 405/4z_1^2z_2 \\ &\quad - 243/4z_1z_2^2 + 27/4z_2^3 - 81/4z_1^2 - 81/2z_1z_2 - 27/4z_2^2 - 9/2z_1 - 9/2z_2. \end{aligned}$$

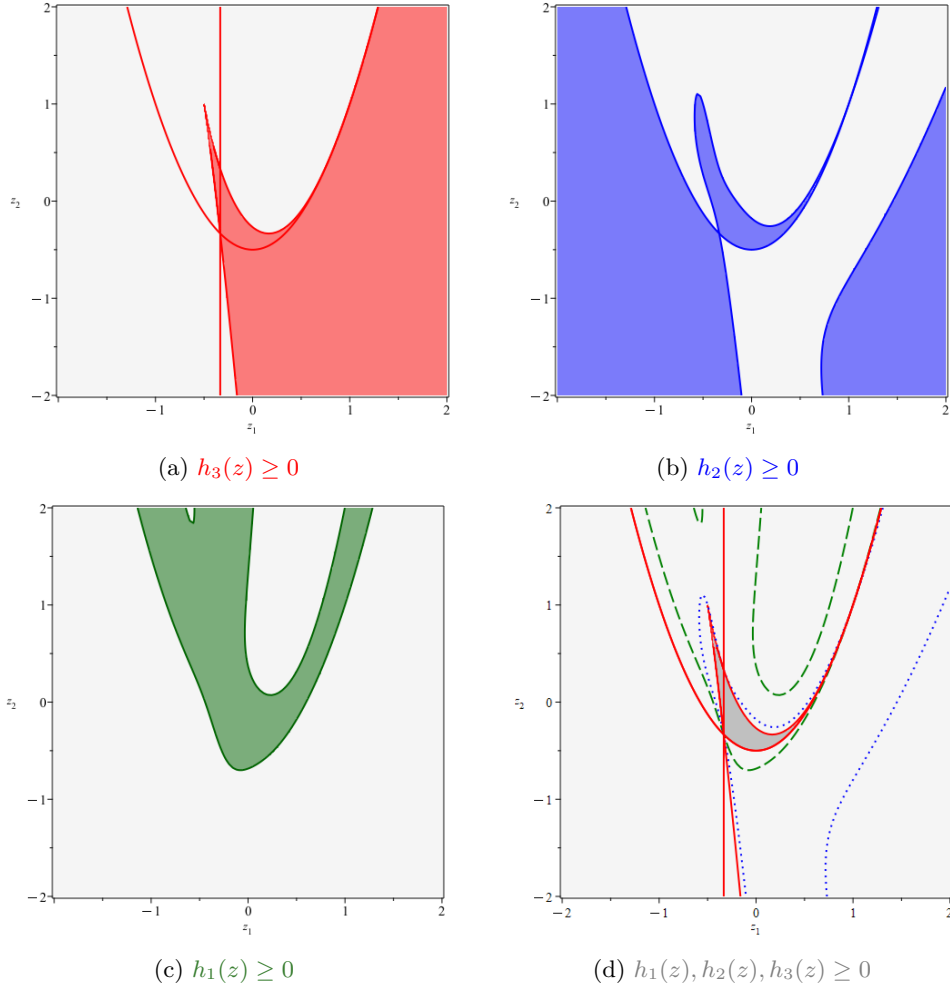


Figure 10: The semi-algebraic sets defined by the coefficients of the characteristic polynomial of $H(z)$.

The vertices are

$$\text{Vertex}_1 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_1/3 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1/3 \rangle)) = \vartheta\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}, 1\right) = \left(-\frac{1}{2}, 1\right),$$

$$\text{Vertex}_2 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_2/6 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2/6 \rangle)) = \vartheta(-1, 1) = \left(-\frac{1}{3}, -\frac{1}{3}\right),$$

$$\text{Vertex}_3 := \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1).$$

9 Hermite characterization of $\mathcal{T}_{\mathbb{R}}$ with Chebyshev polynomials

We shall finally prove [Theorem 1.1](#) from the introduction, which characterizes the real \mathbb{T} -orbit space $\mathcal{T}_{\mathbb{R}}$ in terms of the real generalized Chebyshev polynomials \widehat{T}_α from [Proposition 2.8](#).

