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# Polynomial description for the $\mathbb{T}$ -Orbit Spaces of Multiplicative Actions

(for types  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ )

Evelyne Hubert\*, Tobias Metzloff\*, Cordian Riener†

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## Abstract

A finite group with an integer representation has a multiplicative action on the ring of Laurent polynomials, which is induced by a nonlinear action on the complex torus. We study the structure of the associated orbit space as the image of the fundamental invariants. For the Weyl groups of types  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$ , this image is a compact basic semi-algebraic set and we present the defining polynomial inequalities explicitly. We show how orbits correspond to solutions in the complex torus of symmetric polynomial systems and give a characterization of the orbit space as the positivity-locus of a symmetric real matrix polynomial. The resulting domain is the region of orthogonality for a family of generalized Chebyshev polynomials, which have connections to topics such as Fourier analysis and representations of Lie algebras.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
2.1	Linear group actions on $\mathbb{K}[x^\pm]$ . . . . .	4
2.2	Weyl groups . . . . .	6
2.3	Main results . . . . .	7
<b>3</b>	<b>Symmetric polynomial systems</b>	<b>8</b>
3.1	Solutions in $\mathbb{T}^n$ . . . . .	8
3.2	Hermite characterization for real roots . . . . .	10
<b>4</b>	<b>Type <math>\mathcal{A}</math></b>	<b>12</b>
4.1	Orbit polynomials for $\mathcal{A}_{n-1}$ . . . . .	13
4.2	Hermite matrix for $\mathcal{A}_{n-1}$ . . . . .	14
4.3	Example $\mathcal{A}_2$ . . . . .	15
<b>5</b>	<b>Type <math>\mathcal{C}</math></b>	<b>17</b>
5.1	Orbit polynomials for $\mathcal{C}_n$ . . . . .	17
5.2	Hermite matrix for $\mathcal{C}_n$ . . . . .	18
5.3	Example $\mathcal{C}_2$ . . . . .	19
<b>6</b>	<b>Type <math>\mathcal{B}</math></b>	<b>20</b>
6.1	Orbit polynomials for $\mathcal{B}_n$ . . . . .	20
6.2	Hermite matrix for $\mathcal{B}_n$ . . . . .	21
6.3	Example $\mathcal{B}_2$ . . . . .	23
<b>7</b>	<b>Type <math>\mathcal{D}</math></b>	<b>24</b>
7.1	Orbit polynomials for $\mathcal{D}_n$ . . . . .	24
7.2	Hermite matrix for $\mathcal{D}_n$ . . . . .	26
7.3	Example $\mathcal{D}_4$ . . . . .	29
<b>8</b>	<b>Conjecture</b>	<b>30</b>
<b>9</b>	<b>Orthogonality of Chebyshev polynomials</b>	<b>34</b>
9.1	Generalized cosine functions . . . . .	34
9.2	Generalized Chebyshev polynomials . . . . .	35
<b>10</b>	<b>Conclusion</b>	<b>38</b>

## 1 Introduction

The set of all orbits of a compact Lie group with an action on an affine variety is called orbit space. The coordinate ring of this affine variety contains the ring of invariants as a finitely generated subalgebra and the orbit space can be embedded into the variety defined by the syzygy ideal of the fundamental invariants. For a group with a real representation and a linear action, the orbit space has the structure of a semi-algebraic set and the defining polynomial equations and inequalities are given by a result due to Procesi and Schwarz [PS85, Main Theorem]. We consider finite groups with an integer representation and a nonlinear action on the complex torus with the aim to find such a polynomial description of the orbit space. The purpose of this article is to show that for a Weyl group, the orbit space as the image of the complex torus under fundamental invariants is a compact basic semi-algebraic set, and to characterize it as the positivity-locus of an explicit symmetric matrix polynomial. Being able to effectively describe the structure of an orbit space allows for applications in equivariant dynamical systems theory [Gat00, Theorem 4.2.1], symmetry reduction for polynomial optimization [RTAL13, Theorem 5.2], complex analysis [Sja93, Proposition 1.15], differential geometry [Dub98, Lemma 2.1] and quantum systems [GKP13, Section 2].

A finite group with an integer representation  $\mathcal{G}$  has a nonlinear action on the algebraic torus, which leaves the complex  $n$ -torus  $\mathbb{T}^n$  invariant and induces an action on the ring of Laurent polynomials. Given a minimal set of generators of the ring of invariants, so called fundamental invariants, the image of  $\mathbb{T}^n$  under these generators corresponds to the set of orbits  $\mathbb{T}^n/\mathcal{G}$  and is therefore an orbit space, which we call  $\mathbb{T}$ -orbit space of  $\mathcal{G}$ . Such an action is called multiplicative [Lor05] and the invariant theory for this setting is related to representations of Lie algebras [Bou68b, Hum72, Ser01, FH04]. Applications appear in Fourier analysis [Bee91, MKNR12], cubature [Xu00, LX10, LSX12, Xu15, CH18] and interpolation [HS21].

For the Weyl groups of types  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$ , the ring of invariant Laurent polynomials is a polynomial algebra. In these four cases, we find relations between the elementary symmetric polynomials and the fundamental invariants. This gives rise to a symmetric polynomial system with complex variables. We show that a point in the  $\mathbb{T}$ -orbit space corresponds to such a system, having its solutions in  $\mathbb{T}^n$ . This can be characterized by positivity of a Hermite quadratic form.

In Section 2, we formally introduce  $\mathbb{T}$ -orbit spaces of multiplicative actions by first defining the action of a group with an integer representation on the ring of Laurent polynomials and then identifying it with the multiplicative action on a group algebra of an invariant lattice. Here, we fix the notation for fundamental invariants and explain the relevance of the cases where  $\mathcal{G}$  is of type  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  or  $\mathcal{D}$ . Following with Section 3, we establish the necessary tools to study symmetric polynomial systems. For the needed characterizations, we extract Corollaries 3.7 and 3.8.

We then proceed to the main results and give a polynomial description for the  $\mathbb{T}$ -orbit space of  $\mathcal{G}$  in terms of an explicit symmetric matrix polynomial. The Sections 4 to 7 make the correspondence between points in the  $\mathbb{T}$ -orbit space and solutions of symmetric polynomial systems in  $\mathbb{T}^n$  evident. We give explicit formulae for invariants in Propositions 4.1, 5.1, 6.1 and 7.1 and apply the results from Section 3 to obtain Theorems 4.3, 5.3, 6.4 and 7.4. Examples accompany our results.

Motivated by the characterization due to Procesi and Schwarz [PS85], we formulate an alternative characterization in Section 8 with a proof of the necessary condition. We introduce derivations on  $\mathbb{R}[x^\pm]$  and state Conjecture 8.5 for groups which are not necessarily Weyl groups of crystallographic root systems.

The  $\mathbb{T}$ -orbit spaces that we describe here as basic semi-algebraic sets also appear in other contexts of interest to analysis. We show in Theorem 9.6, how the  $\mathbb{T}$ -orbit spaces we characterize in this article are domains of orthogonality for a family of generalized Chebyshev polynomials of the first kind [HW88, Theorem 5.1].

**Example 1.1.**

1. Consider the group  $\mathfrak{S}_2 \times \{\pm 1\}^2$ , where  $\mathfrak{S}_2$  denotes the symmetric group. This is the Weyl group  $\mathcal{C}_2$  with a multiplicative action on  $\mathbb{R}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ . By [Proposition 5.1](#), fundamental invariants of this action are given by

$$\begin{aligned} \Theta_1(x_1, x_2) &= 1/4 (x_1 + x_1^{-1} + x_2 x_1^{-1} + x_2^{-1} x_1) \quad \text{and} \\ \Theta_2(x_1, x_2) &= 1/4 (x_1 + x_1^{-1}) (x_2 x_1^{-1} + x_2^{-1} x_1). \end{aligned}$$

We can show with [Theorem 5.3](#) that the  $\mathbb{T}$ -orbit space as the image of  $\zeta \mapsto (\Theta_1(\zeta), \Theta_2(\zeta))$  for  $\zeta \in \mathbb{T}^2$  is the compact basic semi-algebraic set  $\{z \in \mathbb{R}^2 \mid H(z) \succeq 0\}$ , where

$$H(z) = \begin{bmatrix} -2z_1^2 + z_2 + 1 & -8z_1^3 + 6z_1 z_2 + 2z_1 \\ -8z_1^3 + 6z_1 z_2 + 2z_1 & -32z_1^4 + 8z_1^2 + 32z_1^2 z_2 - 4z_2^2 - 4z_2 \end{bmatrix}$$

2. Further interesting examples, are the Weyl groups  $\mathcal{A}_3$ ,  $\mathcal{B}_3$  and  $\mathcal{C}_3$ . We have  $\mathcal{B}_3 \cong \mathcal{C}_3$ , but we distinguish them for their different actions. Visualizations of the associated  $\mathbb{T}$ -orbit spaces are depicted below. From this example, we can already suspect that the  $\mathbb{T}$ -orbit space of a multiplicative action is in general a compact, nonconvex domain with cusps.

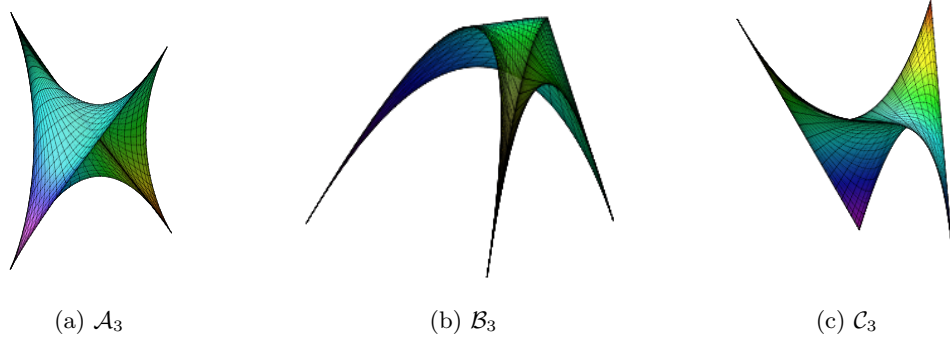


Figure 1: The  $\mathbb{T}$ -orbit spaces of the Weyl groups associated to the 3-dimensional root-systems.

## 2 Preliminaries

Let  $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$  be a field of characteristic 0 and let  $n \in \mathbb{N}$ .

In this section, we discuss the action of a finite group with a representation in  $\text{GL}_n(\mathbb{Z})$  on the ring of Laurent polynomials  $\mathbb{K}[x^\pm] = \mathbb{K}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ , which is identified with the group algebra of an invariant lattice. We motivate the problem of computing the  $\mathbb{T}$ -orbit space as the image of the complex  $n$ -torus under the fundamental invariants and then specify the problem for Weyl groups of crystallographic root systems.

### 2.1 Linear group actions on $\mathbb{K}[x^\pm]$

Let  $\mathfrak{G}$  be a finite group with an integer representation

$$\varrho : \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{Z}). \tag{2.1}$$

Denote by  $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$  the algebraic  $n$ -torus. For  $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$  and a column vector  $\beta = [\beta_1, \dots, \beta_n]^t \in \mathbb{Z}^n$ , define  $x^\beta := x_1^{\beta_1} \dots x_n^{\beta_n}$ . The matrix group  $\mathcal{G} := \varrho(\mathfrak{G})$  has a nonlinear action on

$(\mathbb{C}^*)^n$ , given by monomial maps

$$\begin{aligned} \star : \mathcal{G} \times (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ (B, x) &\mapsto B \star x := (x_1, \dots, x_n)^{B^{-1}} = (x^{B^{-1}_1}, \dots, x^{B^{-1}_n}), \end{aligned} \quad (2.2)$$

where  $B^{-1}_i \in \mathbb{Z}^n$  denotes the  $i$ -th column vector of  $B^{-1} \in \mathcal{G}$  for  $1 \leq i \leq n$ . The coordinate ring of  $(\mathbb{C}^*)^n$  is the ring of multivariate Laurent polynomials  $\mathbb{K}[x^\pm] := \mathbb{K}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ . The monomials of  $\mathbb{K}[x^\pm]$  are all  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ , where  $\alpha \in \mathbb{Z}^n$ , and  $\star$  induces the linear action

$$\begin{aligned} \cdot : \mathcal{G} \times \mathbb{K}[x^\pm] &\rightarrow \mathbb{K}[x^\pm], \\ (B, x^\alpha) &\mapsto B \cdot x^\alpha := x^{B\alpha} \end{aligned} \quad (2.3)$$

on  $\mathbb{K}[x^\pm]$ . Hence, for  $f = \sum_\alpha f_\alpha x^\alpha \in \mathbb{K}[x^\pm]$  and  $B \in \mathcal{G}$ , we write

$$(B \cdot f)(x) = f(x^B) = \sum_\alpha f_\alpha x^{B\alpha}. \quad (2.4)$$

If  $(B \cdot f)(x) = f(x)$  for all  $B \in \mathcal{G}$ , then  $f$  is called  $\mathcal{G}$ -**invariant**. The set of all  $\mathcal{G}$ -invariant Laurent polynomials is a finitely generated, but not necessarily polynomial,  $\mathbb{K}$ -algebra [Lor05, Corollary 3.3.2] denoted  $\mathbb{K}[x^\pm]^\mathcal{G}$ . Define  $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\} \subseteq \mathbb{C}^*$  and denote by  $\mathbb{T}^n$  the complex  $n$ -torus.

**Lemma 2.1.**

1.  $\mathbb{T}^n$  is left invariant by the action  $\star$ , that is  $\mathcal{G} \star \mathbb{T}^n = \mathbb{T}^n$ .
2.  $\mathbb{T}$  is the maximal subset of  $\mathbb{C}^*$ , which is closed with respect to inversion  $x \mapsto 1/x$  and all elements contained satisfy  $1/x = \bar{x}$ .

*Proof.* The first statement follows straightforward from the definition of the action.

$\mathbb{T}$  is closed under inversion. Assume that there exists  $\mathbb{T} \subsetneq X \subseteq \mathbb{C}^*$  and  $x \in X \setminus \mathbb{T}$  with modulus  $0 < r \neq 1$  and argument  $\varphi \in [0, 2\pi)$ . Then  $1/x = 1/r \exp(-\varphi) \neq r \exp(-\varphi) = \bar{x}$ .  $\square$

**Definition 2.2.** Assume that  $\mathbb{K}[x^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \dots, \theta_m]$  for some  $m \in \mathbb{N}$ . Let  $\mathcal{I} \subseteq \mathbb{K}[z]$  be the syzygy ideal of the  $\theta_1, \dots, \theta_m$ .

1. The generators  $\theta_1, \dots, \theta_m \in \mathbb{K}[x^\pm]^\mathcal{G}$  are called **fundamental invariants** of  $\mathcal{G}$ .
2. For  $\alpha \in \mathbb{Z}^n$ , we call

$$\Theta_\alpha(x) := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} x^{B\alpha} \in \mathbb{K}[x^\pm]^\mathcal{G}$$

the **orbit polynomial** associated to  $\alpha$ .

3. Define the map  $\vartheta$  as

$$\begin{aligned} \vartheta : \mathbb{T}^n &\rightarrow \mathcal{V}(\mathcal{I}), \\ x &\mapsto (\theta_1(x), \dots, \theta_m(x)), \end{aligned}$$

where  $\mathcal{V}(\mathcal{I}) \subseteq \mathbb{C}^m$  is the variety of  $\mathcal{I}$ .

A basis for  $\mathbb{K}[x^\pm]^\mathcal{G}$  as a  $\mathbb{K}$ -vector space is given by  $\{\Theta_\alpha \mid \alpha \in \mathbb{Z}^n/\mathcal{G}\}$  [Lor05, Equation (3.4)]. Our goal is to describe the structure of the orbit space for the nonlinear action  $\star$  with fundamental invariants of  $\mathbb{K}[x^\pm]^\mathcal{G}$ . To achieve this, we require the following statement.

**Theorem 2.3.** *The map*

$$\begin{aligned} \mathbb{T}^n/\mathcal{G} &\rightarrow \text{im}(\vartheta), \\ \mathcal{G} \star x &\mapsto \vartheta(x) \end{aligned}$$

*is well defined and bijective.*

*Proof.* We follow the proof of [CLO15, §10, Theorem 10]. For  $x, y \in \mathbb{T}$  with  $\mathcal{G} \star x = \mathcal{G} \star y$ , we have  $\theta_i(x) = \theta_i(y)$  for all  $1 \leq i \leq n$  by definition. Therefore, the map is well defined and surjective.

For injectiveness, assume that  $x, y \in \mathbb{T}$  with  $\mathcal{G} \star x \cap \mathcal{G} \star y = \emptyset$ . Define the set  $X := \mathcal{G} \star x \cup \mathcal{G} \star y \setminus \{y\} \subseteq \mathbb{T}$ . Since  $\mathcal{G}$  is finite,  $X$  is finite and there exists a  $\tilde{f} \in \mathbb{C}[x_1, \dots, x_n]$  with  $\tilde{f}(X) = \{0\}$ ,  $\tilde{f}(y) \neq 0$ . For example

$$\tilde{f} = \prod_{x' \in X} \prod_{i=1}^n (x_i - x'_i)$$

has the desired property. Consider  $\mathbb{C}[x^\pm]$  as a ring extension of  $\mathbb{C}[x_1, \dots, x_n]$  and define

$$f := \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} B \cdot \tilde{f} \in \mathbb{C}[x^\pm]^\mathcal{G}.$$

Then  $f(x) = 0$  and  $f(y) = |\text{Stab}_\mathcal{G}(y)|/|\mathcal{G}| \tilde{f}(y) \neq 0$  by definition, where  $\text{Stab}_\mathcal{G}(y)$  denotes the stabilizer subgroup of  $y$  in  $\mathcal{G}$ . Since  $\mathbb{K}[x^\pm]^\mathcal{G} = \mathbb{K}[\theta_1, \dots, \theta_m]$  and  $\mathbb{K} \subseteq \mathbb{C}$  is a field extension, we have  $\mathbb{C}[x^\pm]^\mathcal{G} = \mathbb{C}[\theta_1, \dots, \theta_m]$ . With  $f(x) \neq f(y)$ , we obtain  $\theta_i(x) \neq \theta_i(y)$  for some  $1 \leq i \leq m$  and thus  $\vartheta(x) \neq \vartheta(y)$ .  $\square$

**Lemma 2.4.**

1.  $\text{im}(\vartheta)$  is compact in  $(\mathbb{C}^*)^m$ .
2.  $\vartheta$  separates the orbits  $\mathbb{T}^n/\mathcal{G}$ , that is for  $x, y \in \mathbb{T}$ ,  $\vartheta(x) = \vartheta(y)$  if and only if  $x \in \mathcal{G} \star y$ .

*Proof.*  $\text{im}(\vartheta) \subseteq \mathcal{V}(\mathcal{I})$  is compact in  $(\mathbb{C}^*)^m$  as the image of the compact set  $\mathbb{T}^n$  under a continuous map. The second statement follows from [Theorem 2.3](#).  $\square$

We conclude that there is a 1 : 1–correspondence between the orbits  $\mathbb{T}^n/\mathcal{G}$  and the image of  $\vartheta$ .

