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Polynomial description for the \mathbb{T} -Orbit Spaces of Multiplicative Actions

(for types \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D})

Evelyne Hubert*, Tobias Metzloff*, Cordian Riener†

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Abstract

A finite group with an integer representation has a multiplicative action on the ring of Laurent polynomials, which is induced by a nonlinear action on the compact torus. We study the structure of the associated orbit space as the image of the fundamental invariants. For the Weyl groups of types \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} , this image is a compact basic semi-algebraic set and we present the defining polynomial inequalities explicitly. We show how orbits correspond to solutions in the compact torus of symmetric polynomial systems and give a characterization of the orbit space as the positivity-locus of a real symmetric matrix polynomial. The resulting domain is the region of orthogonality for two families of generalized Chebyshev polynomials, which have connections to topics such as Fourier analysis and representations of Lie algebras.

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1 Introduction

The set of all orbits of a compact Lie group with an action on an affine variety is called orbit space. The coordinate ring of this affine variety contains the ring of invariants as a finitely generated subalgebra and the orbit space can be embedded into the variety defined by the syzygy ideal of the fundamental invariants. For a group with a linear action on a real representation space, the orbit space is, essentially, a semi-algebraic set and the defining polynomial equations and inequalities are given by a result due to Procesi and Schwarz [PS85, Main Theorem]. Our aim is to find such a polynomial description of the orbit space, when we consider finite groups with an integer representation and a nonlinear action on the compact torus. The purpose of this article is to show that in the case of Weyl groups, the orbit space as the image of the compact torus under fundamental invariants is a compact basic semi-algebraic set, and to characterize it as the positivity-locus of an explicitly known symmetric matrix polynomial. Being able to effectively describe the structure of an orbit space allows for applications in equivariant dynamical systems theory [Gat00, Theorem 4.2.1], symmetry reduction for polynomial optimization [ALRT13, Theorem 5.2], complex analysis [Sja93, Proposition 1.15], differential geometry [Dub98, Lemma 2.1] and quantum systems [GKP13, Section 2].

A finite group with an integer representation \mathcal{G} has a nonlinear action on the algebraic torus, which leaves the maximal compact subgroup, the compact torus \mathbb{T}^n , invariant and induces an action on the ring of Laurent polynomials. Given a set of generators of the ring of invariants, so called fundamental invariants, the image of \mathbb{T}^n under these generators corresponds to the set of orbits \mathbb{T}^n/\mathcal{G} and we call it the \mathbb{T} -orbit space of \mathcal{G} . Such an action is called multiplicative [Lor05] and the invariant theory for this setting is related to representations of complex Lie algebras [Bou68, Hum72, Ser01, FH04]. Recent applications have been studied in the context of Fourier analysis [Bee91, MKNR13], cubature [Xu00, LX10, LSX12, Xu15, CH18] and interpolation [HS21].

For the Weyl groups of types \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} , the ring of invariant Laurent polynomials is a polynomial algebra. In these four cases, we find relations between the elementary symmetric polynomials and the fundamental invariants. This gives rise to a symmetric polynomial system with complex variables. We show that a point in the \mathbb{T} -orbit space corresponds to such a system having all its solutions in \mathbb{T}^n . This property can be characterized by positivity of an explicit Hermite quadratic form.

In Section 2, we formally introduce \mathbb{T} -orbit spaces of multiplicative actions by first defining the action of a group with an integer representation on the ring of Laurent polynomials and then identifying it with the multiplicative action on a group algebra of an invariant lattice. Here, we fix the notation for fundamental invariants and explain the relevance of the four cases \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} . Following with Section 3, we establish the necessary tools to study symmetric polynomial systems, which will later arise from the invariants. This involves Sylvester's characterization of a polynomial having all its roots in a real interval, which was also applied in [Pro78]. For the symmetric polynomial systems, which we shall encounter, we extract Corollaries 3.6 and 3.7.

We then proceed to the main results and give a polynomial description for the \mathbb{T} -orbit space of \mathcal{G} in terms of an explicit symmetric matrix polynomial. In Sections 4 to 7, we work out the correspondence between points in the \mathbb{T} -orbit space and symmetric polynomial systems having all solutions in \mathbb{T}^n . We give explicit formulae for invariants in Propositions 4.1, 5.1, 6.1 and 7.1 and apply the results from Section 3 to obtain Theorems 4.3, 5.3, 6.4 and 7.4. Examples accompany the main results.

Motivated by the characterization due to Procesi and Schwarz [PS85], we formulate an alternative characterization in Section 8 with a proof of the necessary condition. We introduce differentials on the Laurent polynomial ring and state Conjecture 8.5 for groups, which are not necessarily Weyl groups of crystallographic root systems, but have only an integer representation.

The \mathbb{T} -orbit spaces, that we describe here as basic semi-algebraic sets, also appear in other contexts of interest to analysis. We review in Theorem 9.7 that the \mathbb{T} -orbit space is the region of orthogonality for a family of generalized Chebyshev polynomials of the first and second kind.

The following example demonstrates that a nonlinear action on the compact torus admits an orbit space, which needs to be distinguished from the orbit space of a linear action on a real representation space.

Example 1.1.

1. Consider the group \mathfrak{G} of order 8 with presentation $\langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^4 = 1 \rangle$. We fix

$$s_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so that \mathfrak{G} , as a subgroup of the orthogonal group $O_2(\mathbb{R})$, has a linear action on the real representation space \mathbb{R}^2 by matrix multiplication. The \mathfrak{G} -invariant polynomials are those, which are invariant under permutation and sign change of variables, and the ring of invariants is the polynomial algebra

$$\mathbb{Q}[X_1, X_2]^{\mathfrak{G}} = \mathbb{Q}[\pi_1 := X_1^2 + X_2^2, \pi_2 := X_1^2 X_2^2].$$

The image of \mathbb{R}^2 under π_1, π_2 corresponds to the orbit space $\mathbb{R}^2/\mathfrak{G}$. The Gram matrix of the gradients has \mathfrak{G} -invariant entries

$$\langle \nabla \pi_i, \nabla \pi_j \rangle_{ij} = \begin{bmatrix} 4X_1^2 + 4X_2^2 & 8X_1^2 X_2^2 \\ 8X_1^2 X_2^2 & 4X_1^2 X_2^2 (X_1^2 + X_2^2) \end{bmatrix} = \begin{bmatrix} 4\pi_1 & 8\pi_2 \\ 8\pi_2 & 4\pi_1 \pi_2 \end{bmatrix} =: M(\pi_1, \pi_2).$$

By [PS85, Main Theorem], the image of \mathbb{R}^2 under π_1, π_2 is $\{z = (z_1, z_2) \in \mathbb{R}^2 \mid M(z_1, z_2) \succeq 0\}$. Hence, the orbit space $\mathbb{R}^2/\mathfrak{G}$ is identified with a basic semi-algebraic set, that is not compact (Figure 1a).

We will now explain the difference to so called multiplicative actions. The above group \mathfrak{G} is the Weyl group of the B_2 root system with fundamental weights $\binom{1}{0}, \binom{1}{1/2}$. For $W := \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$, the image of the group homomorphism $\mathfrak{G} \rightarrow GL_2(\mathbb{Z}), A \mapsto W^{-1} A W$ is a group denoted \mathcal{G} with generators

$$W^{-1} s_1 W = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad W^{-1} s_2 W = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Instead of \mathbb{R}^2 , we consider the algebraic torus $(\mathbb{C}^*)^2$ and study the nonlinear action

$$\begin{aligned} \mathcal{G} \times (\mathbb{C}^*)^2 &\rightarrow (\mathbb{C}^*)^2, \\ (B, x) &\mapsto x^{B^{-1}} := (x_1^{B_{11}^{-1}} x_2^{B_{21}^{-1}}, x_1^{B_{12}^{-1}} x_2^{B_{22}^{-1}}). \end{aligned}$$

The coordinate ring of the algebraic torus is the bivariate Laurent polynomial ring $\mathbb{Q}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ and the algebraically independent fundamental invariants

$$\theta_1 := \frac{x_1}{4} + \frac{1}{4x_1^{-1}} + \frac{x_2^2}{4x_1} + \frac{x_1}{4x_2^2} \quad \text{and} \quad \theta_2 := \frac{x_2}{4} + \frac{1}{4x_2} + \frac{x_1}{4x_2} + \frac{x_2}{4x_1}$$

generate the ring of \mathcal{G} -invariant Laurent polynomials. We restrict the action to the compact torus \mathbb{T}^2 . In Theorem 2.4, we show that the \mathbb{T} -orbit space \mathbb{T}^2/\mathcal{G} corresponds to the image of \mathbb{T}^2 under θ_1, θ_2 . Furthermore, Theorem 5.3 states that this image is real and consists of all $(z_1, z_2) \in \mathbb{R}^2$, that satisfy $H(z) \succeq 0$ (Figure 1b), where

$$H(z) := \begin{bmatrix} -z_1^2 + 2z_2^2 - z_1 & -4z_1^3 + 12z_1 z_2^2 - 6z_1^2 - 2z_1 \\ -4z_1^3 + 12z_1 z_2^2 - 6z_1^2 - 2z_1 & -16z_1^4 + 64z_1^2 z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1 z_2^2 - 20z_1^2 + 8z_2^2 - 4z_1 \end{bmatrix}.$$

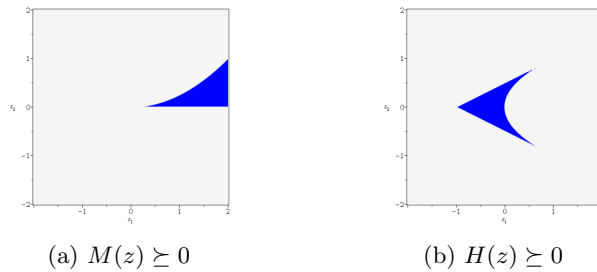


Figure 1: The orbit spaces $\mathbb{R}^2/\mathfrak{G}$ of the linear action and \mathbb{T}^2/\mathcal{G} of the nonlinear action.

The action on the Laurent polynomials is called *multiplicative*, as the weight lattice $\Omega = \langle \binom{1}{0}, \binom{1/2}{1/2} \rangle_{\mathbb{Z}}$ of the B_2 root system is a multiplicative subgroup of the group algebra $\mathbb{Q}[\Omega]$, which we identify with the coordinate ring $\mathbb{Q}[x_1, x_1^{-1}, x_2, x_2^{-1}]$.

2. Further interesting examples are the Weyl groups of the root systems A_3 , B_3 and C_3 . Visualizations of the associated \mathbb{T} -orbit spaces are depicted below.

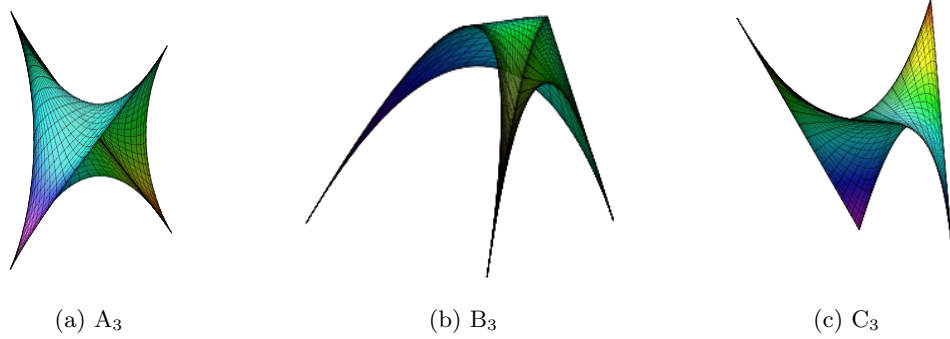


Figure 2: \mathbb{T} -orbit spaces for root systems of rank $n = 3$.

2 Preliminaries

Let $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$ be a field of characteristic 0 and let $n \in \mathbb{N}$.

In this section, we discuss the action of a finite group with a representation in $\mathrm{GL}_n(\mathbb{Z})$ on the ring of Laurent polynomials $\mathbb{K}[x^{\pm}] = \mathbb{K}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, which is identified with the group algebra of a lattice. We motivate the problem of computing the \mathbb{T} -orbit space as the image of the compact n -torus under the fundamental invariants and then specify the problem for Weyl groups of crystallographic root systems.

2.1 Group actions on $\mathbb{K}[x^{\pm}]$

Let \mathfrak{G} be a finite group with an integer representation

$$\rho: \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{Z}). \quad (2.1)$$

Denote by $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$ the algebraic n -torus. For $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$ and a column vector $\alpha = [\alpha_1, \dots, \alpha_n]^t \in \mathbb{Z}^n$, define $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}^*$. The matrix group $\mathcal{G} := \rho(\mathfrak{G})$ has a nonlinear action on $(\mathbb{C}^*)^n$, given by monomial maps

$$\begin{aligned} \star: \mathcal{G} \times (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ (B, x) &\mapsto B \star x := (x_1, \dots, x_n)^{B^{-1}} = (x^{B^{-1}}, \dots, x^{B^{-1}}), \end{aligned} \quad (2.2)$$

where $B_{\cdot i}^{-1} \in \mathbb{Z}^n$ denotes the i -th column vector of $B^{-1} \in \mathcal{G}$ for $1 \leq i \leq n$. The coordinate ring of $(\mathbb{C}^*)^n$ with coefficients in \mathbb{K} is the ring of multivariate Laurent polynomials $\mathbb{K}[x^{\pm}] := \mathbb{K}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. The monomials of $\mathbb{K}[x^{\pm}]$ are all $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}^n$, and \star induces the linear action

$$\begin{aligned} \cdot: \mathcal{G} \times \mathbb{K}[x^{\pm}] &\rightarrow \mathbb{K}[x^{\pm}], \\ (B, x^\alpha) &\mapsto B \cdot x^\alpha := x^{B\alpha} \end{aligned} \quad (2.3)$$

on $\mathbb{K}[x^{\pm}]$. Hence, for $f = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \mathbb{K}[x^{\pm}]$ and $B \in \mathcal{G}$, we write

$$(B \cdot f)(x) = f(x^B) = \sum_{\alpha} f_{\alpha} x^{B\alpha}. \quad (2.4)$$

If $(B \cdot f)(x) = f(x)$ for all $B \in \mathcal{G}$, then f is called \mathcal{G} -invariant. The set of all \mathcal{G} -invariant Laurent polynomials is a finitely generated, but not necessarily polynomial, \mathbb{K} -algebra [Lor05, Corollary 3.3.2] and denoted by $\mathbb{K}[x^\pm]^\mathcal{G}$.

Define $\mathbb{T} := \{x \in \mathbb{C} \mid x\bar{x} = 1\} \subseteq \mathbb{C}^*$, where \bar{x} is the complex conjugate. Note that \mathbb{T} is the maximal compact subgroup of \mathbb{C}^* , which is closed with respect to inversion $x \mapsto 1/x$ and whose elements satisfy $1/x = \bar{x}$. We denote by \mathbb{T}^n the **compact n -torus**.

Lemma 2.1. \mathbb{T}^n is left invariant by the action \star , that is, $\mathcal{G} \star \mathbb{T}^n = \mathbb{T}^n$.

Definition 2.2. Assume that $\mathbb{K}[x^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \dots, \theta_m]$ for some $m \in \mathbb{N}$ and $\theta_1, \dots, \theta_m \in \mathbb{K}[x^\pm]^\mathcal{G}$.

1. The generators $\theta_1, \dots, \theta_m$ are called **fundamental invariants** of \mathcal{G} . We define the map

$$\begin{aligned} \vartheta: \mathbb{T}^n &\rightarrow \mathbb{C}^m, \\ x &\mapsto (\theta_1(x), \dots, \theta_m(x)). \end{aligned}$$

2. For $\alpha \in \mathbb{Z}^n$, we call

$$\Theta_\alpha(x) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B\alpha} \in \mathbb{K}[x^\pm]^\mathcal{G}$$

the **orbit polynomial** associated to α .

A basis for $\mathbb{K}[x^\pm]^\mathcal{G}$ as a \mathbb{K} -vector space is given by $\{\Theta_\alpha \mid \alpha \in \mathbb{Z}^n/\mathcal{G}\}$ [Lor05, Equation (3.4)].

Lemma 2.3. The image $\text{im}(\vartheta)$ of ϑ is compact in \mathbb{C}^m .

Proof. $\text{im}(\vartheta) \subseteq \mathbb{C}^m$ is compact as the image of the compact set \mathbb{T}^n under a continuous map. □

Our goal is to describe with the fundamental invariants of $\mathbb{K}[x^\pm]^\mathcal{G}$ the set of orbits \mathbb{T}^n/\mathcal{G} for the nonlinear action \star . Thanks to the following statement, this is achievable.

Theorem 2.4. The map

$$\begin{aligned} \mathbb{T}^n/\mathcal{G} &\rightarrow \text{im}(\vartheta), \\ \mathcal{G} \star x &\mapsto \vartheta(x) \end{aligned}$$

is well defined and bijective.

Proof. We follow the proof of [CLO15, Chapter 7, §4, Theorem 10]. For $x, y \in \mathbb{T}^n$ with $\mathcal{G} \star x = \mathcal{G} \star y$, we have $\theta_i(x) = \theta_i(y)$ for all $1 \leq i \leq n$ by definition. Therefore, the map is well defined and surjective.

For injectiveness, assume that $x, y \in \mathbb{T}^n$ with $\mathcal{G} \star x \cap \mathcal{G} \star y = \emptyset$. Define the set $X := \mathcal{G} \star x \cup \mathcal{G} \star y \setminus \{y\} \subseteq \mathbb{T}^n$. Since \mathcal{G} is finite, X is finite and there exists $\tilde{f} \in \mathbb{C}[x_1, \dots, x_n]$ with $\tilde{f}(X) = \{0\}$, $\tilde{f}(y) \neq 0$. For example

$$\tilde{f} = \prod_{x' \in X} \prod_{i=1}^n (x_i - x'_i)$$

has the desired property. Consider $\mathbb{C}[x^\pm]$ as a ring extension of $\mathbb{C}[x_1, \dots, x_n]$ and define

$$f := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} B \cdot \tilde{f} \in \mathbb{C}[x^\pm]^\mathcal{G}.$$

Then $f(x) = 0$ and $f(y) = |\text{Stab}_\mathcal{G}(y)|/|\mathcal{G}| \tilde{f}(y) \neq 0$ by definition, where $\text{Stab}_\mathcal{G}(y)$ denotes the stabilizer subgroup of y in \mathcal{G} . Since $\mathbb{K}[x^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \dots, \theta_m]$ and $\mathbb{K} \subseteq \mathbb{C}$ is a field extension, we have $\mathbb{C}[x^\pm]^\mathcal{G} = \mathbb{C}[\theta_1, \dots, \theta_m]$. With $f(x) \neq f(y)$, we obtain $\theta_i(x) \neq \theta_i(y)$ for some $1 \leq i \leq m$ and thus $\vartheta(x) \neq \vartheta(y)$. □

We conclude that there is a one-to-one correspondence between the orbit space \mathbb{T}^n/\mathcal{G} and the image of ϑ . In other words, ϑ separates the orbits of the action of \mathcal{G} by \star on \mathbb{T}^n .

Definition 2.5. We call $\mathcal{T} := \text{im}(\vartheta)$ the \mathbb{T} -orbit space of \mathcal{G} .

The following statement admits a recurrence formula to iteratively compute orbit polynomials and we make use of it in the proofs of [Lemmas 6.2](#) and [7.2](#).