Let $n \in \mathbb{N}$ and R be a root system of type A_{n-1} , B_n , C_n ($n \geq 2$), D_n ($n \geq 4$) or G_2 as in [\[Bou68, Planche I – IV, IX\]](#). In particular, the rank $r := \text{Rank}(R)$ of R is $r = n - 1$ for A_{n-1} , $r = n - 1 = 2$ for G_2 and $r = n$

otherwise. What all of these root systems have in common is that the symmetric group \mathfrak{S}_n is a semi-direct factor of the Weyl group of R , a fact that made the characterization via Hermite quadratic forms in the previous Sections 4 to 8 possible.

The Weyl group has an integer representation $\mathcal{G} \subseteq \mathrm{GL}_r(\mathbb{Z})$ as in Equation (2.1) with respect to the fundamental weights in Equations (4.1) to (8.1), respectively. We denote by $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_r]$ the multivariate polynomial ring. In particular, the size of the corresponding Hermite matrix is n (determined by the symmetric group \mathfrak{S}_n), while the number of coordinates of the \mathbb{T} -orbit space is r (determined by the rank of R).

Theorem 9.1. *Let $H \in \mathbb{Q}[z]^{n \times n}$ be the matrix polynomial with entries*

$$2^{i+j} H_{ij} = -\widehat{T}_{(i+j)e_1} + \sum_{\ell=1}^{\lceil (i+j)/2 \rceil - 1} \left(4 \binom{i+j-2}{\ell-1} - \binom{i+j}{\ell} \right) \widehat{T}_{(i+j-2\ell)e_1} \\ + \frac{1}{2} \begin{cases} 4 \binom{i+j-2}{(i+j)/2-1} - \binom{i+j}{(i+j)/2}, & \text{if } i+j \text{ is even} \\ 0, & \text{if } i+j \text{ is odd} \end{cases}.$$

Then the real \mathbb{T} -orbit space of \mathcal{G} is $\mathcal{T}_{\mathbb{R}} = \{z \in \mathbb{R}^r \mid H(z) \succeq 0\}$.

We need two lemmata involving the orbit polynomials $\Theta_{\alpha} = \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B\alpha}$ for $\alpha \in \mathbb{Z}^r$ from Equation (2.4).

Lemma 9.2. *In $\mathbb{Q}[x^{\pm}]$, define the $2n$ distinct monomials $\{y_1^{\pm 1}, \dots, y_n^{\pm 1}\} := \mathcal{G} \cdot x_1 \cup \mathcal{G} \cdot x_1^{-1}$. Then, for $x \in \mathbb{T}^r$ and $k \in \mathbb{N}$, we have $(y_i(x)^k + y_i(x)^{-k})/2 \in [-1, 1]$ and*

$$\frac{1}{n} \sum_{i=1}^n y_i(x)^k + y_i(x)^{-k} = \Theta_{k e_1}(x) + \Theta_{-k e_1}(x).$$

Proof. This follows immediately from Propositions 4.1, 5.1, 6.1, 7.1 and 8.1. \square

Lemma 9.3. *For $x \in \mathbb{T}^r$ and y_1, \dots, y_n the monomials from Lemma 9.2, let $C \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $(y_i(x) + y_i(x)^{-1})/2$. Then, for $k \in \mathbb{N}$, the trace of the k -th power of C is*

$$\mathrm{Tr}(C^k) = \frac{n}{2^k} \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left(\binom{k}{\ell} (\Theta_{(k-2\ell)e_1}(x) + \Theta_{(2\ell-k)e_1}(x)) \right) + \frac{n}{2^k} \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

