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# Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description

Evelyne Hubert\*, Tobias Metzloff\*<sup>†</sup>, Cordian Riener<sup>‡</sup>

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## Abstract

The Weyl group of a crystallographic root system has a nonlinear action on the compact torus. The orbit space of this action is a compact basic semi-algebraic set. We present a polynomial description of this set for the Weyl groups associated to root systems of type A, B, C, D and G. Our description is given through a polynomial matrix inequality. The novelty lies in an approach via Hermite quadratic forms and a closed form formula for the matrix entries.

The orbit space of the nonlinear Weyl group action is the orthogonality region of generalized Chebyshev polynomials. In this polynomial basis, we show that the matrices obtained for the five types follow the same, surprisingly simple pattern. This is applied to the optimization of trigonometric polynomials with crystallographic symmetries.

Partial results of this article were presented as a poster at the ISSAC 2022 conference [[HMMR22](#)] and in the doctoral thesis [[Met22](#)].

**Keywords:** Multiplicative Invariants, Weyl Groups, Root Systems, Orbit Spaces, Semi-Algebraic Sets, Polynomial Matrix Inequalities, Chebyshev Polynomials

**MSC:** 13A50 14P10 17B22 33C52

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## 1 Introduction

Subgroups of  $GL_n(\mathbb{Z})$  define monomial (a.k.a. multiplicative) actions on the algebraic torus  $(\mathbb{C}^*)^n$  that leave the compact torus  $\mathbb{T}^n := \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \forall 1 \leq i \leq n : |x_i| = 1\}$  invariant [Lor05]. Of special significance among these groups are the integral representations of the Weyl groups. For the Weyl group  $\mathcal{W}$  associated to a root system of type  $A_{n-1}$ ,  $B_n$ ,  $C_n$ ,  $D_n$ , or  $G_2$ , we provide an explicit description of the orbit space  $\mathbb{T}^n/\mathcal{W}$ . It is a compact semi-algebraic set that is given as the locus of positive semi-definiteness of a Hankel matrix with polynomial entries. We give closed form formulae for the entries of these matrices, initially in a monomial basis (Theorems 5.3, 6.3, 7.4, 8.4 and 9.3). In Figure 1 below we depict the resulting semi-algebraic sets for the two and three dimensional cases. Remarkably, for the cases cited above, these matrices have a uniform formula when their entries are expressed in terms of the *generalized Chebyshev polynomials* associated to the root system (Theorem 10.1).

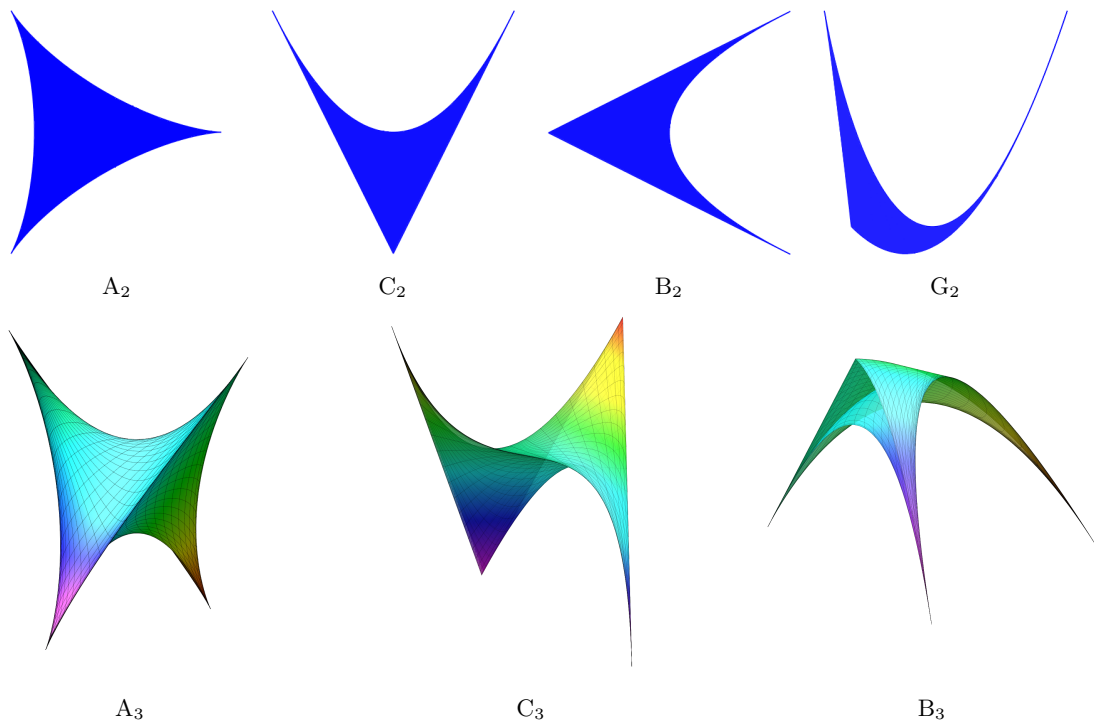


Figure 1: The  $\mathbb{T}$ -orbit space for the irreducible Weyl groups of rank 2 and 3.

We can draw an analogy of our problem with the determination of the orbit space  $\mathbb{R}^n/\mathcal{G}$  of the linear action of a compact group  $\mathcal{G}$  on  $\mathbb{R}^n$ . Explicit formulae of the semi-algebraic set representing  $\mathbb{R}^n/\mathcal{G}$  are also given for the (signed) permutation representations of the symmetric group  $\mathfrak{S}_n$ ,  $\mathfrak{S}_n \times \{\pm 1\}^n$  and  $\mathfrak{S}_n \times \{\pm 1\}^{n-1}$  [Pro78, Tal20]. These formulae are special cases of the encompassing construction of real orbit spaces given in [PS85]. Though the generalization outside of linear actions presented in [PS85, Section 4] could be applied to our cases, our approach is different and its main ingredients from real algebraic geometry are outlined below.