Proposition 2.6. Let $\alpha, \beta \in \mathbb{N}^n$. We have $\Theta_0 = 1$ and

$$|\mathcal{G}\beta| \Theta_\alpha \Theta_\beta = \sum_{\tilde{\beta} \in \mathcal{G}\beta} \Theta_{\alpha+\tilde{\beta}}.$$

2.2 Multiplicative actions and Weyl groups

Let $\Omega \subseteq \mathbb{R}^n$ be an n -dimensional lattice and

$$\pi : \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{R}) \tag{2.5}$$

a real representation of \mathfrak{G} , such that Ω is left invariant by the linear action of $\pi(\mathfrak{G})$ on \mathbb{R}^n by matrix multiplication. For a fixed basis $\{\omega_1, \dots, \omega_n\}$ of Ω , an isomorphism of \mathbb{Z} -modules $\Omega \cong \mathbb{Z}^n$ is

$$\begin{aligned} \mathbb{Z}^n &\rightarrow \Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n, \\ [\alpha_1, \dots, \alpha_n]^t &\mapsto W\alpha = \alpha_1\omega_1 + \dots + \alpha_n\omega_n, \end{aligned} \tag{2.6}$$

where $W \in \mathbb{R}^{n \times n}$ is the matrix with columns $\omega_1, \dots, \omega_n$. We obtain a \mathbb{K} -algebra isomorphism between the group algebra $\mathbb{K}[\Omega]$ with basis $\{\mathbf{e}^\mu \mid \mu \in \Omega\}$ [[Bou75](#), Chapitre VIII, §9.1] and the ring of Laurent polynomials $\mathbb{K}[x^\pm]$ with basis $\{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$, given by

$$\begin{aligned} \mathbb{K}[x^\pm] &\rightarrow \mathbb{K}[\Omega], \\ x^\alpha &\mapsto \mathbf{e}^{\alpha_1\omega_1 + \dots + \alpha_n\omega_n}. \end{aligned} \tag{2.7}$$

Since Ω is left invariant, there exists a representation $\varrho : \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{Z})$, such that $\varrho(g) = W^{-1}\pi(g)W$ for all $g \in \mathfrak{G}$. Then $\mathbb{K}[\Omega]^\mathfrak{G} \cong \mathbb{K}[x^\pm]^\mathfrak{G}$ is finitely generated as a ring. The action of \mathfrak{G} on $\mathbb{K}[\Omega]$ is called **multiplicative**, as the basis $\{\mathbf{e}^\mu \mid \mu \in \Omega\}$ of $\mathbb{K}[\Omega]$ is a multiplicative subgroup of the units in $\mathbb{K}[\Omega]$.

For $\mu = W\alpha \in \Omega$, we say that the orbit polynomial Θ_α is associated to μ . From now on, we denote by $\theta_i := \Theta_{e_i}$ the orbit polynomial associated to $e_i \in \mathbb{Z}^n$, which corresponds to ω_i for $1 \leq i \leq n$.

We deal with lattices, which are obtained from root systems. Therefore, we recall the following definitions.

Definition 2.7. Let V be an n -dimensional \mathbb{R} -vector space with inner product $\langle \cdot, \cdot \rangle$ and R be a finite subset of V . We say that R is a **root system** in V , if the following conditions hold.

1. R spans V and does not contain 0.
2. If $\rho, \tilde{\rho} \in R$, then $s_\rho(\tilde{\rho}) \in R$, where s_ρ is the reflection defined by $s_\rho(u) = u - 2\frac{\langle u, \rho \rangle}{\langle \rho, \rho \rangle}\rho$, $u \in V$.
3. For all $\rho, \tilde{\rho} \in R$, $2\frac{\langle \tilde{\rho}, \rho \rangle}{\langle \rho, \rho \rangle} \in \mathbb{Z}$.
4. For $\rho \in R$ and $c \in \mathbb{R}$, $c\rho \in R$ if and only if $c = \pm 1$.

In many texts, a root system is defined only using the first three of the above conditions and if the fourth condition holds, one speaks of a “reduced” root system. Since we study the group generated by the reflections s_ρ in this article, we only need to deal with such reduced root systems thanks to [[Bou68](#), Chapitre VI, §1,

Proposition 13] and may include the fourth condition in our definition. Less common is also to define a root system without the second “crystallographic” property [Kan01]. The element

$$\rho^\vee = 2 \frac{\rho}{\langle \rho, \rho \rangle}$$

that appears in the definition of the reflection s_ρ is called the **coroot** of $\rho \in R$.

Definition 2.8. *The **Weyl group** of a root system R in V is the subgroup of the orthogonal group, with respect to the inner product $\langle \cdot, \cdot \rangle$, generated by the reflections s_ρ for $\rho \in R$.*

Definition 2.9. *Let R be a root system in V .*

1. A subset $B = \{\rho_1, \dots, \rho_n\}$ of R is a **base** if the following conditions hold.

(a) B is a basis of the vector space V .

(b) Every root $\rho \in R$ can be written as $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$ or $\rho = -\alpha_1 \rho_1 - \dots - \alpha_n \rho_n$ for some $\alpha \in \mathbb{N}^n$.

The elements of B are called **simple roots**.

2. If B is a base, the roots of the form $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$ for $\alpha \in \mathbb{N}^n$ are called the **positive roots** and the set of all positive roots is denoted by R^+ .

3. R contains a unique positive root ρ_0 with maximal coefficients $\alpha \in \mathbb{N}^n$, called the **highest root**.

Existence of the highest root follows from [Bou68, Chapitre VI, §1, Proposition 25]. The Weyl group is generated by the reflections associated to the simple roots [Hum72, Chapter III, §10.3].

The **fundamental Weyl chamber** in V relative to the base $B = \{\rho_1, \dots, \rho_n\}$ is $\mathbb{M} := \{u \in V \mid \langle u, \rho_i \rangle > 0\}$. The closure of \mathbb{M} in V is a fundamental domain for the Weyl group of R [Bou68, Chapitre V, §3, Théorème 2].

A root system defines the following lattice in V . This lattice and related concepts play an important role in the representation theory of semi-simple Lie algebras [Hum72, Ser01, FH04].

Definition 2.10. *Let $B = \{\rho_1, \dots, \rho_n\}$ be a base of R and $B^\vee = \{\rho_1^\vee, \dots, \rho_n^\vee\}$ its dual.*

1. An element μ of V is called a **weight** if

$$\langle \mu, \rho_i^\vee \rangle \in \mathbb{Z}$$

for $1 \leq i \leq n$. The set of weights forms a lattice called the **weight lattice**.

2. The **fundamental weights** are the elements $\{\omega_1, \dots, \omega_n\}$ such that $\langle \omega_i, \rho_j^\vee \rangle = \delta_{i,j}$, $1 \leq i, j \leq n$.

3. A weight μ is **strongly dominant** if $\langle \mu, \rho_i \rangle > 0$ for all $\rho_i \in B$. A weight μ is **dominant** if $\langle \mu, \rho_i \rangle \geq 0$ for all $\rho_i \in B$.

The weight lattice is left invariant under the Weyl group. Note that the set of strongly dominant weights is contained in \mathbb{M} and the set of dominant weights is contained in the closure of \mathbb{M} . The fundamental weights lie on the boundary or “walls” of \mathbb{M} .

For Weyl groups and weight lattices, we have the following specific result about the ring of invariants.

Theorem 2.11. [Bou68, Chapitre VI, §3, Théorème 1] *Let R be a root system. Assume that Ω is the weight lattice and \mathfrak{G} the Weyl group with integer representation \mathcal{G} . Then $\theta_1, \dots, \theta_n$ are algebraically independent and $\mathbb{K}[x^\pm]^\mathcal{G}$ is the polynomial algebra*

$$\mathbb{K}[x^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \dots, \theta_n].$$

The converse “ $\mathbb{K}[x^\pm]^\mathcal{G}$ polynomial algebra $\Rightarrow \mathfrak{G}$ Weyl group” of [Theorem 2.11](#) is also true [[Far86](#)]. The above theorem implies that the fundamental invariants of \mathcal{G} are orbit polynomials as in [Definition 2.2](#) and that $m = n$. Therefore, \mathbb{T} -orbit spaces satisfy the next property.

Lemma 2.12. *Under the assumptions of [Theorem 2.11](#), the following statements hold.*

1. For $z = (z_1, \dots, z_n) \in \mathcal{T}$, we have $|z_i| \leq 1$ for $1 \leq i \leq n$.
2. If $-I_n \in \mathcal{G}$, then $\mathcal{T} \subseteq [-1, 1]^n \subseteq \mathbb{R}^n$.

Proof. Let $z = \vartheta(x)$ for some $x \in \mathbb{T}^n$. Then $|z_i| = |\theta_i(x)| \leq \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} |x^{Be_i}| = 1$.

Furthermore, if $-I_n \in \mathcal{G}$ we have $\overline{\theta_i(x)} = \theta_i(x^{-I_n}) = \theta_i(x) \in \mathbb{R}$ for all $x \in \mathbb{T}$. Hence by the first statement, \mathcal{T} is contained in the cube $[-1, 1]^n$. \square

Proposition 2.13. *Under the assumptions of [Theorem 2.11](#), there exists a permutation $\sigma \in \mathfrak{S}_n$ of order 2, such that $\theta_i(x^{-I_n}) = \theta_{\sigma(i)}(x)$ for all $1 \leq i \leq n$.*

Proof. Let $B = \{\rho_1, \dots, \rho_n\}$ be a base with fundamental weights $\omega_1, \dots, \omega_n$. We make use of [Equation \(2.6\)](#) and prove that there exists a permutation $\sigma \in \mathfrak{S}_n$, such that $-\omega_{\sigma(i)} \in \mathfrak{G}\omega_i$ for all $1 \leq i \leq n$.

By [[Bou68](#), Chapitre VI, §1 Theorem 2], there exist $A \in \mathfrak{G}, \sigma \in \mathfrak{S}_n$ with $A\rho_i^\vee = -\rho_{\sigma(i)}^\vee$ for all $1 \leq i \leq n$. We have

$$\langle -\omega_{\sigma(i)}, -\rho_{\sigma(j)}^\vee \rangle = \delta_{ij} = \langle \omega_i, \rho_j^\vee \rangle = \langle A\omega_i, A\rho_j^\vee \rangle = \langle A\omega_i, -\rho_{\sigma(j)}^\vee \rangle,$$

because σ is a permutation and the inner product is \mathfrak{G} -invariant. Since $-B^\vee$ is a basis of V , $A\omega_i = -\omega_{\sigma(i)}$. Especially, $-\omega_{\sigma(i)} \in \mathfrak{G}\omega_i \cap \mathfrak{G}\omega_{\sigma^2(i)}$ and so $\mathfrak{G}\omega_i = \mathfrak{G}\omega_{\sigma^2(i)}$ for all $1 \leq i \leq n$. As the closure of \mathbb{M} is a fundamental domain for \mathfrak{G} and $\omega_i, \omega_{\sigma^2(i)}$ lie on the walls of \mathbb{M} , we have $\sigma^2 = 1$. \square

To describe the \mathbb{T} -orbit space of \mathcal{G} as a basic semi-algebraic set in \mathbb{R}^n , we can now make the following adjustment for $1 \leq i \leq n$. For $i = \sigma(i)$, we leave the i -th coordinate of ϑ as it is. For $i < \sigma(i)$, replace the i -th and $\sigma(i)$ -th coordinate of ϑ by $\theta_{i,\mathbb{R}} := (\theta_i + \theta_{\sigma(i)})/2$ and $\theta_{\sigma(i),\mathbb{R}} := (\theta_i - \theta_{\sigma(i)})/(2i)$. The resulting map

$$\begin{aligned} \vartheta_{\mathbb{R}} : \mathbb{T}^n &\rightarrow \mathbb{R}^n, \\ x &\mapsto (\theta_{1,\mathbb{R}}(x), \dots, \theta_{n,\mathbb{R}}(x)) \end{aligned} \tag{2.8}$$

has image $\mathcal{T}_{\mathbb{R}} \subseteq [-1, 1]^n$.

2.3 Contribution

For a finite group with an integer representation, the natural problem we pursue is to describe the image of the compact n -torus \mathbb{T}^n under the fundamental invariants in terms of polynomial inequalities. As seen in [Theorem 2.4](#), this image represents the orbit space \mathbb{T}^n/\mathcal{G} . The main results of this article do this explicitly for Weyl groups of type $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} . We show the existence of an n -dimensional \mathbb{R} -vector space $\mathcal{Z} \subseteq \mathbb{C}^n$ containing \mathcal{T} and a symmetric matrix polynomial $H \in \mathbb{K}[z]^{n \times n}$ with the following property: For all $z \in \mathcal{Z}$, $H(z)$ has real entries and z is contained in the \mathbb{T} -orbit space \mathcal{T} of \mathfrak{G} if and only if $H(z)$ is positive semi-definite. The reason that we specifically consider the Weyl groups of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ is the following.

A root system can be decomposed into irreducible components [[Bou68](#), Chapitre VI, §1, Proposition 6]. This leaves us with nine families, which are denoted by A_{n-1}, B_n, C_n ($n \geq 2$), D_n ($n \geq 4$) for fixed $n \in \mathbb{N}$, as well as E_6, E_7, E_8, F_4 and G_2 [[Bou68](#), Chapitre VI, §4, Théorème 3]. In this article, we study the four infinite families A_{n-1}, C_n, B_n and D_n in [Sections 4 to 7](#) for arbitrary n in this order. The associated Weyl groups are $\mathcal{A}_{n-1}, \mathcal{C}_n, \mathcal{B}_n$ and \mathcal{D}_n . For these cases, we present an explicit matrix polynomials $H \in \mathbb{K}[z]$ in [Theorems 4.3, 5.3, 6.4](#) and [7.4](#).

For arbitrary groups with an integer representation, we present a conjecture in [Section 8](#).

Convention. We work over the field $\mathbb{K} = \mathbb{Q}$. A root system in $V \subseteq \mathbb{R}^n$ is crystallographic, reduced, and irreducible. Its Weyl group \mathfrak{G} is a finite subgroup of $O_n(\mathbb{R})$. Stabilizer subgroups of \mathfrak{G} , respectively its integer representation \mathcal{G} , are denoted by $\text{Stab}_{\mathfrak{G}}(\dots)$, respectively $\text{Stab}_{\mathcal{G}}(\dots)$, where the action by either matrix multiplication, [Equation \(2.2\)](#), or [Equation \(2.3\)](#), is evident from the argument.

3 Symmetric polynomial systems

The four Weyl groups \mathcal{A}_{n-1} , \mathcal{B}_n , \mathcal{C}_n and \mathcal{D}_n each contain a subgroup isomorphic to the symmetric group \mathfrak{S}_n and the formulae we give for some specific invariants involve the elementary symmetric polynomials. This gives rise to a symmetric polynomial system. In this section, we give a characterization for all solutions of such a system to be contained in the compact torus \mathbb{T}^n .

3.1 Solutions in \mathbb{T}^n

For $1 \leq i \leq n$, the polynomial

$$\sigma_i(y_1, \dots, y_n) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} y_j \in \mathbb{Q}[y_1, \dots, y_n] \quad (3.1)$$

is called the i -th elementary symmetric polynomials in n indeterminates. We shall be confronted with the following two types of polynomial systems in unknown y_1, \dots, y_n .

$$\begin{aligned} \text{(I)} \quad \sigma_i(y_1, \dots, y_n) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{C} \\ \text{(II)} \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{R} \end{aligned}$$

The goal of this section is to determine, whether all solutions $y = (y_1, \dots, y_n)$ of system (I), respectively system (II), are contained in \mathbb{T}^n . Recall Vieta's formula

$$\prod_{k=1}^n (x - r_k) = x^n + \sum_{i=1}^n (-1)^i \sigma_i(r_1, \dots, r_n) x^{n-i} \in \mathbb{C}[x] \quad (3.2)$$

for $r_1, \dots, r_n \in \mathbb{C}$.

Lemma 3.1.

1. System (I) always has a solution in \mathbb{C}^n . It is unique up to permutation of coordinates.
2. The map

$$\begin{aligned} \mathbb{C}^* &\rightarrow \mathbb{C}, \\ x &\mapsto (x + x^{-1}) \end{aligned}$$

is surjective and the preimage of $[-2, 2]$ is \mathbb{T} .

3. System (II) always has a solution in $(\mathbb{C}^*)^n$. It is unique up to permutation and inversion of coordinates.

Proof. 1. By [Equation \(3.2\)](#), a solution of system (I) is the vector of roots of a polynomial with coefficients given by the right hand side of system (I). Such a polynomial always has n unique roots in \mathbb{C} .

2. For $r \in \mathbb{C}$, consider the polynomial $p := x^2 - rx + 1 \in \mathbb{C}[x]$. Then 0 is not a root of p and $p = 0$ if and only if $r = x + x^{-1}$, that is r is in the image of the map. If $r \in [-2, 2]$, then $p \in \mathbb{R}[x]$ has discriminant

$(r/2)^2 - 1 \leq 0$ and its two roots are $x, \bar{x} = x, x^{-1} = r/2 \pm i\sqrt{1 - (r/2)^2} \in \mathbb{T}$. On the other hand for $x \in \mathbb{T}$, $x + x^{-1} = x + \bar{x} = 2\Re(x) \in [-2, 2]$.

3. By the second statement, we can write the roots r_1, \dots, r_n of the polynomial with coefficients given by the right hand side of system (II) as $r_i = y_i + y_i^{-1}$ for some $y \in (\mathbb{C}^*)^n$. Then y is a unique solution of (II) up to permutation and inversion. \square

From now on, we speak of “the” solution of system (I), respectively (II).

Proposition 3.2. *For $c_1, \dots, c_n \in \mathbb{R}$, the solution of system (II) is contained in \mathbb{T}^n if and only if all the roots of the univariate polynomial*

$$x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{R}[x]$$

are contained in $[-2, 2]$.

Proof. Let $p := x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{R}[x]$ with roots $r_1, \dots, r_n \in \mathbb{C}$. If $y \in (\mathbb{C}^*)^n$ is the solution of system (II), then for all $1 \leq i \leq n$, $y_i + y_i^{-1}$ is a root of p by Equation (3.2). Applying Lemma 3.1 yields $y \in \mathbb{T}^n$ if and only if $r_1, \dots, r_n \in [-2, 2]$. \square

The Chebyshev polynomial of the first kind associated to $\ell \in \mathbb{N}$ is the unique univariate polynomial T_ℓ with $T_\ell((x+x^{-1})/2) = (x^\ell + x^{-\ell})/2$. The set $\{T_\ell \mid \ell \in \mathbb{N}\}$ is a basis of $\mathbb{R}[x]$ as an \mathbb{R} -vector space. For $0 \neq p \in \mathbb{R}[x]$ or $\mathbb{C}[x]$, denote by $\text{Coeff}(x^\ell, p)$ the coefficient of the monomial x^ℓ in p for $0 \leq \ell \leq \deg(p)$.

Proposition 3.3. *For $c_1, \dots, c_{n-1} \in \mathbb{C}$ with $\bar{c}_i = (-1)^n c_{n-i}$ and $c_0 := (-1)^n c_n := 1$, the solution of system (I) is contained in \mathbb{T}^n if and only if all the roots of the univariate polynomial*

$$T_n(x) + d_1 T_{n-1}(x) + \dots + d_{n-1} T_1(x) + \frac{d_n}{2} T_0(x) \in \mathbb{R}[x] \quad \text{with} \quad d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

are contained in $[-1, 1]$.

Proof. By Equation (3.2), the solution of system (I) is contained in \mathbb{T}^n if and only if all the roots of the polynomial $p := x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{C}[x]$ are contained in \mathbb{T} .