**Definition 2.5.** *We call  $\mathcal{T} := \text{im}(\vartheta)$  the  $\mathbb{T}$ –orbit space of  $\mathcal{G}$ .*

The following statement gives a recurrence formula to iteratively compute orbit polynomials and will be used in the proofs of [Lemmas 6.2](#) and [7.2](#)

**Proposition 2.6.** *Let  $\alpha, \beta \in \mathbb{N}^n$ . We have  $\Theta_0 = 1$  and*

$$|\mathcal{G}| \Theta_\alpha \Theta_\beta = \sum_{\beta' \in \mathcal{G} \beta} \Theta_{\alpha+\beta'}.$$

## 2.2 Weyl groups

Let  $\Omega \subseteq \mathbb{R}^n$  be an  $n$ –dimensional  $\mathfrak{G}$ –lattice and

$$\pi : \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{R}),$$

a real representation of  $\mathfrak{G}$ , such that  $\Omega$  is left invariant by the linear action of  $\pi(\mathfrak{G})$  on  $\mathbb{R}^n$ . For a fixed basis  $\{\omega_1, \dots, \omega_n\}$  of  $\Omega$ , the isomorphism of  $\mathbb{Z}$ -modules  $\Omega \cong \mathbb{Z}^n$  is denoted by

$$\begin{aligned} \mathfrak{q} : \quad \mathbb{Z}^n &\rightarrow \Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n, \\ [\alpha_1, \dots, \alpha_n]^t &\mapsto W\alpha = \alpha_1\omega_1 + \dots + \alpha_n\omega_n, \end{aligned} \quad (2.5)$$

where  $W \in \mathbb{R}^{n \times n}$  is the matrix with columns  $\omega_1, \dots, \omega_n$ . Since  $\Omega$  is left invariant, there exists a representation  $\varrho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{Z})$ , such that  $\mathfrak{q} \circ \varrho(s) = \pi(s) \circ \mathfrak{q}$  for all  $s \in \mathfrak{G}$ . We obtain a  $\mathbb{K}$ -algebra isomorphism between the group algebra  $\mathbb{K}[\Omega]$  with basis  $\{e^\mu \mid \mu \in \Omega\}$  [Bou75] and the ring of Laurent polynomials  $\mathbb{K}[x^\pm]$  with basis  $\{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$ , given by

$$\begin{aligned} \mathbb{K}[x^\pm] &\rightarrow \mathbb{K}[\Omega], \\ x^\alpha &\mapsto e^{\mathfrak{q}(\alpha)} = e^{\alpha_1\omega_1 + \dots + \alpha_n\omega_n}. \end{aligned} \quad (2.6)$$

Then  $\mathbb{K}[\Omega]^\mathfrak{G} \cong \mathbb{K}[x^\pm]^\mathfrak{G}$  is finitely generated as a ring. The action of  $\mathfrak{G}$  on  $\mathbb{K}[\Omega]$  is called multiplicative, as the basis  $\{e^\mu \mid \mu \in \Omega\}$  of  $\mathbb{K}[\Omega]$  is a multiplicative subgroup of the units in  $\mathbb{K}[\Omega]$ .

The orbit polynomial in  $\mathbb{K}[x^\pm]$ , which corresponds to  $\omega = \alpha_1\omega_1 + \dots + \alpha_n\omega_n$ , is  $\Theta_\alpha$ . We denote by  $\theta_i := \Theta_{e_i}$  the orbit polynomial associated to  $e_i \in \mathbb{Z}^n$ , which corresponds to  $\omega_i$  for  $1 \leq i \leq n$ . When  $\Omega$  is the weight lattice of a crystallographic root system, we have the following specific result about the ring of invariants.

**Theorem 2.7.** [Bou68b, §3, Théorème 1] *Let  $\mathfrak{G}$  be the Weyl group of an  $n$ -dimensional crystallographic root system  $R$  and  $\Omega$  be the weight lattice of  $R$ . Consider the corresponding integer representation  $\varrho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{Z})$  and let  $\mathcal{G} = \varrho(\mathfrak{G})$ . Then the orbit polynomials  $\theta_1, \dots, \theta_n$  are algebraically independent and  $\mathbb{K}[x^\pm]^\mathfrak{G}$  is the polynomial algebra*

$$\mathbb{K}[x^\pm]^\mathfrak{G} = \mathbb{K}[\theta_1, \dots, \theta_n].$$

For the definition of a root system and its Weyl group, we refer to [Bou68b, §1, Définition 1]. The converse “ $\mathbb{K}[x^\pm]^\mathfrak{G}$  polynomial algebra  $\Rightarrow$   $\mathfrak{G}$  Weyl group” of Theorem 2.7 also holds [Far86], and the equivalence is analogous to the characterization of pseudoreflection groups by the Chevalley-Shephard-Todd theorem [ST54].

**Remark 2.8.** *For similar results over  $\mathbb{Z}$  or over fields with positive characteristic, one can define the orbit polynomial associated to  $e_i \in \mathbb{Z}^n$  to be  $\tilde{\theta}_i := \sum_{\beta \in \mathcal{G}e_i} x^\beta \in \mathbb{K}[x^\pm]^\mathfrak{G}$ . Then consider the free group ring  $\mathbb{Z}[\Omega]^\mathfrak{G} \cong \mathbb{Z}[x^\pm]^\mathfrak{G}$ , which is the polynomial ring*

$$\mathbb{Z}[x^\pm]^\mathfrak{G} = \mathbb{Z}[\tilde{\theta}_1, \dots, \tilde{\theta}_n].$$

### 2.3 Main results

For a finite group with an integer representation, the natural problem we pursue is to effectively describe the image of the complex  $n$ -torus  $\mathbb{T}^n$  under the fundamental invariants in terms of polynomial inequalities. The aim of this article is to do this for Weyl groups by showing that there exists an  $n$ -dimensional  $\mathbb{R}$ -subspace  $W \subseteq \mathbb{C}^n$  containing  $\mathcal{T}$  and a matrix polynomial  $H \in \mathbb{K}[z]^{n \times n}$ , such that for all  $z \in W$ ,  $H(z)$  is symmetric with real entries and  $z$  is contained in the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of  $\mathfrak{G}$  with respect to  $\vartheta$  if and only if  $H(z)$  is positive semi-definite. The reason that we specifically consider the Weyl groups of type  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  or  $\mathcal{D}$  is the following.

A crystallographic root system can be decomposed into irreducible components [Bou68b, §1.2, Proposition 6] and may be assumed to be reduced [Bou68b, §1.4, Proposition 13]. This leaves us with nine families, which are denoted by  $A_{n-1}$ ,  $B_n$ ,  $C_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 3$ ) for fixed  $n \in \mathbb{N}$ , as well as  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$  [Bou68b, §4, Théorème 3]. In this article, we study the four infinite families  $A_{n-1}$ ,  $C_n$ ,  $B_n$ ,  $D_n$  in Sections 4 to 7 for arbitrary  $n$  in this order. The associated Weyl groups are  $\mathcal{A}_{n-1}$ ,  $\mathcal{C}_n$ ,  $\mathcal{B}_n$ ,  $\mathcal{D}_n$ . For these cases, we present explicit matrix polynomials  $H \in \mathbb{K}[z]$  in Theorems 4.3, 5.3, 6.4 and 7.4.

For arbitrary groups with an integer representation, we present a conjecture in Section 8.



**Convention:** We assume  $\mathbb{K} = \mathbb{Q}$ . A root system is crystallographic, irreducible, reduced and represented over (an  $\mathbb{R}$ -subspace of)  $\mathbb{R}^n$ . Its Weyl group  $\mathfrak{G}$  is a subgroup of  $O_n(\mathbb{R})$ . Stabilizer subgroups of  $\mathfrak{G}$ , respectively its integer representation  $\mathcal{G}$ , are denoted by  $\text{Stab}_{\mathfrak{G}}(\cdot)$ , respectively  $\text{Stab}_{\mathcal{G}}(\cdot)$ , where the action by either matrix multiplication, Equation (2.2) or Equation (2.3) is evident from the argument.

The first observation we make for  $\mathbb{T}$ -orbit spaces is the following fact.

**Lemma 2.9.** *For  $z \in \mathcal{T}$ , we have  $|z_i| \leq 1$  for  $1 \leq i \leq n$ . If  $-I_n \in \mathcal{G}$ , then  $\mathcal{T} \subseteq [-1, 1]^n \subseteq \mathbb{R}^n$ .*

*Proof.* Let  $z = \vartheta(x)$  for some  $x \in \mathbb{T}^n$ . Then  $|z_i| = |\theta_i(x)| \leq 1$ .

Furthermore, if  $-I_n \in \mathcal{G}$  we have  $\overline{\theta_i(x)} = \theta_i(x^{-I_n}) = \theta_i(x) \in \mathbb{R}$  for all  $x \in \mathbb{T}$ . Hence,  $\mathcal{T} \subseteq \mathbb{R}^n$  is contained in the cube  $[-1, 1]^n$ .  $\square$

**Remark 2.10.** *For all  $1 \leq i \leq n$ , there exists a unique  $1 \leq j \leq n$ , such that  $x^{-e_i} = x_i^{-1} \in \mathcal{G} \cdot x_j = \mathcal{G} \cdot x^{e_j}$  for all  $x \in \mathbb{T}^n$ . In other words, there exists a unique permutation  $\sigma \in \mathfrak{S}_n$ , such that  $\theta_i(x^{-I_n}) = \theta_{\sigma(i)}(x)$ . To enforce  $\mathcal{T} \subseteq \mathbb{R}^n$  in the case  $-I_n \notin \mathcal{G}$ , we can replace the  $i$ -th and  $j$ -th coordinate of  $\vartheta$  by  $(\theta_i + i\theta_j)(x)/2$  and  $(\theta_i - i\theta_j)(x)/2$ . The situation occurs for  $\mathcal{A}_{n-1}$  and, when  $n$  is odd, also for  $\mathcal{D}_n$ .*

### 3 Symmetric polynomial systems

The four Weyl groups  $\mathcal{A}_{n-1}$ ,  $\mathcal{B}_n$ ,  $\mathcal{C}_n$  and  $\mathcal{D}_n$  each contain a subgroup isomorphic to the symmetric group  $\mathfrak{S}_n$  and formulae for some specific invariants involve the elementary symmetric polynomials. This gives rise to a symmetric polynomial system. In this section, we give a characterization for all solutions of such a system being contained in  $\mathbb{T}^n$ .

#### 3.1 Solutions in $\mathbb{T}^n$

**Definition 3.1.** *The polynomials*

$$\sigma_i(y_1, \dots, y_n) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} y_j \in \mathbb{Q}[y_1, \dots, y_n]$$

for  $1 \leq i \leq n$  are called the **elementary symmetric polynomials** in  $n$  indeterminates.

We shall be confronted with the following two types of polynomial systems in unknown  $y_1, \dots, y_n$ .

$$\begin{aligned} \text{(I)} \quad \sigma_i(y_1, \dots, y_n) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{C} \\ \text{(II)} \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) &= (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{R} \end{aligned}$$

The goal of this section is to determine, whether all solutions  $y = (y_1, \dots, y_n)$  of system (I), respectively system (II), are contained in the complex  $n$ -torus  $\mathbb{T}^n$ . Recall Vieta's formula

$$\prod_{k=1}^n (x - r_k) = x^n + \sum_{i=1}^n (-1)^i \sigma_i(r_1, \dots, r_n) x^{n-i} \in \mathbb{C}[x] \tag{3.1}$$

for  $r_1, \dots, r_n \in \mathbb{C}$ .

**Lemma 3.2.**

1. System (I) always has a solution in  $\mathbb{C}^n$ . It is unique up to permutation of coordinates.

2. The map

$$\begin{aligned} \mathbb{C}^* &\rightarrow \mathbb{C}, \\ x &\mapsto (x + x^{-1}) \end{aligned}$$

is surjective and the preimage of  $[-2, 2]$  is  $\mathbb{T}$ .

3. System (II) always has a solution in  $(\mathbb{C}^*)^n$ . It is unique up to permutation and inversion of coordinates.

*Proof.* 1. By Equation (3.1), a solution of system (I) is the vector of roots of a polynomial with coefficients given by the right hand side of system (I). Such a polynomial always has  $n$  unique roots in  $\mathbb{C}$ .

2. For  $r \in \mathbb{C}$ , consider the polynomial  $p := x^2 - rx + 1 \in \mathbb{R}[x]$ . Then 0 is not a root of  $p$  and  $r = x + 1/x$  is in the image of the map. If  $r \in [-2, 2]$ , then  $p$  has discriminant  $(r/2)^2 - 1 \leq 0$  and its two roots are  $x, \bar{x} = x, x^{-1} = r/2 \pm i\sqrt{1 - (r/2)^2} \in \mathbb{T}$ . On the other hand for  $x \in \mathbb{T}$ ,  $x + x^{-1} = x + \bar{x} = 2\Re(x) \in [-2, 2]$ .

3. By the second statement, we can now always write the roots of the polynomial with coefficients given by the right hand side of system (II) as  $r_i = y_i + y_i^{-1}$ , which uniquely defines a solution up to permutation and inversion.  $\square$

As a convention, we speak of “the” solution of system (I), respectively (II), from now on.

**Proposition 3.3.** For  $c_1, \dots, c_n \in \mathbb{R}$ , the solution of system (II) is contained in  $\mathbb{T}^n$  if and only if all the roots of the univariate polynomial

$$x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{R}[x]$$

are contained in  $[-2, 2]$ .

*Proof.* Let  $p := x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{R}[x]$  with roots  $r_1, \dots, r_n \in \mathbb{C}$ . If  $y \in (\mathbb{C}^*)^n$  is the solution of system (II), then for all  $1 \leq i \leq n$ ,  $y_i + y_i^{-1}$  is a root of  $p$  by Equation (3.1). Applying Lemma 3.2 yields  $y \in \mathbb{T}^n$  if and only if  $r_1, \dots, r_n \in [-2, 2]$ .  $\square$

The Chebyshev polynomial of the first kind associated to  $\ell \in \mathbb{N}$  is the unique univariate polynomial  $T_\ell$  with  $T_\ell((x+x^{-1})/2) = (x^\ell + x^{-\ell})/2$ . The set  $\{T_\ell \mid \ell \in \mathbb{N}\}$  is a basis of  $\mathbb{R}[x]$  as an  $\mathbb{R}$ -vector space. For  $0 \neq p \in \mathbb{R}[x]$  or  $\mathbb{C}[x]$ , denote by  $\text{Coeff}(x^\ell, p)$  the coefficient of the monomial  $x^\ell$  in  $p$  for  $0 \leq \ell \leq \deg(p)$ .

**Proposition 3.4.** For  $c_1, \dots, c_{n-1} \in \mathbb{C}$  with  $\bar{c}_i = (-1)^n c_{n-i}$  and  $c_0 := (-1)^n c_n := 1$ , the solution of system (I) is contained in  $\mathbb{T}^n$  if and only if all the roots of the univariate polynomial

$$T_n(x) + d_1 T_{n-1}(x) + \dots + d_{n-1} T_1(x) + \frac{d_n}{2} T_0(x) \in \mathbb{R}[x] \quad \text{with} \quad d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

are contained in  $[-1, 1]$ .

*Proof.* By Equation (3.1), the solution of system (I) is contained in  $\mathbb{T}^n$  if and only if all the roots of the polynomial  $p := x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{C}[x]$  are contained in  $\mathbb{T}$ .

Set  $\tilde{p} := x^n + \bar{c}_1 x^{n-1} + \dots + \bar{c}_n \in \mathbb{C}[x]$ . The roots of  $p$  are nonzero, because  $p(0) = c_n = (-1)^n \neq 0$ . Since  $\tilde{p}(x) = (-x)^n p(x^{-1})$ , the roots of  $p\tilde{p} \in \mathbb{C}[x]$  are the union of the roots of  $p$  and their inverses. Especially, all the roots of  $p\tilde{p}$  are contained in  $\mathbb{T}$ . The coefficients of  $p\tilde{p}$  satisfy

$$\text{Coeff}(x^\ell, p\tilde{p}) = \sum_{i=0}^{\ell} c_{n-i} \bar{c}_{n-\ell+i} = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} = \begin{cases} \sum_{i=0}^{\ell} \text{Coeff}(x^{n-i}, \tilde{p}) \text{Coeff}(x^{n-\ell+i}, p) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}) \\ \sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \underbrace{(\bar{c}_i c_{\ell-i} + \bar{c}_{\ell-i} c_i)}_{\in \mathbb{R}} + \begin{cases} c_{\ell/2} \bar{c}_{\ell/2}, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases} \end{cases} \in \mathbb{R}$$

for  $0 \leq \ell \leq n$ . Thus,  $\text{Coeff}(x^\ell, p\tilde{p}) = d_\ell$  and  $p\tilde{p}$  is palindromic with real coefficients. Now, we can write

$$p\tilde{p} = \sum_{\ell=1}^n d_{n-\ell}(x^{n+\ell} + x^{n-\ell}) + d_n = 2x^n \left( \sum_{\ell=1}^n d_{n-\ell} T_\ell \left( \frac{x+x^{-1}}{2} \right) + \frac{d_n}{2} T_0 \left( \frac{x+x^{-1}}{2} \right) \right) =: 2x^n g \left( \frac{x+x^{-1}}{2} \right).$$

By [Lemma 3.2](#), we have  $x \in \mathbb{T}$  is a root of  $p\tilde{p}$  if and only if  $(x+x^{-1})/2 \in [-1, 1]$  is a root of  $g$ .  $\square$

### 3.2 Hermite characterization for real roots

Let  $p, q \in \mathbb{R}[x]$  be monic univariate polynomials. The multiplication by  $q$  in the  $\mathbb{R}$ -algebra  $\mathbb{R}[x]/\langle p \rangle$  is the homomorphism

$$m_q : \mathbb{R}[x]/\langle p \rangle \rightarrow \mathbb{R}[x]/\langle p \rangle, [f] \mapsto [qf], \quad (3.2)$$

where  $[f]$  denotes the coset of  $f$  modulo  $\langle p \rangle$ . By [[CLO98](#), Chapter 2, Proposition 4.2], we have

$$m_{q_1+q_2} = m_{q_1} + m_{q_2} \quad \text{and} \quad m_{q_1 q_2} = m_{q_1} \circ m_{q_2} \quad (3.3)$$

for any  $q_1, q_2 \in \mathbb{R}[x]$ .

When  $p$  has degree  $n$ , the dimension of  $\mathbb{R}[x]/\langle p \rangle$  as an  $\mathbb{R}$ -vector space is also  $n$ . A basis is given by  $\{1, x, \dots, x^{n-1}\}$  and if  $p = x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$ , then the matrix of  $m_x$  in this basis is the companion matrix

$$\begin{bmatrix} 0 & 0 & -c_n \\ 1 & \ddots & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & 1 & -c_1 \end{bmatrix} \quad (3.4)$$

of  $p$ , since  $x x^i = x^{i+1}$  for  $0 \leq i \leq n-2$  and  $x x^{n-1} = x^n \equiv -c_1 x^{n-1} - \dots - c_{n-1} x - c_n$  modulo  $\langle p \rangle$ . On the other hand, the univariate Chebyshev polynomials of the first kind up to degree  $n-1$  also form a basis  $\{T_0, T_1, \dots, T_{n-1}\}$  of  $\mathbb{R}[x]/\langle p \rangle$ . If  $p = T_n + d_1 T_{n-1} + \dots + d_{n-1} T_1 + d_n/2 T_0 \in \mathbb{R}[x]$ , then the matrix of  $m_x$  in this basis is

$$\begin{bmatrix} 0 & 1/2 & 0 & -d_n/4 \\ 1 & 0 & \ddots & \vdots \\ & 1/2 & \ddots & 1/2 & -d_3/2 \\ & & \ddots & 0 & (1-d_2)/2 \\ 0 & & & 1/2 & -d_1/2 \end{bmatrix}, \quad (3.5)$$

where the rows and columns are indexed by  $T_0, T_1, \dots, T_{n-1}$ . The entries in the columns originate from the recurrence formula  $x T_0 = T_1$  and  $2x T_j = T_{j+1} + T_{j-1}$  for  $1 \leq j \leq n-1$ . Especially,

$$2x T_{n-1} = T_n + T_{n-2} \equiv -d_1 T_{n-1} + (1-d_2) T_{n-2} - d_3 T_{n-3} - \dots - d_{n-1} T_1 - d_n/2 T_0 \quad (3.6)$$

modulo  $\langle p \rangle$  yields the last column.

**Example 3.5.** Let  $p = x^3 - c x^2 + \bar{c} x - 1 \in \mathbb{C}[x]$  with  $c \in \mathbb{C}$  and roots  $r_1, r_2, r_3 \in \mathbb{T}$ , such that  $r_1 \cdot r_2 \cdot r_3 = 1$ .