Proof. The trace of the k -th power of C is the sum of the k -th powers of the eigenvalues. Thus,

$$\mathrm{Tr}(C^k) = \sum_{i=1}^n \left(\frac{y_i(x) + y_i(x)^{-1}}{2} \right)^k = \frac{1}{2^k} \sum_{i=1}^n \left(\sum_{\ell=0}^k \binom{k}{\ell} y_i(x)^{k-\ell} y_i(x)^{-\ell} \right) \\ = \frac{1}{2^k} \sum_{i=1}^n \left(\sum_{\ell=0}^{\lceil k/2 \rceil - 1} \binom{k}{\ell} y_i(x)^{k-2\ell} + \sum_{\ell=\lfloor k/2 \rfloor + 1}^k \binom{k}{\ell} y_i(x)^{k-2\ell} + \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \right) \\ = \frac{1}{2^k} \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left(\binom{k}{\ell} \sum_{i=1}^n (y_i(x)^{k-2\ell} + y_i(x)^{2\ell-k}) \right) + \frac{n}{2^k} \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

Applying Lemma 9.2 yields the formula. \square

Proof of Theorem 9.1. The \mathbb{T} -orbit space of \mathcal{G} is by definition $\mathcal{T} = \{\vartheta(x) \mid x \in \mathbb{T}^r\}$. According to Theorems 4.3, 5.3, 6.4, 7.4 and 8.3, we have $\mathcal{T} = \{\tilde{z} \in \mathcal{Z} \mid H(\tilde{z}) \succeq 0\}$, where \mathcal{Z} is an \mathbb{R} -vector space with $\dim(\mathcal{Z}) = r$ and $H \in \mathbb{Q}[z]^{n \times n}$ has entries $H_{ij} = \text{Tr}(C^{i+j-2}) - \text{Tr}(C^{i+j})$ for a certain matrix $C \in \mathbb{Q}[z]^{n \times n}$.

For $x \in \mathbb{T}^r$ and $\tilde{z} := \vartheta(x) \in \mathcal{T} \subseteq \mathcal{Z}$, it is now essential to note that $C(\tilde{z}) \in \mathbb{R}^{n \times n}$ has eigenvalues $(y_i(x) + y_i(x)^{-1})/2 \in [-1, 1]$ with y_i as in Lemma 9.2. This follows from the construction, by which C is the matrix of a multiplication map, see Section 3.2. In particular, we obtain a formula for the entries $H(\tilde{z})_{ij}$ with Lemma 9.3. In the basis of (real) generalized Chebyshev polynomials from Definition 2.6 and Proposition 2.8, we have

$$\Theta_{(k-2\ell)e_1}(x) + \Theta_{(2\ell-k)e_1}(x) = T_{(k-2\ell)e_1}(\vartheta(x)) + T_{(2\ell-k)e_1}(\vartheta(x)) = 2\widehat{T}_{(k-2\ell)e_1}(\vartheta_{\mathbb{R}}(x)).$$

We now substitute $\tilde{z} \mapsto z := \vartheta_{\mathbb{R}}(x) \in \mathcal{T}_{\mathbb{R}} \subseteq \mathbb{R}^n$. For $1 \leq i, j \leq n$ and $k := i + j$, the entry $H(z)_{ij}$ is

$$\begin{aligned} & \text{Tr}(C(\tilde{z})^{k-2}) - \text{Tr}(C(\tilde{z})^k) \\ &= \frac{2n}{2^k} \left(4 \sum_{\ell=0}^{\lceil k/2 \rceil - 2} \left(\binom{k-2}{\ell} \widehat{T}_{(k-2(\ell+1))e_1}(z) \right) - \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left(\binom{k}{\ell} \widehat{T}_{(k-2\ell)e_1}(z) \right) \right) \\ & \quad + \frac{n}{2^k} \begin{cases} 4 \binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \\ &= \frac{2n}{2^k} \left(4 \sum_{\ell=1}^{\lceil k/2 \rceil - 1} \left(\binom{k-2}{\ell-1} \widehat{T}_{(k-2\ell)e_1}(z) \right) - \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left(\binom{k}{\ell} \widehat{T}_{(k-2\ell)e_1}(z) \right) \right) \\ & \quad + \frac{n}{2^k} \begin{cases} 4 \binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \\ &= \frac{2n}{2^k} \left(-\widehat{T}_{ke_1}(z) + \sum_{\ell=1}^{\lceil k/2 \rceil - 1} \left(4 \binom{k-2}{\ell-1} - \binom{k}{\ell} \right) \widehat{T}_{(k-2\ell)e_1}(z) \right) + \frac{n}{2^k} \begin{cases} 4 \binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}. \end{aligned}$$