Set $\tilde{p} := x^n + \bar{c}_1 x^{n-1} + \dots + \bar{c}_n \in \mathbb{C}[x]$. The roots of p are nonzero, because $p(0) = c_n = (-1)^n \neq 0$. Since $\tilde{p}(x) = (-x)^n p(x^{-1})$, the roots of $p\tilde{p} \in \mathbb{C}[x]$ are the union of the roots of p and their inverses. Especially, all the roots of $p\tilde{p}$ are contained in \mathbb{T} if and only if the roots of p are. The coefficients of $p\tilde{p}$ satisfy

$$\text{Coeff}(x^\ell, p\tilde{p}) = \sum_{i=0}^{\ell} c_{n-i} \bar{c}_{n-\ell+i} = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} = \begin{cases} \sum_{i=0}^{\ell} \text{Coeff}(x^{n-i}, \tilde{p}) \text{Coeff}(x^{n-\ell+i}, p) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}) \\ \sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \underbrace{(\bar{c}_i c_{\ell-i} + \bar{c}_{\ell-i} c_i)}_{\in \mathbb{R}} + \begin{cases} c_{\ell/2} \bar{c}_{\ell/2}, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases} \end{cases} \in \mathbb{R}$$

for $0 \leq \ell \leq n$. Thus, $\text{Coeff}(x^\ell, p\tilde{p}) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}) = d_\ell \in \mathbb{R}$ and we can write

$$p\tilde{p} = \sum_{\ell=1}^n d_{n-\ell} (x^{n+\ell} + x^{n-\ell}) + d_n = 2x^n \left(\sum_{\ell=1}^n d_{n-\ell} T_\ell \left(\frac{x+x^{-1}}{2} \right) + \frac{d_n}{2} T_0 \left(\frac{x+x^{-1}}{2} \right) \right) =: 2x^n g \left(\frac{x+x^{-1}}{2} \right).$$

By Lemma 3.1, we have $x \in \mathbb{T}$ is a root of $p\tilde{p}$ if and only if $(x+x^{-1})/2 \in [-1, 1]$ is a root of g . \square

3.2 Characterization of real roots via Hermite quadratic forms

Let $p, q \in \mathbb{R}[x]$ be monic univariate polynomials. The multiplication by q in the \mathbb{R} -algebra $\mathbb{R}[x]/\langle p \rangle$ is the homomorphism

$$m_q : \mathbb{R}[x]/\langle p \rangle \rightarrow \mathbb{R}[x]/\langle p \rangle, \quad (3.3)$$

$$[f] \mapsto [qf].$$

By [CLO05, Chapter 2, Proposition 4.2], we have for $q_1, q_2 \in \mathbb{R}[x]$

$$m_{q_1+q_2} = m_{q_1} + m_{q_2} \quad \text{and} \quad m_{q_1 q_2} = m_{q_1} \circ m_{q_2}. \quad (3.4)$$

If $p = x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$, then the matrix of m_x in the basis $\{1, x, \dots, x^{n-1}\}$ of $\mathbb{R}[x]/\langle p \rangle$ is the companion matrix

$$\begin{bmatrix} 0 & 0 & \dots & -c_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & & 1 & -c_1 \end{bmatrix} \quad (3.5)$$

of p , since $x x^i = x^{i+1}$ for $0 \leq i \leq n-2$ and $x x^{n-1} = x^n \equiv -c_1 x^{n-1} - \dots - c_{n-1} x - c_n \pmod{\langle p \rangle}$. On the other hand, the univariate Chebyshev polynomials of the first kind up to degree $n-1$ also form a basis $\{T_0, T_1, \dots, T_{n-1}\}$ of $\mathbb{R}[x]/\langle p \rangle$. If $p = T_n + d_1 T_{n-1} + \dots + d_{n-1} T_1 + d_n/2 T_0 \in \mathbb{R}[x]$ and $n \geq 3$, then the matrix of m_x in this basis is

$$\begin{bmatrix} 0 & 1/2 & & & 0 & -d_n/4 \\ 1 & 0 & \ddots & & & -d_{n-1}/2 \\ & 1/2 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3/2 \\ & & & \ddots & 0 & (1-d_2)/2 \\ 0 & & & & 1/2 & -d_1/2 \end{bmatrix}, \quad (3.6)$$

where the rows and columns are indexed by T_0, T_1, \dots, T_{n-1} . The entries in the columns originate from the recurrence formula $x T_0 = T_1$ and $2x T_j = T_{j+1} + T_{j-1}$ for $1 \leq j \leq n-1$. Especially,

$$2x T_{n-1} = T_n + T_{n-2} \equiv -d_1 T_{n-1} + (1-d_2) T_{n-2} - d_3 T_{n-3} - \dots - d_{n-1} T_1 - d_n/2 T_0 \pmod{\langle p \rangle} \quad (3.7)$$

yields the last column.

Example 3.4. Let $p = x^3 - c x^2 + \bar{c} x - 1 \in \mathbb{C}[x]$ with $c \in \mathbb{C}$ and roots $r_1, r_2, r_3 \in \mathbb{T}$, such that $r_1 \cdot r_2 \cdot r_3 = 1$. Following the proof of Proposition 3.3, we consider the palindromic polynomial $p\bar{p} \in \mathbb{R}[x]$ with

$$\begin{aligned} & \frac{1}{2x^3} p\bar{p} \\ &= \frac{1}{2x^3} \left((x^6 + 1) - (c + \bar{c})(x^5 + x) + (c\bar{c} + c + \bar{c})(x^4 + x^2) - \frac{c^2 + \bar{c}^2 + 2}{2} x^3 \right) \\ &= \frac{x^3 + x^{-3}}{2} - (c + \bar{c}) \frac{x^2 + x^{-2}}{2} + (c\bar{c} + c + \bar{c}) \frac{x + x^{-1}}{2} - \frac{c^2 + \bar{c}^2 + 2}{2} \\ &= T_3 \left(\frac{x + x^{-1}}{2} \right) - (c + \bar{c}) T_2 \left(\frac{x + x^{-1}}{2} \right) + (c\bar{c} + c + \bar{c}) T_1 \left(\frac{x + x^{-1}}{2} \right) - \frac{c^2 + \bar{c}^2 + 2}{2} T_0 \left(\frac{x + x^{-1}}{2} \right) \\ &= g \left(\frac{x + x^{-1}}{2} \right). \end{aligned}$$

The matrix of the multiplication by x in $\mathbb{R}[x]/\langle g \rangle$ in the basis of Chebyshev polynomials is

$$\begin{bmatrix} 0 & 1/2 & (c^2 + \bar{c}^2 + 2)/4 \\ 1 & 0 & (1 - c\bar{c} - c - \bar{c})/2 \\ 0 & 1/2 & (c + \bar{c})/2 \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$

To characterize univariate polynomial with roots in $[-2, 2]$, respectively $[-1, 1]$, we use the next statement, which is an application of Sturm's version for Sylvester's theorem.

Theorem 3.5. *Let $p \in \mathbb{R}[x]$ be a monic univariate polynomial of degree n and let m_x be the associated multiplication by x in $\mathbb{R}[x]/\langle p \rangle$. For $0 < a \in \mathbb{R}$, all the roots of p are contained in $[-a, a]$ if and only if the Hermite quadratic form*

$$H_q(p) : \mathbb{R}[x]/\langle p \rangle \rightarrow \mathbb{R}, [f] \mapsto \text{Tr}(m_q f^2), \quad \text{with } q = (a - x)(a + x) \in \mathbb{R}[x],$$

is positive semi-definite.

Proof. Let $H \in \mathbb{R}^{n \times n}$ be the symmetric matrix associated to $H_q(p)$ for a fixed basis of $\mathbb{R}[x]/\langle p \rangle$. Denote by N_+ , respectively N_- , the number of strictly positive, respectively negative, eigenvalues of H , counting multiplicities. By [BPR06, Theorem 4.57] or also [CLO05, Chapter 2, Theorem 5.2], rank and signature of $H_q(p)$ are

$$\begin{aligned} N_+ + N_- &= \text{Rank}(H_q(p)) = |\{x \in \mathbb{C} \mid p(x) = 0, q(x) \neq 0\}|, \\ N_+ - N_- &= \text{Sign}(H_q(p)) = \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) > 0\}|}_{=: n_+} - \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) < 0\}|}_{=: n_-}. \end{aligned}$$

If all roots of p are contained in $[-a, a]$, then $n_- = 0$ and thus $N_+ + N_- = \text{Rank}(H_q(p)) = \text{Sign}(H_q(p)) = n_+ = N_+ - N_-$. Therefore, $N_- = 0$ and all eigenvalues of H are nonnegative, that is, $H_q(p)$ is positive semi-definite.

For the converse, assume that $H_q(p)$ is positive semi-definite. Then $N_- = 0$ and $N_+ = \text{Sign}(H_q(p)) = n_+ - n_- \leq \text{Rank}(H_q(p)) = N_+$, that is, $n_+ - n_- = \text{Rank}(H_q(p))$. On the other hand, $\text{Rank}(H_q(p)) \geq n_+ + n_-$. Hence, $n_- = 0$ and $\text{Rank}(H_q(p)) = n_+$ implies that all the roots of p are real and contained in $[-a, a]$. \square

Note that the matrix H in the proof of [Theorem 3.5](#) does not depend on the choice of a basis for $\mathbb{R}[x]/\langle p \rangle$, as the trace is basis-invariant. As corollaries, we obtain explicit characterizations for solutions of symmetric polynomial systems in \mathbb{T}^n .

Corollary 3.6. *Let $n \geq 3$. For $c_1, \dots, c_n \in \mathbb{C}$ with $\bar{c}_i = (-1)^n c_{n-i}$ for $1 \leq i \leq n-1$ and $c_0 := (-1)^n c_n := 1$, the solution of system (I) is contained in \mathbb{T}^n if and only if the matrix $H \in \mathbb{R}^{n \times n}$ with entries*

$$H_{ij} = \text{Tr}(C^{i+j-2} - C^{i+j}), \quad \text{where}$$

$$C = \begin{bmatrix} 0 & 1/2 & & & 0 & -d_n/4 \\ 1 & 0 & \ddots & & & -d_{n-1}/2 \\ & 1/2 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3/2 \\ & & & \ddots & 0 & (1 - d_2)/2 \\ 0 & & & & 1/2 & -d_1/2 \end{bmatrix} \in \mathbb{R}^{n \times n} \quad \text{and} \quad d_\ell = \sum_{i=0}^{\ell} c_i \bar{c}_{\ell-i} \quad \text{for } 1 \leq \ell \leq n,$$

is positive semi-definite.

Proof. H is the matrix associated to the Hermite quadratic form $H_q(p)$ from [Theorem 3.5](#) with $q = 1 - x^2$ in the basis $\{1, x, x^2, \dots, x^{n-1}\}$. Indeed, by [Equation \(3.4\)](#), the entries of the associated matrix are

$$\mathrm{Tr}(m_q x^{i-1} x^{j-1}) = \mathrm{Tr}(m_{x^{i+j-2-x^{i+j}}}) = \mathrm{Tr}(m_x^{i+j-2} - m_x^{i+j})$$

for $1 \leq i, j \leq n$. Since the trace is independent of the basis for $\mathbb{R}[x]/\langle p \rangle$, we can consider the matrix of m_x in the basis of univariate Chebyshev polynomials of the first kind, which according to [Equation \(3.6\)](#) is C . The statement now follows from [Proposition 3.3](#). \square

Analogously, we obtain the following statement for systems of type (II).

Corollary 3.7. *For $c_1, \dots, c_n \in \mathbb{R}$, the solution of system (II) is contained in \mathbb{T}^n if and only if the matrix $H \in \mathbb{R}^{n \times n}$ with entries*

$$H_{ij} = \mathrm{Tr}(4C^{i+j-2} - C^{i+j}), \quad \text{where}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & -c_n \\ 1 & & 0 & -c_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

is positive semi-definite.

4 Type \mathcal{A}

In [Sections 4 to 7](#) we study the Weyl groups \mathfrak{G} of types \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} . First, we give an explicit formula for some of the orbit polynomials in terms of the elementary symmetric polynomials. This induces a symmetric system of type (I), respectively (II). Afterwards, we present an explicit symmetric matrix polynomial H to characterize the solution of this symmetric system in \mathbb{T}^n . Such a solution then corresponds to a point in \mathcal{T} , the image of ϑ , representing the \mathbb{T} -orbit space. An example to visualize the 2-dimensional case accompanies the main result of each section, except \mathcal{D} .

Convention. We abuse notation and speak of “the orbit polynomial Θ_μ associated to $\mu = W\alpha \in \Omega$ ” with $\alpha \in \mathbb{Z}^n$, to refer to the orbit polynomial Θ_α associated to α . Furthermore for $f \in \mathbb{Q}[x^\pm]$ and $\mathfrak{G} \in \{\mathcal{A}_{n-1}, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n\}$, we write $\mathfrak{G} \cdot f$ for the orbit $\mathcal{G} \cdot f$ of f with respect to the action from [Equation \(2.3\)](#).

For \mathcal{A}_{n-1} , we shall encounter a system of type (I) in n indeterminates. To be consistent with the notation of [Section 3](#), we consider \mathcal{A}_{n-1} as a root system of rank $n - 1$. In particular, we denote the Laurent polynomial ring and the polynomial ring in $n - 1$ indeterminates by $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}]$ and $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_{n-1}]$.

4.1 Fundamental invariants for \mathcal{A}_{n-1}

The group \mathfrak{S}_n acts on \mathbb{R}^n by permutation of coordinates and leaves the subspace $V := \{u \in \mathbb{R}^n \mid u_1 + \dots + u_n = 0\}$ invariant. The root system \mathcal{A}_{n-1} given in [[Bou68](#), Planche I] is a root system of rank $n - 1$ in V with simple roots and fundamental weights

$$\rho_i = e_i - e_{i+1} \quad \text{and} \quad \omega_i = \sum_{j=1}^i e_j - \frac{i}{n} \sum_{j=1}^n e_j = \frac{1}{n} \underbrace{[n-i, \dots, n-i]}_{i \text{ times}} \underbrace{[-i, \dots, -i]}_{n-i \text{ times}} \quad (4.1)$$

for $1 \leq i \leq n - 1$. The Weyl group of \mathcal{A}_{n-1} is $\mathcal{A}_{n-1} \cong \mathfrak{S}_n$ and the graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-2} \text{ --- } s_{n-1} \quad (4.2)$$

is the associated Coxeter diagram, where the $s_i := s_{\rho_i}$ are the reflections from [Definition 2.7](#), which generate \mathcal{A}_{n-1} . Thus, $-\omega_{n-i} \in \mathcal{A}_{n-1}\omega_i$ and the orbit $\mathcal{A}_{n-1}\omega_i$ has cardinality $\binom{n}{i}$ for $1 \leq i \leq n-1$.

The orbit polynomials associated to the fundamental weights of A_{n-1} and the elementary symmetric polynomials from [Equation \(3.1\)](#) satisfy the following relations.

Proposition 4.1. *In $\mathbb{Q}[x^\pm]$, define the monomials*

$$y_1 = x_1, \quad y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n-1 \quad \text{and} \quad y_n = x_{n-1}^{-1}.$$

Then

$$\sigma_i(y_1(x), \dots, y_n(x)) = \binom{n}{i} \theta_i(x)$$

for $1 \leq i \leq n-1$.

Proof. We first show that the orbit of the monomial $x^{e_1} = x_1$ is $\mathcal{A}_{n-1} \cdot x_1 = \{y_1, \dots, y_n\}$, and then infer the orbit of the x_i .

It follows from [Equation \(4.1\)](#), that $W e_1 = \omega_1 = (n e_1 - \sum_{j=1}^n e_j)/n$ and

$$\omega_i - \omega_{i-1} = \sum_{j=1}^i e_j - \frac{i}{n} \sum_{j=1}^n e_j - \sum_{j=1}^{i-1} e_j + \frac{i-1}{n} \sum_{j=1}^n e_j = \frac{1}{n} \left(n e_i - \sum_{j=1}^n e_j \right)$$

for $2 \leq i \leq n-1$. Hence, $\omega_i - \omega_{i-1}$ is obtained from ω_1 by permutation of coordinates. Furthermore, $-\omega_{n-1} = \omega_1 = (n e_n - \sum_{j=1}^n e_j)/n \in \mathcal{A}_{n-1}\omega_1$. Thus, we have $\{\omega_1, \omega_2 - \omega_1, \dots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\} \subseteq \mathcal{A}_{n-1}\omega_1$. Since these n weights are distinct and the orbit of ω_1 has cardinality n , we also have “ \supseteq ”. The monomials in $\mathbb{Q}[x^\pm]$, which correspond to $\omega_1, \omega_2 - \omega_1, \dots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}$ are y_1, \dots, y_n .

For $1 \leq i \leq n-1$, we have $x^{e_i} = x_i = y_1 \dots y_i$. Since the linear action of \mathcal{A}_{n-1} on $\mathbb{Q}[x^\pm]$ by [Equation \(2.3\)](#) permutes the monomials y_i , we have $\mathcal{A}_{n-1} \cdot x_i = \{\prod_{j \in J} y_j \mid J \subseteq \{1, \dots, n\}, |J| = i\}$. Hence,

$$\theta_i = \frac{|\text{Stab}_{\mathcal{A}_{n-1}}(x_i)|}{|\mathcal{A}_{n-1}|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} y_j = \frac{1}{|\mathcal{A}_{n-1} \cdot x_i|} \sigma_i(y_1, \dots, y_n) = \binom{n}{i}^{-1} \sigma_i(y_1, \dots, y_n).$$

□

4.2 Hermite matrix for \mathcal{A}_{n-1}

We introduce the subset $\mathbb{T}_1^n := \{y = (y_1, \dots, y_n) \in \mathbb{T}^n \mid y_1 \dots y_n = 1\}$ of the compact torus. The proof of the following fact is straightforward.

Proposition 4.2. *The map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^{n-1} &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-1} x_{n-2}^{-1}, x_{n-1}^{-1}) \end{aligned}$$

is injective and the preimage of \mathbb{T}_1^n is \mathbb{T}^{n-1} .

We state the main result for Weyl groups of type \mathcal{A} . Note that for $n = 2$, we are in the univariate case $\mathcal{A}_1 \cong \mathfrak{S}_2 \cong \{\pm 1\}$, where $\mathcal{T} = [-1, 1]$.

Theorem 4.3. Let $n \geq 3$. Consider the \mathbb{R} -vector space $\mathcal{Z} := \{z \in \mathbb{C}^{n-1} \mid \forall 1 \leq j \leq n-1 : \bar{z}_j = z_{n-j}\}$ and define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by

$$H(z)_{ij} = \text{Tr}((C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & 1/2 & & 0 & -d_n(z)/4 \\ 1 & 0 & \ddots & & -d_{n-1}(z)/2 \\ & 1/2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3(z)/2 \\ & & & \ddots & 0 & (1-d_2(z))/2 \\ 0 & & & & 1/2 & -d_1(z)/2 \end{bmatrix}, \quad \text{where}$$

$$d_\ell(z) = (-1)^\ell \left(\binom{n}{\ell} (z_\ell + z_{n-\ell}) + \sum_{i=1}^{\ell-1} \binom{n}{i} \binom{n}{\ell-i} z_i z_{n-\ell+i} \right) \quad \text{for } 1 \leq \ell \leq n-1 \quad \text{and}$$

$$d_n(z) = (-1)^n \left(2 + \sum_{i=1}^{n-1} \binom{n}{i}^2 z_i^2 \right).$$

For all $z \in \mathcal{Z}$, $H(z) \in \mathbb{R}^{n \times n}$ and $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{C}^{n-1}$ and define $c_0 := (-1)^n c_n := 1$, $c_i := (-1)^i \binom{n}{i} z_i \in \mathbb{C}$ for $1 \leq i \leq n-1$ as well as $d_\ell := d_\ell(z)$ for $1 \leq \ell \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$ and fix $x \in \mathbb{T}^{n-1}$, such that $\theta_i(x) = z_i$. By [Proposition 4.1](#), the solution of

$$(I) \quad \sigma_i(y_1, \dots, y_n) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is given by $y = \psi(x) \in \mathbb{T}_1^n$, where ψ is the map from [Proposition 4.2](#). Note that $\theta_j(x)$ and $\theta_{n-j}(x)$ are complex conjugates, because $-\omega_j \in \mathcal{A}_{n-1} \omega_{n-j}$. Therefore, $z \in \mathcal{Z}$ and $d_\ell = \sum_{i=0}^{\ell} c_i \bar{c}_{\ell-i} \in \mathbb{R}$ yields the last column of $C(z)$. Applying [Corollary 3.6](#) with coefficients d_ℓ yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $z \in \mathcal{Z}$ with $H(z) \succeq 0$. By [Corollary 3.6](#), the solution y of system (I) is contained in \mathbb{T}^n and satisfies $y_1 \dots y_n = (-1)^n c_n = 1$, that is, $y \in \mathbb{T}_1^n$. Let $x \in \mathbb{T}^{n-1}$ be the unique preimage of y under ψ . Then

$$z_i = (-1)^i \binom{n}{i}^{-1} c_i = \binom{n}{i}^{-1} \sigma_i(y_1, \dots, y_n) = \theta_i(x)$$

by [Proposition 4.1](#) and thus $z = \vartheta(x) \in \mathcal{T}$. □

We finish this subsection with a remark on the embedding of \mathcal{T} in \mathbb{R}^{n-1} and on the degree of the entries of the Hermite matrix polynomial.