Following the proof of [Proposition 3.4](#), we consider the palindromic polynomial  $p\tilde{p} \in \mathbb{R}[x]$  with

$$\begin{aligned}
 & \frac{1}{2x^3} p\tilde{p} \\
 &= \frac{1}{2x^3} \left( (x^6 + 1) - (c + \bar{c})(x^5 + x) + (c\bar{c} + c + \bar{c})(x^4 + x^2) - \frac{c^2 + \bar{c}^2 + 2}{2} x^3 \right) \\
 &= \frac{x^3 + x^{-3}}{2} - (c + \bar{c}) \frac{x^2 + x^{-2}}{2} + (c\bar{c} + c + \bar{c}) \frac{x + x^{-1}}{2} - \frac{c^2 + \bar{c}^2 + 2}{2} \\
 &= T_3 \left( \frac{x + x^{-1}}{2} \right) - (c + \bar{c}) T_2 \left( \frac{x + x^{-1}}{2} \right) + (c\bar{c} + c + \bar{c}) T_1 \left( \frac{x + x^{-1}}{2} \right) - \frac{c^2 + \bar{c}^2 + 2}{2} T_0 \left( \frac{x + x^{-1}}{2} \right) \\
 &= g \left( \frac{x + x^{-1}}{2} \right).
 \end{aligned}$$

The matrix of the multiplication by  $x$  in  $\mathbb{R}[x]/\langle g \rangle$  in the basis of univariate Chebyshev polynomials of the first kind is

$$\begin{bmatrix} 0 & 1/2 & (c^2 + \bar{c}^2 + 2)/4 \\ 1 & 0 & (1 - c\bar{c} - c - \bar{c})/2 \\ 0 & 1/2 & (c + \bar{c})/2 \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$

To characterize roots of univariate polynomials in  $[-2, 2]$ , respectively  $[-1, 1]$ , we use the next statement, which is an application for Sturm's version of Sylvester's theorem.

**Theorem 3.6.** *Let  $p \in \mathbb{R}[x]$  be a monic univariate polynomial of degree  $n$  and let  $m_x$  be the associated multiplication by  $x$  in  $\mathbb{R}[x]/\langle p \rangle$ . For  $a \in \{1, 2\}$ , all the roots of  $p$  are contained in  $[-a, a]$  if and only if the Hermite quadratic form*

$$H_q(p) : \mathbb{R}[x]/\langle p \rangle \rightarrow \mathbb{R}, [f] \mapsto \text{Trace}(m_q f^2), \quad \text{with } q = (a - x)(a + x) \in \mathbb{R}[x],$$

is positive semi-definite.

*Proof.* Let  $H \in \mathbb{R}^{n \times n}$  be the symmetric matrix associated to  $H_q(p)$  for a fixed basis of  $\mathbb{R}[x]/\langle p \rangle$ . Denote by  $N_+$ , respectively  $N_-$ , the number of strictly positive, respectively negative, eigenvalues of  $H$ , counting multiplicities. By [\[BPR06, Corollary 4.44 and Theorem 4.57\]](#), rank and signature of  $H_q(p)$  are

$$\begin{aligned}
 N_+ + N_- &= \text{Rank}(H_q(p)) = |\{x \in \mathbb{C} \mid p(x) = 0, q(x) \neq 0\}| & \text{and} \\
 N_+ - N_- &= \text{Sign}(H_q(p)) = \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) > 0\}|}_{=: n_+} - \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) < 0\}|}_{=: n_-}.
 \end{aligned}$$

If all roots of  $p$  are contained in  $[-a, a]$ , then  $n_- = 0$  and thus  $N_+ + N_- = \text{Rank}(H_q(p)) = \text{Sign}(H_q(p)) = n_+ = N_+ - N_-$ . Therefore,  $N_- = 0$  and all eigenvalues of  $H$  are nonnegative, that is  $H_q(p)$  is positive semi-definite.

For the converse, assume that  $H_q(p)$  is positive semi-definite. Then  $N_- = 0$  and  $N_+ = \text{Sign}(H_q(p)) = n_+ - n_- \leq \text{Rank}(H_q(p)) = N_+$ , that is  $n_+ - n_- = \text{Rank}(H_q(p))$ . On the other hand,  $\text{Rank}(H_q(p)) \geq n_+ + n_-$ . Hence,  $n_- = 0$  and  $\text{Rank}(H_q(p)) = n_+$  implies that all the roots of  $p$  are real and contained in  $[-a, a]$ .  $\square$

Note that the matrix  $H$  in the proof of [Theorem 3.6](#) does not depend on the choice of a basis for  $\mathbb{R}[x]/\langle p \rangle$ , as the trace is basis-invariant. As corollaries, we obtain explicit characterizations for solutions of symmetric polynomial systems in  $\mathbb{T}$ .

**Corollary 3.7.** For  $c_1, \dots, c_n \in \mathbb{R}$ , the solution of system (II) is contained in  $\mathbb{T}^n$  if and only if the matrix  $H \in \mathbb{R}^{n \times n}$  with entries

$$H_{ij} = \text{Trace}(4C^{i+j-2} - C^{i+j}) \quad \text{for } 1 \leq i, j \leq n, \quad \text{where}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & -c_n \\ 1 & & 0 & -c_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

is positive semi-definite.

*Proof.* The matrix  $H$  is the matrix associated to the Hermite quadratic form  $H_q(p)$  from [Theorem 3.6](#) with  $a = 2$  in the basis  $\{1, x, \dots, x^{n-1}\}$ . Indeed, by [Equation \(3.3\)](#), the entries of the associated matrix are

$$\text{Trace}(m_q x^{i-1} x^{j-1}) = \text{Trace}(m_{4x^{i+j-2}-x^{i+j}}) = \text{Trace}(4m_x^{i+j-2} - m_x^{i+j}) = H_{ij}$$

for  $1 \leq i, j \leq n$ . □

Analogously, we obtain the following statement for systems of type (I) with  $a = 1$ .

**Corollary 3.8.** For  $c_1, \dots, c_n \in \mathbb{C}$  with  $\overline{c_i} = (-1)^n c_{n-i}$  for  $1 \leq i \leq n-1$  and  $c_0 := (-1)^n c_n = 1$ , the solution of system (I) is contained in  $\mathbb{T}^n$  if and only if the matrix  $H \in \mathbb{R}^{n \times n}$  with entries

$$H_{ij} = \text{Trace}(C^{i+j-2} - C^{i+j}), \quad \text{where}$$

$$C = \begin{bmatrix} 0 & 1/2 & & 0 & -d_n/4 \\ 1 & 0 & \ddots & & \vdots \\ & 1/2 & \ddots & 1 & -d_3/2 \\ & & \ddots & 0 & (1-d_2)/2 \\ 0 & & & 1 & -d_1/2 \end{bmatrix} \in \mathbb{R}^{n \times n} \quad \text{and} \quad d_\ell = \sum_{i=0}^{\ell} c_i \overline{c_{\ell-i}},$$

is positive semi-definite.

In our examples, we will characterize  $H \succeq 0$  via the coefficients of the characteristic polynomial to give a proper visualization.

**Proposition 3.9.** [[BPR06](#), [Theorem 4.58](#)] Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix with characteristic polynomial

$$\text{Det}(xI_n - A) = x^n + \sum_{i=1}^n (-1)^i a_i x^{n-i} \in \mathbb{R}[x].$$

Then  $A$  is positive semi-definite if and only if  $a_i \geq 0$  for all  $1 \leq i \leq n$ .

## 4 Type $\mathcal{A}$

In [Sections 4 to 7](#) we study the four cases where  $\mathfrak{G}$  is of type  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  or  $\mathcal{D}$ . First, we give an explicit formula for the orbit polynomials in terms of the elementary symmetric polynomials. This induces a symmetric system of type (I), respectively (II). Afterwards, we present an explicit symmetric matrix polynomial  $H$  to characterize the solution of this symmetric system in  $\mathbb{T}$ , that then corresponds to an orbit as the imagepoint of  $\vartheta$ . An example to visualize the 2-dimensional case accompanies the main result of each section, except for

$D_n$ . In the proofs, we avoid arguments such as “similar/analogous to a previous case”, so that all sections are self contained.

**Convention:** We abuse notation for the sake of simplicity and speak of “the orbit polynomial  $\Theta_\mu$  associated to  $\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n \in \Omega$ ” with  $\alpha \in \mathbb{Z}^n$ , to refer to the orbit polynomial  $\Theta_\alpha$  associated to  $\alpha$ . Furthermore for  $f \in \mathbb{Q}[x^\pm]$  and  $\mathfrak{G} \in \{\mathcal{A}_{n-1}, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n\}$ , we write  $\mathfrak{G} \cdot f$  for the orbit  $\varrho(\mathfrak{G}) \cdot f$  of  $f$  under the integer representation.

For  $\mathcal{A}_{n-1}$ , we will encounter a system of type (I) in  $n$  indeterminates. To be consistent with the notation of Section 3, we consider  $A_{n-1}$  as a root system of rank  $n - 1$ . In particular, we denote the Laurent polynomial ring and the polynomial ring in  $n - 1$  indeterminates by  $\mathbb{Q}[x^\pm] = \mathbb{Q}[x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}]$  and  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_{n-1}]$ .

#### 4.1 Orbit polynomials for $\mathcal{A}_{n-1}$

The Weyl group of the root system  $A_{n-1}$  is  $\mathcal{A}_{n-1} \cong \mathfrak{S}_n$ . A representation can be given over the  $(n - 1)$ -dimensional  $\mathbb{R}$ -vector space  $V := \{u \in \mathbb{R}^n \mid u_1 + \dots + u_n = 0\}$ .  $\mathfrak{S}_n$  acts on  $V$  by permutation of coordinates. By [Bou68b, Planche I], the simple roots and fundamental weights in this representation are

$$\rho_i = e_i - e_{i+1} \quad \text{and} \quad \omega_i = \sum_{j=1}^i e_j - \frac{i}{n} \sum_{k=1}^n e_k = \frac{1}{n} \underbrace{[n-i, \dots, n-i]}_{i \text{ times}} \underbrace{[-i, \dots, -i]}_{n-i \text{ times}}. \quad (4.1)$$

The graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-2} \text{ --- } s_{n-1} \quad (4.2)$$

is the Coxeter diagram of the Weyl group  $\mathcal{A}_{n-1}$ , where  $s_i \in O_n(\mathbb{R})$  is the permutation matrix of the  $i$ -th and  $(i + 1)$ -th coordinate for  $1 \leq i \leq n$ , leaving  $V$  invariant and generating  $WeylA$ . We have  $-\omega_i \in \mathcal{A}_{n-1} \omega_{n-i}$  for  $1 \leq i \leq n - 1$ .

The orbit polynomials associated to the fundamental weights of  $A_{n-1}$  and the elementary symmetric polynomials from Definition 3.1 satisfy the following relations.

**Proposition 4.1.** *In  $\mathbb{Q}[x^\pm]$ , define the monomials*

$$y_1 = x_1, \quad y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n - 1 \quad \text{and} \quad y_n = x_{n-1}^{-1}.$$

Then

$$\sigma_i(y_1(x), \dots, y_n(x)) = \binom{n}{i} \theta_i(x)$$

for  $1 \leq i \leq n - 1$ .

*Proof.* We first show that the orbit of the monomial  $x^{e_1} = x_1$  corresponding to  $\omega_1$  is  $\mathcal{A}_{n-1} \cdot x_1 = x_1 \{y_1, \dots, y_n\}$ , and then infer the orbit of  $x_i$ .

The eigenvectors of the reflection  $s_i$  are  $\rho_i$  with eigenvalue  $-1$  and  $\omega_j$  for  $i \neq j$  with eigenvalue  $1$ . Therefore, the stabilizer of  $\omega_1$  is the subgroup generated by  $s_2, \dots, s_{n-1}$ . Removing  $s_1$  from the Coxeter diagram in Section 4.1 yields  $\text{Stab}_{\mathcal{A}_{n-1}}(\omega_1) \cong \mathcal{A}_{n-2}$  and the index of  $\text{Stab}_{\mathcal{A}_{n-1}}(\omega_1)$  in  $\mathcal{A}_{n-1}$  is  $n$ . Thus,  $\mathcal{A}_{n-1} \omega_1$  consists of exactly  $n$  elements. Assume that  $\mathcal{A}_{n-1} / \text{Stab}_{\mathcal{A}_{n-1}}(\omega_1) = \{[h_1], \dots, [h_n]\}$  are the cosets. Then the orbit of  $\omega_1$  under  $\mathcal{A}_{n-1}$  is given by  $\mathcal{A}_{n-1} \omega_1 = \{h_1 \cdot \omega_1, \dots, h_n \cdot \omega_1\}$ . Choose  $h_i \in \mathcal{A}_{n-1}$  such that  $h_i$  permutes the 1-st and  $i$ -th coordinate. Then  $h_1 = \text{id}_V$ ,  $h_i = s_1 \cdots s_{i-1}$  for  $2 \leq i \leq n$  and we have

$$h_1 \omega_1 = \omega_1, h_2 \omega_1 = \omega_2 - \omega_1, \dots, h_{n-1} \omega_1 = \omega_{n-1} - \omega_{n-2} \quad \text{and} \quad h_n \omega_1 = -\omega_{n-1}.$$

Those are elements of the orbit  $\mathcal{A}_{n-1} \omega_1$  and, beyond that, contained in the lattice of weights  $\Omega$ . Hence, they correspond to  $n$  monomials in  $\mathbb{Q}[x^\pm]$ , which are precisely the  $y_i$ .

For  $1 \leq i \leq n-1$ , write  $x_i = y_1 \dots y_i$ . Since the action of  $\mathcal{A}_{n-1}$  on  $\mathbb{Q}[x^\pm]$  by “ $\cdot$ ” permutes the  $y_i$  and is linear, the monomials in  $\theta_i$  are all choices of products of  $i$  monomials among  $y_1, \dots, y_n$ . This is the elementary symmetric polynomials from [Definition 3.1](#) on the left hand side. The coefficient of this monomial in  $\theta_i$  is by [Definition 2.2](#) given by  $1/|\mathcal{A}_{n-1} \omega_i| = \binom{n}{i}^{-1}$ .  $\square$

## 4.2 Hermite matrix for $\mathcal{A}_{n-1}$

Introduce the subset of the complex  $n$ -torus  $\mathbb{T}_1^n := \{y = (y_1, \dots, y_n) \in \mathbb{T}^n \mid y_1 \dots y_n = 1\}$ . The proof of the following fact is straightforward.

**Proposition 4.2.** *The map*

$$\begin{aligned} \psi: (\mathbb{C}^*)^{n-1} &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-1} x_{n-2}^{-1}, x_{n-1}^{-1}) \end{aligned}$$

is injective and the preimage of  $\mathbb{T}_1^n$  is  $\mathbb{T}^{n-1}$ .

We state the main result for Weyl groups of type  $\mathcal{A}$ .

**Theorem 4.3.** *Consider the  $\mathbb{R}$ -vector space  $W := \{z \in \mathbb{C}^{n-1} \mid \forall 1 \leq j \leq n-1 : \bar{z}_j = z_{n-j}\}$  with dimension  $n-1$  and define the matrix  $H \in \mathbb{R}[z]^{n \times n}$  by*

$$H(z)_{ij} = \text{Trace}((C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & 1/2 & & 0 & -f_n(z)/4 \\ 1 & 0 & \ddots & & -f_{n-1}(z)/2 \\ & 1/2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 & -f_3(z)/2 \\ & & & \ddots & 0 & (1-f_2(z))/2 \\ 0 & & & & 1/2 & -f_1(z)/2 \end{bmatrix}, \quad \text{where } c_i = \binom{n}{i} \quad \text{and}$$

$$f_\ell(z) = (-1)^\ell \left( c_\ell z_\ell + c_\ell z_{n-\ell} + \sum_{i=1}^{\ell-1} c_i c_{\ell-i} z_i z_{n-\ell+i} \right) \quad \text{for } 1 \leq \ell \leq n-1,$$

$$f_n(z) = (-1)^n \left( 1 + \frac{1}{2} \sum_{i=1}^{n-1} c_i^2 z_i^2 \right).$$

For all  $z \in W$ ,  $H(z) \in \mathbb{R}^{n \times n}$  and  $\mathcal{T} = \{z \in W \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{C}^{n-1}$  and set  $\tilde{c}_i := (-1)^i c_i z_i \in \mathbb{C}$  for  $1 \leq i \leq n$ . Then

$$f_\ell(z) = (-1)^\ell \sum_{i=0}^{\ell} c_i c_{\ell-i} z_i z_{n-\ell+i} = \sum_{i=0}^{\ell} \tilde{c}_i \overline{\tilde{c}_{\ell-i}} =: d_\ell$$

yield the last column of  $C(z)$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$  and fix  $x \in \mathbb{T}^{n-1}$ , such that  $\theta_i(x) = z_i$ . Note that  $\theta_j(x)$  and  $\theta_{n-j}(x)$  are complex conjugates, because  $-\omega_j \in \mathcal{A}_{n-1} \omega_{n-j}$ . Therefore,  $\bar{z}_j = z_{n-j}$  and thus  $z \in W$ . Now let  $z_n := z_0 := 1$ . By [Proposition 4.1](#), the solution of

$$(I) \quad \sigma_i(y_1, \dots, y_n) = (-1)^i \tilde{c}_i \quad \text{for } 1 \leq i \leq n$$

is given by  $y = \psi(x) \in \mathbb{T}^n$ , where  $\psi$  is the map from [Proposition 4.2](#). Applying [Corollary 3.8](#) with coefficients  $d_\ell \in \mathbb{R}$  yields  $H(z) \succeq 0$ .

For “ $\succeq$ ” on the other hand, assume  $z \in W$  with  $H(z) \succeq 0$ . By [Corollary 3.8](#), the solution  $y$  of system (I) is contained in  $\mathbb{T}^n$  and satisfies  $y_1 \dots y_n = (-1)^n \tilde{c}_n = 1$ . Let  $x \in \mathbb{T}^{n-1}$  be the unique preimage of  $y$  under  $\psi$ . Then

$$z_i = (-1)^i \frac{\tilde{c}_i}{c_i} = \frac{1}{c_i} \sigma_i(\psi(x)_1, \dots, \psi(x)_n) = \theta_i(x)$$

by [Proposition 4.1](#) and thus  $z \in \mathcal{T}$ . □

We finish this subsection with a remark on the embedding of the image in  $\mathbb{R}^{n-1}$  and on the degree of the entries of the Hermite matrix polynomial.

#### Remark 4.4.

1. In the case of  $\mathcal{A}_{n-1}$ , we have  $\mathcal{T} \not\subseteq \mathbb{R}^{n-1}$ . Instead, we consider the embedding in  $\mathbb{R}^{n-1}$  as the corresponding image  $\mathcal{T}_{\mathbb{R}}$  of the map

$$\begin{aligned} \vartheta_{\mathbb{R}} : \mathbb{T}^{n-1} &\rightarrow \mathbb{R}^{n-1}, x \mapsto (\theta_{1,\mathbb{R}}(x), \dots, \theta_{n-1,\mathbb{R}}(x)) \quad \text{with} \\ \theta_{i,\mathbb{R}}(x) &= \frac{\theta_i(x) + \theta_{n-i}(x)}{2}, \quad \theta_{n-i,\mathbb{R}}(x) = \frac{\theta_i(x) - \theta_{n-i}(x)}{2i} \quad \text{for } 1 \leq i \leq \lfloor (n-1)/2 \rfloor \quad \text{and} \\ \theta_{n/2,\mathbb{R}}(x) &= \theta_{n/2}(x), \quad \text{when } n \text{ is even.} \end{aligned}$$

2. The entries of the last column of the matrix polynomial  $C(z)$  have degree 2, except  $C(z)_{nn} = -f_1(z)$ , which has degree 1. All other entries are constant. Hence, the degree of  $H(z)_{ij}$  is  $i+j$ . The determinant of  $H(z)$  has degree  $4n - 2$  for  $n \geq 3$ .

### 4.3 Example $\mathcal{A}_2$

In this subsection, we further investigate the case  $n = 3$  and give a visualization for the embedding of the image of  $\vartheta$  in  $\mathbb{R}^2$ . The proof of the next statement is straightforward.

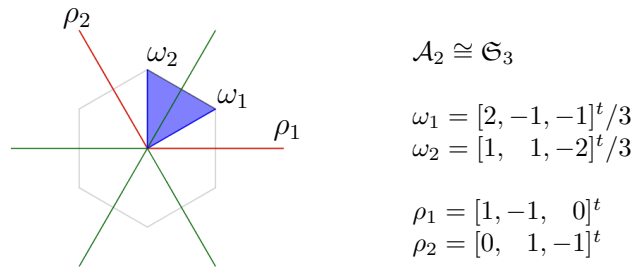


Figure 2: The root system  $\mathcal{A}_2$  in  $\mathbb{R}^3$  with fundamental weights  $\omega_1, \omega_2$  and simple roots  $\rho_1, \rho_2$ .

In [Figure 2](#), the hexagon is the image of the blue shaded triangle with vertices  $0, \omega_1, \omega_2$  under reflection by  $\mathcal{A}_2 \cong \mathfrak{S}_3$ .