In both approaches the orbit space is given as the image of  $\mathbb{T}^n$  by a fundamental set of invariants. We show that a point  $z$  is in this image if and only if all the solutions of a certain polynomial system belong to  $\mathbb{T}^n$ . The polynomial system is given in terms of the elementary symmetric functions and thus, by Vieta's theorem, the problem boils down to characterizing when all the roots of a univariate polynomial belong to the interval  $[-1, 1]$ . Our Hankel matrix stems from the application of Sturm's theorem [CLO05] (a.k.a. Sylvester inertia

principle [Pro78]).

The Weyl groups  $\mathcal{W}$  associated to the root systems of type  $A_{n-1}$ ,  $B_n$ ,  $C_n$ , or  $D_n$  are semi-direct products  $\mathfrak{S}_n \times \{\pm 1\}^k$ , where  $\mathfrak{S}_n$  is the symmetric group and  $k$  is respectively 0,  $n$ ,  $n$  or  $n-1$ ; For  $G_2$ ,  $(n, k) = (3, 1)$ . The first idea of our constructions is to write the elementary symmetric functions, in some invariants of the monomial action of  $\{\pm 1\}^k$ , in terms of a fundamental set of invariants for the whole Weyl group (and not the contrary). Several sets of fundamental invariants are known explicitly [Bou68]. The treatment for  $C_n$  is the most straightforward and the other cases treated derived from there. The Weyl groups associated with the remaining exceptional (crystallographic) root systems,  $F_4$ ,  $E_6$ ,  $E_7$ , and  $E_8$ , do not entertain a simple decomposition in terms of symmetric group and hence we do not foresee how to apply our approach there.

Our original motivation was to propose a symmetry reduction scheme for the optimization of trigonometric polynomials, as they appear in bounding the chromatic number of some infinite Cayley graphs [HMMR23]. Yet the regions we describe as full dimensional semi-algebraic sets already appeared as of interest in analysis. For instance they are the only Gaussian cubature domains known in all dimensions and all degrees [LSX08, LX10, MP11, LSX12, Xu15, HHM16]. The nodes of these cubatures are known explicitly and provide suitable sampling points [MNR13]. This is all connected to the fact that these are the regions of orthogonality of the generalized Chebyshev polynomials [HW88]. These polynomials have entered the area of multivariate special functions with the work of Koornwinder [Koo74], MacDonald [Mac90] and beyond [DL80, EL82, Bee91]. In this paper, we only draw on their definition through the ring of invariants of the Weyl groups. Yet their interpretation via discrete Fourier analysis based on full dimensional lattices with symmetry should entail further analytical interest.

This article is organized as follows. In Section 2 we introduce, through an example, the general context in which we were drawn to investigate the problem solved here: optimization of trigonometric polynomials with symmetry. Then Section 3 contains mostly standard material from the theory of root systems and lattices from [Bou68] and helps us settling the notations. The following Section 4 is our toolbox for characterizing when all solutions of the polynomial systems we shall encounter are contained in the compact torus. These tools are based on the material in [CLO05]. In Sections 5 to 9 we treat in turn the cases  $A_{n-1}$ ,  $C_n$ ,  $B_n$ ,  $D_n$ , and  $G_2$  in a monomial basis. Finally, in Section 10, we give the unified formula for all cases in their respective Chebyshev basis.

The Maple package `GENERALIZEDCHEBYSHEV`<sup>1</sup> supports all the constructions and formulae we present in this paper. In particular all the computations and the figures in this article were produced with this package, and the plotting tools of Maple.

## 2 Motivation

Let  $\Omega$  be a full-dimensional lattice in a finite-dimensional real vector space  $V$  with inner product  $\langle \cdot, \cdot \rangle$ . Denote by  $\Lambda$  the dual lattice of  $\Omega$  and let  $f : V \rightarrow \mathbb{R}$  be an  $L^2$ -integrable  $\Lambda$ -periodic function. Then  $f$  has a Fourier expansion

$$f : V \rightarrow \mathbb{R}, u \mapsto \sum_{\mu \in \Omega} f_{\mu} \exp(-2\pi i \langle \mu, u \rangle), \quad (2.1)$$

with coefficients  $f_{\mu} = \overline{f_{-\mu}} \in \mathbb{C}$ , see [DM85, CS99]. Since  $f$  is continuous and the periodicity domain  $V/\Lambda$  is compact, the global minimum

$$f^* := \min_{u \in V/\Lambda} f(u) \quad (2.2)$$

exists. If all but finitely many coefficients  $f_{\mu}$  are zero, then we call  $f$  a **trigonometric polynomial**. Trigonometric polynomials are good  $L^2$ -approximations for  $\Lambda$ -periodic functions [MNR13]. The trigonometric optimization problem is to find the value  $f^*$  and arises, for example, in filter design [Dum07], computation of spectral bounds [BDOV14], and optimal power flow [BFW08, JM18].

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<sup>1</sup><https://github.com/TobiasMetzloff/GeneralizedChebyshev>.

In joint work with Philippe Moustrou [HMMR23], the authors address the trigonometric optimization problem for trigonometric polynomials with Weyl group symmetry. The idea is to rewrite  $f$  as a classical polynomial on the  $\mathbb{T}$ -orbit space of the multiplicative Weyl group action on  $V/\Lambda$ . Having an explicit description of this orbit space as a basic semi-algebraic set allows us to use the sums of square/moment hierarchy for global optimization [Las01, Lau09, BPT13].