Remark 4.4.

1. In the case of \mathcal{A}_{n-1} , we have $\mathcal{T} \not\subseteq \mathbb{R}^{n-1}$, because $-I_n \notin \mathcal{A}_{n-1}$. Following [Equation \(2.8\)](#), we consider the image $\mathcal{T}_{\mathbb{R}}$ of the map

$$\begin{aligned} \vartheta_{\mathbb{R}} : \mathbb{T}^{n-1} &\rightarrow \mathbb{R}^{n-1}, x \mapsto (\theta_{1,\mathbb{R}}(x), \dots, \theta_{n-1,\mathbb{R}}(x)) \quad \text{with} \\ \theta_{i,\mathbb{R}}(x) &= \frac{\theta_i(x) + \theta_{n-i}(x)}{2}, \quad \theta_{n-i,\mathbb{R}}(x) = \frac{\theta_i(x) - \theta_{n-i}(x)}{2i} \quad \text{for } 1 \leq i \leq \lfloor (n-1)/2 \rfloor \quad \text{and} \\ \theta_{n/2,\mathbb{R}}(x) &= \theta_{n/2}(x), \quad \text{when } n \text{ is even.} \end{aligned}$$

2. The entries of the last column of the matrix polynomial $C(z)$ have degree 2, except $C(z)_{nn} = -f_1(z)$, which has degree 1. All other entries are constant. Hence, the degree of $H(z)_{ij}$ is $i+j$. The determinant of $H(z)$ has degree $4n - 2$ for $n \geq 3$.

4.3 Example \mathcal{A}_2

In this subsection, we further investigate the case $n = 3$ and give a visualization for the embedding of the \mathbb{T} -orbit space in \mathbb{R}^2 .

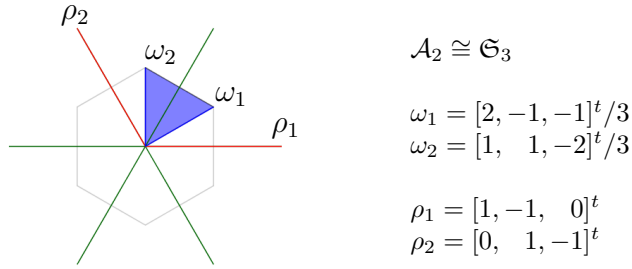


Figure 3: The root system \mathcal{A}_2 in $V \subseteq \mathbb{R}^3$ with fundamental weights ω_1, ω_2 and simple roots ρ_1, ρ_2 .

In Figure 3, the hexagon is the orbit of the blue shaded triangle with vertices $0, \omega_1, \omega_2$ under \mathcal{A}_2 .

Let $z_1, z_2 \in \mathbb{R}$ and $z = (z_1 + iz_2, z_1 - iz_2) \in \mathcal{Z}$. Then the matrix $C(z) \in \mathbb{R}[z]^{3 \times 3}$ from Theorem 4.3 is

$$C(z) = \begin{bmatrix} 0 & 1/2 & (1 + 9z_1^2 - 9z_2^2)/2 \\ 1 & 0 & (1 - 9z_1^2 - 9z_2^2 - 6z_1)/2 \\ 0 & 1/2 & 3z_1 \end{bmatrix}. \quad (4.3)$$

We already computed this matrix (up to a substitution) in Example 3.4. Following Theorem 4.3, define the matrix $H(z) \in \mathbb{R}[z]^{3 \times 3}$ with entries $(H(z))_{ij} = 4 \operatorname{Tr}(C(z)^{i+j-2}) - \operatorname{Tr}(C(z)^{i+j})$. Then (z_1, z_2) is contained in $\mathcal{T}_{\mathbb{R}}$ if and only if $H(z)$ is positive semi-definite. Assume that $\operatorname{Det}(xI_3 - H(z)) = x^3 - h_2(z)x^2 + h_1(z)x - h_0(z) \in \mathbb{R}[x]$ is the characteristic polynomial of $H(z)$ for some $h_i(z) \in \mathbb{R}[z]$. By [BPR06, Theorem 4.58], $H(z)$ is positive semi-definite if and only if $h_i(z) \geq 0$ for $0 \leq i \leq 2$.

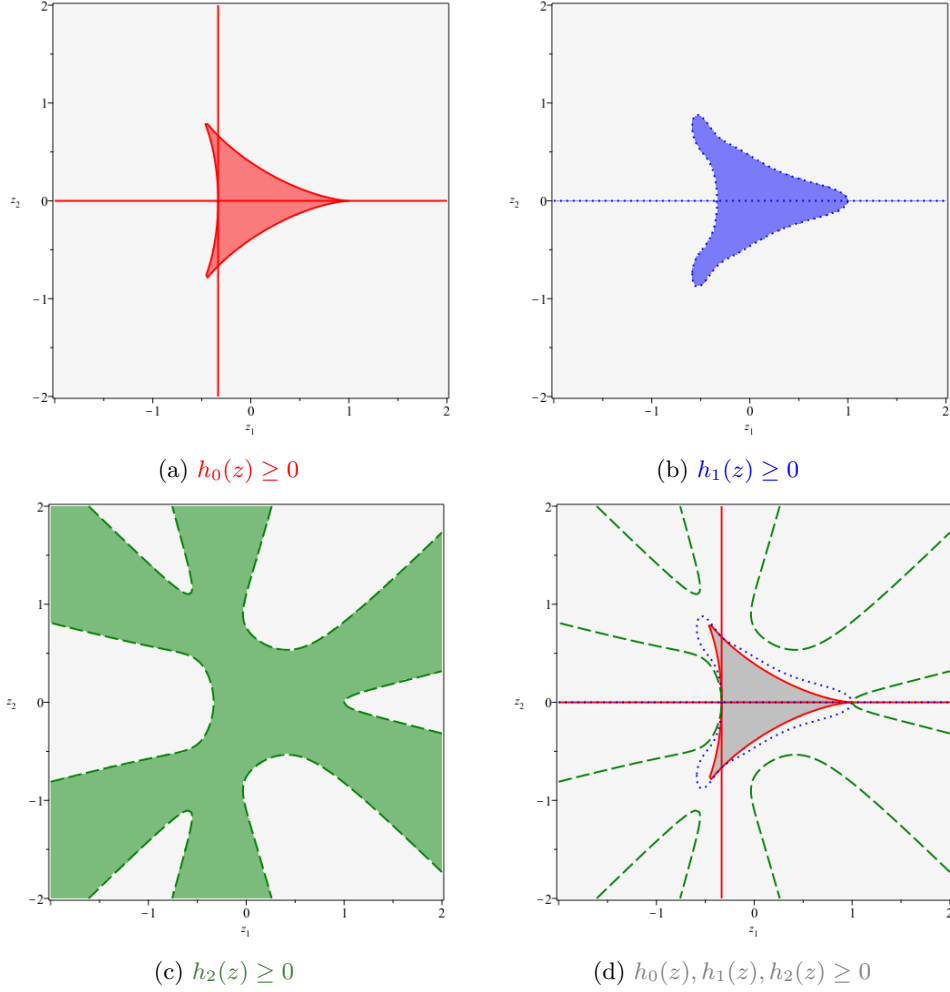


Figure 4: Vanishing points and positivity regions for the coefficients of the characteristic polynomial of $H(z)$.

Of particular interest are the vertices, which correspond to the fundamental weights and the origin. They are given by

$$\text{Vertex}_1 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_1 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{4}{3}\pi i\right), \exp\left(-\frac{2}{3}\pi i\right)\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_2 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{2}{3}\pi i\right), \exp\left(-\frac{4}{3}\pi i\right)\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_3 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta_{\mathbb{R}}(1, 1) = (1, 0).$$

We visualize the problem “ $(z_1, z_2) \in \mathcal{T}_{\mathbb{R}}$?” by evaluating h_0, h_1, h_2 at z . In the above images, a solid red line, blue dots and green dashes indicate the varieties of these three polynomials. The plots indicate an invariance under the dihedral groups \mathfrak{D}_3 . Let $g_1(z) := z_1^2 + z_2^2, g_2(z) := z_1(z_1^2 - 3z_2^2)$. Then $\mathbb{R}[z]^{\mathfrak{D}_3} = \mathbb{R}[g_1(z), g_2(z)]$

and we have

$$\begin{aligned}
 h_0(z) &= -\text{Coeff}(t^0, \text{Det}(tI_3 - H(z))) \quad (\text{solid}) \\
 &= 2187/64 z_2^4 (3z_1 + 1)^2 (-3z_1^4 - 6z_1^2 z_2^2 - 3z_2^4 + 8z_1^3 - 24z_1 z_2^2 - 6z_1^2 - 6z_2^2 + 1) \\
 &= 2187/64 z_2^4 (3z_1 + 1)^2 \underbrace{(-6g_1(z) - 3g_1(z)^2 + 8g_2(z) + 1)}_{\mathfrak{D}_3\text{-invariant}}, \\
 h_1(z) &= \text{Coeff}(t^1, \text{Det}(tI_3 - H(z))) \quad (\text{dots}) \\
 &= 243/256 z_2^2 (-243z_1^8 - 972z_1^6 z_2^2 - 1458z_1^4 z_2^4 - 972z_1^2 z_2^6 - 243z_2^8 + 324z_1^7 - 1620z_1^5 z_2^2 \\
 &\quad - 4212z_1^3 z_2^4 - 2268z_1 z_2^6 - 432z_1^6 - 2052z_1^4 z_2^2 - 5400z_1^2 z_2^4 - 324z_2^6 + 180z_1^5 - 3384z_1^3 z_2^2 \\
 &\quad - 684z_1 z_2^4 + 18z_1^4 - 804z_1^2 z_2^2 + 42z_2^4 + 76z_1^3 + 404z_1 z_2^2 - 8z_1^2 - 108z_2^2 + 60z_1 + 25), \\
 h_2(z) &= -\text{Coeff}(t^2, \text{Det}(tI_3 - H(z))) \quad (\text{dash}) \\
 &= 1/32 (-729z_1^6 + 1458z_1^5 + (10935z_2^2 - 1215)z_1^4 + (-2916z_2^2 + 540)z_1^3 + 351z_2^2 + 63 \\
 &\quad + (-10935z_2^4 + 1458z_2^2 - 135)z_1^2 + (-4374z_2^4 + 972z_2^2 + 18)z_1 + 729z_2^6 - 1215z_2^4).
 \end{aligned}$$

For a generic point $z \in \mathcal{Z}$, $H(z)$ has rank 3. We observe an intersection of all three varieties in Vertex_3 , in which case the rank of $H(\text{Vertex}_3)$ vanishes. This also occurs at

$$\vartheta_{\mathbb{R}} \left(\exp \left(-2\pi i \langle \omega_1, \frac{\omega_1 + \omega_2}{2} \rangle \right), \exp \left(-2\pi i \langle \omega_2, \frac{\omega_1 + \omega_2}{2} \rangle \right) \right) = \vartheta_{\mathbb{R}}(\exp(-\pi i), \exp(\pi i)) = \left(-\frac{1}{3}, 0 \right).$$

Furthermore, the rank of both $H(\text{Vertex}_1)$, $H(\text{Vertex}_2)$ is 1, we only have an intersection of “ $h_0(z) = 0$ ” with “ $h_1(z) = 0$ ”. Two more intersection of “ $h_0(z) = 0$ ” with “ $h_1(z) = 0$ ” lie at $(-1/3, 2/3)$ and $(-1/3, -2/3)$. Every other point on the boundary of $\mathcal{T}_{\mathbb{R}}$ admits rank 2.

5 Type \mathcal{C}

In [Sections 5 to 7](#), we will encounter symmetric systems similarly to [Section 4](#), however they will be of type (II) and therefore, we dealt with the type \mathcal{A} first. The dimension of the root system and the number of indeterminates of the symmetric system are from now on both n .

5.1 Fundamental invariants for \mathcal{C}_n

The groups \mathfrak{S}_n and $\{\pm 1\}^n$ act on \mathbb{R}^n by permutation of coordinates, and multiplication of coordinates by ± 1 . The root system \mathcal{C}_n given in [[Bou68](#), Planche III] is a root system in \mathbb{R}^n with simple roots and fundamental weights

$$\rho_i = e_i - e_{i+1}, \quad \rho_n = 2e_n \quad \text{and} \quad \omega_i = e_1 + \dots + e_i. \quad (5.1)$$

for $1 \leq i \leq n$. The Weyl group of \mathcal{C}_n is $\mathcal{C}_n \cong \mathfrak{S}_n \times \{\pm 1\}^n$ and the graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-1} \text{ ===== } s_n \quad (5.2)$$

is the associated Coxeter diagram, where the $s_i := s_{\rho_i}$ are the reflections from [Definition 2.7](#), which generate \mathcal{C}_n . We have $-I_n \in \mathcal{C}_n$ and thus, $-\omega_i \in \mathcal{C}_n \omega_i$. Furthermore, the orbit $\mathcal{C}_n \omega_i$ has cardinality $2^i \binom{n}{i}$ for $1 \leq i \leq n$.

The orbit polynomials associated to the fundamental weights of \mathcal{C}_n and the elementary symmetric polynomials from [Equation \(3.1\)](#) satisfy the following relations.

Proposition 5.1. *In $\mathbb{Q}[x^{\pm}]$, define the monomials*

$$y_1 = x_1 \quad \text{and} \quad y_k = x_k x_{k-1}^{-1} \quad \text{for} \quad 2 \leq k \leq n.$$

Then

$$\sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) = 2^i \binom{n}{i} \theta_i(x)$$

for $1 \leq i \leq n$.

Proof. We follow the proof of [Proposition 4.1](#).

It follows from [Equation \(5.1\)](#), that $\pm\omega_1 = \pm e_1$ and $\pm(\omega_i - \omega_{i-1}) = \pm e_i$ for $2 \leq i \leq n$, which is obtained from ω_1 by permutation and sign change of coordinates. Thus, $\{\pm\omega_1, \pm(\omega_2 - \omega_1), \dots, \pm(\omega_n - \omega_{n-1})\} \subseteq \mathcal{C}_n \omega_1$. Since these $2n$ weights are distinct and the orbit of ω_1 has cardinality $2n$, we also have “ \supseteq ”. The corresponding monomials in $\mathbb{Q}[x^\pm]$ are given by y_1, \dots, y_n and their inverses.

For $1 \leq i \leq n$, we have $x_i = y_1 \dots y_i$. Since the linear action of \mathcal{C}_n on $\mathbb{Q}[x^\pm]$ by [Equation \(2.3\)](#) permutes and inverts the y_i , we have $\mathcal{C}_n \cdot x_i = \{\prod_{j \in J} y_j^{\delta_j} \mid J \subseteq \{1, \dots, n\}, |J| = i, \delta \in \{\pm 1\}^J\}$. Thus,

$$\theta_i = \frac{|\text{Stab}_{\mathcal{C}_n}(x_i)|}{|\mathcal{C}_n|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \sum_{\delta \in \{\pm 1\}^J} \prod_{j \in J} y_j^{\delta_j} = \frac{1}{|\mathcal{C}_n \cdot x_i|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} (y_j + y_j^{-1})$$

with $|\mathcal{C}_n \cdot x_i| = 2^i \binom{n}{i}$ yields the statement. \square

5.2 Hermite matrix for \mathcal{C}_n

The proof of the next statement is straightforward.

Proposition 5.2. *The map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_n x_{n-1}^{-1}) \end{aligned}$$

is bijective and the preimage of \mathbb{T}^n is \mathbb{T}^n .

We state the main result of this section, which we applied in [Example 1.1](#).

Theorem 5.3. *Define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by*

$$H(z)_{ij} = \text{Tr}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix}, \quad \text{where } c_i(z) = (-2)^i \binom{n}{i} z_i \quad \text{for } 1 \leq i \leq n.$$

Then $\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{R}^n$ and set $c_i := c_i(z) \in \mathbb{R}$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. By [Proposition 5.1](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is $y = \psi(x) \in \mathbb{T}^n$, where ψ is the map from [Proposition 5.2](#). Applying [Corollary 3.7](#) yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $H(z) \succeq 0$. By [Corollary 3.7](#), the solution y of the above system (II) is contained in \mathbb{T}^n . Let $x \in \mathbb{T}^n$ be the unique preimage of y under ψ . Then $z_i = \theta_i(x)$ and so $z = \vartheta(x)$ is contained in \mathcal{T} . \square

We finish this subsection with a remark on the degree of the entries of the Hermite matrix polynomial.

Remark 5.4. *The entries of the last column of the matrix polynomial $C(z)$ from [Theorem 5.3](#) have degree 1, all other entries are constant. The degree of $H(z)_{ij}$ is $i + j$ and the determinant of $H(z)$ has degree $2n$.*

5.3 Example \mathcal{C}_2

In this subsection, we further investigate the case $n = 2$.

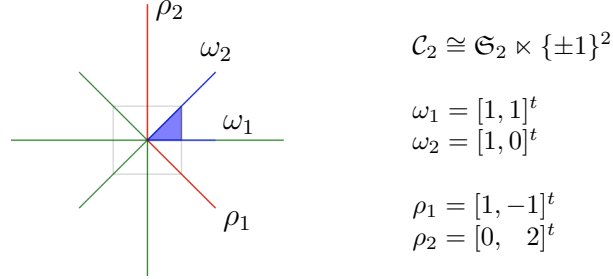


Figure 5: The root system \mathcal{C}_2 in \mathbb{R}^2 with fundamental weights ω_1, ω_2 and simple roots ρ_1, ρ_2 .

In Figure 5, the square is the orbit of the blue shaded triangle with vertices $0, \omega_1/2, \omega_2/2$ under reflection by \mathcal{C}_2 .

Let $z = (z_1, z_2) \in \mathbb{R}^2$. The matrix $C(z) \in \mathbb{Q}[z]^{2 \times 2}$ from Theorem 5.3 evaluated in z is

$$C(z) = \begin{bmatrix} 0 & -4z_2 \\ 1 & 4z_1 \end{bmatrix}. \quad (5.3)$$

Then z is contained in \mathcal{T} if and only if the resulting Hermite matrix

$$H(z) = 8 \begin{bmatrix} -2z_1^2 + z_2 + 1 & -8z_1^3 + 6z_1z_2 + 2z_1 \\ -8z_1^3 + 6z_1z_2 + 2z_1 & -32z_1^4 + 8z_1^2 + 32z_1^2z_2 - 4z_2^2 - 4z_2 \end{bmatrix} \quad (5.4)$$

is positive semi-definite, which is equivalent to its determinant and trace being nonnegative. The varieties of these two polynomials in z_1, z_2 are depicted below.