Let  $z_1, z_2 \in \mathbb{R}$  and  $z = (z_1 + iz_2, z_1 - iz_2) \in W$ . Then the matrix  $C(z) \in \mathbb{R}[z]^{3 \times 3}$  from [Theorem 4.3](#) is

$$C(z) = \begin{bmatrix} 0 & 1/2 & (1 + 9z_1^2 - 9z_2^2)/2 \\ 1 & 0 & (1 - 9z_1^2 - 9z_2^2 - 6z_1)/2 \\ 0 & 1/2 & 3z_1 \end{bmatrix}. \quad (4.3)$$



We already computed this matrix (up to a substitution) in [Example 3.5](#). Following [Theorem 4.3](#), define the matrix  $H(z) \in \mathbb{R}[z]^{3 \times 3}$  with entries  $(H(z))_{ij} = 4 \operatorname{Trace}(C(z)^{i+j-2}) - \operatorname{Trace}(C(z)^{i+j})$ . Then  $z$  is contained in  $\mathcal{T}$  if and only if  $H(z)$  is positive semi-definite. Assume that  $\operatorname{Det}(x I_3 - H(z)) = x^3 - h_2(z)x^2 + h_1(z)x - h_0(z) \in \mathbb{R}[x]$  is the characteristic polynomial of  $H(z)$  for some  $h_i(z) \in \mathbb{R}[z]$ . By [Proposition 3.9](#),  $H(z)$  is positive semi-definite if and only if  $h_i(z) \geq 0$  for  $0 \leq i \leq 2$ .

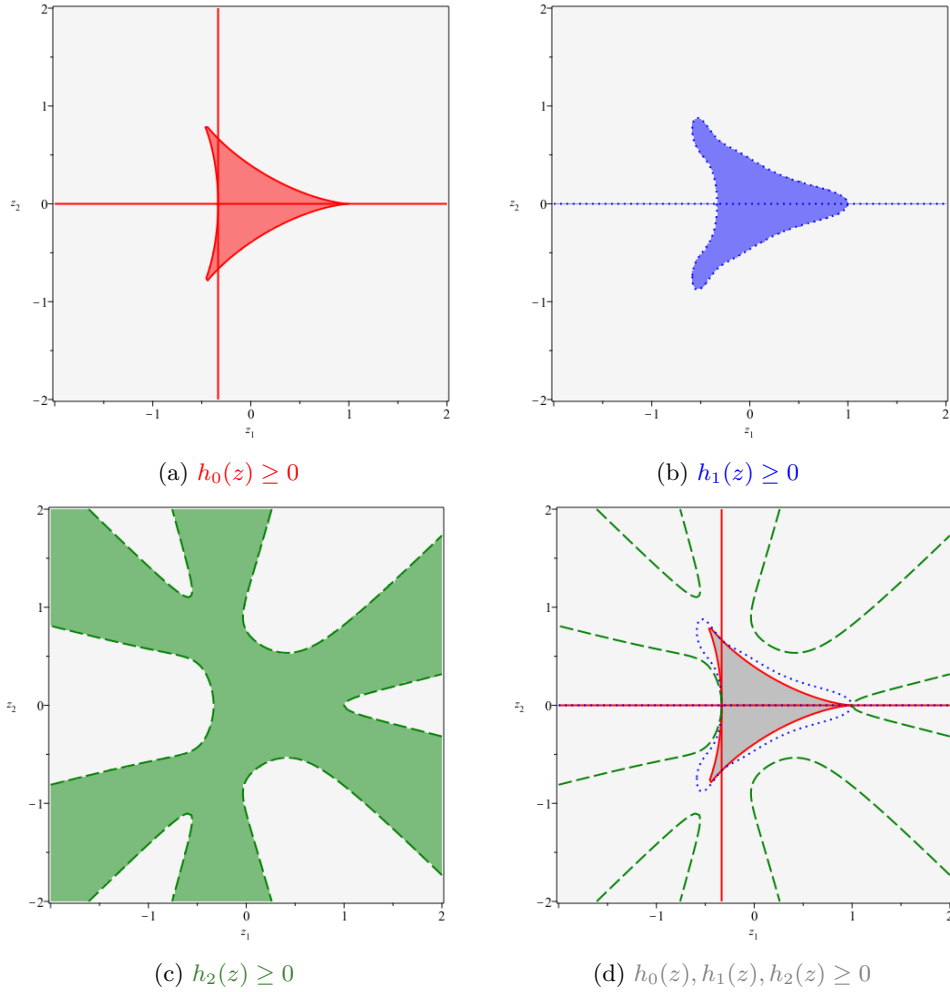


Figure 3: Vanishing points and positivity regions for the coefficients of the characteristic polynomial of  $H(z)$ .

Of particular interest are the vertices, which correspond to the fundamental weights and the origin. They are given by

$$\text{Vertex}_1 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_1 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{4}{3}\pi i\right), \exp\left(-\frac{2}{3}\pi i\right)\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_2 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, \omega_2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2 \rangle)) = \vartheta_{\mathbb{R}}\left(\exp\left(-\frac{2}{3}\pi i\right), \exp\left(-\frac{4}{3}\pi i\right)\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_3 := \vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta_{\mathbb{R}}(1, 1) = (1, 0) \quad .$$

We visualize the problem “ $z \in \mathcal{T}$ ?” by evaluating  $h_0, h_1, h_2$  at  $z$ . In the above images, a solid red line, blue dots and green dashes indicate the varieties of these three polynomials. The plots indicate an invariance under the dihedral groups  $\mathfrak{D}_3$ . Let  $g_1(z) := z_1^2 + z_2^2, g_2(z) := z_1(z_1^2 - 3z_2^2)$ . Then  $\mathbb{R}[z]^{\mathfrak{D}_3} = \mathbb{R}[g_1(z), g_2(z)]$  and we have

$$\begin{aligned}
 h_0(z) &= -\text{Coeff}(t^0, \text{Det}(tI_3 - H(z))) \quad (\text{solid}) \\
 &= 2187/64 z_2^4 (3z_1 + 1)^2 (-3z_1^4 - 6z_1^2 z_2^2 - 3z_2^4 + 8z_1^3 - 24z_1 z_2^2 - 6z_1^2 - 6z_2^2 + 1) \\
 &= 2187/64 z_2^4 (3z_1 + 1)^2 \underbrace{(-6g_1(z) - 3g_1(z)^2 + 8g_2(z) + 1)}_{\mathfrak{D}_3\text{-invariant}}, \\
 h_1(z) &= \text{Coeff}(t^1, \text{Det}(tI_3 - H(z))) \quad (\text{dots}) \\
 &= 243/256 z_2^2 (-243z_1^8 - 972z_1^6 z_2^2 - 1458z_1^4 z_2^4 - 972z_1^2 z_2^6 - 243z_2^8 + 324z_1^7 - 1620z_1^5 z_2^2 \\
 &\quad - 4212z_1^3 z_2^4 - 2268z_1 z_2^6 - 432z_1^6 - 2052z_1^4 z_2^2 - 5400z_1^2 z_2^4 - 324z_2^6 + 180z_1^5 - 3384z_1^3 z_2^2 \\
 &\quad - 684z_1 z_2^4 + 18z_1^4 - 804z_1^2 z_2^2 + 42z_2^4 + 76z_1^3 + 404z_1 z_2^2 - 8z_1^2 - 108z_2^2 + 60z_1 + 25), \\
 h_2(z) &= -\text{Coeff}(t^2, \text{Det}(tI_3 - H(z))) \quad (\text{dash}) \\
 &= 1/32 (-729z_1^6 + 1458z_1^5 + (10935z_2^2 - 1215)z_1^4 + (-2916z_2^2 + 540)z_1^3 + 351z_2^2 + 63 \\
 &\quad + (-10935z_2^4 + 1458z_2^2 - 135)z_1^2 + (-4374z_2^4 + 972z_2^2 + 18)z_1 + 729z_2^6 - 1215z_2^4).
 \end{aligned}$$

For a generic point  $z \in W$ ,  $H(z)$  has rank 3. We observe an intersection of all three varieties in  $\text{Vertex}_3$ , in which case the rank of  $H(\text{Vertex}_3)$  vanishes. This also occurs at

$$\vartheta_{\mathbb{R}} \left( \exp \left( -2\pi i \langle \omega_1, \frac{\omega_1 + \omega_2}{2} \rangle \right), \exp \left( -2\pi i \langle \omega_2, \frac{\omega_1 + \omega_2}{2} \rangle \right) \right) = \vartheta_{\mathbb{R}}(\exp(-\pi i), \exp(\pi i)) = \left( -\frac{1}{3}, 0 \right)$$

Furthermore, the rank of both  $H(\text{Vertex}_1), H(\text{Vertex}_2)$  is 1, we only have an intersection of “ $h_0(z) = 0$ ” and “ $h_1(z) = 0$ ”. Two more intersection of “ $h_0(z) = 0$ ” and “ $h_1(z) = 0$ ” lie at  $(-1/3, 2/3)$  and  $(-1/3, -2/3)$ . Every other point in the union of the boundary of  $\mathcal{T}$  and the varieties “ $z_1 = -1/3$ ”, “ $z_2 = 0$ ” admits rank 2. The interior points admit full rank 3.

## 5 Type $\mathcal{C}$

In [Sections 5 to 7](#), we will encounter symmetric systems similarly to [Section 4](#), however they will be of type (II) and therefore, we dealt with the type  $\mathcal{A}$  first. The dimension of the root system and the number of indeterminates of the symmetric system are from now on both  $n$ .

### 5.1 Orbit polynomials for $\mathcal{C}_n$

The Weyl group of the root system  $\mathcal{C}_n$  is  $\mathcal{C}_n \cong \mathfrak{S}_n \times \{\pm 1\}^n$ . A representation can be given over  $\mathbb{R}^n$ .  $\mathfrak{S}_n$  acts on  $\mathbb{R}^n$  by permutation of coordinates and  $\{\pm 1\}^n$  acts by multiplication of coordinates with  $\pm 1$ . By [[Bou68b](#), Planche III], the simple roots and fundamental weights in this representation are

$$\rho_i = e_i - e_{i+1}, \quad \rho_n = 2e_n \quad \text{and} \quad \omega_i = e_1 + \dots + e_i. \quad (5.1)$$

The graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-1} \text{ ===== } s_n \quad (5.2)$$

is the Coxeter diagram of the Weyl group  $\mathcal{C}_n$ , where  $s_i \in O_n(\mathbb{R})$  is the permutation matrix of the  $i$ -th and  $(i+1)$ -th coordinate for  $1 \leq i \leq n-1$  and  $s_n = \text{diag}(1, \dots, 1, -1)$ , generating  $\mathcal{C}_n$ . We have  $-I_n \in \mathcal{C}_n$ .

The orbit polynomials associated to the fundamental weights of  $\mathcal{C}_n$  and the elementary symmetric polynomials from [Definition 3.1](#) satisfy the following relations.

**Proposition 5.1.** *In  $\mathbb{Q}[x^{\pm}]$ , define the monomials*

$$y_1 = x_1 \quad \text{and} \quad y_k = x_k x_{k-1}^{-1} \quad \text{for} \quad 2 \leq k \leq n.$$

Then

$$\sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) = 2^i \binom{n}{i} \theta_i(x)$$

for  $1 \leq i \leq n$ .

*Proof.* We follow the idea of the proof for [Proposition 4.1](#).

It follows from the Coxeter diagram in [Equation \(5.2\)](#) that  $\text{Stab}_{\mathcal{C}_n}(\omega_1) \cong \mathcal{C}_{n-1}$  and the index of  $\text{Stab}_{\mathcal{C}_n}(\omega_1)$  in  $\mathcal{C}_n$  is  $2n$ . Assume that  $\mathcal{C}_n/\text{Stab}_{\mathcal{C}_n}(\omega_1) = \{[h_1], \dots, [h_{2n}]\}$  are the cosets. Then the orbit of  $\omega_1$  under  $\mathcal{C}_n$  is given by  $\mathcal{C}_n \omega_1 = \{h_1 \cdot \omega_1, \dots, h_{2n} \cdot \omega_1\}$ . Choose  $h_1 = \text{id}_{\mathbb{R}^n}$  and  $h_i = s_1 \cdots s_{i-1}$  for  $2 \leq i \leq n$ . Then

$$h_1 \omega_1 = \omega_1 = e_1, h_2 \omega_1 = \omega_2 - \omega_1 = e_2, \dots, h_n \omega_1 = \omega_n - \omega_{n-1} = e_n.$$

Those are elements of the orbit  $\mathfrak{S}_n \omega_1 \subseteq \mathcal{C}_n \omega_1$ . The missing elements are  $h_{n+i} \omega_1 := -I_n h_i \omega_1$  for  $1 \leq i \leq n$ , given by the subgroup  $\{\pm 1\}^n$ . The  $y_i, y_i^{-1}$  are the corresponding monomials in  $\mathbb{Q}[x^\pm]$ .

$\mathcal{C}_n$  is isomorphic to the semi-direct product of  $\mathfrak{S}_n$  with  $\{\pm 1\}^n$  and  $\{\pm 1\}^n$  is normal in  $\mathcal{C}_n$ . Therefore, we have  $\mathbb{Q}[x^\pm]^{\mathcal{C}_n} = (\mathbb{Q}[x^\pm]^{\mathfrak{S}_n})^{\{\pm 1\}^n}$ . Since  $\mathfrak{S}_n \cong \mathcal{A}_{n-1}$  and  $x_i = y_1 \dots y_i$ , the generators of  $\mathbb{Q}[x^\pm]^{\mathfrak{S}_n}$  are given by [Proposition 4.1](#). The action of the diagonal group  $\{\pm 1\}^n$  on  $\mathbb{Q}[x^\pm]$  has fundamental invariants  $y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}$ . With  $|\{\pm 1\}^n \omega_i| = 2^i$  we obtain the statement.  $\square$

## 5.2 Hermite matrix for $\mathcal{C}_n$

The proof of the next statement is straightforward.

**Proposition 5.2.** *The map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_n x_{n-1}^{-1}) \end{aligned}$$

is bijective and the preimage of  $\mathbb{T}^n$  is  $\mathbb{T}^n$ .

We state the main result of this section, which we applied in [Example 1.1](#).

**Theorem 5.3.** *Define the matrix  $H \in \mathbb{Q}[z]^{n \times n}$  by*

$$H(z)_{ij} = \text{Trace}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \cdots & 0 & (-1)^{n+1} c_n z_n \\ 1 & & 0 & (-1)^n c_{n-1} z_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & c_1 z_1 \end{bmatrix}, \quad \text{where } c_i = 2^i \binom{n}{i}.$$

Then  $\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{R}^n$  and set  $\tilde{c}_i := (-1)^i c_i z_i \in \mathbb{R}$  for  $1 \leq i \leq n$ . Then  $(-1)^{i+1} c_i z_i = -\tilde{c}_i$  are the entries of the last column of  $C(z)$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$  for  $1 \leq i \leq n$ . By [Proposition 5.1](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i \tilde{c}_i \quad \text{for } 1 \leq i \leq n$$

is  $y = \psi(x) \in \mathbb{T}^n$ , where  $\psi$  is the map from [Proposition 5.2](#). Applying [Corollary 3.7](#) yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $H(z) \succeq 0$ . By [Corollary 3.7](#), the solution  $y$  of the above system (II) is contained in  $\mathbb{T}^n$ . Let  $x \in \mathbb{T}^n$  be the unique preimage of  $y$  under  $\psi$ . Then  $z_i = \theta_i(x)$  and so  $z$  is contained in  $\mathcal{T}$ .  $\square$

We finish this subsection with a remark on the degree of the entries of the Hermite matrix polynomial.

**Remark 5.4.** *The entries of the last column of the matrix polynomial  $C(z)$  from [Theorem 5.3](#) have degree 1, all other entries are constant. The degree of  $H(z)_{ij}$  is  $i + j$  and the determinant of  $H(z)$  has degree  $2n$ .*

### 5.3 Example $\mathcal{C}_2$

In this subsection, we further investigate the case  $n = 2$ .

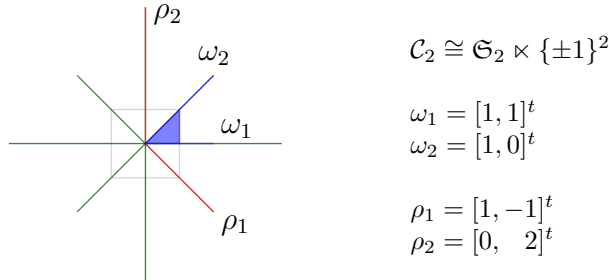


Figure 4: The root system  $\mathcal{C}_2$  in  $\mathbb{R}^2$  with fundamental weights  $\omega_1, \omega_2$  and simple roots  $\rho_1, \rho_2$ .

In [Figure 4](#), the square is the image of the blue shaded triangle with vertices  $0, \omega_1/2, \omega_2/2$  under reflection by  $\mathcal{C}_2 = \mathfrak{S}_2 \times \{\pm 1\}^2$ .

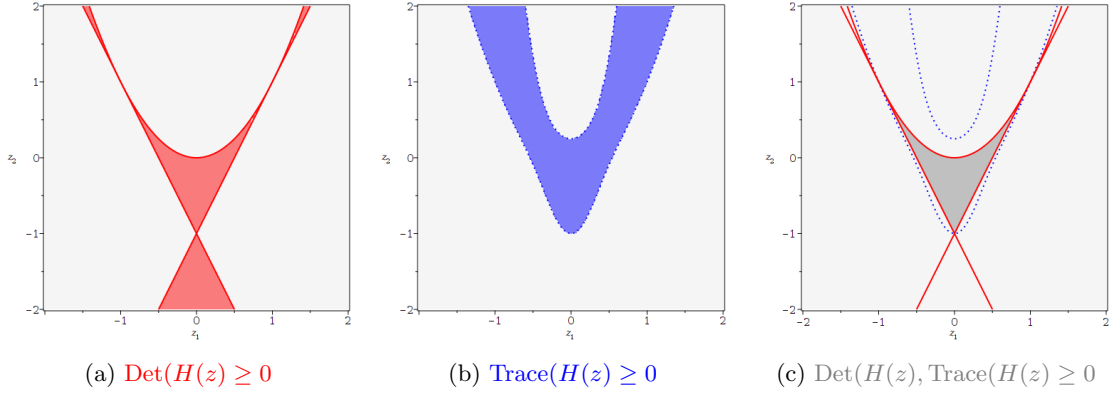
Let  $z = (z_1, z_2) \in \mathbb{R}^2$ . The matrix  $C(z) \in \mathbb{Q}[z]^{2 \times 2}$  from [Theorem 5.3](#) evaluated in  $z$  is

$$C(z) = \begin{bmatrix} 0 & -4z_2 \\ 1 & 4z_1 \end{bmatrix}. \tag{5.3}$$

Then  $z$  is contained in  $\mathcal{T}$  if and only if the resulting Hermite matrix

$$H(z) = 8 \begin{bmatrix} -2z_1^2 + z_2 + 1 & -8z_1^3 + 6z_1z_2 + 2z_1 \\ -8z_1^3 + 6z_1z_2 + 2z_1 & -32z_1^4 + 8z_1^2 + 32z_1^2z_2 - 4z_2^2 - 4z_2 \end{bmatrix} \tag{5.4}$$

is positive semi-definite. By [Proposition 3.9](#),  $H(z) \succeq 0$  is equivalent to its determinant and trace being nonnegative. The varieties of these two polynomials in  $z_1, z_2$  are depicted below.


 Figure 5: Vanishing points and positivity regions for determinant and trace of  $H(z)$ .

$$\begin{aligned}
 \text{Det}(H(z)) &= 256(-4z_1^4 + z_1^2 z_2^2 + 6z_1^2 z_2 - z_2^3 + z_1^2 - 2z_2^2 - z_2) && \text{(solid)} \\
 \text{Trace}(H(z)) &= 8(-32z_1^4 + 32z_1^2 z_2 + 6z_1^2 - 4z_2^2 - 3z_2 + 1) && \text{(dots)}
 \end{aligned}$$

We observe three intersections of “ $\text{Det}(H(z)) = 0$ ” (red solid line) and “ $\text{Trace}(H(z)) = 0$ ” (blue dots) in the the vertices

$$\begin{aligned}
 \text{Vertex}_1 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \frac{\omega_1}{2} \rangle), \exp(-2\pi i \langle \omega_2, \frac{\omega_1}{2} \rangle)) = \vartheta(-1, -1) = (0, -1) \\
 \text{Vertex}_2 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \frac{\omega_2}{2} \rangle), \exp(-2\pi i \langle \omega_2, \frac{\omega_2}{2} \rangle)) = \vartheta(-1, 1) = (-1, 1) \\
 \text{Vertex}_3 &:= \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1)
 \end{aligned}$$

The shape of this domain is dictated by the determinant, but from the positivity condition one can observe that the trace is also required. Alternatively the inequation given by the trace could be replaced by the constraint that the orbit space is contained in the square  $[-1, 1]^n$ .