**Example 2.1.** Let  $\Omega = \mathbb{Z}^2$  be the square lattice in  $V = \mathbb{R}^2$  and consider the trigonometric polynomial

$$\begin{aligned} f(u) := & \exp(-2\pi i \langle \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u \rangle) + \exp(-2\pi i \langle \pm \begin{bmatrix} 0 \\ -1 \end{bmatrix}, u \rangle) \\ & + \frac{1}{2} \left( \exp(-2\pi i \langle \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u \rangle) + \exp(-2\pi i \langle \pm \begin{bmatrix} 1 \\ -1 \end{bmatrix}, u \rangle) \right. \\ & \quad \left. - \exp(-2\pi i \langle \pm \begin{bmatrix} 2 \\ 2 \end{bmatrix}, u \rangle) - \exp(-2\pi i \langle \pm \begin{bmatrix} 2 \\ -2 \end{bmatrix}, u \rangle) \right) \\ & - \frac{3}{4} \left( \exp(-2\pi i \langle \pm \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u \rangle) + \exp(-2\pi i \langle \pm \begin{bmatrix} 2 \\ -1 \end{bmatrix}, u \rangle) \right. \\ & \quad \left. + \exp(-2\pi i \langle \pm \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u \rangle) + \exp(-2\pi i \langle \pm \begin{bmatrix} 1 \\ -2 \end{bmatrix}, u \rangle) \right). \end{aligned}$$

The group  $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$  acts on  $\mathbb{R}^2$  by permutation and sign change of coordinates. Note that  $f$  is invariant under the substitution  $u \mapsto s(u)$  whenever  $s \in \mathcal{W}$ .

By Bourbaki's Theorem 3.4, the set of trigonometric polynomials, which are left invariant under this substitution, is a polynomial algebra. This means that we can fix a set of fundamental invariants  $\{\theta_1, \theta_2\}$  and a map  $\vartheta : u \mapsto (\theta_1(u), \theta_2(u))$ , so that  $f(u)$  can be rewritten uniquely as a polynomial  $g(\vartheta(u))$ . In particular, the minimum is

$$\begin{aligned} f^* &= \min_{u \in \mathbb{R}^2} f(u) = \min_{u \in \mathbb{R}^2} g(\vartheta(u)) = \min_{z \in \vartheta(\mathbb{R}^2)} g(z) \\ &= \min_z 16 z_1^2 - 12 z_1 z_2 - 8 z_2^2 + 10 z_1 - 6 z_2 - 2, \\ & \text{s.t. } z = (z_1, z_2) \in \mathbb{R}^2 \text{ with } \begin{bmatrix} -4 z_1^2 + 2 z_2 + 2 & -8 z_1^3 + 6 z_1 z_2 + 2 z_1 \\ -8 z_1^3 + 6 z_1 z_2 + 2 z_1 & -16 z_1^4 + 16 z_1^2 z_2 + 4 z_1^2 - 2 z_2^2 - 2 z_2 \end{bmatrix} \succeq 0, \end{aligned}$$

where  $\succeq 0$  means that the matrix is positive semi-definite. The rewriting process from  $f(u)$  to  $g(z) = 8 z_1^2 - \dots$  is straightforward once a set of fundamental invariants is fixed, see Example 3.7. The matrix is obtained with our formula in Theorem 6.3 and describes the new feasible region in Figure 2.

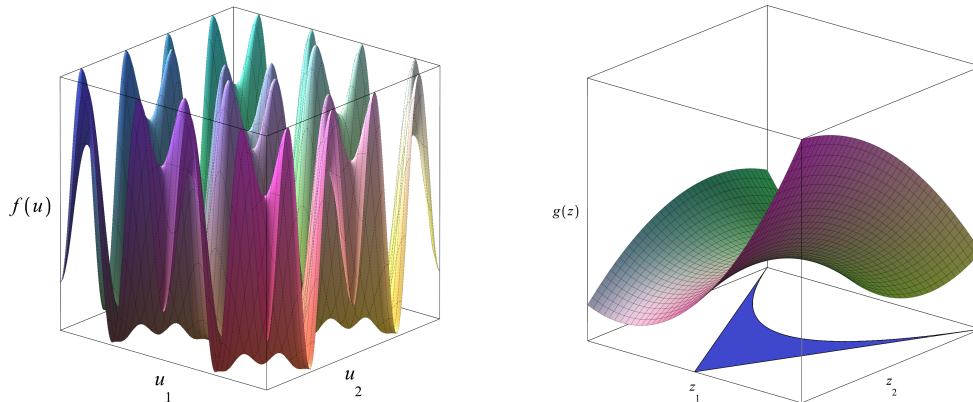


Figure 2: The graphs of  $f$  (left) and  $g$  with the new feasible region (right), see Example 2.1.

### 3 The $\mathbb{T}$ -orbit space of a Weyl group

A key theorem from multiplicative invariant theory states that, given a finite group of integer matrices with a multiplicative action, the invariant ring is a polynomial algebra if and only if the group is a Weyl group. Classical references for this statement are the book of Bourbaki [Bou68] and the papers by Steinberg [Ste75] and Farkas [Far84]. In this section, we fix the terminology and notation: We revisit the theory of Weyl groups from [Bou68] and of their multiplicative actions on compact tori from [Lor05]. This admits a definition of generalized Chebyshev polynomials, which will later turn out to be the suitable basis for our explicit description of the associated orbit space. Finally, we show that the orbit space can always be assumed to be a real compact set.

#### 3.1 $\mathbb{T}$ -orbit spaces

We recall the theory of multiplicative invariants from [Lor05]. Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space with inner product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ ,  $\mathcal{W}$  be a finite subgroup of  $\mathrm{GL}(V)$  and  $\Omega \subseteq V$  be a full-dimensional  $\mathcal{W}$ -invariant lattice, that is,  $\Omega \otimes_{\mathbb{Z}} \mathbb{R} = V$  and  $\mathcal{W}\Omega = \Omega$ .

We fix a  $\mathbb{Z}$ -basis  $\{\omega_1, \dots, \omega_n\}$  for  $\Omega$  and denote by  $\varphi : \mathbb{Z}^n \rightarrow \Omega$  the  $\mathbb{Z}$ -module isomorphism that takes the standard basis vector  $e_i$  to  $\omega_i$ . Then  $\mathcal{W}$  can be represented as  $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ , where the group elements  $s \in \mathcal{W}$  are replaced by matrices  $B$ : The action on a column vector  $\alpha \in \mathbb{Z}^n$  is given by matrix vector multiplication  $B\alpha := \varphi^{-1}(s(\varphi(\alpha))) = \alpha_1 s(\omega_1) + \dots + \alpha_n s(\omega_n)$ .