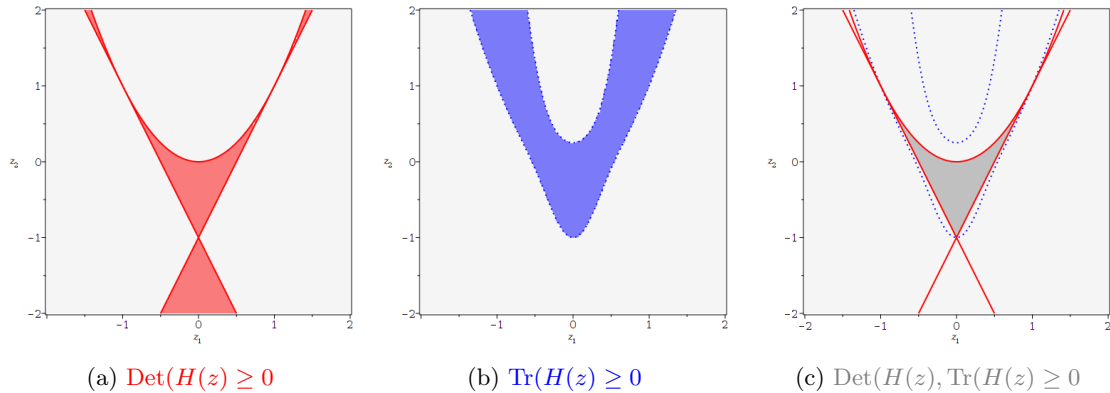


Figure 6: Vanishing points and positivity regions for determinant and trace of $H(z)$.

$$\begin{aligned} \text{Det}(H(z)) &= 256(z_2 - z_1^2)(2z_1 + z_2 + 1)(2z_1 - z_2 - 1) && \text{(solid)} \\ \text{Tr}(H(z)) &= 8(-32z_1^4 + 32z_1^2z_2 + 6z_1^2 - 4z_2^2 - 3z_2 + 1) && \text{(dots)} \end{aligned}$$

We observe three intersections of “ $\text{Det}(H(z)) = 0$ ” (red solid line) and “ $\text{Tr}(H(z)) = 0$ ” (blue dots) in the

the vertices

$$\text{Vertex}_1 := \vartheta(\exp(-2\pi i \langle \omega_1, \frac{\omega_1}{2} \rangle), \exp(-2\pi i \langle \omega_2, \frac{\omega_1}{2} \rangle)) = \vartheta(-1, -1) = (0, -1)$$

$$\text{Vertex}_2 := \vartheta(\exp(-2\pi i \langle \omega_1, \frac{\omega_2}{2} \rangle), \exp(-2\pi i \langle \omega_2, \frac{\omega_2}{2} \rangle)) = \vartheta(-1, 1) = (-1, 1)$$

$$\text{Vertex}_3 := \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1)$$

The shape of this domain is dictated by the determinant, but from the positivity condition one can observe that the trace is also required. Alternatively the inequation given by the trace could be replaced by the constraint that the orbit space is contained in the square $[-1, 1]^n$.

6 Type \mathcal{B}

The Weyl group of B_n is isomorphic to that of C_n and we find a similar formula for the orbit polynomials in [Proposition 6.1](#) with a deviation at σ_n that leads to a different Hermite matrix in [Theorem 6.4](#).

6.1 Fundamental invariants for \mathcal{B}_n

Similar to the case C_n , the root system B_n given in [[Bou68](#), Planche II] is a root system in \mathbb{R}^n and the simple roots and fundamental weights are

$$\rho_i = e_i - e_{i+1}, \quad \rho_n = e_n \quad \text{and} \quad \omega_i = e_1 + \dots + e_i, \quad \omega_n = (e_1 + \dots + e_n)/2. \quad (6.1)$$

for $1 \leq i \leq n$. The Weyl group of B_n is $\mathcal{B}_n \cong C_n \cong \mathfrak{S}_n \times \{\pm 1\}^n$ and the graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-1} \text{ ===== } s_n \quad (6.2)$$

is the associated Coxeter diagram, where the $s_i := s_{\rho_i}$ are the reflections from [Definition 2.7](#), which generate \mathcal{B}_n . We have $-I_n \in \mathcal{B}_n$ and thus, $-\omega_i \in C_n \omega_i$. Furthermore, the orbit $C_n \omega_i$ has cardinality $2^i \binom{n}{i}$ for $1 \leq i \leq n$.

The orbit polynomials associated to the fundamental weights of B_n and the elementary symmetric polynomials from [Equation \(3.1\)](#) satisfy the following relations.

Proposition 6.1. *In $\mathbb{Q}[x^\pm]$, define the monomials*

$$y_1 = x_1, y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n-1 \quad \text{and} \quad y_n = x_n^2 x_{n-1}^{-1}.$$

Then

$$\sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) = 2^i \binom{n}{i} \theta_i(x) \quad \text{for } 1 \leq i \leq n-1 \quad \text{and}$$

$$\sigma_n(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) = 2^n \Theta_{2\omega_n}(x).$$

Proof. Since $\mathcal{B}_n \cong C_n$, the proof is analogous to the one of [Proposition 5.1](#), when we replace ω_n with $2\omega_n$. Then the corresponding monomial in $\mathbb{Q}[x^\pm]$ is $x^{2e_n} = x_n^2$. \square

The orbit polynomial $\Theta_{2\omega_n}$ is \mathcal{B}_n -invariant. Hence, it can be expressed as a polynomial in the fundamental invariants of $\mathbb{Q}[x^\pm]^{\mathcal{B}_n}$. We give the explicit equation in the next lemma.

Lemma 6.2. *In $\mathbb{Q}[x^\pm]$, the orbit polynomial associated to $2\omega_n \in \Omega$ satisfies*

$$\Theta_{2\omega_n} = 2^n \theta_n^2 - \sum_{j=1}^{n-1} \binom{n}{j} \theta_j - 1.$$

Proof. The cardinality of the orbit $\mathcal{B}_n \omega_n$ is 2^n . Let $\mu \in \mathcal{B}_n \omega_n$ and distinguish between the following three cases.

1. If $\mu = \omega_n$, then $\Theta_{\omega_n+\mu} = \Theta_{2\omega_n}$ is the term on the left-hand side of the statement, for which we search an explicit formula.
2. If $\mu = -\omega_n$, then $\Theta_{\omega_n+\mu} = \Theta_0 = 1$.
3. For any other $\mu \in \mathcal{B}_n \omega_n$, there exists $1 \leq j \leq n-1$, such that μ contains exactly j positive coordinates. Therefore, $\mu + \omega_n$ has exactly j nonzero entries and is contained in the orbit of ω_j under \mathcal{B}_n , that is, $\Theta_{\mu+\omega_n} = \Theta_{\omega_j}$. The number of μ , for which this is the case, is $\binom{n}{j}$.

From this and [Proposition 2.6](#), we conclude that

$$2^n \theta_n^2 = \sum_{\mu \in \mathcal{B}_n \omega_n} \Theta_{\omega_n+\mu} = \Theta_{2\omega_n} + 1 + \sum_{j=1}^{n-1} \binom{n}{j} \theta_j$$

to obtain the equation for $\Theta_{\omega_n+\mu}$. □

6.2 Hermite matrix for \mathcal{B}_n

Proposition 6.3. *Define the map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-1} x_{n-2}^{-1}, x_n^2 x_{n-1}^{-1}). \end{aligned}$$

1. ψ is surjective. The preimage of \mathbb{T}^n under ψ is \mathbb{T}^n .
2. Every point in \mathbb{T}^n has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ with $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$ and $x_n = -x'_n$.
3. If $x, x' \in \mathbb{T}^n$ with $\psi(x) = \psi(x')$ and $x_n = -x'_n$, then $\theta_n(x) = -\theta_n(x')$.

Proof. 1. and 2. For all $y \in (\mathbb{C}^*)^n$, there exists $x \in (\mathbb{C}^*)^n$ with $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-1} = y_{n-1} x_{n-2}$ and $x_n^2 = y_n x_{n-1}$. Thus, x is a preimage of y under ψ and uniquely determined by y up to a sign in the last coordinate. We have $y \in \mathbb{T}^n$ if and only if $x \in \mathbb{T}^n$.

3. We deduce from the Coxeter diagram in [Equation \(6.2\)](#) that $\text{Stab}_{\mathcal{B}_n}(\omega_n) \cong \mathcal{A}_{n-1} \cong \mathfrak{S}_n$. Hence, $\mathcal{B}_n \omega_n = \{\pm 1\}^n \omega_n$ and by [Proposition 6.1](#) this orbit consists of elements

$$\mu = \frac{\epsilon_1}{2} e_1 + \dots + \frac{\epsilon_n}{2} e_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-1} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_n}{2} (2\omega_n - \omega_{n-1}) = \epsilon_n \omega_n + \nu \in \Omega$$

with $\epsilon_i \in \{\pm 1\}$ for $1 \leq i \leq n$ and $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-1}$. Let $\alpha, \beta \in \mathbb{Z}^n$ be the coordinates of μ, ν in Ω . Then $\beta_n = 0$ and the monomial in θ_n corresponding to μ is $x^\alpha = x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}} x_n^{\epsilon_n}$. Thus, x^α is linear in x_n . Since every monomial in θ_n can be written in terms of such β, ϵ , θ_n is linear in x_n and with x, x' from the hypothesis we have $\theta_n(x) = -\theta_n(x')$. □

We state the main result for Weyl groups of type \mathcal{B} .

Theorem 6.4. *Define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by*

$$H(z)_{ij} = \text{Tr}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix}, \quad \text{where } c_i(z) = (-2)^i \binom{n}{i} z_i \quad \text{for } 1 \leq i \leq n-1 \quad \text{and}$$

$$c_n(z) = (-2)^n \left(2^n z_n^2 - \sum_{i=1}^{n-1} \binom{n}{i} z_i - 1 \right).$$

Then $\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{R}^n$ and set $c_i := c_i(z) \in \mathbb{R}$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. By [Proposition 6.1](#) and [Lemma 6.2](#), the solution of

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is $y = \psi(x) \in \mathbb{T}^n$, where ψ is the map from [Proposition 6.3](#). Applying [Corollary 3.7](#) yields $H(z) \succeq 0$.

For “ \supseteq ” on the other hand, assume $H(z) \succeq 0$. By [Corollary 3.7](#), the solution y of system (II) with coefficients c_i is contained in \mathbb{T}^n . According to [Proposition 6.3](#), y has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ under ψ with $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$ and $x_n = -x'_n$. We have $z_i = \theta_i(x) = \theta_i(x')$ for $1 \leq i \leq n-1$ and $z_n^2 = \theta_n(x)^2 = \theta_n(x')^2$ with $\theta_n(x) = -\theta_n(x')$. Therefore, $z_n = \theta_n(x)$ or $z_n = \theta_n(x') = -\theta_n(x)$ and thus, z is contained in \mathcal{T} . \square

Remark 6.5. *The entries of the last column of the matrix polynomial $C(z)$ from [Theorem 6.4](#) have degree 1, except $C(z)_{1n}$, which has degree 2. All other entries are constant. The degree of $H(z)_{ij}$ is $i + j$ and the degree of the determinant of $H(z)$ is $3n$.*

6.3 Example \mathcal{B}_2

In this subsection, we further investigate the case $n = 2$.

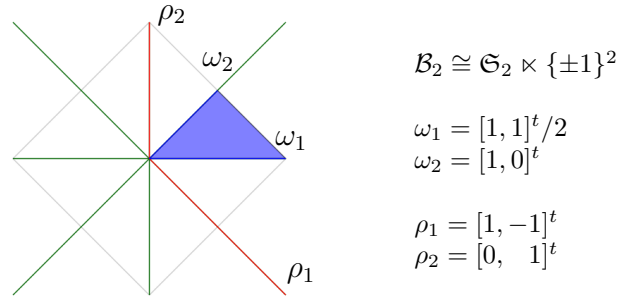


Figure 7: The root system \mathcal{B}_2 in \mathbb{R}^2 with fundamental weights ω_1, ω_2 and simple roots ρ_1, ρ_2 .

In [Figure 7](#), the square is the orbit of the blue shaded triangle with vertices $0, \omega_1, \omega_2$ under reflection by \mathcal{B}_2 .

Let $z = (z_1, z_2) \in \mathbb{R}^2$. The matrix $C(z) \in \mathbb{Q}[z]^{2 \times 2}$ from [Theorem 6.4](#) evaluated in z is

$$C(z) = \begin{bmatrix} 0 & -16z_2^2 + 8z_1 + 4 \\ 1 & 4z_1 \end{bmatrix}. \quad (6.3)$$

Then z is contained in \mathcal{T} if and only if the resulting Hermite matrix

$$H(z) = 16 \begin{bmatrix} -z_1^2 + 2z_2^2 - z_1 & -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 \\ -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 & -16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 20z_1^2 + 8z_2^2 - 4z_1 \end{bmatrix}$$

is positive semi-definite, which is equivalent to its determinant and trace being nonnegative. The varieties of these two polynomials in z_1, z_2 are depicted below.

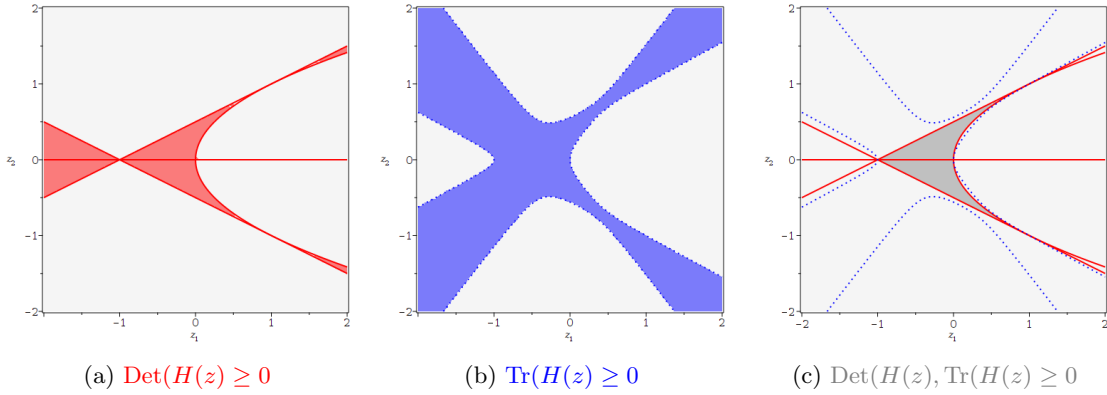


Figure 8: Vanishing points and positivity regions for determinant and trace of $H(z)$.

$$\begin{aligned} \text{Det}(H(z)) &= 4096z_2^2(z_1 - z_2^2)(z_1 + 2z_2 + 1)(z_1 - 2z_2 + 1) && \text{(solid)} \\ \text{Tr}(H(z)) &= 16(-16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 21z_1^2 + 10z_2^2 - 5z_1) && \text{(dots)} \end{aligned}$$

The \mathbb{T} -orbit space in the B_2 -case is obtained from the C_2 -case in [Figure 6](#) by swapping z_1, z_2 . Apart from a rotation, we also observe that the rank of $H(z)$ reduces from 2 to 1 on the line “ $z_2 = 0$ ”. Furthermore, there is a fourth intersection of the varieties of the determinant and the trace at $z = (0, 0)$.

7 Type \mathcal{D}

The Weyl group \mathcal{D}_n is isomorphic to a subgroup of $\mathcal{C}_n \cong \mathcal{B}_n$ of index 2, which lead to a symmetric system of type (II). The explicit formula between fundamental invariants and the elementary symmetric polynomials is similar to [Proposition 5.1](#), but with deviations in σ_{n-1} and σ_n that additionally depend on the parity of n . If n is odd, then the orbit space of \mathcal{D}_n contains complex points, but a real description is possible with a change of variables as in [Equation \(2.8\)](#).

7.1 Fundamental invariants for \mathcal{D}_n

The groups \mathfrak{S}_n and $\{\pm 1\}_+^n := \{\epsilon \in \{\pm 1\}^n \mid \epsilon_1 \dots \epsilon_n = 1\}$ act on \mathbb{R}^n by permutation of coordinates, and multiplication of coordinates by ± 1 , where only an even amount of sign changes is admissible. The root system D_n given in [[Bou68](#), Planche IV] is a root system in \mathbb{R}^n with simple roots and fundamental weights

$$\begin{aligned} \rho_i &= e_i - e_{i+1}, & \rho_n &= e_{n-1} + e_n & \text{and} \\ \omega_i &= e_1 + \dots + e_i, & \omega_{n-1} &= (e_1 + \dots + e_{n-1} - e_n)/2, & \omega_n &= (e_1 + \dots + e_n)/2. \end{aligned} \quad (7.1)$$

The Weyl group of D_n is $\mathcal{D}_n \cong \mathfrak{S}_n \times \{\pm 1\}_+^n$ and the graph

$$\begin{array}{ccccccc}
 & & & & & & s_{n-1} \\
 & & & & & & / \\
 s_1 & \text{---} & s_2 & \text{---} & \dots & \text{---} & s_{n-2} \\
 & & & & & & \backslash \\
 & & & & & & s_n
 \end{array} \tag{7.2}$$

is the associated Coxeter diagram, where the $s_i := s_{\rho_i}$ are the reflections from [Definition 2.7](#), which generate \mathcal{D}_n . For all $1 \leq i \leq n$, we have $-\omega_i \in \mathcal{D}_n \omega_i$, except when n is odd, where $-\omega_{n-1} \in \mathcal{D}_n \omega_n$. Furthermore, the orbit $\mathcal{D}_n \omega_i$ has cardinality $2^i \binom{n}{i}$ for $1 \leq i \leq n-2$ and $|\mathcal{D}_n \omega_{n-1}| = |\mathcal{D}_n \omega_n| = 2^{n-1}$.

The orbit polynomials associated to the fundamental weights of D_n and the elementary symmetric polynomials from [Equation \(3.1\)](#) satisfy the following relations.

Proposition 7.1. *In $\mathbb{Q}[x^\pm]$, define the monomials*

$$y_1 = x_1, \quad y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n-2, \quad y_{n-1} = x_n x_{n-1} x_{n-2}^{-1} \quad \text{and} \quad y_n = x_n x_{n-1}^{-1}.$$

Then

$$\begin{aligned}
 \sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^i \binom{n}{i} \theta_i(x) \quad \text{for } 1 \leq i \leq n-2, \\
 \sigma_{n-1}(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^{n-1} n \Theta_{\omega_{n-1} + \omega_n}(x) \quad \text{and} \\
 \sigma_n(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^{n-1} (\Theta_{2\omega_{n-1}}(x) + \Theta_{2\omega_n}(x)).
 \end{aligned}$$

Proof. We follow the proof of [Proposition 4.1](#).

It follows from [Equation \(7.1\)](#), that $\pm\omega_1 = \pm e_1$ and

$$\begin{aligned}
 \pm(\omega_i - \omega_{i-1}) &= \pm e_i \quad \text{for } 2 \leq i \leq n-2, \\
 \pm(\omega_n + \omega_{n-1} - \omega_{n-2}) &= \pm e_{n-1} \quad \text{and} \\
 \pm(\omega_n - \omega_{n-1}) &= \pm e_n,
 \end{aligned}$$

are obtained from ω_1 by permutation and admissible sign changes of coordinates. Those $2n$ weights are distinct and thus form the orbit $\mathcal{D}_n \omega_1$. The corresponding monomials in $\mathbb{Q}[x^\pm]$ are given by y_1, \dots, y_n and their inverses.

For $1 \leq i \leq n-2$, we have $x_i = y_1 \dots y_i$. Then the statement for θ_i is proven analogously to [Proposition 5.1](#). With $x_n x_{n-1} = y_1 \dots y_{n-1}$, we obtain the equation for $\Theta_{\omega_{n-1} + \omega_n}$ as well.

Finally, apply $x_{n-1}^2 = y_1 \dots y_{n-1} y_n^{-1}$ and $x_n^2 = y_1 \dots y_{n-1} y_n$ to obtain

$$2^{n-1} (\Theta_{2\omega_{n-1}} + \Theta_{2\omega_n}) = \sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = -1}} y^\epsilon + \sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = 1}} y^\epsilon = \sum_{\epsilon \in \{\pm 1\}^n} y^\epsilon = \prod_{i=1}^n (y_i + y_i^{-1}),$$

where $y^\epsilon := y_1^{\epsilon_1} \dots y_n^{\epsilon_n}$. This proves the last equation. \square

The orbit polynomials associated to $\omega_n + \omega_{n-1}$, $2\omega_{n-1}$ and $2\omega_n \in \Omega$ are invariant. Hence, they can be expressed in terms of the generators θ_i of $\mathbb{Q}[x^\pm]^{\mathcal{D}_n}$. The formula depends on the parity of n as follows.