## 6 Type $\mathcal{B}$

The Weyl group of  $B_n$  is isomorphic to that of  $C_n$  and we find a similar formula for the orbit polynomials in [Proposition 6.1](#) with a deviation at  $\sigma_n$  that leads to a different Hermite matrix [Theorem 6.4](#).

### 6.1 Orbit polynomials for $\mathcal{B}_n$

The Weyl group of the root system  $B_n$  is  $\mathcal{B}_n \cong \mathfrak{S}_n \times \{\pm 1\}^n \cong C_n$ . A representation can be given over  $\mathbb{R}^n$ . By [\[Bou68b, Planche II\]](#), the simple roots and fundamental weights in this representation are

$$\rho_i = e_i - e_{i+1}, \quad \rho_n = e_n \quad \text{and} \quad \omega_i = e_1 + \dots + e_i, \quad \omega_n = (e_1 + \dots + e_n)/2. \quad (6.1)$$

The graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-1} \text{ === } s_n \quad (6.2)$$

is the Coxeter diagram of  $\mathcal{B}_n$ , where  $s_i \in O_n(\mathbb{R})$  is the permutation matrix of the  $i$ -th and  $(i+1)$ -th coordinate for  $1 \leq i \leq n-1$  and  $s_n = \text{diag}(1, \dots, 1, -1)$ , generating  $\mathcal{B}_n$ . We have  $-I_n \in \mathcal{B}_n$ .

The orbit polynomials associated to the fundamental weights of  $B_n$  and the elementary symmetric polynomials from [Definition 3.1](#) satisfy the following relations.

**Proposition 6.1.** *In  $\mathbb{Q}[x^\pm]$ , define the monomials*

$$y_1 = x_1, y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n-1 \quad \text{and} \quad y_n = x_n^2 x_{n-1}^{-1}.$$

Then

$$\begin{aligned} \sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^i \binom{n}{i} \theta_i(x) \quad \text{for } 1 \leq i \leq n-1 \quad \text{and} \\ \sigma_n(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^n \Theta_{2\omega_n}(x). \end{aligned}$$

*Proof.* Since  $\mathcal{B}_n \cong \mathcal{C}_n$ , the proof is analogous to the one of [Proposition 5.1](#), when we replace  $\omega_n$  with  $2\omega_n$ . Then the corresponding monomial in  $\mathbb{Q}[x^\pm]$  is  $x^{2e_n} = x_n^2$ .  $\square$

The orbit polynomial  $\Theta_{2\omega_n}$  is  $\mathcal{B}_n$ -invariant. Hence, it can be expressed as a polynomial in the fundamental invariants of  $\mathbb{Q}[x^\pm]^{\mathcal{B}_n}$ . We give the explicit equation in the next lemma.

**Lemma 6.2.** *In  $\mathbb{Q}[x^\pm]$ , the orbit polynomial associated to  $2\omega_n \in \Omega$  satisfies*

$$\Theta_{2\omega_n} = 2^n \theta_n^2 - \sum_{j=1}^{n-1} \binom{n}{j} \theta_j - 1.$$

*Proof.* The cardinality of the orbit  $\mathcal{B}_n \omega_n$  is  $2^n$ . Let  $\mu \in \mathcal{B}_n \omega_n$  and distinguish between the following three cases.

1. If  $\mu = \omega_n$ , then  $\Theta_{\omega_n + \mu} = \Theta_{2\omega_n}$  is the term on the left-hand side of the statement, for which we search an explicit formula.
2. If  $\mu = -\omega_n$ , then  $\Theta_{\omega_n + \mu} = \Theta_0 = 1$ .
3. For any other  $\mu \in \mathcal{B}_n \omega_n$ , there exists a  $1 \leq j \leq n-1$ , such that  $\mu$  contains exactly  $j$  positive coordinates. Therefore,  $\mu + \omega_n$  has exactly  $j$  nonzero entries and is contained in the orbit of  $\omega_j$  under  $\mathcal{B}_n$ , i.e.  $\Theta_{\mu + \omega_n} = \Theta_{\omega_j}$ . The number of  $\mu$ , for which this is the case, is  $\binom{n}{j}$ .

From this and [Proposition 2.6](#), we conclude that

$$2^n \theta_n^2 = \sum_{\mu \in \mathcal{B}_n \omega_n} \Theta_{\omega_n + \mu} = \Theta_{2\omega_n} + 1 + \sum_{j=1}^{n-1} \binom{n}{j} \theta_j$$

to obtain the equation for  $\Theta_{\omega_n + \mu}$ .  $\square$

## 6.2 Hermite matrix for $\mathcal{B}_n$

**Proposition 6.3.** *Define the map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-1} x_{n-2}^{-1}, x_n^2 x_{n-1}^{-1}). \end{aligned}$$

1.  $\psi$  is surjective. The preimage of  $\mathbb{T}^n$  under  $\psi$  is  $\mathbb{T}^n$ .
2. Every point in  $\mathbb{T}^n$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  with  $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$  and  $x_n = -x'_n$ .

3. If  $x, x' \in \mathbb{T}^n$  with  $\psi(x) = \psi(x')$  and  $x_n = -x'_n$ , then  $\theta_n(x) = -\theta_n(x')$ .

*Proof.* 1. and 2. For all  $y \in (\mathbb{C}^*)^n$ , there exists  $x \in (\mathbb{C}^*)^n$  with  $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-1} = y_{n-1} x_{n-2}$  and  $x_n^2 = y_n x_{n-1}$ . Thus,  $x$  is a preimage of  $y$  under  $\psi$  and uniquely determined by  $y$  up to a sign in the last coordinate. We have  $y \in \mathbb{T}^n$  if and only if  $x \in \mathbb{T}^n$ .

3. We deduce from the Coxeter diagram in Equation (6.2) that  $\text{Stab}_{\mathcal{B}_n}(\omega_n) \cong \mathcal{A}_{n-1} \cong \mathfrak{S}_n$ . Hence,  $\mathcal{B}_n \omega_n = \{\pm 1\}^n \omega_n$  and by Proposition 6.1 this orbit consists of elements

$$\mu = \frac{\delta_1}{2} e_1 + \dots + \frac{\delta_n}{2} e_n = \frac{\delta_1}{2} \omega_1 + \sum_{i=2}^{n-1} \frac{\delta_i}{2} (\omega_i - \omega_{i-1}) + \frac{\delta_n}{2} (2\omega_n - \omega_{n-1}) = \delta_n \omega_n + \nu \in \Omega$$

with  $\delta_i \in \{\pm 1\}$  for  $1 \leq i \leq n$  and  $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-1}$ . Let  $\alpha, \beta \in \mathbb{Z}^n$  be the coordinates of  $\mu, \nu$  in  $\Omega$ . Then  $\beta_n = 0$  and the monomial in  $\theta_n$  corresponding to  $\mu$  is  $x^\alpha = x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}} x_n^{\delta_n}$ . Thus,  $x^\alpha$  is linear in  $x_n$ . Since every monomial in  $\theta_n$  can be written in terms of such  $\beta, \delta$ ,  $\theta_n$  is linear in  $x_n$  and with  $x, x'$  from the hypothesis we have  $\theta_n(x) = -\theta_n(x')$ .  $\square$

We state the main result for Weyl groups of type  $\mathcal{B}$ .

**Theorem 6.4.** Define the matrix  $H \in \mathbb{Q}[z]^{n \times n}$  by

$$H(z)_{ij} = \text{Trace}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \dots & 0 & (-1)^{n+1} c_n f_n(z) \\ 1 & & 0 & (-1)^n c_{n-1} z_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & c_1 z_1 \end{bmatrix}, \quad \text{where } c_i = 2^i \binom{n}{i} \quad \text{and}$$

$$f_n(z) = c_n z_n^2 - \sum_{i=1}^{n-1} \binom{n}{i} z_i - 1.$$

Then  $\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{R}^n$  and set  $\tilde{c}_i := (-1)^i c_i z_i \in \mathbb{R}$  for  $1 \leq i \leq n-1$ ,  $\tilde{c}_n := (-1)^n c_n f_n(z)$ . Then  $(-1)^{i+1} c_i z_i = -\tilde{c}_i$  for  $1 \leq i \leq n-1$  and  $(-1)^{n+1} c_n f_n(z) = -\tilde{c}_n$  are the entries of the last column of  $C(z)$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists a  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$  for  $1 \leq i \leq n$ . By Proposition 6.1 and Lemma 6.2, the solution of

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i \tilde{c}_i \quad \text{for } 1 \leq i \leq n$$

is  $y = \psi(x) \in \mathbb{T}^n$ , where  $\psi$  is the map from Proposition 6.3. Applying Corollary 3.7 yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $H(z) \succeq 0$ . Then by Corollary 3.7, the solution  $y$  of system (II) with coefficients  $\tilde{c}_i$  is contained in  $\mathbb{T}^n$ . By Proposition 6.3,  $y$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  under  $\psi$  with  $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$  and  $x_n = -x'_n$ . Then  $z_i = \theta_i(x) = \theta_i(x')$  for  $1 \leq i \leq n-1$  and  $z_n^2 = \theta_n(x)^2 = \theta_n(x')^2$  with  $\theta_n(x) = -\theta_n(x')$ . Therefore,  $z_n = \theta_n(x)$  or  $z_n = \theta_n(x') = -\theta_n(x)$  and thus  $z$  is contained in  $\mathcal{T}$ .  $\square$

**Remark 6.5.** The entries of the last column of the matrix polynomial  $C(z)$  from Theorem 6.4 have degree 1, except  $C(z)_{1n}$ , which has degree 2. All other entries are constant. The degree of  $H(z)_{ij}$  is  $i + j$  and the degree of the determinant of  $H(z)$  is  $3n$ .

### 6.3 Example $\mathcal{B}_2$

In this subsection, we further investigate the case  $n = 2$ .

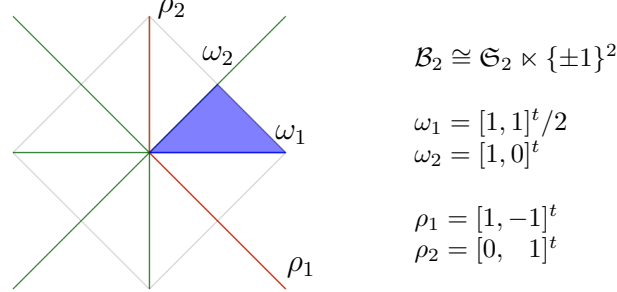


Figure 6: The root system  $\mathcal{B}_2$  in  $\mathbb{R}^2$  with fundamental weights  $\omega_1, \omega_2$  and simple roots  $\rho_1, \rho_2$ .

In **Figure 6**, the square is the image of the blue shaded triangle with vertices  $0, \omega_1, \omega_2$  under reflection by  $\mathcal{B}_2 = \mathfrak{S}_2 \times \{\pm 1\}^2$ .

Let  $z = (z_1, z_2) \in \mathbb{R}^2$ . The matrix  $C(z) \in \mathbb{Q}[z]^{2 \times 2}$  from **Theorem 6.4** evaluated in  $z$  is

$$C(z) = \begin{bmatrix} 0 & -16z_2^2 + 8z_1 + 4 \\ 1 & 4z_1 \end{bmatrix}. \quad (6.3)$$

Then  $z$  is contained in  $\mathcal{T}$  if and only if the resulting Hermite matrix

$$H(z) = 16 \begin{bmatrix} -z_1^2 + 2z_2^2 - z_1 & -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 \\ -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 & -16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 20z_1^2 + 8z_2^2 - 4z_1 \end{bmatrix}$$

is positive semi-definite. By **Proposition 3.9**,  $H(z) \succeq 0$  is equivalent to its determinant and trace being nonnegative. The varieties of these two polynomials in  $z_1, z_2$  are depicted below.

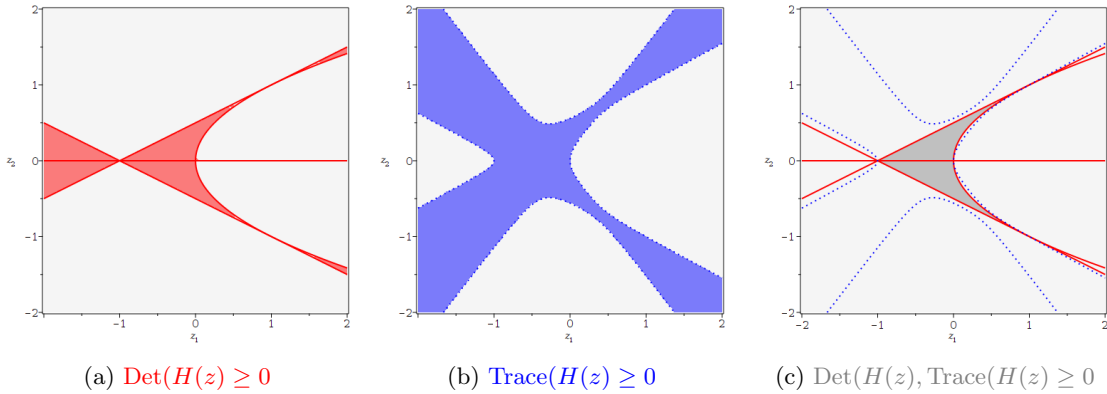


Figure 7: Vanishing points and positivity regions for determinant and trace of  $H(z)$ .

$$\begin{aligned} \text{Det}(H(z)) &= 4096(z_1^2z_2^4 - 4z_2^6 - z_1^3z_2^2 + 6z_1z_1^4 - 2z_1^2z_2^2 + z_2^4 - z_1z_2^2) && \text{(solid)} \\ \text{Trace}(H(z)) &= 16(-16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 21z_1^2 + 10z_2^2 - 5z_1) && \text{(dots)} \end{aligned}$$

The  $\mathcal{B}_2$  case is similar, but not identical to  $\mathcal{C}_2$  from **Section 5.3**. Apart from a rotation, we also observe that the rank of  $H(z)$  reduces from 2 to 1 on the line “ $z_2 = 0$ ”. Furthermore, there is a fourth intersection of the varieties of the determinant and the trace at  $z = (0, 0)$ .



## 7 Type $\mathcal{D}$

The Weyl group  $\mathcal{D}_n$  is isomorphic to a subgroup of  $\mathcal{C}_n \cong \mathcal{B}_n$  of index 2, which leads us to a symmetric system of type (II). The explicit formula between the orbit polynomials and the elementary symmetric polynomials is similar to [Proposition 5.1](#), but with deviations for  $\sigma_{n-1}$  and  $\sigma_n$  that additionally depend on the parity of  $n$ . If  $n$  is odd, then the orbit space of  $\mathcal{D}_n$  contains complex points, but a real description is possible with a change of variables similar to the case of  $\mathcal{A}_{n-1}$  in [Remark 4.4](#).

### 7.1 Orbit polynomials for $\mathcal{D}_n$

The Weyl group of the root system  $\mathcal{D}_n$  is  $\mathcal{D}_n \cong \mathfrak{S}_n \times \{\pm 1\}_+^n$ . A representation can be given over  $\mathbb{R}^n$ .  $\mathfrak{S}_n$  acts on  $\mathbb{R}^n$  by permutation of coordinates and  $\{\pm 1\}_+^n$  acts by multiplication of coordinates with  $\pm 1$ , where only an even amount of sign changes by  $-1$  is admissible. By [\[Bou68b, Planche IV\]](#), the simple roots and fundamental weights in this representation are

$$\begin{aligned} \rho_i &= e_i - e_{i+1}, & \rho_n &= e_{n-1} + e_n & \text{and} \\ \omega_i &= e_1 + \dots + e_i, & \omega_{n-1} &= (e_1 + \dots + e_{n-1} - e_n)/2, & \omega_n &= (e_1 + \dots + e_n)/2. \end{aligned} \quad (7.1)$$

The graph

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-2} \begin{cases} \nearrow s_{n-1} \\ \searrow s_n \end{cases} \quad (7.2)$$

is the Coxeter diagram of the Weyl group  $\mathcal{D}_n$ , where  $s_i$  is the permutation matrix of the  $i$ -th and  $(i+1)$ -th coordinate for  $1 \leq i \leq n-1$  and  $s_n$  is  $\text{diag}(1, \dots, 1, -1, -1)$  times the permutation matrix of the  $(n-1)$ -th and  $n$ -th coordinate, generating  $\mathcal{D}_n$ . For all  $1 \leq i \leq n$ , we have  $-\omega_i \in \mathfrak{G}\omega_i$ , except when  $n$  is odd, where  $-\omega_{n-1} \in \mathfrak{G}\omega_n$ .

The orbit polynomials associated to the fundamental weights of  $\mathcal{D}_n$  and the elementary symmetric polynomials from [Definition 3.1](#) satisfy the following relations.

**Proposition 7.1.** *In  $\mathbb{Q}[x^\pm]$ , define the monomials*

$$y_1 = x_1, \quad y_k = x_k x_{k-1}^{-1} \quad \text{for } 2 \leq k \leq n-2, \quad y_{n-1} = x_n x_{n-1} x_{n-2}^{-1} \quad \text{and} \quad y_n = x_n x_{n-1}^{-1}.$$

Then

$$\begin{aligned} \sigma_i(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^i \binom{n}{i} \theta_i(x) \quad \text{for } 1 \leq i \leq n-2, \\ \sigma_{n-1}(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^{n-1} n \Theta_{\omega_{n-1} + \omega_n}(x) \quad \text{and} \\ \sigma_n(y_1(x) + y_1(x)^{-1}, \dots, y_n(x) + y_n(x)^{-1}) &= 2^{n-1} (\Theta_{2\omega_{n-1}}(x) + \Theta_{2\omega_n}(x)). \end{aligned}$$

*Proof.* We follow the idea of the proof for [Proposition 4.1](#).

It follows from the Coxeter diagram in [Equation \(7.2\)](#) that  $\text{Stab}_{\mathcal{D}_n}(\omega_1) \cong \mathcal{D}_{n-1}$  and the index of  $\text{Stab}_{\mathcal{D}_n}(\omega_1)$  in  $\mathcal{D}_n$  is  $2n$ . Assume that  $\mathcal{D}_n/\text{Stab}_{\mathcal{D}_n}(\omega_1) = \{[h_1], \dots, [h_{2n}]\}$  are the cosets. Then the orbit of  $\omega_1$  under  $\mathcal{C}_n$  is given by  $\mathcal{D}_n \omega_1 = \{h_1 \cdot \omega_1, \dots, h_{2n} \cdot \omega_1\}$ . Choose  $h_1 = \text{id}_{\mathbb{R}^n}$  and  $h_i = s_1 \cdots s_{i-1}$  for  $2 \leq i \leq n$ . Then

$$\begin{aligned} h_1 \omega_1 &= \omega_1 = e_1, h_2 \omega_1 = \omega_2 - \omega_1 = e_2, \dots, h_{n-2} \omega_1 = \omega_{n-2} - \omega_{n-3} = e_{n-2} \\ \text{and } h_{n-1} \omega_1 &= \omega_n + \omega_{n-1} - \omega_{n-2} = e_{n-1}, h_n \omega_1 = \omega_n - \omega_{n-1} = e_n. \end{aligned}$$

Those are elements of the orbit  $\mathfrak{S}_n \omega_1 \subseteq \mathcal{D}_n \omega_1$ . The missing elements are  $h_{n+i} \omega_1 := -h_i \omega_1$  for  $1 \leq i \leq n$ , given by the subgroup  $\{\pm 1\}_+^n$ . This can be achieved by an even amount of sign changes, because  $h_i \omega_1$  has exactly one nonzero coordinate. The  $y_i, y_i^{-1}$  are the corresponding monomials in  $\mathbb{Q}[x^\pm]$ .