Denote by  $\Lambda := \{\lambda \in V \mid \forall \mu \in \Omega : \langle \mu, \lambda \rangle \in \mathbb{Z}\}$  the dual lattice of  $\Omega$ . The periodicity domain of a trigonometric polynomial as in Equation (2.1) is the compact additive group  $V/\Lambda$ . Furthermore, the **compact  $n$ -torus**  $\mathbb{T}^n := \{x = (x_1, \dots, x_n) \in \mathbb{C}^n \mid \forall 1 \leq i \leq n : |x_i| = 1\}$  is a multiplicative group. A group isomorphism between  $V/\Lambda$  and  $\mathbb{T}^n$  is given by  $u + V \mapsto (\exp(-2\pi i \langle \omega_i, u \rangle))_{1 \leq i \leq n}$ .

The complexification of  $\mathbb{T}^n$  is the **algebraic  $n$ -torus**  $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$ . For  $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$  and a column vector  $\alpha = [\alpha_1, \dots, \alpha_n]^t \in \mathbb{Z}^n$ , we define  $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}^*$ . Then

$$\begin{aligned} \star : \mathcal{W} \times (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ (B, x) &\mapsto B \star x := x^{B^{-1}} := (x^{B^{-1}_1}, \dots, x^{B^{-1}_n}) \end{aligned} \quad (3.1)$$

is a nonlinear action, where  $B^{-1}_i \in \mathbb{Z}^n$  denotes the  $i$ -th column vector of the inverse matrix  $B^{-1} \in \mathcal{W}$ . The orbits are denoted by  $\mathcal{W} \star x$  and the action  $\star$  leaves the compact  $n$ -torus invariant, that is  $\mathcal{W} \star \mathbb{T}^n = \mathbb{T}^n$ . We refer to  $\star$  as the **multiplicative action** of  $\mathcal{W}$  on the (compact) torus.

The coordinate ring of  $(\mathbb{C}^*)^n$  is the multivariate Laurent polynomial ring  $\mathbb{C}[x^\pm] := \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ . The monomials in  $\mathbb{C}[x^\pm]$  are the  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$  with  $\alpha \in \mathbb{Z}^n$  and form the multiplicative unit group of  $\mathbb{C}[x^\pm]$ . The nonlinear action  $\star$  induces a linear action on  $\mathbb{C}[x^\pm]$ , given by its values on the monomial basis

$$\begin{aligned} \cdot : \mathcal{W} \times \mathbb{C}[x^\pm] &\rightarrow \mathbb{C}[x^\pm], \\ (B, x^\alpha) &\mapsto B \cdot x^\alpha := x^{B\alpha}. \end{aligned} \quad (3.2)$$

For  $f = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \mathbb{C}[x^{\pm}]$ ,  $B \in \mathcal{W}$  and  $x \in (\mathbb{C}^*)^n$ , we have  $(B \cdot f)(x) = \sum_{\alpha} f_{\alpha} x^{B\alpha} = f(B^{-1} \star x)$ . The orbits are denoted by  $\mathcal{W} \cdot f$ . If  $\mathcal{W} \cdot f = \{f\}$ , then  $f$  is called  **$\mathcal{W}$ -invariant**. The set of all  $\mathcal{W}$ -invariant Laurent polynomials is a subring denoted by  $\mathbb{C}[x^\pm]^{\mathcal{W}}$ . The **orbit polynomial** associated to  $\alpha \in \mathbb{Z}^n$  is

$$\Theta_{\alpha} := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha} \in \mathbb{C}[x^\pm]^{\mathcal{W}}. \quad (3.3)$$

If  $\tilde{\alpha} \in \mathcal{W}\alpha$ , then  $\Theta_{\tilde{\alpha}} = \Theta_{\alpha}$ . The distinct  $\Theta_{\alpha}$  form a basis for  $\mathbb{C}[x^\pm]^{\mathcal{W}}$  as a  $\mathbb{C}$ -vector space [Lor05, Eq. (3.4)]. We have  $\Theta_0 = 1$  and, for  $\alpha, \beta \in \mathbb{Z}^n$ ,

$$\Theta_{\alpha} \Theta_{\beta} = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} \Theta_{\alpha+B\beta}. \quad (3.4)$$

As a  $\mathbb{C}$ -algebra,  $\mathbb{C}[x^\pm]^\mathcal{W}$  is finitely generated [Lor05, Coro. 3.3.2]. A minimal set of generators is called a **set of fundamental invariants** for  $\mathbb{C}[x^\pm]^\mathcal{W}$ .

**Proposition 3.1.** *Let  $\{\theta_1, \dots, \theta_m\} \subseteq \mathbb{C}[x^\pm]^\mathcal{W}$  be a set of fundamental invariants. The map*

$$\begin{aligned} \mathbb{T}^n/\mathcal{W} &\rightarrow \mathbb{C}^m, \\ \mathcal{W} \star x &\mapsto (\theta_1(x), \dots, \theta_m(x)) \end{aligned}$$

*is well-defined and injective.*

*Proof.* For  $x, y \in \mathbb{T}^n$  with  $\mathcal{W} \star x = \mathcal{W} \star y$  and  $1 \leq i \leq m$ , we have  $\theta_i(x) = \theta_i(y)$  by definition. Therefore, the map is well-defined.

For injectiveness, we follow the proof of [CLO15, Ch. 7, §4, Thm. 10] for linear actions: Assume that  $x, y \in \mathbb{T}^n$  with  $\mathcal{W} \star x \neq \mathcal{W} \star y$ . Then  $\mathcal{W} \star x \cap \mathcal{W} \star y = \emptyset$  and  $X := \mathcal{W} \star x \cup \mathcal{W} \star y \setminus \{y\} \subseteq \mathbb{T}^n$  is a finite set. We define a polynomial

$$\tilde{f} := \prod_{x' \in X} \prod_{i=1}^n (x_i - x'_i) \in \mathbb{C}[x_1, \dots, x_n],$$

so that  $\tilde{f}(X) = \{0\}$  and  $\tilde{f}(y) \neq 0$ . By letting  $\mathcal{W}$  act linearly on  $f$  as a Laurent polynomial, we may define

$$f := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} B \cdot \tilde{f} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\mathcal{W}.$$