Lemma 7.2. *In $\mathbb{Q}[x^\pm]$, the following equations hold.*

1. If n is even, then

$$\Theta_{\omega_{n-1}+\omega_n} = \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1},$$

$$\Theta_{2\omega_{n-1}} = 2^{n-1} \theta_{n-1}^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1 \quad \text{and}$$

$$\Theta_{2\omega_n} = 2^{n-1} \theta_n^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1.$$

2. If n is odd, then

$$\Theta_{\omega_{n-1}+\omega_n} = \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j} - \frac{1}{n},$$

$$\Theta_{2\omega_{n-1}} = 2^{n-1} \theta_{n-1}^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1} \quad \text{and}$$

$$\Theta_{2\omega_n} = 2^{n-1} \theta_n^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}.$$

Proof. We have $|\mathcal{D}_n \omega_{n-1}| = |\mathcal{D}_n \omega_n| = 2^{n-1}$. The equations can be obtained from the recurrence formula [Proposition 2.6](#) and the following combinatorial steps.

1. Assume that n is even. We first prove the equation for $\Theta_{2\omega_n}$ in detail. For $0 \leq j \leq n/2$, consider $\mu \in \mathcal{D}_n \omega_n$ with $2j$ positive coordinates. There are precisely $\binom{n}{2j}$ such elements in $\mathcal{D}_n \omega_n$ and an odd amount of positive coordinates is not possible. We distinguish three cases. If $j = 0$, then $\omega_{n-1} + \mu = 0$, and if $j = n/2$, then $\omega_n + \mu = 2\omega_n$. Otherwise, $\omega_n + \mu$ has $2j$ nonzero coordinates and must therefore be contained in $\mathcal{D}_n \omega_{2j}$. All in all, we obtain

$$2^{n-1} \theta_n^2 = \sum_{\mu \in \mathcal{D}_n \omega_n} \Theta_{\omega_n + \mu} = \Theta_{2\omega_n} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} + 1.$$

Next for $1 \leq j \leq n/2$, consider $\mu \in \mathcal{D}_n \omega_{n-1}$ with $2j-1$ positive coordinates. Then

$$\omega_{n-1} + \mu \begin{cases} = 0, & \text{if } j = 1, \quad \mu_n = \frac{1}{2} \\ = 2\omega_{n-1}, & \text{if } j = \frac{n}{2}, \quad \mu_n = -\frac{1}{2} \\ \in \mathcal{D}_n \omega_{2j}, & \text{otherwise} \end{cases} \quad \text{and} \quad \omega_n + \mu \begin{cases} \in \mathcal{D}_n (\omega_n + \omega_{n-1}), & \text{if } j = \frac{n}{2} \\ \in \mathcal{D}_n \omega_{2j-1}, & \text{otherwise} \end{cases}.$$

After counting the number of possibilities in each case, we obtain the two equations

$$2^{n-1} \theta_{n-1}^2 = 1 + \Theta_{2\omega_{n-1}} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} \quad \text{and} \quad 2^{n-1} \theta_{n-1} \theta_n = n \Theta_{\omega_n + \omega_{n-1}} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1}.$$

2. Now assume that n is odd. For $1 \leq j \leq (n+1)/2$, consider $\mu \in \mathcal{D}_n \cdot \omega_n$ with $2j-1$ positive coordinates. If $j = (n+1)/2$ then $\omega_n + \mu = 2\omega_n$. Otherwise, $\omega_n + \mu$ has $2j-1$ nonzero coordinates. The equation for $\Theta_{2\omega_n}$ is

$$2^{n-1} \theta_n^2 = \Theta_{2\omega_n} + \sum_{j=1}^{(n-1)/2} \binom{n}{2j-1} \theta_{2j-1}.$$

Finally for $0 \leq j \leq (n-1)/2$, consider $\mu \in \mathcal{D}_n \omega_{n-1}$ with $2j$ positive coordinates. Then

$$\omega_{n-1} + \mu \begin{cases} = 2\omega_{n-1}, & \text{if } j = \frac{n-1}{2}, \quad \mu_n = -\frac{1}{2} \\ \in \mathcal{D}_n \omega_{2j+1}, & \text{otherwise} \end{cases} \quad \text{and} \quad \omega_n + \mu \begin{cases} = 0, & \text{if } j = 0 \\ \in \mathcal{D}_n (\omega_n + \omega_{n-1}), & \text{if } j = \frac{n-1}{2} \\ \in \mathcal{D}_n \omega_{2j}, & \text{otherwise} \end{cases}.$$

After counting the number of possibilities in each case, we obtain the two equations

$$2^{n-1} \theta_{n-1}^2 = \Theta_{2\omega_{n-1}} + \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1} \quad \text{and} \quad 2^{n-1} \theta_{n-1} \theta_n = 1 + n \Theta_{\omega_n + \omega_{n-1}} + \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j}.$$

This completes the proof. \square

7.2 Hermite matrix for \mathcal{D}_n

Proposition 7.3. *Define the map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-2} x_{n-3}^{-1}, x_n x_{n-1} x_{n-2}^{-1}, x_n x_{n-1}^{-1}). \end{aligned}$$

1. ψ is surjective. The preimage of \mathbb{T}^n under ψ is \mathbb{T}^n .
2. Every point in \mathbb{T}^n has exactly two distinct preimages $x, x' \in \mathbb{T}^n$ with $x' = (x_1, \dots, x_{n-2}, -x_{n-1}, -x_n)$.
3. If $x \neq x' \in \mathbb{T}^n$ with $\psi(x) = \psi(x')$, then $\theta_i(x) = \theta_i(x')$ for $1 \leq i \leq n-2$ and $\theta_{n-1}(x) = -\theta_{n-1}(x')$, $\theta_n(x) = -\theta_n(x')$.
4. For all $x \in \mathbb{T}$, there exists $\tilde{x} \in \mathbb{T}$, such that $\theta_i(x) = \theta_i(\tilde{x})$ for $1 \leq i \leq n-2$ and $\theta_{n-1}(x) = \theta_n(\tilde{x})$, $\theta_n(x) = \theta_{n-1}(\tilde{x})$.

Proof. 1. and 2. For all $y \in (\mathbb{C}^*)^n$, there exists $x \in (\mathbb{C}^*)^n$ with $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-2} = y_{n-2} x_{n-3}$ and $x_{n-1}^2 = y_{n-1}^{-1} y_{n-1} x_{n-2}, x_n = y_n x_{n-1}$. Hence, x is uniquely determined by y up to a sign in x_{n-1}, x_n . We have $y \in \mathbb{T}^n$ if and only if $x \in \mathbb{T}^n$.

3. We deduce from the Coxeter diagram in Equation (5.2) that $\text{Stab}_{\mathcal{D}_n}(\omega_n) \cong \text{Stab}_{\mathcal{D}_n}(\omega_{n-1}) \cong \mathcal{A}_{n-1} \cong \mathfrak{S}_n$. Hence, $\mathcal{D}_n \omega_{n-1} = \{\pm 1\}_+^n \omega_{n-1}$, respectively $\mathcal{D}_n \omega_n = \{\pm 1\}_+^n \omega_n$, and by Proposition 7.1, this orbit consists of elements

$$\begin{aligned} \mu &= \frac{\epsilon_1}{2} e_1 + \dots + \frac{\epsilon_n}{2} e_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-2} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_{n-1}}{2} (\omega_n + \omega_{n-1} - \omega_{n-2}) + \frac{\epsilon_n}{2} (\omega_n - \omega_{n-1}) \\ &= \frac{\epsilon_{n-1} + \epsilon_n}{2} \omega_n + \frac{\epsilon_{n-1} - \epsilon_n}{2} \omega_{n-1} + \nu \in \Omega \end{aligned}$$

with $\epsilon_i \in \{\pm 1\}$ for $1 \leq i \leq n$ satisfying

$$\epsilon_1 \dots \epsilon_n = \begin{cases} -1, & \text{if } \mu \in \mathcal{D}_n \omega_{n-1} \\ 1, & \text{if } \mu \in \mathcal{D}_n \omega_n \end{cases}$$

and $\nu \in \mathbb{Z} \omega_1 \oplus \dots \oplus \mathbb{Z} \omega_{n-2}$. Let $\alpha, \beta \in \mathbb{Z}^n$ be the coordinate vectors of μ, ν in Ω . Then $\beta_{n-1} = \beta_n = 0$ and the monomial in θ_{n-1} , respectively θ_n , corresponding to μ is

$$x^\alpha = x_1^{\beta_1} \dots x_{n-2}^{\beta_{n-2}} x_{n-1}^{(\epsilon_{n-1} - \epsilon_n)/2} x_n^{(\epsilon_{n-1} + \epsilon_n)/2}$$

with $(\epsilon_{n-1} \pm \epsilon_n)/2 \in \{-1, 0, 1\}$. Therefore, x^α is linear in x_{n-1} and independent of x_n or vice versa. With x, x' from the hypothesis we have $x^\alpha = -(x')^\alpha$. Since every monomial in θ_{n-1} , respectively θ_n , can be written in terms of such β, ϵ , we obtain $\theta_{n-1}(x) = -\theta_{n-1}(x')$ as well as $\theta_n(x) = -\theta_n(x')$. Furthermore, $\theta_i(x) = \theta_i(x')$ holds according to [Proposition 7.1](#), since $y_k(x) = y_k(x')$ for all $1 \leq k \leq n$.

4. Following the proof for the third statement, this holds for $\tilde{x} := (x_1, \dots, x_{n-2}, x_n, x_{n-1})$. \square

We state the main result for Weyl groups of type \mathcal{D} .

Theorem 7.4. *Consider the n -dimensional \mathbb{R} -vector space*

$$\mathcal{Z} := \begin{cases} \mathbb{R}^n, & \text{if } n \text{ is even} \\ \{z \in \mathbb{C}^n \mid z_1, \dots, z_{n-2} \in \mathbb{R}, \bar{z}_n = z_{n-1}\}, & \text{if } n \text{ is odd} \end{cases}$$

and define the matrix $H \in \mathbb{Q}[z]^{n \times n}$ by

$$H(z)_{ij} = \text{Tr}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix}, \quad \text{where } c_i(z) = (-2)^i \binom{n}{i} z_i \quad \text{for } 1 \leq i \leq n-2 \quad \text{and}$$

$$c_{n-1}(z) = (-2)^{n-1} \begin{cases} 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} z_{2j-1}, & \text{if } n \text{ is even} \\ 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-3)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is odd} \end{cases},$$

$$c_n(z) = (-2)^n \begin{cases} 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is even} \\ 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} z_{2j+1}, & \text{if } n \text{ is odd} \end{cases}.$$

For all $z \in \mathcal{Z}$, $H(z) \in \mathbb{R}^{n \times n}$ and $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$.

Proof. Let $z \in \mathbb{C}^n$ and set $c_i := c_i(z)$ for $1 \leq i \leq n$.

To show “ \subseteq ”, assume that $z \in \mathcal{T}$. Then there exists $x \in \mathbb{T}^n$, such that $\theta_i(x) = z_i$ for $1 \leq i \leq n$. Furthermore, we have $z \in \mathcal{Z}$ and $c_i \in \mathbb{R}$. By [Proposition 7.1](#) and [Lemma 7.2](#), the solution of

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i \tilde{c}_i \quad \text{for } 1 \leq i \leq n$$

is $y = \psi(x) \in \mathbb{T}^n$, where ψ is the map from [Proposition 7.3](#). Applying [Corollary 3.7](#) yields $H(z) \succeq 0$.

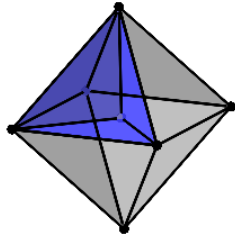
For “ \supseteq ” on the other hand, assume $z \in \mathcal{Z}$ with $H(z) \succeq 0$. Hence, $c_i \in \mathbb{R}$ and by [Corollary 3.7](#), the solution y of the above system (II) is contained in \mathbb{T}^n . According to [Proposition 7.3](#), y has two distinct preimages $x, x' \in \mathbb{T}^n$. We have $z_i = \theta_i(x) = \theta_i(x')$ for $1 \leq i \leq n-2$ and $\theta_{n-1}(x) = -\theta_{n-1}(x')$, $\theta_n(x) = -\theta_n(x')$. Furthermore, $z_{n-1}^2 + z_n^2 = \theta_{n-1}(x)^2 + \theta_n(x)^2 = \theta_{n-1}(x')^2 + \theta_n(x')^2$ and $z_{n-1} z_n = \theta_{n-1}(x) \theta_n(x) = \theta_{n-1}(x') \theta_n(x')$. Therefore, $\{z_{n-1}, z_n\} \in \{\{\theta_{n-1}(x), \theta_n(x)\}, \{\theta_{n-1}(x'), \theta_n(x')\}\}$. If $z_{n-1} = \theta_{n-1}(x)$, $z_n = \theta_n(x)$, then $z = \vartheta(x)$. Otherwise by [Proposition 7.3](#), there exists \tilde{x} , such that $z_{n-1} = \theta_{n-1}(\tilde{x})$, $z_n = \theta_n(\tilde{x})$ and $z = \vartheta(\tilde{x})$. An analogous argument applies to x' and thus, z is contained in \mathcal{T} . \square

Remark 7.5.

1. If n is odd, we have $-\omega_{n-1} \in \mathcal{D}_n \omega_n$. In this case, $\mathcal{T} \not\subseteq \mathbb{R}^n$. We follow [Equation \(2.8\)](#) and substitute $z_n = \Re(\tilde{z}_n)$ and $z_{n-1} = \Im(\tilde{z}_n)$. This leaves us with the substitutions $z_n z_{n-1} \mapsto z_n^2 + z_{n-1}^2$ in $c_{n-1}(z)$ and $z_n^2 + z_{n-1}^2 \mapsto 2(z_n^2 - z_{n-1}^2)$ in $c_n(z)$. The image of this new map $\vartheta_{\mathbb{R}}$ is contained in $[-1, 1]^n$.
2. The entries $H(z)_{ij} \in \mathbb{Q}[z]$ have degree $i + j$. Independent of the parity of n , the determinant of $H(z)$ has degree $3n + 1$.

7.3 Example \mathcal{D}_4

In this subsection, we further investigate the case $n = 4$.



$$\mathcal{D}_4 \cong \mathfrak{S}_4 \times \{\pm 1\}_+^4$$

$$\begin{aligned} \omega_1 &= [1, 0, 0, 0]^t \\ \omega_2 &= [1, 1, 0, 0]^t \\ \omega_3 &= [1, 1, 1, -1]^t/2 \\ \omega_4 &= [1, 1, 1, 1]^t/2 \end{aligned}$$

Figure 9: A partial projection of the root system \mathcal{D}_4 in \mathbb{R}^4 with fundamental weights $\omega_1, \omega_2, \omega_3, \omega_4$.

The 4-simplex with vertices $0, \omega_1, \omega_2/2, \omega_3, \omega_4$ is mapped under reflection by $\mathcal{D}_4 \cong \mathfrak{S}_4 \times \{\pm 1\}_+^4$ to an icositetrachoron. This is a polytope in \mathbb{R}^4 with boundary consisting of 24 octahedral cells. In [Figure 9](#), we depict a projection of the unique cell with center $\omega_2/2$ and three out of six vertices $\omega_1, \omega_3, \omega_4$ to \mathbb{R}^3 . The blue shaded region is the 3-simplex with projected vertices $\omega_1, \omega_2/2, \omega_3, \omega_4$.

Let $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$. Then the matrix $H(z)$ from [Theorem 7.4](#) has determinant

$$\begin{aligned} \text{Det}(H(z)) = & 4294967296 (z_3 - z_4)^2 (z_3 + z_4)^2 (-512 z_1^3 z_3^3 z_4^3 - 432 z_1^4 z_3^4 - 96 z_1^4 z_3^2 z_4^2 - 432 z_1^4 z_4^4 \\ & + 864 z_1^3 z_2 z_3^3 z_4 + 864 z_1^3 z_2 z_3 z_4^3 + 144 z_1^2 z_2^2 z_3^2 z_4^2 - 96 z_1^2 z_3^4 z_4^2 - 96 z_1^2 z_3^2 z_4^4 \\ & + 864 z_1 z_2 z_3^3 z_4^3 - 432 z_3^4 z_4^4 - 384 z_1^5 z_3 z_4 + 864 z_1^4 z_2 z_3^2 + 864 z_1^4 z_2 z_4^2 - 1440 z_1^3 z_2^2 z_3 z_4 \\ & + 96 z_1^3 z_3^3 z_4 + 96 z_1^3 z_3 z_4^3 - 216 z_1^2 z_2^3 z_3^2 - 216 z_1^2 z_2^3 z_4^2 + 864 z_1^2 z_2 z_3^4 + 576 z_1^2 z_2 z_4^4 \\ & + 864 z_1^2 z_2 z_4^4 - 1440 z_1 z_2^2 z_3^3 z_4 - 1440 z_1 z_2^2 z_3 z_4^3 - 384 z_1 z_3^5 z_4 + 96 z_1 z_3^3 z_4^3 - 384 z_1 z_3 z_4^5 \\ & - 216 z_2^3 z_3^2 z_4^2 + 864 z_2 z_3^4 z_4^2 + 864 z_2 z_3^2 z_4^4 + 64 z_2^6 - 288 z_1^4 z_2^2 + 192 z_1^4 z_3^2 + 192 z_1^4 z_4^2 \\ & + 288 z_1^3 z_2 z_3 z_4 + 324 z_1^2 z_2^2 - 1872 z_1^2 z_2^2 z_3^2 - 1872 z_1^2 z_2^2 z_4^2 + 192 z_1^2 z_3^4 - 240 z_1^2 z_3^2 z_4^2 \\ & + 192 z_1^2 z_4^4 + 2376 z_1 z_2^3 z_3 z_4 + 288 z_1 z_2 z_3^3 z_4 + 288 z_1 z_2 z_3 z_4^3 + 324 z_1^4 z_2^3 + 324 z_1^2 z_4^2 \\ & - 288 z_2^2 z_3^4 - 1872 z_2^2 z_3^2 z_4^2 - 288 z_2^2 z_4^4 + 64 z_2^6 + 192 z_3^4 z_4^2 + 192 z_3^2 z_4^4 + 64 z_4^6 - 288 z_1^4 z_2 \\ & + 192 z_1^3 z_3 z_4 + 864 z_1^2 z_3^2 - 792 z_1^2 z_2 z_3^2 - 792 z_1^2 z_2 z_4^2 + 792 z_1 z_2^2 z_3 z_4 + 192 z_1 z_3^3 z_4 \\ & + 192 z_1 z_3 z_4^3 - 486 z_2^3 + 864 z_2^3 z_3^2 + 864 z_2^3 z_4^2 - 288 z_2 z_3^4 - 792 z_2 z_3^2 z_4^2 - 288 z_2 z_4^4 - 48 z_3^4 \\ & + 576 z_1^2 z_2^2 - 96 z_1^2 z_3^2 - 96 z_1^2 z_4^2 - 72 z_1 z_2 z_3 z_4 - 729 z_2^4 + 576 z_2^2 z_3^2 + 576 z_2^2 z_4^2 - 48 z_3^4 \\ & - 96 z_3^2 z_4^2 - 48 z_4^4 + 144 z_1^2 z_2 - 24 z_1 z_3 z_4 - 432 z_2^3 + 144 z_2 z_3^2 + 144 z_2 z_4^2 + 12 z_1^2 - 126 z_2^2 \\ & + 12 z_3^2 + 12 z_4^2 - 18 z_2 - 1). \end{aligned}$$

8 Conjecture

In this section, we formulate the conjecture that the \mathbb{T} -orbit space can be computed as the negativity-locus of another matrix polynomial, which is closer to the characterization of the real orbit space for the linear action of a compact Lie group [PS85, Main Theorem].