For  $1 \leq i \leq n-2$ , write  $x_i = y_1 \dots y_i$ . Then the statement for  $\theta_i$  can be proven analogous to [Proposition 5.1](#). When we replace  $\omega_{n-1}$  with  $\omega_{n-1} + \omega_n$ , the corresponding monomial is  $x_{n-1} x_n$  and we obtain the equation for  $\Theta_{\omega_{n-1} + \omega_n}$  as well.

Finally, apply  $x_{n-1}^2 = y_1 \dots y_{n-1} y_n^{-1}$  and  $x_n^2 = y_1 \dots y_{n-1} y_n$  to get

$$2^{n-1} (\Theta_{2\omega_{n-1}}(x) + \Theta_{2\omega_n}(x)) = \sum_{\substack{\delta \in \{\pm 1\}^n \\ \delta_1 \dots \delta_n = -1}} y(x)^\delta + \sum_{\substack{\delta \in \{\pm 1\}^n \\ \delta_1 \dots \delta_n = 1}} y(x)^\delta = \sum_{\delta \in \{\pm 1\}^n} y(x)^\delta = \prod_{i=1}^n y_i(x) + y_i(x)^{-1},$$

which is the last equation.  $\square$

The orbit polynomials associated to  $\omega_n + \omega_{n-1}$ ,  $2\omega_{n-1}$  and  $2\omega_n \in \Omega$  are invariant. Hence, they can be expressed in terms of the generators  $\theta_i$  of  $\mathbb{Q}[x^\pm]^{\mathcal{D}^n}$ . The formula depends on the parity of  $n$  as follows.

**Lemma 7.2.** *In  $\mathbb{Q}[x^\pm]$ , the following equations hold.*

1. *If  $n$  is even, then*

$$\begin{aligned} \Theta_{\omega_{n-1} + \omega_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1}, \\ \Theta_{2\omega_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1 \quad \text{and} \\ \Theta_{2\omega_n} &= 2^{n-1} \theta_n^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1. \end{aligned}$$

2. *If  $n$  is odd, then*

$$\begin{aligned} \Theta_{\omega_{n-1} + \omega_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j} - \frac{1}{n}, \\ \Theta_{2\omega_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1} \quad \text{and} \\ \Theta_{2\omega_n} &= 2^{n-1} \theta_n^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}. \end{aligned}$$

*Proof.* We have  $|\mathcal{D}_n \omega_{n-1}| = |\mathcal{D}_n \omega_n| = 2^{n-1}$ . The equations can be obtained from the recurrence formula [Proposition 2.6](#), the representation over  $\mathbb{R}^n$  and the following combinatorial steps.

1. Assume that  $n$  is even. We first prove the equation for  $\Theta_{2\omega_n}$  in detail. For  $0 \leq j \leq n/2$ , consider  $\mu \in \mathcal{D}_n \omega_n$  with  $2j$  positive coordinates. There are precisely  $\binom{n}{2j}$  such elements in  $\mathcal{D}_n \omega_n$  and an odd amount of positive coordinates is not possible. We distinguish three cases. If  $j = 0$ , then  $\omega_{n-1} + \mu = 0$ , and if  $j = n/2$ , then  $\omega_n + \mu = 2\omega_n$ . Otherwise,  $\omega_n + \mu$  has  $2j$  nonzero coordinates and must therefore be contained in  $\mathcal{D}_n \omega_{2j}$ . All in all, we obtain

$$2^{n-1} \theta_n^2 = \sum_{\mu \in \mathcal{D}_n \omega_n} \Theta_{\omega_n + \mu} = \Theta_{2\omega_n} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} + 1.$$

Next for  $1 \leq j \leq n/2$ , consider  $\mu \in \mathcal{D}_n \omega_{n-1}$  with  $2j - 1$  positive coordinates. Then

$$\omega_{n-1} + \mu \begin{cases} = 0, & \text{if } j = 1, \quad \mu_n = \frac{1}{2} \\ = 2\omega_{n-1}, & \text{if } j = \frac{n}{2}, \quad \mu_n = -\frac{1}{2} \\ \in \mathcal{D}_n \omega_{2j}, & \text{else} \end{cases} \quad \text{and} \quad \omega_n + \mu \begin{cases} \in \mathcal{D}_n (\omega_n + \omega_{n-1}), & \text{if } j = \frac{n}{2} \\ \in \mathcal{D}_n \omega_{2j-1}, & \text{else} \end{cases}.$$

After counting the number of possibilities in each case, we obtain the two equations

$$2^{n-1} \theta_{n-1}^2 = 1 + \Theta_{2\omega_{n-1}} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} \quad \text{and} \quad 2^{n-1} \theta_{n-1} \theta_n = n \Theta_{\omega_n + \omega_{n-1}} + \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1}.$$

2. Now assume that  $n$  is odd. For  $1 \leq j \leq (n+1)/2$ , consider  $\mu \in \mathcal{D}_n \cdot \omega_n$  with  $2j - 1$  positive coordinates. If  $j = (n+1)/2$  then  $\omega_n + \mu = 2\omega_n$ . Otherwise,  $\omega_n + \mu$  has  $2j - 1$  nonzero coordinates. The equation for  $\Theta_{2\omega_n}$  is

$$2^{n-1} \theta_n^2 = \Theta_{2\omega_n} + \sum_{j=1}^{(n-1)/2} \binom{n}{2j-1} \theta_{2j-1}.$$

Finally for  $0 \leq j \leq (n-1)/2$ , consider  $\mu \in \mathcal{D}_n \omega_{n-1}$  with  $2j$  positive coordinates. Then

$$\omega_{n-1} + \mu \begin{cases} = 2\omega_{n-1}, & \text{if } j = \frac{n-1}{2}, \quad \mu_n = -\frac{1}{2} \\ \in \mathcal{D}_n \omega_{2j+1}, & \text{else} \end{cases} \quad \text{and} \quad \omega_n + \mu \begin{cases} = 0, & \text{if } j = 0 \\ \in \mathcal{D}_n (\omega_n + \omega_{n-1}), & \text{if } j = \frac{n-1}{2} \\ \in \mathcal{D}_n \omega_{2j}, & \text{else} \end{cases}.$$

After counting the number of possibilities in each case, we obtain the two equations

$$2^{n-1} \theta_{n-1}^2 = \Theta_{2\omega_{n-1}} + \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1} \quad \text{and} \quad 2^{n-1} \theta_{n-1} \theta_n = 1 + n \Theta_{\omega_n + \omega_{n-1}} + \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j}.$$

This completes the proof. □

## 7.2 Hermite matrix for $\mathcal{D}_n$

**Proposition 7.3.** *Define the map*

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ x &\mapsto (x_1, x_2 x_1^{-1}, \dots, x_{n-2} x_{n-3}^{-1}, x_n x_{n-1} x_{n-2}^{-1}, x_n x_{n-1}^{-1}). \end{aligned}$$

1.  $\psi$  is surjective. The preimage of  $\mathbb{T}^n$  under  $\psi$  is  $\mathbb{T}^n$ .
2. Every point in  $\mathbb{T}^n$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  with  $x' = (x_1, \dots, x_{n-2}, -x_{n-1}, -x_n)$ .
3. If  $x \neq x' \in \mathbb{T}^n$  with  $\psi(x) = \psi(x')$ , then  $\theta_i(x) = \theta_i(x')$  for  $1 \leq i \leq n-2$  and  $\theta_{n-1}(x) = -\theta_{n-1}(x')$ ,  $\theta_n(x) = -\theta_n(x')$ .
4. For all  $x \in \mathbb{T}$ , there exists a  $\tilde{x} \in \mathbb{T}$ , such that  $\theta_i(x) = \theta_i(\tilde{x})$  for  $1 \leq i \leq n-2$  and  $\theta_{n-1}(x) = \theta_n(\tilde{x})$ ,  $\theta_n(x) = \theta_{n-1}(\tilde{x})$ .

*Proof.* 1. and 2. For all  $y \in (\mathbb{C}^*)^n$ , there exists  $x \in (\mathbb{C}^*)^n$  with  $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-2} = y_{n-2} x_{n-3}$  and  $x_{n-1}^2 = y_n^{-1} y_{n-1} x_{n-2}, x_n = y_n x_{n-1}$ . Hence,  $x$  is uniquely determined by  $y$  up to a sign in  $x_{n-1}, x_n$ . We have  $y \in \mathbb{T}^n$  if and only if  $x \in \mathbb{T}^n$ .

3. We deduce from the Coxeter diagram in Equation (5.2) that  $\text{Stab}_{\mathcal{D}_n}(\omega_n) \cong \text{Stab}_{\mathcal{D}_n}(\omega_{n-1}) \cong \mathcal{A}_{n-1} \cong \mathfrak{S}_n$ . Hence,  $\mathcal{D}_n \omega_{n-1} = \{\pm 1\}_+^n \omega_{n-1}$ , respectively  $\mathcal{D}_n \omega_n = \{\pm 1\}_+^n \omega_n$ , and by Proposition 7.1 this orbit consists of elements

$$\begin{aligned} \mu &= \frac{\delta_1}{2} e_1 + \dots + \frac{\delta_n}{2} e_n = \frac{\delta_1}{2} \omega_1 + \sum_{i=2}^{n-2} \frac{\delta_i}{2} (\omega_i - \omega_{i-1}) + \frac{\delta_{n-1}}{2} (\omega_n + \omega_{n-1} - \omega_{n-2}) + \frac{\delta_n}{2} (\omega_n - \omega_{n-1}) \\ &= \frac{\delta_{n-1} + \delta_n}{2} \omega_n + \frac{\delta_{n-1} - \delta_n}{2} \omega_{n-1} + \nu \in \Omega \end{aligned}$$

with  $\delta_i \in \{\pm 1\}$  for  $1 \leq i \leq n$  satisfying

$$\delta_1 \dots \delta_n = \begin{cases} -1, & \text{if } \mu \in \mathcal{D}_n \omega_{n-1} \\ 1, & \text{if } \mu \in \mathcal{D}_n \omega_n \end{cases}$$

and  $\nu \in \mathbb{Z} \omega_1 \oplus \dots \oplus \mathbb{Z} \omega_{n-2}$ . Let  $\alpha, \beta \in \mathbb{Z}^n$  be the coordinate vectors of  $\mu, \nu$  in  $\Omega$ . Then  $\beta_{n-1} = \beta_n = 0$  and the monomial in  $\theta_{n-1}$ , respectively  $\theta_n$ , corresponding to  $\mu$  is

$$x^\alpha = x_1^{\beta_1} \dots x_{n-2}^{\beta_{n-2}} x_{n-1}^{(\delta_{n-1} - \delta_n)/2} x_n^{(\delta_{n-1} + \delta_n)/2}$$

with  $(\delta_{n-1} \pm \delta_n)/2 \in \{-1, 0, 1\}$ . Therefore,  $x^\alpha$  is linear in  $x_{n-1}$  and independent of  $x_n$  or vice versa. With  $x, x'$  from the hypothesis we have  $x^\alpha = -(x')^\alpha$ . Since every monomial in  $\theta_{n-1}$ , respectively  $\theta_n$ , can be written in terms of such  $\beta, \delta$ , we obtain  $\theta_{n-1}(x) = -\theta_{n-1}(x')$  as well as  $\theta_n(x) = -\theta_n(x')$ . Furthermore,  $\theta_i(x) = \theta_i(x')$  holds according to Proposition 7.1, since  $y_k(x) = y_k(x')$  for all  $1 \leq k \leq n$ .

4. Following the proof for the third statement, this holds for  $\tilde{x} := (x_1, \dots, x_{n-2}, x_n, x_{n-1})$ . □

We state the main result for Weyl groups of type  $\mathcal{D}$ .

**Theorem 7.4.** *Consider the  $n$ -dimensional  $\mathbb{R}$ -vector space*

$$W := \begin{cases} \mathbb{R}^n, & \text{if } n \text{ is even} \\ \{z \in \mathbb{C}^n \mid z_1, \dots, z_{n-2} \in \mathbb{R}, \bar{z}_n = z_{n-1}\}, & \text{if } n \text{ is odd} \end{cases}$$

and define the matrix  $H \in \mathbb{Q}[z]^{n \times n}$  by

$$H(z)_{ij} = \text{Trace}(4(C(z))^{i+j-2} - (C(z))^{i+j}) \quad \text{with}$$

$$C(z) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{n+1} c_n & f_n(z) \\ 1 & & & 0 & (-1)^n c_{n-1} & f_{n-1}(z) \\ & \ddots & & & (-1)^{n-1} c_{n-2} z_{n-2} & \\ & & \ddots & & \vdots & \\ 0 & & & 1 & c_1 z_1 & \end{bmatrix}, \quad \text{where } c_i = 2^i \binom{n}{i} \quad \text{and}$$

$$f_{n-1}(z) = \begin{cases} 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} z_{2j-1}, & \text{if } n \text{ is even} \\ 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-3)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is odd} \end{cases},$$

$$f_n(z) = \begin{cases} 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is even} \\ 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} z_{2j+1}, & \text{if } n \text{ is odd} \end{cases}.$$

For all  $z \in W$ ,  $H(z) \in \mathbb{R}^{n \times n}$  and  $\mathcal{T} = \{z \in W \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{R}^n$  and set  $\tilde{c}_i := (-1)^i c_i z_i \in \mathbb{R}$  for  $1 \leq i \leq n-2$ ,  $\tilde{c}_{n-1} := (-1)^{n-1} c_{n-1} f_{n-1}(z)$ ,  $\tilde{c}_n := (-1)^n c_n f_n(z)$ . Then  $(-1)^{i+1} c_i z_i = -\tilde{c}_i$  for  $1 \leq i \leq n-2$ ,  $(-1)^n c_{n-1} f_{n-1}(z) = -\tilde{c}_{n-1}$ ,  $(-1)^{n+1} c_n f_n(z) = -\tilde{c}_n$  are the entries of the last column of  $C(z)$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists a  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$ . By [Proposition 7.1](#) and [Lemma 7.2](#), the solution of

$$(II) \quad \sigma_i(y_1 + y_1^{-1}, \dots, y_n + y_n^{-1}) = (-1)^i \tilde{c}_i \quad \text{for } 1 \leq i \leq n$$

is  $y = \psi(x) \in \mathbb{T}^n$ , where  $\psi$  is the map from [Proposition 7.3](#). Applying [Corollary 3.7](#) yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $H(z) \succeq 0$  and fix  $\tilde{c}_i$  as above. Then by [Corollary 3.7](#), the solution  $y$  of the above system (II) is contained in  $\mathbb{T}^n$ . According to [Proposition 7.3](#),  $y$  has two distinct preimages  $x, x' \in \mathbb{T}^n$ . Then  $z_i = \theta_i(x) = \theta_i(x')$  for  $1 \leq i \leq n-2$  and  $z_{n-1}^2 + z_n^2 = \theta_{n-1}(x)^2 + \theta_n(x)^2 = \theta_{n-1}(x')^2 + \theta_n(x')^2$  as well as  $z_{n-1} z_n = \theta_{n-1}(x) \theta_n(x) = \theta_{n-1}(x') \theta_n(x')$  with  $\theta_{n-1}(x) = -\theta_{n-1}(x')$ ,  $\theta_n(x) = -\theta_n(x')$ . Therefore,  $\{z_{n-1}, z_n\} \in \{\{\theta_{n-1}(x), \theta_n(x)\}, \{\theta_{n-1}(x'), \theta_n(x')\}\}$ . Assume that we are in the first case. If  $z_{n-1} = \theta_{n-1}(x)$ ,  $z_n = \theta_n(x)$ , then  $z = \vartheta(x)$ . Otherwise by [Proposition 7.3](#), there exists  $\tilde{x}$ , such that  $z_{n-1} = \theta_{n-1}(\tilde{x})$ ,  $z_n = \theta_n(\tilde{x})$  and  $z = \vartheta(\tilde{x})$ . A similar argument holds for  $x'$ . Therefore,  $z$  is contained in  $\mathcal{T}$ .  $\square$

### Remark 7.5.

1. Concerning points  $\tilde{z} \in \mathcal{T} \subseteq W$ , we are in a similar situation as in the  $\mathcal{A}_{n-1}$  case:  $-\omega_{n-1} \in \mathcal{D}_n \omega_n$  leads to  $W \not\subseteq \mathbb{R}^n$ , when  $n$  is odd. In order to obtain a point  $z \in \mathbb{R}^n$ , that corresponds to  $\tilde{z}$ , we substitute  $z_n = \Re(\tilde{z}_n)$  and  $z_{n-1} = \Im(\tilde{z}_n)$ . This leaves us with the substitutions  $z_n z_{n-1} \mapsto z_n^2 + z_{n-1}^2$  in  $f_{n-1}(z)$  and  $z_n^2 + z_{n-1}^2 \mapsto 2(z_n^2 - z_{n-1}^2)$  in  $f_n(z)$ .
2. The entries  $H(z)_{ij} \in \mathbb{Q}[z]$  have degree  $i + j$ . Independent of the parity of  $n$ , the determinant of  $H(z)$  has degree  $3n + 1$ .

### 7.3 Example $\mathcal{D}_4$

In this subsection, we further investigate the case  $n = 4$ .



Figure 8: A partial projection of the root system  $D_4$  in  $\mathbb{R}^4$  with fundamental weights  $\omega_1, \omega_2, \omega_3, \omega_4$ .

The 4-simplex with vertices  $0, \omega_1, \omega_2/2, \omega_3, \omega_4$  is mapped under reflection by  $\mathcal{D}_4 = \mathfrak{S}_4 \times \{\pm 1\}_+^4$  to an icositetrachoron. This is a polytope in  $\mathbb{R}^4$  with boundary consisting of 24 octahedral cells. In Figure 8, we depict a projection of the unique cell with center  $\omega_2/2$  and three out of six vertices  $\omega_1, \omega_3, \omega_4$  to  $\mathbb{R}^3$ . The blue shaded region is the 3-simplex with projected vertices  $\omega_1, \omega_2/2, \omega_3, \omega_4$ .

Let  $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ . Then the matrix  $H(z)$  from Theorem 7.4 has determinant

$$\begin{aligned} \text{Det}(H(z)) = & -4294967296(z_3^2 - 8z_3z_4 + z_4^2 + 3z_1)(-z_3^2 - 8z_3z_4 - z_4^2 + 3z_1)(-32768z_1^3z_3^3z_4^3 \\ & - 432z_1^4z_4^3 + 48288z_1^4z_3^2z_4^2 - 432z_1^4z_4^4 + 3456z_1^3z_2z_3^3z_4 + 3456z_1^3z_2z_3z_4^3 \\ & + 2304z_1^2z_2^2z_3^2z_4^2 - 1536z_1^2z_3^4z_4^2 - 1536z_1^2z_3^2z_4^4 + 55296z_1z_2z_3^3z_4^3 - 110592z_3^4z_4^4 \\ & - 24576z_1^5z_3z_4 - 432z_1^4z_2z_3^2 - 432z_1^4z_2z_4^2 - 7488z_1^3z_2^2z_3z_4 + 1536z_1^3z_3^3z_4 \\ & + 1536z_1^3z_3z_4^3 - 216z_1^2z_2^3z_3^2 - 216z_1^2z_2^3z_4^2 + 864z_1^2z_2z_3^4 - 78912z_1^2z_2z_3^2z_4^2 \\ & + 864z_1^2z_2z_4^4 - 5760z_1z_2^2z_3^3z_4 - 5760z_1z_2^2z_3z_4^3 - 1536z_1z_3^5z_4 + 218112z_1z_3^3z_4^3 \\ & - 1536z_1z_3z_4^5 - 3456z_2^3z_3^2z_4^2 + 13824z_2z_3^4z_4^2 + 13824z_2z_3^2z_4^4 + 4096z_1^6 + 2196z_1^4z_2^2 \\ & - 168z_1^4z_3^2 - 168z_1^4z_4^2 + 38304z_1^3z_2z_3z_4 + 324z_1^2z_4^4 + 288z_1^2z_2^2z_3^2 + 288z_1^2z_2^2z_4^2 \\ & + 768z_1^2z_3^3 - 163968z_1^2z_3^2z_4^2 + 768z_1^2z_4^4 + 12096z_1z_2^3z_3z_4 - 9216z_1z_2z_3^3z_4 \\ & - 9216z_1z_2z_3z_4^3 + 324z_2^4z_3^3 + 324z_2^4z_4^2 - 288z_2^2z_3^4 - 21312z_2^2z_3^2z_4^2 - 288z_2^2z_4^4 + 64z_3^6 \\ & + 192z_3^4z_4^2 + 192z_3^2z_4^4 + 64z_4^6 - 6228z_1^4z_2 + 54912z_1^3z_3z_4 - 3186z_1^2z_2^3 + 720z_1^2z_2z_3^2 \\ & + 720z_1^2z_2z_4^2 + 18720z_1z_2^2z_3z_4 + 768z_1z_3^3z_4 + 768z_1z_3z_4^3 - 486z_2^5 + 864z_2^3z_3^2 + 864z_2^3z_4^2 \\ & - 288z_2z_3^4 - 4032z_2z_3^2z_4^2 - 288z_2z_4^4 - 6843z_1^4 - 3528z_1^2z_2^2 - 384z_1^2z_3^2 - 384z_1^2z_4^2 \\ & + 2304z_1z_2z_3z_4 - 729z_2^4 + 576z_2^2z_3^2 + 576z_2^2z_4^2 - 48z_3^4 - 96z_3^2z_4^2 - 48z_4^4 - 234z_1^2z_2 \\ & - 96z_1z_3z_4 - 432z_2^3 + 144z_2z_3^2 + 144z_2z_4^2 + 48z_1^2 - 126z_2^2 + 12z_3^2 + 12z_4^2 - 18z_2 - 1). \end{aligned}$$

## 8 Conjecture

In this section, we formulate the conjecture that the  $\mathbb{T}$ -orbit space of  $\mathfrak{G}$  can be computed as the negativity-locus of another matrix polynomial, which is closer to the characterization [PS85, Main Theorem] for the linear action of a compact Lie group on a polynomial ring.