Since  $\{\theta_1, \dots, \theta_m\}$  is a set of fundamental invariants, there exists a polynomial  $g \in \mathbb{C}[z_1, \dots, z_m]$ , such that

$$g(\theta_1(x), \dots, \theta_m(x)) = f(x) = 0 \neq \tilde{f}(y)/|\mathcal{W} \star y| = f(y) = g(\theta_1(y), \dots, \theta_m(y)).$$

In particular, we have  $(\theta_1(x), \dots, \theta_m(x)) \neq (\theta_1(y), \dots, \theta_m(y))$ . □

For a fixed set  $\{\theta_1, \dots, \theta_m\} \subseteq \mathbb{C}[x^\pm]^\mathcal{W}$  of fundamental invariants, we define the associated map

$$\begin{aligned} \vartheta : (\mathbb{C}^*)^n &\rightarrow \mathbb{C}^m, \\ x &\mapsto (\theta_1(x), \dots, \theta_m(x)). \end{aligned} \tag{3.5}$$

**Definition 3.2.** *We call  $\mathcal{T} := \vartheta(\mathbb{T}^n)$  the  $\mathbb{T}$ -orbit space of the multiplicative action of  $\mathcal{W}$  on  $\mathbb{T}^n$  w.r.t.  $\vartheta$ .*

## 3.2 Root systems and Weyl groups

The groups and lattices that we are interested in arise from root systems. We take the definitions from [Bou68, Ch. VI]. Again,  $V$  is a finite-dimensional  $\mathbb{R}$ -vector space with inner product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ . A subset  $R \subseteq V$  is called a **root system** in  $V$ , if the following conditions hold.

R1 The set  $R$  is finite, spans  $V$  as an  $\mathbb{R}$ -vector space and does not contain 0.

R2 If  $\rho, \tilde{\rho} \in R$ , then  $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$ , where  $\rho^\vee := \frac{2\rho}{\langle \rho, \rho \rangle}$ .

R3 If  $\rho, \tilde{\rho} \in R$ , then  $s_\rho(\tilde{\rho}) \in R$ , where  $s_\rho$  is the reflection  $s_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$  for  $u \in V$ .

Furthermore,  $R$  is said to be **reduced**, if the following condition holds.

R4 For  $\rho \in R$  and  $c \in \mathbb{R}$ , we have  $c\rho \in R$  if and only if  $c = \pm 1$ .



We assume that the “reduced” property R4 is always fulfilled when we speak of a “root system”. The distinction between root systems and roots of polynomials is assumed to be clear from the context.

The **Weyl group**  $\mathcal{W}$  of  $R$  is the group generated by the reflections  $s_\rho$  for  $\rho \in R$ . This is a finite subgroup of the orthogonal group on  $V$  with respect to the inner product  $\langle \cdot, \cdot \rangle$ . The orbits are denoted by  $\mathcal{W}u$  for  $u \in V$ . The **rank** of  $R$ , respectively  $\mathcal{W}$ , is the dimension of  $V$ .

A **weight** of  $R$  is an element  $\mu \in V$ , such that, for all  $\rho \in R$ , we have  $\langle \mu, \rho^\vee \rangle \in \mathbb{Z}$ . The **weight lattice**  $\Omega$  is the set of all weights and a full-dimensional  $\mathcal{W}$ -lattice.

Let  $\dim(V) = n$  and  $B = \{\rho_1, \dots, \rho_n\} \subseteq R$  be a subset with the following properties.

B1 The set  $B$  is an  $\mathbb{R}$ -vector space basis for  $V$ .

B2 For every  $\rho \in R$ , there exists  $\alpha \in \mathbb{N}^n$ , such that  $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$  or  $\rho = -\alpha_1 \rho_1 - \dots - \alpha_n \rho_n$ .

Such a set  $B$  always exists and is called **base** [Bou68, Ch. VI, §1, Thm. 3]. We define the **fundamental weights** as the linearly independent  $\omega_1, \dots, \omega_n \in \Omega$ , such that, for  $1 \leq i, j \leq n$ , we have  $\langle \omega_i, \rho_j^\vee \rangle = \delta_{i,j}$ .

**Proposition 3.3.** *Denote by  $\mathfrak{S}_n$  the symmetric group on  $n$  points. There exists a permutation  $\sigma \in \mathfrak{S}_n$  of order at most 2, such that, for all  $1 \leq i \leq n$ , we have  $-\omega_i \in \mathcal{W}\omega_{\sigma(i)}$ .*

*Proof.* The statement follows from the existence of the longest Weyl group element  $s_0 \in \mathcal{W}$ : For the proof, it suffices to know that  $s_0$  is a reflection that takes a fixed base  $\{\rho_1, \dots, \rho_n\}$  to  $\{-\rho_1, \dots, -\rho_n\}$ , see [Bou68, Ch. VI, §1, Prop. 17, Coro. 3]. Hence, there is a permutation  $\sigma \in \mathfrak{S}_n$  with  $s_0(\rho_i) = -\rho_{\sigma(i)}$ . Since  $s_0^2 = \text{Id}_V$  and the inner product is  $\mathcal{W}$ -invariant,  $\sigma$  is of order at most 2 and thus

$$-s_0(\omega_i) = \sum_{j=1}^n \langle -s_0(\omega_i), \rho_j^\vee \rangle \omega_j = \sum_{j=1}^n \langle \omega_i, -s_0(\rho_j^\vee) \rangle \omega_j = \sum_{j=1}^n \langle \omega_i, \rho_{\sigma(j)}^\vee \rangle \omega_j = \omega_{\sigma(i)}$$

is also a fundamental weight. □

Assume that  $V = V^{(1)} \oplus \dots \oplus V^{(k)}$  is the direct sum of proper orthogonal subspaces and that, for each  $1 \leq i \leq k$ , there is a root system  $R^{(i)}$  in  $V^{(i)}$ . Then  $R^{(1)} \cup \dots \cup R^{(k)}$  is a root system in  $V$  called the **direct sum** of the  $R^{(i)}$ . If a root system is not the direct sum of at least two root systems, then it is called **irreducible** [Bou68, Ch. VI, §1.2].