8.1 Euler derivations

We return to the general case of a finite group with an integer representation \mathcal{G} as in Equation (2.1). Assume that $\mathbb{Q}[x^\pm]^\mathcal{G} = \mathbb{Q}[\theta_1, \dots, \theta_m]$ for some $n \leq m \in \mathbb{N}$. On $\mathbb{Q}[x^\pm]$, define the **Euler derivation** $x_i \partial / \partial x_i$ with $x_i \partial / \partial x_i(x_i) = x_i$ and $x_i \partial / \partial x_i(x_j) = 0$ for $1 \leq i \neq j \leq n$ as well as the associated gradient $\tilde{\nabla} := [x_1 \partial / \partial x_1, \dots, x_n \partial / \partial x_n]^t$. We fix a symmetric positive definite matrix $S \in \mathbb{Q}^{n \times n}$ with $B^t S B = S$ for all $B \in \mathcal{G}$. For example, $S = 1/|\mathcal{G}| \sum_{B \in \mathcal{G}} B^t B$ has the desired property. For $F, F' \in \mathbb{Q}[x^\pm]^n$, write $\langle F, F' \rangle_S := F^t S F'$. This is an S -induced inner product, which is \mathcal{G} -invariant. First, we show that the S -induced inner product of two gradients of \mathcal{G} -invariant Laurent polynomials is again \mathcal{G} -invariant.

Proposition 8.1. *Let $f, f' \in \mathbb{Q}[x^\pm]^\mathcal{G}$.*

1. For all $B \in \mathcal{G}$, $B(\tilde{\nabla} f)(x^B) = \tilde{\nabla} f(x)$.
2. We have $\langle \tilde{\nabla} f, \tilde{\nabla} f' \rangle_S = (\tilde{\nabla} f)^t S (\tilde{\nabla} f') \in \mathbb{Q}[x^\pm]^\mathcal{G}$.

Proof. We obtain the first statement from

$$\tilde{\nabla} f(x) = \tilde{\nabla} (B \cdot f)(x) = B \left[x^{B \cdot 1} \frac{\partial f}{\partial x_1}(x^B), \dots, x^{B \cdot n} \frac{\partial f}{\partial x_n}(x^B) \right]^t = B(\tilde{\nabla} f)(x^B),$$

where we applied the hypothesis that f is \mathcal{G} -invariant and then the chain rule. Therefore,

$$\begin{aligned} (B \cdot ((\tilde{\nabla} f)^t S (\tilde{\nabla} f')))(x) &= ((\tilde{\nabla} f)(x^B))^t S ((\tilde{\nabla} f')(x^B)) \\ &= (B^{-1} \tilde{\nabla} f(x))^t S (B^{-1} \tilde{\nabla} f'(x)) \\ &= (\tilde{\nabla} f(x))^t (B^{-1})^t S B^{-1} (\tilde{\nabla} f'(x)) \\ &= ((\tilde{\nabla} f)^t S (\tilde{\nabla} f'))(x) \end{aligned}$$

proves the second statement. □

For $f \in \mathbb{Q}[x^\pm]$, define $\widehat{f} \in \mathbb{Q}[x^\pm]$ by $\widehat{f}(x) = f(x^{-I_n})$.

Proposition 8.2. *Let $f \in \mathbb{Q}[x^\pm]$.*

1. We have $\tilde{\nabla} \widehat{f}(x) = -(\widehat{\tilde{\nabla} f})(x)$.
2. If $f \in \mathbb{Q}[x^\pm]^\mathcal{G}$, then $\widehat{f} \in \mathbb{Q}[x^\pm]^\mathcal{G}$.

Proof. By the chain rule, we have

$$\tilde{\nabla} \widehat{f}(x) = - \left[x^{(-I_n) \cdot 1} \frac{\partial f}{\partial x_1}(x^{-I_n}), \dots, x^{(-I_n) \cdot n} \frac{\partial f}{\partial x_n}(x^{-I_n}) \right]^t = -(\tilde{\nabla} f)(x^{-I_n}) = -(\widehat{\tilde{\nabla} f})(x).$$

The second statement follows from the fact that for all $B \in \mathcal{G}$, $(B \cdot \widehat{f})(x) = f(x^{-B}) = f(x^{-I_n}) = \widehat{f}(x)$, if f is \mathcal{G} -invariant. □

Lemma 8.3. For $f \in \mathbb{Q}[x^\pm]^\mathcal{G}$ and $x \in \mathbb{T}^n$, $\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) \leq 0$.

Proof. By [Proposition 8.2](#), we have

$$\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) = -(\tilde{\nabla} f(x))^t S(\widehat{(\tilde{\nabla} f)}(x)) = -(\tilde{\nabla} f(x))^t S((\tilde{\nabla} f)(x^{-I_n})).$$

Since $x \in \mathbb{T}^n$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) = (x_1^{-1}, \dots, x_n^{-1}) = x^{-I_n}$ and thus,

$$\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) = -(\tilde{\nabla} f(x))^t S((\tilde{\nabla} f)(\bar{x})) = -(\tilde{\nabla} f(x))^t S(\overline{(\tilde{\nabla} f)(x)}) \leq 0$$

completes the proof. □

We can now state a necessary condition for points in \mathbb{C}^n to be contained in the \mathbb{T} -orbit space of \mathcal{G} given by the image of \mathbb{T}^n under ϑ .

Corollary 8.4. Define the matrix $\tilde{M} \in (\mathbb{Q}[x^\pm]^\mathcal{G})^{m \times m}$ with entries $\tilde{M}_{ij} = \langle \tilde{\nabla} \theta_i, \tilde{\nabla} \widehat{\theta}_j \rangle_S$. For $x \in \mathbb{T}^n$, $\tilde{M}(x)$ is Hermitian negative semi-definite.

Proof. Let $J := [\tilde{\nabla} \theta_1 \mid \dots \mid \tilde{\nabla} \theta_m]$ be the Jacobian transpose of ϑ with respect to the Euler derivations and assume that $S = C^t C$ is the Cholesky decomposition of S for an upper triangular matrix $C \in \mathbb{Q}^{n \times n}$. We write $\tilde{M}(x) = J(x)^t S \widehat{J}(x)$. Since $x \in \mathbb{T}^n$, $\bar{x} = x^{-I_n}$ and consequently $\widehat{J}(x) = -\overline{J(x)}$. Then

$$\tilde{M}(x) = J(x)^t C^t C J(x) = -(C J(x))^t (\overline{C J(x)})$$

is Hermitian negative semi-definite. □

8.2 Characterization of the \mathbb{T} -orbit space

The following characterization is similar to the one we used in the introductory [Example 1.1](#) for the linear case in the sense that we compute derivations of the fundamental invariants, but now consider a negativity locus.

Conjecture 8.5. Let $M \in \mathbb{Q}[z_1, \dots, z_m]^{m \times m}$, such that $\tilde{M}(x) = M(\theta_1(x), \dots, \theta_m(x))$. Furthermore, let $\mathcal{I} \subseteq \mathbb{Q}[z]$ be the ideal of relations among the $\theta_1, \dots, \theta_m$ and denote by $\mathcal{V}(\mathcal{I}) \subseteq \mathbb{C}^m$ the variety of \mathcal{I} .

If $z \in \mathcal{V}(\mathcal{I})$ is such that $M(z)$ is Hermitian negative semi-definite, then $z \in \mathcal{T}$.

With [Corollary 8.4](#), we obtain a necessary and sufficient condition and therefore a characterization of the \mathbb{T} -orbit space as a basic semi-algebraic set. There are upsides and downsides for using M instead of the Hermite matrix polynomial H for Weyl groups. Generically, the matrix M admits the same number of necessary polynomial inequalities, respectively one less in the case of \mathcal{A}_{n-1} . There is to our knowledge no closed expression for M . Therefore, one needs to express the entries of $M(\theta_1, \dots, \theta_n)$ as polynomials in the fundamental invariants.

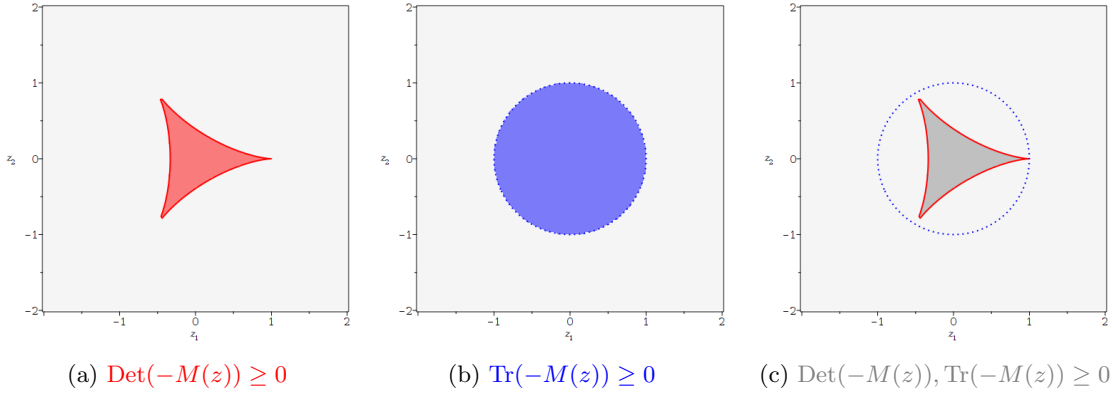
Remark 8.6. The entry H_{ij} of the Hermite matrix polynomial has degree $i+j$ for $1 \leq i, j \leq n$. [Remarks 4.4, 5.4, 6.5 and 7.5](#) state that the degree of the determinant of H can be expected to be linear in n . In general, the degree of the entries M_{ij} can not be predicted. For $\mathcal{G} \cong \mathcal{D}_4$, we have $\deg(M_{22}) = 3$ and $\deg(\text{Det}(M)) = 9$.

To support [Conjecture 8.5](#), we give a visualization of the two-dimensional cases for the groups, which we treated in the main results of this article.

Example 8.7.

1. Let $z = (z_1 + iz_2, z_1 - iz_2) \in \mathbb{C}^2$ and consider $\mathfrak{G} = \mathcal{A}_2$. The matrix $H(z) \in \mathbb{Q}[z]^{3 \times 3}$ from [Theorem 4.3](#), which characterizes the \mathbb{T} -orbit space of \mathcal{A}_2 , was computed in [Section 4.3](#), so that we obtain a description of \mathcal{T} in terms of polynomials $h_0(z), h_1(z), h_2(z) \in \mathbb{Q}[z]$. The Hermitian matrix $M(z)$ from [Conjecture 8.5](#) is

$$M(z) = \frac{2}{3} \begin{bmatrix} z_1^2 + z_2^2 - 1 & 2(z_1 + iz_2)^2 - 2(z_1 - iz_2)^2 \\ 2(z_1 - iz_2)^2 - 2(z_1 + iz_2)^2 & z_1^2 + z_2^2 - 1 \end{bmatrix} \in \mathbb{Q}[z]^{2 \times 2}.$$


 Figure 10: Vanishing points and positivity regions for determinant and trace of $M(z)$ in the case \mathcal{A}_2 .

As in [Section 4.3](#), we observe a \mathfrak{D}_3 -invariance

$$\begin{aligned} \text{Det}(-M(z)) &= \frac{1}{9}(-12z_1^4 - 24z_1^2z_2^2 - 12z_2^4 + 32z_1^3 - 96z_1z_2^2 - 24z_1^2 - 24z_2^2 + 4) && \text{(solid)} \\ &= \frac{4}{9}(-3g_1(z)^2 - 6g_1(z) + 8g_2(z) + 1), \\ \text{Tr}(-M(z)) &= -\frac{4}{3}(z_1^2 + z_2^2 - 1) && \text{(dots)} \\ &= -\frac{4}{3}(g_1(z) - 1), \end{aligned}$$

where $g_1(z) := z_1^2 + z_2^2$, $g_2(z) := z_1(z_1^2 - 3z_2^2)$ generate the ring of \mathfrak{D}_3 -invariant polynomials. One can observe that $\text{Det}(-M(z))$ divides $h_0(z) = \text{Det}(H(z))$ and the quotient $(3z_1 + 1)^2 z_2^4$ (up to a positive scalar) is a square.

2. The property “ $\text{Det}(-M(z))$ divides $\text{Det}(H(z))$ in $\mathbb{Q}[z]$ ” is also true for the other two-dimensional cases. We have

$$\begin{aligned} \mathcal{A}_2: \quad \text{Det}(H(z)) &= \frac{19683}{256}(3z_1 + 1)^2 z_2^4 \text{Det}(-M(z)) && \text{(solid)} , \\ \mathcal{B}_2: \quad \text{Det}(H(z)) &= \frac{16384}{9} z_2^2 \text{Det}(-M(z)) && \text{(solid)} , \\ \mathcal{C}_2: \quad \text{Det}(H(z)) &= \frac{1024}{9} \text{Det}(-M(z)) && \text{(solid)} , \end{aligned}$$

which are depicted in [Figure 11](#).

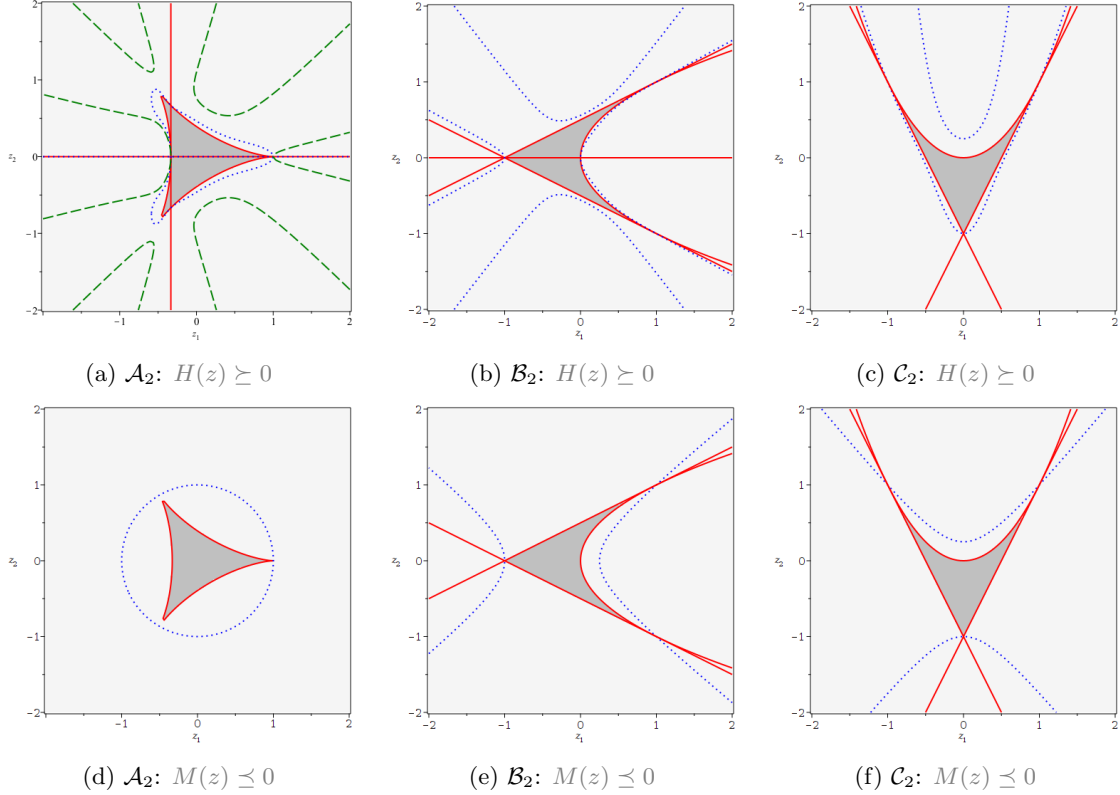


Figure 11: The regions “ $H(z) \succeq 0$ ” and “ $M(z) \preceq 0$ ” for $\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2$.

9 Orthogonality of Chebyshev polynomials

The content of this section was the original motivation to study \mathbb{T} -orbit spaces. In [HMMR22], we apply the results of this article to solve an optimization problem for symmetric trigonometric functions as a polynomial optimization problem on the \mathbb{T} -orbit space. A specific application is the computation of the spectral bound for the chromatic number of an infinite graph.

The image \mathcal{T} of ϑ that we describe in this article also appears in other contexts of interest to analysis. The equations were known for specific examples such as the deltoid in Figure 4 and Steiner’s hypocycloid in Figure 2. These domains contain sampling points with logarithmic Lebesgue number [MKNR13, Theorem 1.24] and minimal cubature points [LSX12, HMP16]. They are also the region of orthogonality for a family of generalized Chebyshev polynomials [EL82, HW88, LU13]. As far as we know, we give in this paper their first explicit general characterization as semi-algebraic sets.

In this section, we review the orthogonality of generalized Chebyshev polynomials in terms of multiplicative invariants and characterize the determinant of the matrix M from Conjecture 8.5.

9.1 Generalized cosine functions

Assume that \mathfrak{G} is the Weyl group of a rank n root system R , which is represented over \mathbb{R}^n equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle$. \mathfrak{G} is the subgroup of the orthogonal matrix group $O_n(\mathbb{R})$, that is generated by the orthogonal reflections s_ρ for $\rho \in R$. The **affine Weyl group** is the semi-direct product of \mathfrak{G} with the Abelian group of translations with respect to the lattice of coroots Λ , that is the \mathbb{Z} -module generated

by \mathbb{R}^\vee [Bou68, Chapitre VI, §2, Proposition 1]. Assume that ρ_1, \dots, ρ_n form a base of \mathbb{R} and that ρ_0 is the highest root. For μ in the weight lattice Ω , define the map

$$\begin{aligned} \mathfrak{e}_\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \exp(-2\pi i \langle \mu, u \rangle). \end{aligned} \quad (9.1)$$

Lemma 9.1.

1. A fundamental domain for $\mathfrak{G} \times \Lambda$ is the closed alcove

$$\Delta := \{u \in \mathbb{R}^n \mid \langle u, \rho_0 \rangle \leq 1, \langle u, \rho_1 \rangle \geq 0, \dots, \langle u, \rho_n \rangle \geq 0\}.$$

and $\mathfrak{G} \Delta = \bigcup_{A \in \mathfrak{G}} A \Delta$ is the closure of a fundamental domain for the group of translations Λ .

2. $\{\mathfrak{e}_\mu \mid \mu \in \Omega\}$ is an orthonormal basis for both

- the Λ -periodic locally square integrable function $L^2(\mathbb{R}^n/\Lambda)$ and
- the square integrable functions on $\mathfrak{G} \Delta$

with respect to the inner product

$$(f, g) \mapsto \frac{1}{|\mathfrak{G}| \text{Vol}(\Delta)} \int_{\mathfrak{G} \Delta} f(u) \overline{g(u)} \, du,$$

where $\text{Vol}(\Delta)$ is the Lebesgue measure of Δ in \mathbb{R}^n .

Proof. The first assertion of 1. is [Bou68, Chapitre VI, §2.1]. The second assertion of 1. follows from [LX10, Equation (3.3)].

The second statement is a consequence of [Fug74, Section 5]. \square

Definition 9.2. The *generalized cosine function* associated to $\mu \in \Omega$ is the \mathfrak{G} -invariant Λ -periodic function

$$\begin{aligned} \mathfrak{c}_\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} \mathfrak{e}_{A\mu}(u). \end{aligned}$$

The *generalized sine function* associated to $\mu \in \Omega$ is the \mathfrak{G} -anti-invariant Λ -periodic function

$$\begin{aligned} \mathfrak{s}_\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} \text{Det}(A) \mathfrak{e}_{A\mu}(u). \end{aligned}$$

Let $\mu, \nu \in \Omega$. By Lemma 9.1 and the \mathfrak{G} -invariance, we have

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} \mathfrak{c}_\mu \overline{\mathfrak{c}_\nu} \, du = \begin{cases} \frac{|\text{Stab}_{\mathfrak{G}}(\mu)|}{|\mathfrak{G}|}, & \text{if } \mu \in \mathfrak{G} \nu \\ 0, & \text{otherwise} \end{cases}. \quad (9.2)$$

For $\alpha \in \mathbb{N}^n \setminus ([1, \dots, 1]^t + \mathbb{N}^n)$, $\mu = W\alpha$ lies on a wall of a Weyl chamber [Bou68, Chapitre V, §1.4] and thus $\mathfrak{s}_\mu = 0$. For $\alpha \in [1, \dots, 1]^t + \mathbb{N}^n$ on the other hand, $\mu = W\alpha$ is strongly dominant and we have $|\text{Stab}_{\mathfrak{G}}(\mu)| = 1$. Hence, for strongly dominant weights $\mu, \nu \in \Omega$,

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} \mathfrak{s}_\mu \overline{\mathfrak{s}_\nu} \, du = \begin{cases} \frac{1}{|\mathfrak{G}|}, & \text{if } \mu \in \mathfrak{G} \nu \\ 0, & \text{otherwise.} \end{cases} \quad (9.3)$$

We show that the image of the generalized cosine functions associated to the fundamental weights is precisely the \mathbb{T} -orbit space \mathcal{T} as in Definition 2.5.