Dividing by $2n$ does not change whether H is positive semi-definite in z and so we obtain the formula. \square

Remark 9.4. The matrix polynomial $H \in \mathbb{Q}[z]^{n \times n}$ from Theorem 9.1 follows the pattern

$$\begin{bmatrix} \frac{\widehat{T}_0 - \widehat{T}_{2e_1}}{4} & \frac{\widehat{T}_{e_1} - \widehat{T}_{3e_1}}{8} & \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \dots \\ \frac{\widehat{T}_{e_1} - \widehat{T}_{3e_1}}{8} & \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \dots \\ \frac{\widehat{T}_0 - \widehat{T}_{4e_1}}{16} & \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \frac{5\widehat{T}_{e_1} - \widehat{T}_{3e_1} - 3\widehat{T}_{5e_1} - \widehat{T}_{7e_1}}{128} & \dots \\ \frac{2\widehat{T}_{e_1} - \widehat{T}_{3e_1} - \widehat{T}_{5e_1}}{32} & \frac{2\widehat{T}_0 + \widehat{T}_{2e_1} - 2\widehat{T}_{4e_1} - \widehat{T}_{6e_1}}{64} & \frac{5\widehat{T}_{e_1} - \widehat{T}_{3e_1} - 3\widehat{T}_{5e_1} - \widehat{T}_{7e_1}}{128} & \frac{5\widehat{T}_0 + 4\widehat{T}_{2e_1} - 4\widehat{T}_{4e_1} - 4\widehat{T}_{6e_1} - \widehat{T}_{8e_1}}{256} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

1. H only depends on the \widehat{T}_{α} , where $\alpha \in \mathbb{N}^r$ is a multiple of e_1 . An entry of H is a linear combination of at most $n + 1$ such polynomials. The largest possible degree of an entry is $2n = \deg(\widehat{T}_{2ne_1})$.
2. H has Hankel structure, that is, $H_{ij} = H_{kl}$ whenever $i + j = k + \ell$. Therefore, the number of distinct entries is at most $2n - 1$.
3. Denote by $H^{(n)}$ the matrix H for fixed n . Then the leading principal submatrices of H have the structure of $H^{(k)}$ for $k \leq n$.
4. If \mathbb{R} is of type B_n , C_n or G_2 , then $\mathcal{T}_{\mathbb{R}} = \mathcal{T}$ and $\widehat{T}_{ke_i}(z) = T_{ke_i}(z)$ is the usual generalized Chebyshev polynomial of the first kind from Definition 2.6. The same holds for D_n when n is even. When n is odd on the other hand, then $T_{ke_i}(z) = \widehat{T}_{ke_i}(\tilde{z})$ with $\frac{z_{n-1} + z_n}{2} = \tilde{z}_{n-1}$ and $\frac{z_{n-1} - z_n}{2i} = \tilde{z}_n$. Finally, for A_{n-1} , we have $\frac{T_{ke_i}(z) + T_{-ke_i}(z)}{2} = \frac{T_{ke_i}(z) + T_{ke_{n-i}}(z)}{2} = \widehat{T}_{ke_i}(\tilde{z})$ with $\frac{z_{n-i} + z_i}{2} = \tilde{z}_i$ and $\frac{z_{n-i} - z_i}{2i} = \tilde{z}_{n-i}$.

Conclusion and future work

We have characterized the \mathbb{T} -orbit space of a Weyl group acting nonlinearly on the compact torus as a compact basic semi-algebraic set. The characterization is given through a polynomial matrix inequality and holds for the Weyl groups of type A_{n-1} , B_n , C_n , D_n and G_2 . It is the first explicit characterization for nonlinear actions in the sense that we have given a closed formula for the matrix. By expressing the entries in the basis of generalized Chebyshev polynomials, we have shown that these matrices have a common structure.

For applications in graph theory [BDdV14] and optimization [HMMR23], it is desirable to give such a closed formula also for the remaining cases $E_{6,7,8}$ and F_4 . We will address those in a future work.

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