Every root system can be uniquely decomposed into irreducible ones [Bou68, Ch. VI, §1, Prop. 6, 7]. There exist nine families of irreducible root systems, which are denoted  $A_{n-1}, B_n, C_n, D_n, E_6, E_7, E_8, F_4$  and  $G_2$ , see [Bou68, Ch. VI, §4, Thm. 3] and [Bou68, Planches I – IX].

### 3.3 Chebyshev polynomials

For the remainder of this article,  $\mathcal{W}$  is the Weyl group of a root system. We fix a set of fundamental weights  $\{\omega_1, \dots, \omega_n\}$  as a  $\mathbb{Z}$ -basis of the weight lattice  $\Omega$ . Thus, we may write  $\mathcal{W} \subseteq \text{GL}_n(\mathbb{Z})$ . We already know that  $\mathbb{C}[x^\pm]^\mathcal{W}$  is finitely generated, but we need to fix a set of generators in order to define the  $\mathbb{T}$ -orbit space.

**Theorem 3.4.** [Bou68, Ch. VI, §3, Thm. 1] *The orbit polynomials  $\Theta_{e_1}, \dots, \Theta_{e_n}$  are algebraically independent and  $\{\Theta_{e_1}, \dots, \Theta_{e_n}\} \subseteq \mathbb{C}[x^\pm]^\mathcal{W}$  is a set of fundamental invariants.*

For  $1 \leq i \leq n$ , we denote by  $\theta_i := \Theta_{e_i}$  from now on the orbit polynomial associated to  $e_i \in \mathbb{Z}^n$ . Let  $\vartheta$  be the associated map from Equation (3.5) with  $n = m$ . By Proposition 3.1, the  $\mathbb{T}$ -orbit space  $\mathcal{T} = \vartheta(\mathbb{T}^n)$  of the multiplicative action of  $\mathcal{W}$  on  $\mathbb{T}^n$  w.r.t.  $\vartheta$  is in bijection to  $\mathbb{T}^n/\mathcal{W}$ . Since the  $\theta_i$  are algebraically independent, the image  $\mathcal{T}$  does not lie in any proper subvariety and is full-dimensional in an  $n$ -dimensional  $\mathbb{R}$ -vector space, that we specify later.

To describe  $\mathcal{T}$ , it suffices to consider irreducible root systems. Indeed, the Weyl group  $\mathcal{W}$  of a root system  $R$  is the outer product of the Weyl groups corresponding to the irreducible components, see the discussion before [Bou68, Ch. VI, §1, Prop. 5]. Hence, if  $R = R^{(1)} \cup \dots \cup R^{(k)}$  is a direct sum of irreducible root systems  $R^{(i)}$ , then the  $\mathbb{T}$ -orbit space is a Cartesian product  $\mathcal{T} = \mathcal{T}^{(1)} \times \dots \times \mathcal{T}^{(k)}$  of  $\mathbb{T}$ -orbit spaces  $\mathcal{T}^{(i)}$ .

By [Theorem 3.4](#),  $\mathbb{C}[x^\pm]^\mathcal{W} = \mathbb{C}[\theta_1, \dots, \theta_n]$  is a polynomial algebra. Hence, for every  $\alpha \in \mathbb{Z}^n$ , there is a unique polynomial  $T_\alpha \in \mathbb{C}[z_1, \dots, z_n]$ , so that the orbit polynomial  $\Theta_\alpha$  can be written as

$$\Theta_\alpha(x) = T_\alpha(\theta_1(x), \dots, \theta_n(x)). \quad (3.6)$$

The polynomial  $T_\alpha$  can be computed recursively via the recurrence formula from [Equation \(3.4\)](#). Note that the monomials occurring in  $\Theta_\alpha$  all have the same coefficient, namely  $1/|\mathcal{W}\alpha| \in \mathbb{Q}$ . In particular,  $T_\alpha$  has rational coefficients. To describe the  $\mathbb{T}$ -orbit space, we can therefore work over the multivariate polynomial ring  $\mathbb{Q}[z] := \mathbb{Q}[z_1, \dots, z_n]$ .

**Definition 3.5.** *The **generalized Chebyshev polynomial of the first kind** associated to  $\alpha \in \mathbb{Z}^n$  is the unique  $T_\alpha \in \mathbb{Q}[z]$  from [Equation \(3.6\)](#).*

For every  $\tilde{\alpha} \in \mathbb{Z}^n$ , there exists a unique  $\alpha \in \mathbb{N}^n$ , such that  $\tilde{\alpha} \in \mathcal{W}\alpha$  [Bou68, Ch. VI, §1, Thm. 2]. In particular,  $\{T_\alpha \mid \alpha \in \mathbb{N}^n\}$  is a  $\mathbb{Q}$ -vector space basis for  $\mathbb{Q}[z]$ . By the definition, we have  $T_0 = 1, T_{e_1} = z_1, \dots, T_{e_n} = z_n$ . For higher degrees,  $T_\alpha$  depends on the type of the root system.

**Example 3.6.** *For  $n = 1$ , there is only one integral Weyl group representation, namely  $\mathcal{W} = \{\pm 1\}$ . Thus,  $T_\ell((x + x^{-1})/2) = (x^\ell + x^{-\ell})/2$  defines the univariate Chebyshev polynomial associated to  $\ell \in \mathbb{N}$ .*

*The generalized Chebyshev polynomials of the first kind for type  $A_2$  are*

$$\begin{aligned} T_{02} &= 3z_2^2 - 2z_1, T_{11} = 3/2 z_1 z_2 - 1/2, T_{20} = 3z_1^2 - 2z_2, T_{03} = 9z_2^3 - 9z_1 z_2 + 1, \\ T_{12} &= 9/2 z_1 z_2^2 - 3z_1^2 - 1/2 z_2, T_{21} = -3z_2^2 - 1/2 z_1 + 9/2 z_1^2 z_2, T_{30} = 9z_1^3 - 9z_1 z_2 + 1, \dots \end{aligned}$$