Lemma 9.3. *We have $\mathcal{T} = \mathbf{c}(\Delta) = \mathbf{c}(\mathbb{R}^n)$, where $\mathbf{c}(u) := (\mathbf{c}_{\omega_1}(u), \dots, \mathbf{c}_{\omega_n}(u)) \in \mathbb{C}^n$ for $u \in \mathbb{R}^n$.*

Proof. Note that for $\mu \in \Omega, \alpha \in \mathbb{Z}^n$ with $W\alpha = \mu$, we have $\mathbf{c}_\mu = \Theta_\alpha \circ (\mathbf{e}_{\omega_1}, \dots, \mathbf{e}_{\omega_n})$. Indeed, $\mathbf{e}_{\omega_1}^{\alpha_1} \dots \mathbf{e}_{\omega_n}^{\alpha_n} = \mathbf{e}_\mu$, and if $A \in \mathfrak{G}, B \in \mathfrak{G}$ with $A = WBW^{-1}$ and $\beta := B\alpha \in \mathbb{Z}^n$, then $\mathbf{e}_{A\mu} = \mathbf{e}_{\omega_1}^{\beta_1} \dots \mathbf{e}_{\omega_n}^{\beta_n}$. Now, the map

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{T}^n, \\ u &\mapsto (\mathbf{e}_{\omega_1}(u), \dots, \mathbf{e}_{\omega_n}(u)) \end{aligned}$$

is surjective, because the fundamental weights form a basis of \mathbb{R}^n . Hence, $\mathbf{c}(\mathbb{R}^n) = \vartheta(\mathbb{T}^n) = \mathcal{T}$. We have $\mathbf{c}(\mathbb{R}^n) = \mathbf{c}(\Delta)$, because \mathbf{c}_{ω_j} is \mathfrak{G} -invariant, Λ -periodic for all $1 \leq j \leq n$, and, by Lemma 9.1, Δ is a fundamental domain for $\mathfrak{G} \times \Lambda$. \square

9.2 Generalized Chebyshev polynomials

Recall that $\mathbb{Q}[x^\pm]^\mathfrak{G} = \mathbb{Q}[\theta_1, \dots, \theta_n]$ from Theorem 2.11. For $\alpha \in \mathbb{Z}^n$, there thus exists a unique multivariate polynomial $T_\alpha \in \mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$, such that $T_\alpha(\theta_1, \dots, \theta_n) = \Theta_\alpha$. We call $\{T_\alpha \in \mathbb{Q}[z] \mid \alpha \in \mathbb{Z}^n\}$ the **generalized Chebyshev polynomials of the first kind** associated to the root system R .

If $\mu \in \Omega$ is such that $W\alpha = \mu$, then $\mathbf{c}_\mu = \Theta_\alpha \circ (\mathbf{e}_{\omega_1}, \dots, \mathbf{e}_{\omega_n}) = T_\alpha \circ (\mathbf{c}_{\omega_1}, \dots, \mathbf{c}_{\omega_n})$. Thus, Definition 9.2 is a generalization of the univariate case $T_\ell(\cos(u)) = \cos(\ell u)$ for $\ell \in \mathbb{N}$, that corresponds to the root system A_1 .

The second kind Chebyshev polynomials are defined through the character polynomials. Thanks to Weyl's character formula [Bou75, Chapitre VIII, §9, Théorème 1], those are given by

$$\Xi_\alpha(x) = \frac{\Upsilon_{\alpha+\delta}(x)}{\Upsilon_\delta(x)} \in \mathbb{Q}[x^\pm]^\mathfrak{G} \quad (9.4)$$

for $\alpha \in \mathbb{Z}^n$, where $\delta := [1, \dots, 1]^t \in \mathbb{Z}^n$ and $\Upsilon_\alpha := \sum_{B \in \mathfrak{G}} \text{Det}(B) x^{B\alpha} \in \mathbb{Q}[x^\pm]$ is anti-invariant. The anti-invariant Laurent polynomials form a free $\mathbb{Q}[x^\pm]^\mathfrak{G}$ -module of rank 1, which is generated by Υ_δ . Υ_δ is also known as the ‘‘Weyl denominator’’. The **generalized Chebyshev polynomials of the second kind** associated to the root system R are the unique $\{U_\alpha \in \mathbb{Q}[z] \mid \alpha \in \mathbb{Z}^n\}$ satisfying $U_\alpha(\theta_1, \dots, \theta_n) = \Xi_\alpha$. Hence, for $W\alpha = \mu \in \Omega$ and $\omega_0 := W\delta = \omega_1 + \dots + \omega_n$,

$$U_\alpha \circ (\mathbf{c}_{\omega_1}, \dots, \mathbf{c}_{\omega_n}) = \frac{\Upsilon_{\alpha+\delta}}{\Upsilon_\delta} \circ (\mathbf{e}_{\omega_1}, \dots, \mathbf{e}_{\omega_n}) = \frac{\mathfrak{s}_{\mu+\omega_0}}{\mathfrak{s}_{\omega_0}}. \quad (9.5)$$

In the classical univariate case $\mathfrak{G} = A_1 = \{\pm 1\}$, we have

$$\Xi_\ell(x) = \frac{x^{\ell+1} - x^{-\ell-1}}{x - x^{-1}} = x^\ell + x^{\ell-1} + \dots + x^{1-\ell} + x^{-\ell} \quad (9.6)$$

and thus $U_\ell(\cos(u)) = \sin((\ell+1)u)/\sin(u)$ for $\ell \in \mathbb{N}$.

Lemma 9.4. *We have $\Upsilon_{-\delta}(x) = \pm \Upsilon_\delta(x)$ and $\Upsilon_\delta(\mathbb{T}^n)^2 \subseteq \mathbb{R}$.*

Proof. Recall the proof of Proposition 2.13, where we showed the existence of $A \in \mathfrak{G}, \sigma \in \mathfrak{S}_n$, such that $A\omega_i = -\omega_{\sigma(i)}$ for all $1 \leq i \leq n$. Then

$$-W\delta = -\sum_{i=1}^n \omega_i = \sum_{i=1}^n -\omega_{\sigma(i)} = A \sum_{i=1}^n \omega_i = AW\delta$$

and thus $\Upsilon_{-\delta}(x) = \text{Det}(A) \Upsilon_\delta(x)$. Especially, $\Upsilon_\delta(x)$ is either real or pure imaginary for $x \in \mathbb{T}^n$ and so its square is real. \square

We shall review now that \mathcal{T} is the domain of orthogonality of the generalized Chebyshev polynomials and show that the weight of orthogonality can be given in terms of Υ_δ .

Lemma 9.5. *Let $J := [\tilde{\nabla} \theta_1 | \dots | \tilde{\nabla} \theta_n] \in \mathbb{Q}[x^\pm]^{n \times n}$. Then $\text{Det}(J) \in \mathbb{Q}[x^\pm]$ is anti-invariant with*

$$\text{Det}(J) = \frac{\prod_{i=1}^n |\text{Stab}_{\mathcal{G}}(e_i)|}{|\mathcal{G}|^n} \Upsilon_\delta.$$

Proof. By [Proposition 8.1](#), we have $J(x^B) = B J(x)$ for any $B \in \mathcal{G}$. Hence, $\text{Det}(J)$ is anti-invariant in $\mathbb{Q}[x^\pm]$ and thus there exists a unique $f \in \mathbb{Q}[x^\pm]^\mathcal{G}$, such that $\text{Det}(J) = f \Upsilon_\delta$. We show that x^δ is the highest monomial of $\text{Det}(J)$ in the partial ordering given in [\[Bou75, Chapitre VIII, §6, Lemma 2\]](#). Note that

$$J_{ij} = [\tilde{\nabla} \theta_1 | \dots | \tilde{\nabla} \theta_n]_{ij} = x_i \partial / \partial x_i \theta_j = \frac{|\text{Stab}_{\mathcal{G}}(e_j)|}{|\mathcal{G}|} x_i \frac{\partial}{\partial x_i} \sum_{\alpha \in \mathcal{G} e_j} x^\alpha = \frac{|\text{Stab}_{\mathcal{G}}(e_j)|}{|\mathcal{G}|} x_i \frac{\partial}{\partial x_i} (x_j + \sum_{\alpha \prec e_j} c_\alpha x^\alpha),$$

where $c_\alpha \in \{0, 1\}$ and $\alpha \prec e_j$ if and only if $W(e_i - \alpha)$ is a sum of positive roots. The diagonal elements of the matrix J have maximal monomial x_i , while the non-diagonal entries have maximal monomial x^α for some $\alpha \prec e_j$. It follows that there exist coefficient $c_\alpha \in \mathbb{N}$, such that

$$\text{Det}(J) = \frac{\prod_{i=1}^n |\text{Stab}_{\mathcal{G}}(e_i)|}{|\mathcal{G}|^n} \left(\underbrace{x_1 \dots x_n}_{=x^\delta} + \sum_{\alpha \prec \delta} c_\alpha x^\alpha \right) = f \Upsilon_\delta.$$

Thus, f must be the constant, which is the given quotient that appears as the coefficient of x^δ in the determinant. \square

On the set of square integrable functions on $\mathfrak{G} \triangle$ with basis $\{\mathfrak{e}_\mu \mid \mu \in \Omega\}$, we define the gradient $\nabla := [\partial / \partial u_1, \dots, \partial / \partial u_n]^\dagger$. The relation to the gradient of Euler derivations $\tilde{\nabla}$ is given by the next statement.

Lemma 9.6. *We have*

$$[\nabla \mathfrak{e}_{\omega_1} | \dots | \nabla \mathfrak{e}_{\omega_n}](u) = 2\pi i W J(\mathfrak{e}_{\omega_1}(u), \dots, \mathfrak{e}_{\omega_n}(u)),$$

where $J = [\tilde{\nabla} \theta_1 | \dots | \tilde{\nabla} \theta_n] \in \mathbb{Q}[x^\pm]^{n \times n}$.

Proof. This is a straightforward computation from the definition

$$\begin{aligned} [\nabla \mathfrak{e}_{\omega_1} | \dots | \nabla \mathfrak{e}_{\omega_n}]_{ij}(u) &= \frac{\partial \mathfrak{e}_{\omega_j}}{\partial u_i}(u) = 2\pi i \sum_{k=1}^n (\omega_k)_i x_k \frac{\partial \theta_j}{\partial x_k}(\mathfrak{e}_{\omega_1}(u), \dots, \mathfrak{e}_{\omega_n}(u)) \\ &= 2\pi i \sum_{k=1}^n W_{ik} J_{kj}(\mathfrak{e}_{\omega_1}(u), \dots, \mathfrak{e}_{\omega_n}(u)) = 2\pi i (W J(\mathfrak{e}_{\omega_1}(u), \dots, \mathfrak{e}_{\omega_n}(u)))_{ij}. \end{aligned}$$

\square

The generalized Chebyshev polynomials satisfy the following orthogonality property. This was also shown for instance in [\[HW88, Theorem 5.1\]](#).

Theorem 9.7. *For $\alpha, \beta \in \mathbb{N}^n$,*

$$\int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} |\phi(z)|^{-1/2} dz = \begin{cases} 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G} e_i| \frac{\text{Vol}(\Delta) |\text{Stab}_{\mathcal{G}}(\alpha)|}{|\mathcal{G}|}, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases}$$

and

$$\int_{\mathcal{T}} U_\alpha(z) \overline{U_\beta(z)} |\phi(z)|^{1/2} dz = \begin{cases} 2\pi |\text{Det}(W)| \frac{\text{Vol}(\Delta)}{|\mathcal{G}|}, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases},$$

where the weight function is defined via

$$\phi(\theta_1, \dots, \theta_n) = (\Upsilon_\delta)^2 \in \mathbb{Q}[x^\pm]^\mathcal{G}$$

with $\delta = [1, \dots, 1]^t \in \mathbb{Z}^n$.

Proof. Assume that $W\alpha = \mu, W\beta = \nu \in \Omega$. By [Lemma 9.3](#) and the definition of generalized Chebyshev polynomials of the first kind, we can conduct the transformation

$$\begin{aligned} \int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} |\phi(z)|^{-1/2} dz &= \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} \frac{|\text{Det}([\nabla \mathbf{c}_{\omega_1} | \dots | \nabla \mathbf{c}_{\omega_n}](u))|}{|\Upsilon_\delta(\mathbf{e}_{\omega_1}(u), \dots, \mathbf{e}_{\omega_n}(u))|} du \\ &= \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} \frac{|\mathcal{G}|^n}{\prod_{i=1}^n |\text{Stab}_{\mathcal{G}}(e_i)|} \frac{|2\pi i \text{Det}(W) \text{Det}(J)|}{|\text{Det}(J)|} (\mathbf{e}_{\omega_1}(u), \dots, \mathbf{e}_{\omega_n}(u)) du \\ &= \prod_{i=1}^n |\mathcal{G} e_i| |2\pi i \text{Det}(W)| \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} du. \end{aligned}$$

According to [Equation \(9.2\)](#),

$$\int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} (\phi(z))^{-1/2} dz = 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G} e_i| \text{Vol}(\Delta) \frac{|\text{Stab}_{\mathcal{G}}(\alpha)|}{|\mathcal{G}|}$$

if $\mu \in \mathfrak{G}\nu$ and 0 otherwise.

The result on generalized Chebyshev polynomials of the second kind is analogous with

$$U_\alpha(\mathbf{c}(u)) \overline{U_\beta(\mathbf{c}(u))} = \frac{\mathfrak{s}_{\mu+\omega_0}(u)}{\mathfrak{s}_{\omega_0}} \frac{\overline{\mathfrak{s}_{\nu+\omega_0}(u)}}{\overline{\mathfrak{s}_{\omega_0}}} = \frac{\mathfrak{s}_{\mu+\omega_0}(u) \overline{\mathfrak{s}_{\nu+\omega_0}(u)}}{|\Upsilon_\delta(\mathbf{e}_{\omega_1}(u), \dots, \mathbf{e}_{\omega_n}(u))|^2}$$

and then applying [Equation \(9.3\)](#). □

Remark 9.8. In [Section 8](#) we make use of $\tilde{M} = J^t S \hat{J} \in (\mathbb{Q}[x^\pm]^\mathcal{G})^{m \times m}$ to give the equations of the \mathbb{T} -orbit space. According to [Proposition 2.13](#) in the present case of Weyl groups, the fundamental invariants $\theta_1, \dots, \theta_n$ are such that there exists a permutation $\sigma \in \mathfrak{S}_n$ with $\theta_i = \theta_{\sigma(i)}$. Hence, $\text{Det}(\hat{J}) = \text{sgn}(\sigma) \text{Det}(J)$ and thus the determinant of the matrix M from [Conjecture 8.5](#) is a scalar multiple of ϕ in $\mathbb{Q}[z]$.

Example 9.9.

1. In the univariate case \mathcal{A}_1 , we have $\mathcal{T} = [-1, 1]$ and $\Upsilon_\delta = x - x^{-1}$. Therefore,

$$(\Upsilon_\delta)^2 = x^2 - 2 + x^{-2} = 2T_2\left(\frac{x+x^{-1}}{2}\right) - 2T_0\left(\frac{x+x^{-1}}{2}\right) = 4T_1\left(\frac{x+x^{-1}}{2}\right)^2 - 4T_0\left(\frac{x+x^{-1}}{2}\right)$$

and $\phi(z) = 4(z^2 - 1)$.

2. For \mathcal{A}_2 , \mathcal{T} is the compact basic semi-algebraic set in [Figure 4](#) and

$$\begin{aligned} (\Upsilon_\delta)^2 &= (-x_1 x_2 - x_1/x_2^2 - x_2/x_1^2 + 1/(x_1 x_2) + x_1^2/x_2 + x_2^2/x_1)^2 \\ &= 6\Theta_{2\omega_1+2\omega_2} - 6\Theta_{3\omega_1} - 6\Theta_{3\omega_2} + 12\Theta_{\omega_1+\omega_2} - 6\Theta_0. \end{aligned}$$

We have $\phi(z) = 81z_1^2 z_2^2 - 108z_1^3 - 108z_2^3 + 162z_1 z_2 - 27 = 243/3 \text{Det}(M)$.

3. For \mathcal{B}_2 , \mathcal{T} is the compact basic semi-algebraic set in *Figure 8* and

$$\begin{aligned} (\Upsilon_\delta)^2 &= (-x_1/x_2^3 - x_2/x_1^2 + 1/(x_1 x_2) + x_1^2/x_2^3 + x_2^3/x_1^2 - x_1^2/x_2 - x_2^3/x_1 + x_1 x_2)^2 \\ &= -8\Theta_{4\omega_2} - 8\Theta_{3\omega_1} + 8\Theta_{2\omega_1+2\omega_2} + 16\Theta_{\omega_1+2\omega_2} - 8\Theta_{2\omega_2} - 8\Theta_{\omega_1} + 8\Theta_0. \end{aligned}$$

We have $\phi(z) = 256 z_1^2 z_2^2 - 1024 z_2^4 - 256 z_1^3 + 1536 z_1 z_2^2 - 512 z_1^2 + 256 z_2^2 - 256 z_1 = 1024/9 \text{Det}(M)$.

4. For \mathcal{C}_2 , \mathcal{T} is the compact basic semi-algebraic set in *Figure 6* and

$$\begin{aligned} (\Upsilon_\delta)^2 &= (-x_1/x_2^2 - x_2/x_1^3 + 1/(x_1 x_2) + x_1^3/x_2^2 + x_2^2/x_1^3 - x_1^3/x_2 - x_2^2/x_1 + x_1 x_2)^2 \\ &= -8\Theta_{4\omega_1} + 8\Theta_{2\omega_1+2\omega_2} + 16\Theta_{2\omega_1+\omega_2} - 8\Theta_{3\omega_2} - 8\Theta_{2\omega_1} - 8\Theta_{\omega_2} + 8\Theta_0. \end{aligned}$$

We have $\phi(z) = -1024 z_1^4 + 256 z_1^2 z_2^2 + 1536 z_1^2 z_2 - 256 z_2^3 + 256 z_1^2 - 512 z_2^2 - 256 z_2 = 256 \text{Det}(M)$.

5. In the main results *Theorems 4.3, 5.3, 6.4* and *7.4*, we have presented the matrix H in the standard monomial basis. In the basis of generalized Chebeshev polynomials of the first kind, this matrix is “much sparser”, in the sense that the entries consist of less terms with smaller coefficients. For \mathcal{D}_4 , the matrix is

$$H = 8 \begin{bmatrix} T_0 - T_{2\omega_1} & T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} \\ T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & 2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1} \\ T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & 2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1} & 5T_{\omega_1} - T_{3\omega_1} - 3T_{5\omega_1} - T_{7\omega_1} \\ 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & 2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1} & 5T_{\omega_1} - T_{3\omega_1} - 3T_{5\omega_1} - T_{7\omega_1} & 5T_0 + 4T_{2\omega_1} - 4T_{4\omega_1} - 4T_{6\omega_1} - T_{8\omega_1} \end{bmatrix}.$$

10 Conclusion

To our knowledge, we have given the first explicit and general characterization for the orbit space of a nonlinear action of a Weyl group on the compact torus. The fact that the \mathbb{T} -orbit space is a compact basic semi-algebraic set in these cases is already in itself a new result.

Future work revolves around finding such explicit descriptions also for the Weyl groups associated to root systems of types E, F, G. Using a computer algebra system, such characterizations can be given, but a unifying theory for all root systems is desirable. With a conjecture we support computations for these remaining cases, and beyond that, for any nonlinear action of a finite group on the compact torus.

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