$\mathfrak{G}$  is not necessarily the Weyl group of a root system in this section, but simply a group with an integer representation  $\mathcal{G}$  as in Equation (2.1). Assume that  $\mathbb{Q}[x^\pm]^\mathcal{G} = \mathbb{Q}[\theta_1, \dots, \theta_m]$  for some  $n \leq m \in \mathbb{N}$ . On  $\mathbb{Q}[x^\pm]$ , define the **Euler derivation**  $x_i \partial / \partial x_i$  with  $x_i \partial / \partial x_i(x_i) = x_i$  and  $x_i \partial / \partial x_i(x_j) = 0$  for  $1 \leq i \neq j \leq n$  as well as the associated gradient  $\tilde{\nabla} := [x_1 \partial / \partial x_1, \dots, x_n \partial / \partial x_n]^t$ . We fix a symmetric positive definite matrix  $S \in \mathbb{Q}^{n \times n}$  with  $B^t S B = S$  for all  $B \in \mathcal{G}$ . For example  $S = 1/|\mathcal{G}| \sum_{B \in \mathcal{G}} B^t B$  has the desired property. For  $f, f' \in \mathbb{Q}[x^\pm]^n$ , write  $\langle f, f' \rangle_S := f^t S f'$ .

We give a necessary condition for points to be contained in the image  $\mathcal{T}$ . First, we show that the  $S$ -induced  $\mathcal{G}$ -invariant inner product of two gradients of  $\mathcal{G}$ -invariant Laurent polynomials is again  $\mathcal{G}$ -invariant.

**Proposition 8.1.** *Let  $f, f' \in \mathbb{Q}[x^\pm]^\mathcal{G}$ .*

1. For all  $B \in \mathcal{G}$ ,  $B(\tilde{\nabla} f)(x^B) = \tilde{\nabla} f(x)$ .
2. We have  $\langle \tilde{\nabla} f, \tilde{\nabla} f' \rangle_S = (\tilde{\nabla} f)^t S (\tilde{\nabla} f') \in \mathbb{Q}[x^\pm]^\mathcal{G}$ .

*Proof.* We obtain the first statement from

$$\tilde{\nabla} f(x) = \tilde{\nabla} (B \cdot f)(x) = B \left[ x^{B \cdot 1} \frac{\partial f}{\partial x_1}(x^B), \dots, x^{B \cdot n} \frac{\partial f}{\partial x_n}(x^B) \right]^t = B(\tilde{\nabla} f)(x^B),$$

where we applied the hypothesis that  $f$  is  $\mathcal{G}$ -invariant and then the chain rule. Therefore,

$$\begin{aligned} (B \cdot ((\tilde{\nabla} f)^t S (\tilde{\nabla} f')))(x) &= ((\tilde{\nabla} f)(x^B))^t S ((\tilde{\nabla} f')(x^B)) \\ &= (B^{-1} \tilde{\nabla} f(x))^t S (B^{-1} \tilde{\nabla} f'(x)) \\ &= (\tilde{\nabla} f(x))^t (B^{-1})^t S B^{-1} (\tilde{\nabla} f'(x)) \\ &= ((\tilde{\nabla} f)^t S (\tilde{\nabla} f'))(x) \end{aligned}$$

proves the second statement. □

For  $f \in \mathbb{Q}[x^\pm]$ , define  $\widehat{f} \in \mathbb{Q}[x^\pm]$  by  $\widehat{f}(x) = f(x^{-I_n})$ .

**Proposition 8.2.** *Let  $f \in \mathbb{Q}[x^\pm]$ .*

1. We have  $\tilde{\nabla} \widehat{f}(x) = -(\widehat{\tilde{\nabla} f})(x)$ .
2. If  $f \in \mathbb{Q}[x^\pm]^\mathcal{G}$ , then  $\widehat{f} \in \mathbb{Q}[x^\pm]^\mathcal{G}$ .

*Proof.* By the chain rule, we have

$$\tilde{\nabla} \widehat{f}(x) = - \left[ x^{(-I_n) \cdot 1} \frac{\partial f}{\partial x_1}(x^{-I_n}), \dots, x^{(-I_n) \cdot n} \frac{\partial f}{\partial x_n}(x^{-I_n}) \right]^t = -(\tilde{\nabla} f)(x^{-I_n}) = -(\widehat{\tilde{\nabla} f})(x).$$

The second statement follows from the fact that for all  $B \in \mathcal{G}$ ,  $(B \cdot \widehat{f})(x) = f(x^{-B}) = f(x^{-I_n}) = \widehat{f}(x)$ , if  $f$  is  $\mathcal{G}$ -invariant. □

**Lemma 8.3.** *For  $f \in \mathbb{Q}[x^\pm]^\mathcal{G}$  and  $x \in \mathbb{T}^n$ ,  $\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) \leq 0$ .*

*Proof.* By [Proposition 8.2](#), we have

$$\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) = -(\tilde{\nabla} f(x))^t S(\widehat{(\tilde{\nabla} f)}(x)) = -(\tilde{\nabla} f(x))^t S((\tilde{\nabla} f)(x^{-I_n})).$$

Since  $x \in \mathbb{T}^n$ ,  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) = (x_1^{-1}, \dots, x_n^{-1}) = x^{-I_n}$  and thus,

$$\langle \tilde{\nabla} f, \tilde{\nabla} \widehat{f} \rangle_S(x) = -(\tilde{\nabla} f(x))^t S((\tilde{\nabla} f)(\bar{x})) = -(\tilde{\nabla} f(x))^t \overline{S(\tilde{\nabla} f(x))} \leq 0$$

completes the proof.  $\square$

We can now state a necessary condition for points in  $\mathbb{C}^n$  to be contained in the  $\mathbb{T}$ -orbit space of  $\mathfrak{G}$  given by the image of  $\mathbb{T}^n$  under  $\vartheta$ .

**Corollary 8.4.** *Define the matrix  $\tilde{M} \in (\mathbb{Q}[x^\pm]^\mathcal{G})^{m \times m}$  with entries  $\tilde{M}_{ij} = \langle \tilde{\nabla} \theta_i, \tilde{\nabla} \widehat{\theta}_j \rangle_S$ . For  $x \in \mathbb{T}^n$ ,  $\tilde{M}(x)$  is Hermitian negative semi-definite.*

*Proof.* Let  $\tilde{\nabla} \vartheta = [\tilde{\nabla} \theta_1 \mid \dots \mid \tilde{\nabla} \theta_m]$  be the Jacobian transpose of  $\vartheta$  with respect to the Euler derivations and assume that  $S = C^t C$  is the Cholesky decomposition of  $S$  for an upper triangular matrix  $C \in \mathbb{Q}^{n \times n}$ . We write  $\tilde{M}(x) = (\tilde{\nabla} \vartheta(x))^t S(\tilde{\nabla} \widehat{\vartheta}(x))$ . Since  $x \in \mathbb{T}^n$ ,  $\bar{x} = x^{-I_n}$  and consecutively  $\tilde{\nabla} \widehat{\vartheta}(x) = -\tilde{\nabla} \vartheta(x)$ . Then

$$\tilde{M}(x) = (\tilde{\nabla} \vartheta(x))^t C^t C(\tilde{\nabla} \widehat{\vartheta}(x)) = -(C \tilde{\nabla} \vartheta(x))^t \overline{(C \tilde{\nabla} \vartheta(x))}$$

is Hermitian negative semi-definite.  $\square$

**Conjecture 8.5.** *Let  $M \in \mathbb{Q}[z]^{m \times m}$ , such that  $\tilde{M} = M(\theta_1, \dots, \theta_m) \in (\mathbb{Q}[x^\pm]^\mathcal{G})^{m \times m}$ . Furthermore, let  $\mathcal{I} \subseteq \mathbb{Q}[z]$  be the ideal of relations among the  $\theta_1, \dots, \theta_m$  and denote by  $\mathcal{V}(\mathcal{I}) \subseteq \mathbb{C}^m$  the variety of  $\mathcal{I}$ .*

*If  $z \in \mathcal{V}(\mathcal{I})$  is such that  $M(z)$  is Hermitian negative semi-definite, then  $z \in \mathcal{T}$ .*

With [Corollary 8.4](#), we obtain a necessary and sufficient condition and therefore a characterization of the  $\mathbb{T}$ -orbit space as a basic semi-algebraic set. There are upsides and downsides for using  $M$  instead of the Hermite matrix polynomial  $H$  for Weyl groups. Generically, the matrix  $M$  admits the same number of necessary polynomial inequalities, respectively one less in the case of  $\mathcal{A}_{n-1}$ . There is to our knowledge no closed expression for  $M$ . Therefore, one needs to express the entries of  $M(\theta_1, \dots, \theta_n)$  as polynomials in the fundamental invariants.

**Remark 8.6.** *The entry  $H_{ij}$  of the Hermite matrix polynomial has degree  $i + j$ , whereas all entries of  $M$  are expected to have degree 2 and the determinant of  $M$  has degree at most  $2n$ . Computations indicate that this bound is sharp. [Remarks 4.4](#), [5.4](#), [6.5](#) and [7.5](#) state that the degree of the determinant of  $H$  can also be expected to be linear in  $n$ . We observe*

$$\deg(H) = \begin{bmatrix} 2 & 3 & \cdots & n+1 \\ 3 & 4 & \cdots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \cdots & 2n \end{bmatrix}, \quad \deg(\text{Det}(H)) = \begin{cases} 4n-2, & \text{if } \mathfrak{G} = \mathcal{A}_{n-1} \\ 3n, & \text{if } \mathfrak{G} = \mathcal{B}_n \\ 2n, & \text{if } \mathfrak{G} = \mathcal{C}_n \\ 3n+1, & \text{if } \mathfrak{G} = \mathcal{D}_n \end{cases},$$

$$\deg(M) = \begin{bmatrix} 2 & 2 & \cdots & 2 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2 \end{bmatrix}, \quad \deg(\text{Det}(M)) = 2n.$$

To support [Conjecture 8.5](#), we give a visualization of the two-dimensional cases for Weyl groups.



**Example 8.7.**

1. Let  $z = (z_1 + iz_2, z_1 - iz_2) \in \mathbb{C}^2$  and consider  $\mathfrak{G} = \mathcal{A}_2$ . The matrix  $H(z) \in \mathbb{Q}[z]^{3 \times 3}$  from [Theorem 4.3](#), which characterizes the  $\mathbb{T}$ -orbit space of  $\mathcal{A}_2$ , was computed in [Section 4.3](#), so that we obtain a description of  $\mathcal{T}$  in terms of polynomials  $h_0(z), h_1(z), h_2(z) \in \mathbb{Q}[z]$ . The Hermitian matrix  $M(z)$  from [Conjecture 8.5](#) is

$$M(z) = \frac{2}{3} \begin{bmatrix} z_1^2 + z_2^2 - 1 & 2(z_1 + iz_2)^2 - 2(z_1 - iz_2) \\ 2(z_1 - iz_2)^2 - 2(z_1 + iz_2) & z_1^2 + z_2^2 - 1 \end{bmatrix} \in \mathbb{Q}[z]^{2 \times 2}.$$

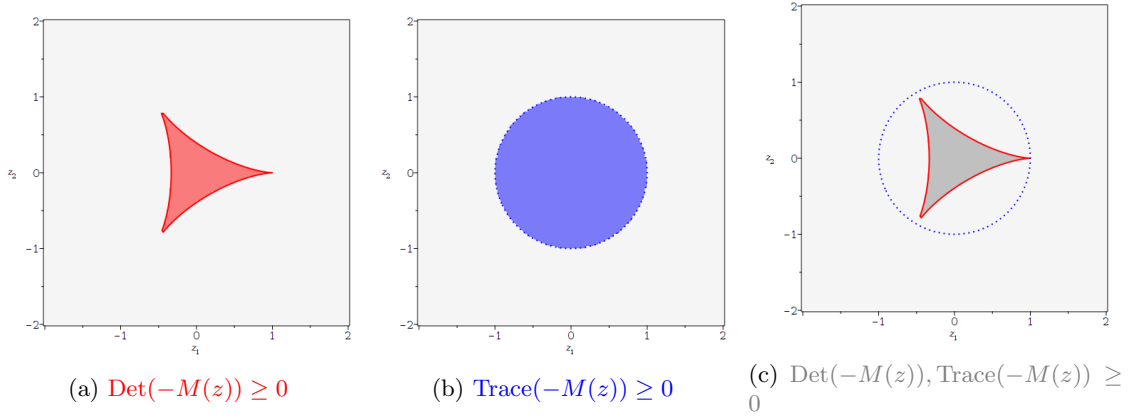


Figure 9: Vanishing points and positivity regions for determinant and trace of  $M(z)$ .

As in [Section 4.3](#), we observe a  $\mathfrak{D}_3$ -invariance: With  $g_1(z) := z_1^2 + z_2^2$ ,  $g_2(z) := z_1(z_1^2 - 3z_2^2)$  we have  $\mathbb{Q}[z]^{\mathfrak{D}_3} = \mathbb{R}[g_1, g_2]$  and

$$\begin{aligned} \text{Det}(-M(z)) &= \frac{1}{9} (-12 z_1^4 - 24 z_1^2 z_2^2 - 12 z_2^4 + 32 z_1^3 - 96 z_1 z_2^2 - 24 z_1^2 - 24 z_2^2 + 4) && \text{(solid)} \\ &= \frac{4}{9} (-3 g_1(z)^2 - 6 g_1(z) + 8 g_2(z) + 1), \\ \text{Trace}(-M(z)) &= -\frac{4}{3} (z_1^2 + z_2^2 - 1) && \text{(dots)} \\ &= -\frac{4}{3} (g_1(z) - 1). \end{aligned}$$

One can observe that  $\text{Det}(-M(z))$  divides  $h_0(z) = \text{Det}(H(z))$  and the quotient  $(3z_1 + 1)^2 z_2^4$  (up to a positive scalar) is a square.

2. The property “ $\text{Det}(-M(z))$  divides  $\text{Det}(H(z))$  in  $\mathbb{Q}[z]$ ” is also true for the other two-dimensional cases. We have

$$\begin{aligned} \mathcal{A}_2: \quad \text{Det}(H(z)) &= \frac{19683}{256} (3z_1 + 1)^2 z_2^4 \text{Det}(-M(z)) && \text{(solid)} \quad , \\ \mathcal{B}_2: \quad \text{Det}(H(z)) &= \frac{16384}{9} z_2^2 \text{Det}(-M(z)) && \text{(solid)} \quad , \\ \mathcal{C}_2: \quad \text{Det}(H(z)) &= \frac{1024}{9} \text{Det}(-M(z)) && \text{(solid)} \quad . \end{aligned}$$

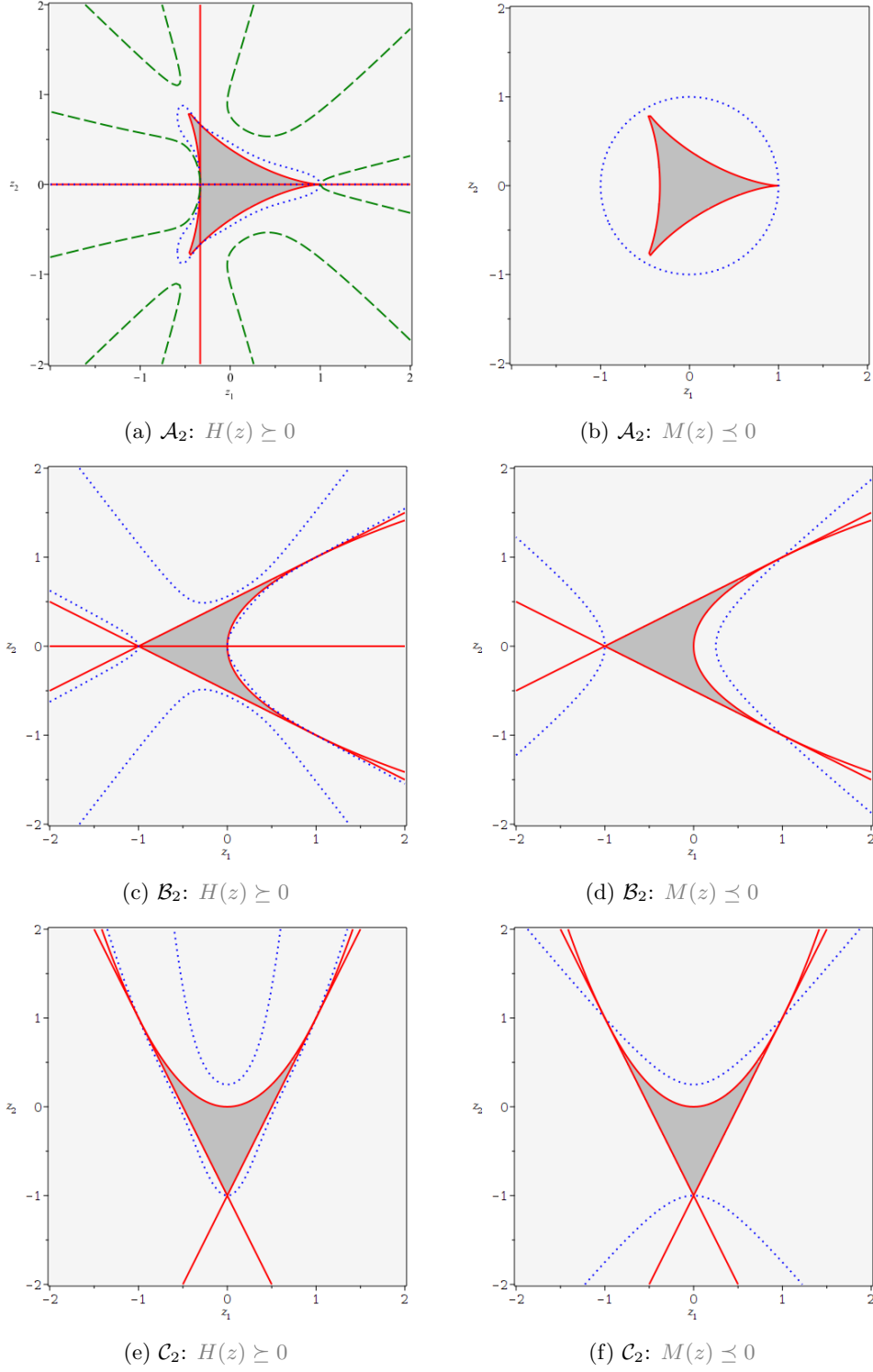


Figure 10: The regions “ $H(z) \succeq 0$ ” and “ $M(z) \preceq 0$ ” for  $\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2$ .

## 9 Orthogonality of Chebyshev polynomials

The image  $\mathcal{T}$  of  $\mathcal{V}$  that we describe in this article also appears in other contexts of interest to analysis. The equations were known for specific examples such as the deltoid in [Figure 3](#) and Steiner’s hypocycloid in [Figure 1](#). These domains contain sampling points with logarithmic Lebesgue number [[MKNR12](#), Theorem 1.24] and minimal cubature points [[LSX12](#), [HMP16](#)]. They are also the domain of orthogonality for a family of generalized Chebyshev polynomials [[EL82](#), [HW88](#), [LU13](#)]. As far as we know, we have given their first explicit general characterization as semi-algebraic sets.