*The generalized Chebyshev polynomials of the first kind for type  $C_2$  are*

$$\begin{aligned} T_{02} &= -8z_1^2 + 4z_2^2 + 4z_2 + 1, T_{11} = 2z_1 z_2 - z_1, T_{20} = 4z_1^2 - 2z_2 - 1, T_{03} = -48z_1^2 z_2 + 16z_2^3 + 24z_2^2 + 9z_2, \\ T_{12} &= -16z_1^3 + 8z_1 z_2^2 + 6z_1 z_2 + 3z_1, T_{21} = 8z_1^2 z_2 - 4z_2^2 - 3z_2, T_{30} = 8z_1^2 z_2 - 4z_2^2 - 3z_2, \dots \end{aligned}$$

**Example 3.7.** *In [Example 2.1](#), we claim that the rewriting process from a trigonometric polynomial with Weyl group symmetry to a classical one is straightforward: From the classification in [Bou68, Planches I–IX], we know that the group  $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$  is the Weyl group of a root system in  $V = \mathbb{R}^2$  of type  $C_2$  with weight lattice  $\Omega = \mathbb{Z}^2$  and fundamental weights  $\omega_1 = [1, 0]^t$  and  $\omega_2 = [1, 1]^t$ .*

*Now, after substitution  $u \mapsto x := (x_1, x_2) := (\exp(-2\pi i \langle \omega_1, u \rangle), \exp(-2\pi i \langle \omega_2, u \rangle))$ , we redefine  $f$  to be*

$$\begin{aligned} f &:= x^{\pm e_1} + x^{\pm(e_1 - e_2)} + \frac{1}{2} \left( x^{\pm e_2} + x^{\pm(2e_1 - e_2)} - x^{\pm 2e_2} - x^{\pm(4e_1 - 2e_2)} \right) \\ &\quad - \frac{3}{4} \left( x^{\pm(e_1 + e_2)} + x^{\pm(3e_1 - e_2)} - x^{\pm(-e_1 + 2e_2)} - x^{\pm(3e_1 - 2e_2)} \right) \in \mathbb{C}[x^\pm]^\mathcal{W}. \end{aligned}$$

*We collect the terms with exponent  $\alpha \in \mathbb{N}^2$ . The remaining exponents are contained in the orbit  $\mathcal{W}\alpha$  and thus occur as exponents in the associated orbit polynomial  $\Theta_\alpha$ . Specifically,*

$$\begin{aligned} f &= x^{e_1} + \frac{1}{2} (x^{e_2} - x^{2e_2}) - \frac{3}{4} x^{e_1 + e_2} + \dots \\ &= |\mathcal{W}e_1| \Theta_{e_1}(x) + \frac{|\mathcal{W}e_2|}{2} (\Theta_{e_2}(x) - \Theta_{2e_2}(x)) - \frac{3|\mathcal{W}(e_1 + e_2)|}{4} \Theta_{e_1 + e_2} \\ &= 4 \Theta_{e_1}(x) + 2 (\Theta_{e_2}(x) - \Theta_{2e_2}(x)) - 6 \Theta_{e_1 + e_2}(x). \end{aligned}$$

With  $\Theta_\alpha(x) = T_\alpha(\vartheta(x))$  and  $z := \vartheta(x)$ , the minimum is

$$f^* = \min_{x \in \mathbb{T}^2} f(x) = \min_{z \in \mathcal{T}} 4T_{e_1}(z) + 2(T_{e_2}(z) - T_{2e_2}(z)) - 2T_{e_1+e_2}(z) = \min_{z \in \mathcal{T}} 16z_1^2 - 12z_1z_2 - 8z_2^2 + 10z_1 - 6z_2 - 2.$$

The new feasible region  $\mathcal{T}$  is the  $\mathbb{T}$ -orbits space of the multiplicative action of  $\mathfrak{S}_2 \times \{\pm 1\}^2$  on  $\mathbb{T}^2$  w.r.t.  $\vartheta$ .

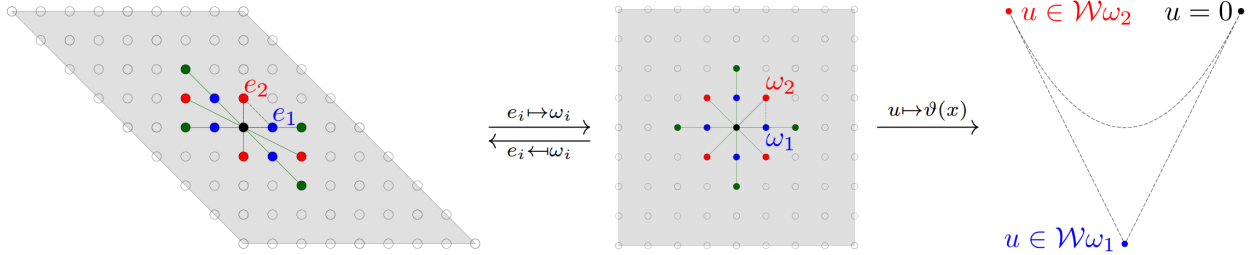


Figure 3: The root system  $C_2$  admits a quadratic weight lattice (middle). Note that the change of basis  $\omega_i \mapsto e_i$  is not the identity (left). The circles  $\circ$  are the weights and the green dots  $\bullet$  are the roots. The blue dots  $\bullet$  and red dots  $\bullet$  are the orbits of the fundamental weights and  $\bullet$  is the origin. The  $\mathbb{T}$ -orbit space is indicated by a dashed line (right).

### 3.4 Real embedding

We want to describe  $\mathcal{T}$  as a compact semi-algebraic set in  $\mathbb{R}^n$ . For  $x \in \mathbb{T}^n$  and  $1 \leq i \leq n$ , we observe

$$\overline{\theta_i(x)} = \theta_i(-I_n \star x) = (-I_n \cdot \theta_i)(x) = \theta_{\sigma(i)}(x), \quad (3.7)$$

where  $\sigma \in \mathfrak{S}_n$  is the permutation from [Proposition 3.3](#). Hence, if  $-I_n \notin \mathcal{W} \subseteq \text{GL}_n(\mathbb{Z})$ , or equivalently, if  $\sigma$  is not trivial, then  $\mathcal{T}$  is not a real set as the image of  $\mathbb{T}^n$  under  $x \mapsto \vartheta(x) = (\theta_1(x), \dots, \theta_n(x))$ . The irreducible root systems, for which this is the case, are both  $A_{n-1}$  and  $D_{2n-1}$  whenever  $n \geq 3$  as well as  $E_6$ .