### 9.1 Generalized cosine functions

Assume that  $\mathfrak{G}$  is the Weyl group of a rank  $n$  root system  $R$ , which is represented over  $\mathbb{R}^n$  equipped with the Euclidean scalar product  $\langle \cdot, \cdot \rangle$ .  $\mathfrak{G}$  is the subgroup of the orthogonal matrix group  $O_n(\mathbb{R})$ , that is generated by the orthogonal reflections

$$s_\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \text{with } \rho^\vee := \frac{2}{\langle \rho, \rho \rangle} \rho$$

$$u \mapsto u - \langle \rho^\vee, u \rangle \rho \tag{9.1}$$

for  $\rho \in R$ . The **affine Weyl group** is the semi-direct product of  $\mathfrak{G}$  with the group of translations, with respect to the lattice of coroots  $\Lambda$ , that is the  $\mathbb{Z}$ -module generated by  $R^\vee := \{\rho^\vee \mid \rho \in R\}$ . Assume that  $\rho_1, \dots, \rho_n$  form a base of  $R$  and that  $\rho_0$  is the highest root. For  $\mu$  in the weight lattice  $\Omega$ , define the map

$$e_\mu : \mathbb{R}^n \rightarrow \mathbb{C},$$

$$u \mapsto \exp(-2\pi i \langle \mu, u \rangle). \tag{9.2}$$

**Lemma 9.1.**

1. A fundamental domain for  $\mathfrak{G} \ltimes \Lambda$  is the closed alcove

$$\Delta := \{u \in \mathbb{R}^n \mid \langle u, \rho_0 \rangle \leq 1, \langle u, \rho_1 \rangle \geq 0, \dots, \langle u, \rho_n \rangle \geq 0\}.$$

and  $\mathfrak{G} \Delta = \bigcup_{A \in \mathfrak{G}} A \Delta$  is a fundamental domain for the group of translations  $\Lambda$ .

2.  $\{e_\mu \mid \mu \in \Omega\}$  is an orthonormal basis for both

- the  $\Lambda$ -periodic locally square integrable function  $L^2(\mathbb{R}^n/\Lambda)$  and
- the square integrable functions on  $\mathfrak{G} \Delta$

with respect to the inner product

$$(f, g) \mapsto \frac{1}{|\mathfrak{G}| \text{Vol}(\Delta)} \int_{\mathfrak{G} \Delta} f(u) \overline{g(u)} \, du,$$

where  $\text{Vol}(\Delta)$  is the Lebesgue measure of  $\Delta$  in  $\mathbb{R}^n$ .

*Proof.* The first statement follows from [[Bou68b](#), §2, Proposition 2 and Proposition 5].

The second statement is a consequence of [[Fug74](#), Section 5]. □

**Definition 9.2.** The **generalized cosine function** associated to  $\mu \in \Omega$  is the  $\mathfrak{G}$ -invariant  $\Lambda$ -periodic function

$$c_\mu : \mathbb{R}^n \rightarrow \mathbb{C},$$

$$u \mapsto \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} e_{A\mu}(u).$$

The *generalized sine function* associated to  $\mu \in \Omega$  is the  $\mathfrak{G}$ -anti-invariant  $\Lambda$ -periodic function

$$\begin{aligned} \mathfrak{s}_\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} \text{Det}(A) e_{A\mu}(u). \end{aligned}$$

Let  $\mu, \nu \in \Omega$ . By [Lemma 9.1](#) and the  $\mathfrak{G}$ -invariance, we have

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} \mathfrak{c}_\mu \overline{\mathfrak{c}_\nu} du = \begin{cases} \frac{|\text{Stab}_{\mathfrak{G}}(\mu)|}{|\mathfrak{G}|}, & \text{if } \mu \in \mathfrak{G}\nu \\ 0, & \text{else} \end{cases}. \quad (9.3)$$

If  $\alpha \in \mathbb{Z}^n$  such that  $W\alpha = \mu$  and  $\alpha \in \mathbb{N}^n \setminus ([1, \dots, 1]^t + \mathbb{N}^n)$ , then  $\mu$  lies on a wall of a Weyl chamber [[Bou68a](#), §1.4] and thus  $\mathfrak{s}_\mu = 0$ . For  $\alpha \in [1, \dots, 1]^t + \mathbb{N}^n$  on the other hand,  $\mu$  is called strongly dominant and we have  $|\text{Stab}_{\mathfrak{G}}(\mu)| = 1$ . Furthermore, for strongly dominant weights  $\mu, \nu \in \Omega$ ,

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} \mathfrak{s}_\mu \overline{\mathfrak{s}_\nu} du = \begin{cases} \frac{1}{|\mathfrak{G}|}, & \text{if } \mu \in \mathfrak{G}\nu \\ 0, & \text{else} \end{cases}. \quad (9.4)$$

We show that the image of the generalized cosine functions associated to the fundamental weights is precisely the  $\mathbb{T}$ -orbit space  $\mathcal{T}$ .

**Lemma 9.3.** *We have  $\mathcal{T} = \mathfrak{c}(\Delta) = \mathfrak{c}(\mathbb{R}^n)$ , where  $\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \dots, \mathfrak{c}_{\omega_n}(u)) \in \mathbb{C}^n$  for  $u \in \mathbb{R}^n$ .*

*Proof.* We have  $\mathfrak{c}(\Delta) = \mathfrak{c}(\mathbb{R}^n)$ , because  $\mathfrak{c}_{\omega_j}$  is  $\mathfrak{G}$ -invariant and  $\Lambda$ -periodic for all  $1 \leq j \leq n$ . Fix an index  $j$ . Since the fundamental weights  $\omega_1, \dots, \omega_n$  form a basis of  $\mathbb{R}^n$ , we have  $u \in \mathbb{R}^n$  if and only if  $\exp(-2\pi i \langle \omega_i, u \rangle) \in \mathbb{T}$  for all  $1 \leq i \leq n$ . For  $A \in \mathfrak{G}$ , let  $\alpha^{(A)} \in \mathbb{Z}^n$  be the vector of coordinates of  $A\omega_j$  in  $\Omega$ . Hence,

$$\theta_j(\zeta) = \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} \prod_{i=1}^n \zeta_i^{\alpha_i^{(A)}} \quad \text{and} \quad \mathfrak{c}_{\omega_j}(u) = \frac{1}{|\mathfrak{G}|} \sum_{A \in \mathfrak{G}} \prod_{i=1}^n (\exp(-2\pi i \langle \omega_i, u \rangle))^{\alpha_i^{(A)}}.$$

Given  $u \in \mathbb{R}^n$ , choose  $\zeta_i = \exp(-2\pi i \langle \omega_i, u \rangle) \in \mathbb{T}$  to obtain  $\theta_j(\zeta) = \mathfrak{c}_{\omega_j}(u)$ .

On the other hand, if  $\zeta \in \mathbb{T}^n$  has entries  $\zeta_i = \exp(-2\pi i \varphi_i)$  for  $\varphi_i \in [0, 1)$ , then the unique solution  $u \in \mathbb{R}^n$  of the linear equations  $\langle \omega_i, u \rangle = \varphi_i$  for  $1 \leq i \leq n$  is the preimage of  $\vartheta(\zeta) \in \mathcal{T}$ .  $\square$

## 9.2 Generalized Chebyshev polynomials

Assume that  $\mathfrak{G}$  has the integer representation  $\mathcal{G}$  from [Equation \(2.1\)](#) and  $\mathbb{Q}[x^\pm]^\mathcal{G} = \mathbb{Q}[\theta_1, \dots, \theta_n]$  as in [Theorem 2.7](#). For  $\alpha \in \mathbb{Z}^n$ , there exists a unique multivariate polynomial  $T_\alpha \in \mathbb{Q}[z]$ , such that  $\Theta_\alpha = T_\alpha(\theta_1, \dots, \theta_n)$ . We call  $\{T_\alpha \in \mathbb{Q}[z] \mid \alpha \in \mathbb{Z}^n\}$  the **generalized Chebyshev polynomials of the first kind** associated to the root system  $R$ .

If  $\mu \in \Omega$  is such that  $W\alpha = \mu$ , then  $\mathfrak{c}_\mu = \theta_\alpha \circ (e_{\omega_1}, \dots, e_{\omega_n}) = T_\alpha \circ (\mathfrak{c}_{\omega_1}, \dots, \mathfrak{c}_{\omega_n})$ . Thus, [Definition 9.2](#) is a generalization of the univariate case  $T_\ell(\cos(u)) = \cos(\ell u)$  for  $\ell \in \mathbb{N}$ .

The second kind Chebyshev polynomials are defined through the character polynomials. Thanks to Weyl's character formula [[Bou75](#), §9, Théorème 1], those are given by

$$\Xi_\alpha = \frac{\Upsilon_{\alpha+\delta}}{\Upsilon_\delta} \in \mathbb{Q}[x^\pm]^\mathcal{G} \quad (9.5)$$

for  $\alpha \in \mathbb{Z}^n$ , where  $\delta := [1, \dots, 1]^t$  and  $\Upsilon_\alpha := \sum_{B \in \mathcal{G}} \text{Det}(B) x^{B\alpha} \in \mathbb{Q}[x^\pm]$  is anti-invariant. The anti-invariant polynomials form a free  $\mathbb{Q}[x^\pm]^\mathcal{G}$ -module of rank 1, which is generated by  $\Upsilon_\delta$ . As we shall review,

$\Upsilon_\delta$  also determines the weight of orthogonality. The **generalized Chebyshev polynomials of the second kind** associated to the root system  $R$  are the unique  $\{U_\alpha \in \mathbb{Q}[z] \mid \alpha \in \mathbb{Z}^n\}$  satisfying  $U_\alpha(\theta_1, \dots, \theta_n) = \Xi_\alpha$ . Hence, for  $W\alpha = \mu \in \Omega$  and  $\omega_0 := W\delta = \omega_1 + \dots + \omega_n$ ,

$$U_\alpha \circ (\mathbf{c}_{\omega_1}, \dots, \mathbf{c}_{\omega_n}) = \frac{\Upsilon_{\alpha+\delta}}{\Upsilon_\delta} \circ (e_{\omega_1}, \dots, e_{\omega_n}) = \frac{\mathfrak{s}_{\mu+\omega_0}}{\mathfrak{s}_{\omega_0}}. \quad (9.6)$$

In the classical univariate case  $\mathfrak{G} = \mathcal{A}_1$ , we have

$$\Xi_\ell = \frac{x^{\ell+1} - x^{-\ell-1}}{x - x^{-1}} = x^\ell + x^{\ell-1} + \dots + x^{1-\ell} + x^{-\ell} \quad (9.7)$$

and thus  $U_\ell(\cos(u)) = \sin((\ell+1)u)/\sin(u)$  for  $\ell \in \mathbb{N}$ .

We shall review now that  $\mathcal{T}$  is the domain of orthogonality of the generalized Chebyshev polynomials and show that the weight of orthogonality can be given in terms of  $\Upsilon_\delta$ , also known as the ‘‘Weyl denominator’’.

**Lemma 9.4.** *Let  $J := [\tilde{\nabla} \theta_1 \mid \dots \mid \tilde{\nabla} \theta_n] \in \mathbb{Q}[x^\pm]^{n \times n}$ . Then  $\text{Det}(J) \in \mathbb{Q}[x^\pm]$  is anti-invariant with*

$$\text{Det}(J) = \frac{\prod_{i=1}^n |\text{Stab}_{\mathfrak{G}}(e_i)|}{|\mathfrak{G}|^n} \Upsilon_\delta.$$

*Proof.* By [Proposition 8.1](#), we have  $J(x^B) = B J(x)$  for any  $B \in \mathfrak{G}$ . Hence,  $\text{Det}(J)$  is anti-invariant in  $\mathbb{Q}[x^\pm]$  and there exists a unique  $f \in \mathbb{Q}[x^\pm]^\mathfrak{G}$ , such that  $\text{Det}(J) = f \Upsilon_\delta$ . We show that  $x^\delta$  is the highest monomial of  $\text{Det}(J)$  in the partial ordering given in [\[Bou75, §6.2\]](#). Note that

$$J_{ij} = \left[ \tilde{\nabla} \theta_1 \mid \dots \mid \tilde{\nabla} \theta_n \right]_{ij} = x_i \partial / \partial x_i \theta_j = \frac{|\text{Stab}_{\mathfrak{G}}(e_j)|}{|\mathfrak{G}|} x_i \frac{\partial}{\partial x_i} \sum_{\alpha \in \mathfrak{G} e_j} x^\alpha = \frac{|\text{Stab}_{\mathfrak{G}}(e_j)|}{|\mathfrak{G}|} x_i \frac{\partial}{\partial x_i} \left( x_j + \sum_{\alpha \prec e_j} c_\alpha x^\alpha \right),$$

where  $c_\alpha \in \{0, 1\}$  and  $\alpha \prec e_j$  if and only if  $e_i - \alpha \neq 0$  is the vector of coordinates in  $\Omega$  for a sum of positive roots [\[Bou75, §6, Lemma 2\]](#). Therefore, there exist coefficient  $c_\alpha \in \mathbb{N}$ , such that

$$\text{Det}(J) = \frac{\prod_{i=1}^n |\text{Stab}_{\mathfrak{G}}(e_i)|}{|\mathfrak{G}|^n} \left( \underbrace{x_1 \cdots x_n}_{=x^\delta} + \sum_{\alpha \prec \delta} c_\alpha x^\alpha \right) = f \Upsilon_\delta.$$

Thus,  $f$  must be the constant, which is the given quotient.  $\square$

For the square integrable functions on  $\mathfrak{G} \triangle$  with basis  $\{e_\mu \mid \mu \in \Omega\}$ , define the gradient  $\nabla := [\partial/\partial u_1, \dots, \partial/\partial u_n]^t$ . The relation to the gradient of Euler derivations  $\tilde{\nabla}$  is given by the next statement.

**Lemma 9.5.** *We have*

$$[\nabla \mathbf{c}_{\omega_1} \mid \dots \mid \nabla \mathbf{c}_{\omega_n}](u) = 2\pi i W J(e_{\omega_1}(u), \dots, e_{\omega_n}(u)),$$

where  $W$  is the matrix with columns  $\omega_1, \dots, \omega_n$ .

*Proof.* This is a straightforward computation from the definition

$$\begin{aligned} [\nabla \mathbf{c}_{\omega_1} \mid \dots \mid \nabla \mathbf{c}_{\omega_n}]_{ij}(u) &= \frac{\partial \mathbf{c}_{\omega_j}}{\partial u_i}(u) = 2\pi i \sum_{k=1}^n (\omega_k)_i x_k \partial / \partial x_k \theta_j(e_{\omega_1}(u), \dots, e_{\omega_n}(u)) \\ &= 2\pi i \sum_{k=1}^n W_{ik} J_{kj}(e_{\omega_1}(u), \dots, e_{\omega_n}(u)) = 2\pi i (W J(e_{\omega_1}(u), \dots, e_{\omega_n}(u)))_{ij}. \end{aligned}$$

$\square$

The generalized Chebyshev polynomials satisfy the following orthogonality property.

**Theorem 9.6.** For  $\alpha, \beta \in \mathbb{N}^n$ ,

$$\int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} (\phi(z))^{-1/2} dz = \begin{cases} 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G} e_i| \frac{\text{Vol}(\Delta) |\text{Stab}_{\mathcal{G}}(\alpha)|}{|\mathcal{G}|}, & \text{if } \alpha = \beta \\ 0, & \text{else} \end{cases}$$

and

$$\int_{\mathcal{T}} U_\alpha(z) \overline{U_\beta(z)} (\phi(z))^{1/2} dz = \begin{cases} 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G} e_i| \frac{\text{Vol}(\Delta)}{|\mathcal{G}|}, & \text{if } \alpha = \beta \\ 0, & \text{else} \end{cases},$$

where the weight function  $\phi \in \mathbb{Q}[z]$  is defined via

$$\phi(\theta_1, \dots, \theta_n) = (\Upsilon_\delta)^2 \in \mathbb{Q}[x^\pm]^\mathcal{G}$$

with  $\delta = [1, \dots, 1]^t \in \mathbb{Z}^n$ .

*Proof.* Assume that  $W\alpha = \mu, W\beta = \nu \in \Omega$ . By [Lemma 9.3](#) and the definition of generalized Chebyshev polynomials of the first kind, we can conduct the transformation

$$\begin{aligned} \int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} (\phi(z))^{-1/2} dz &= \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} \frac{|\text{Det}([\nabla \mathbf{c}_{\omega_1} | \dots | \nabla \mathbf{c}_{\omega_n}](u))|}{|\Upsilon_\delta(e_{\omega_1}(u), \dots, e_{\omega_n}(u))|} du \\ &= \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} \frac{|\mathcal{G}|^n}{\prod_{i=1}^n |\text{Stab}_{\mathcal{G}}(e_i)|} \frac{|2\pi i \text{Det}(W) \text{Det}(J)|}{|\text{Det}(J)|} (e_{\omega_1}(u), \dots, e_{\omega_n}(u)) du \\ &= \prod_{i=1}^n |\mathcal{G} e_i| |2\pi i \text{Det}(W)| \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} du. \end{aligned}$$

According to [Equation \(9.3\)](#),

$$\int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} (\phi(z))^{-1/2} dz = 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G} e_i| \text{Vol}(\Delta) \frac{|\text{Stab}_{\mathcal{G}}(\alpha)|}{|\mathcal{G}|}$$

if  $\mu \in \mathfrak{G}\nu$  and 0 otherwise.

The result on generalized Chebyshev polynomials of the second kind is analogous with

$$\begin{aligned} U_\alpha(\mathbf{c}(u)) \overline{U_\beta(\mathbf{c}(u))} (\phi(\mathbf{c}(u)))^{1/2} &= \frac{\mathfrak{s}_{\mu+\omega_0}(u)}{\mathfrak{s}_{\omega_0}} \frac{\overline{\mathfrak{s}_{\nu+\omega_0}(u)}}{\overline{\mathfrak{s}_{\omega_0}}} |\Upsilon_\delta(e_{\omega_1}(u), \dots, e_{\omega_n}(u))| \\ &= \frac{\mathfrak{s}_{\mu+\omega_0}(u) \overline{\mathfrak{s}_{\nu+\omega_0}(u)}}{|\Upsilon_\delta(e_{\omega_1}(u), \dots, e_{\omega_n}(u))|^2} |\Upsilon_\delta(e_{\omega_1}(u), \dots, e_{\omega_n}(u))| \end{aligned}$$

and then applying [Equation \(9.4\)](#). □

**Remark 9.7.** In [Section 8](#) we make use of  $\tilde{M} = J^t S \hat{J} \in (\mathbb{Q}[x^\pm]^\mathcal{G})^{m \times m}$  to give the equations of  $\mathcal{T}$ . According to [Remark 2.10](#), in the present case of Weyl groups the fundamental invariants  $\theta_1, \dots, \theta_n$  are such that there exists a permutation  $\sigma \in \mathfrak{S}_n$  with  $\hat{\theta}_i = \theta_{\sigma(i)}$ . Hence,  $\text{Det}([\tilde{\nabla} \hat{\theta}_1 | \dots | \tilde{\nabla} \hat{\theta}_n]) = \text{sgn}(\sigma) \text{Det}([\tilde{\nabla} \theta_1 | \dots | \tilde{\nabla} \theta_n])$  and thus the determinant of  $M$  is a scalar multiple of  $\phi$  in  $\mathbb{Q}[z]$ .

## 10 Conclusion

To our knowledge, we have given the first explicit and simultaneously general characterization for the orbits of a nonlinear action of a Weyl group on the complex torus as semi-algebraic sets. This was originally motivated by the problem of determining the chromatic number for infinite graphs, which can be reformulated as a polynomial optimization problem on the  $\mathbb{T}$ -orbit space thanks to the properties of the generalized Chebyshev polynomials. The fact that the  $\mathbb{T}$ -orbit space is a compact basic semi-algebraic set in these cases is a self-contained result. Future work revolves around finding such explicit descriptions also for the Weyl groups of root systems E, F, G. Using a computer algebra system, such characterizations can be given, but a unifying theory for all root systems is desirable. [Conjecture 8.5](#) would extend our concrete results in that sense.

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