We fix this circumstance in a straightforward manner: When  $j = \sigma(j)$ , we set  $\hat{\theta}_j := \theta_j \in \mathbb{C}[x^\pm]$ . When  $j < \sigma(j)$ , we replace the  $j$ -th, respectively  $\sigma(j)$ -th, coordinate of  $\vartheta$  by  $\hat{\theta}_j := (\theta_j + \theta_{\sigma(j)})/2 \in \mathbb{C}[x^\pm]$ , respectively  $\hat{\theta}_{\sigma(j)} := (\theta_j - \theta_{\sigma(j)})/(2i) \in \mathbb{C}[x^\pm]$ . For  $x \in \mathbb{T}^n$ , we have  $\hat{\theta}_j(x) = \Re(\theta_j(x)) \in \mathbb{R}$  and  $\hat{\theta}_{\sigma(j)}(x) = \Im(\theta_j(x)) \in \mathbb{R}$ . Thus, the image of the map

$$\begin{aligned} \hat{\vartheta}: \mathbb{T}^n &\rightarrow \mathbb{R}^n, \\ x &\mapsto (\hat{\theta}_1(x), \dots, \hat{\theta}_n(x)) \end{aligned} \quad (3.8)$$

is contained in the cube  $[-1, 1]^n \subseteq \mathbb{R}^n$ . We denote this image by  $\hat{\mathcal{T}}$  and call it the **real  $\mathbb{T}$ -orbit space** of the multiplicative action of  $\mathcal{W}$  on  $\mathbb{T}^n$ . On  $\hat{\mathcal{T}}$ , we define the **real generalized Chebyshev polynomials**.

**Proposition 3.8.** *Let  $\alpha, \hat{\alpha} \in \mathbb{N}^n$  with  $-\alpha \in \mathcal{W}\hat{\alpha}$ . There exist unique polynomials  $\hat{T}_\alpha, \hat{T}_{\hat{\alpha}} \in \mathbb{Q}[\hat{z}]$ , such that*

$$T_\alpha(\vartheta(x)) = \hat{T}_\alpha(\hat{\vartheta}(x)) + i\hat{T}_{\hat{\alpha}}(\hat{\vartheta}(x)) \quad \text{and} \quad T_{\hat{\alpha}}(\vartheta(x)) = \hat{T}_\alpha(\hat{\vartheta}(x)) - i\hat{T}_{\hat{\alpha}}(\hat{\vartheta}(x)).$$

*Proof.* Assume that  $T_\alpha = \sum_\beta c_\beta z^\beta \in \mathbb{Q}[z]$  for some  $c_\beta \in \mathbb{Q}$  and  $\beta \in \mathbb{N}^n$ . For  $x \in \mathbb{T}^n$ , we observe that

$$T_\alpha(\vartheta(x)) = \sum_\beta c_\beta \prod_{j=1}^n (\Re(\theta_j(x)) + i\Im(\theta_j(x)))^{\beta_j} \quad \text{and} \quad T_{\hat{\alpha}}(\vartheta(x)) = \sum_\beta c_\beta \prod_{j=1}^n (\Re(\theta_{\sigma(j)}(x)) + i\Im(\theta_{\sigma(j)}(x)))^{\beta_j}$$

are complex conjugates.

Furthermore, if  $j = \sigma(j)$ , then  $\Re(\theta_j(x)) = \widehat{\theta}_j(x)$  and  $\Im(\theta_j(x)) = 0$ . Otherwise, our definition of  $\theta_{j,\mathbb{R}}$  implies

$$\Re(\theta_j(x)) = \begin{cases} \widehat{\theta}_j(x), & \text{if } j < \sigma(j) \\ \widehat{\theta}_{\sigma(j)}(x), & \text{if } j > \sigma(j) \end{cases} \quad \text{and} \quad \Im(\theta_j(x)) = \begin{cases} \widehat{\theta}_{\sigma(j)}(x), & \text{if } j < \sigma(j) \\ -\widehat{\theta}_j(x), & \text{if } j > \sigma(j) \end{cases}.$$

Altogether, we obtain

$$\frac{T_\alpha(\vartheta(x)) + T_{\widehat{\alpha}}(\vartheta(x))}{2} = \sum_{\beta} \frac{c_\beta}{2} \prod_{j=\sigma(j)} \widehat{\theta}_j(x)^{\beta_j} \left( \prod_{j < \sigma(j)} \left( \widehat{\theta}_j(x) + i \widehat{\theta}_{\sigma(j)}(x) \right)^{\beta_j} \left( \widehat{\theta}_j(x) - i \widehat{\theta}_{\sigma(j)}(x) \right)^{\beta_{\sigma(j)}} \right. \\ \left. + \prod_{j < \sigma(j)} \left( \widehat{\theta}_j(x) - i \widehat{\theta}_{\sigma(j)}(x) \right)^{\beta_j} \left( \widehat{\theta}_j(x) + i \widehat{\theta}_{\sigma(j)}(x) \right)^{\beta_{\sigma(j)}} \right).$$

The right hand side is a unique polynomial in  $\widehat{\vartheta}(x) = (\widehat{\theta}_1(x), \dots, \widehat{\theta}_n(x))$ , denoted by  $\widehat{T}_\alpha$ . Since the left hand side is real for every  $x \in \mathbb{T}^n$ , the coefficient of  $\widehat{T}_\alpha$  in front of  $i$  must be 0. Hence, we have  $\widehat{T}_\alpha \in \mathbb{Q}[z]$ . Similarly, by computing  $(T_\alpha - T_{\widehat{\alpha}})/(2i)$ , we obtain  $\widehat{T}_{\widehat{\alpha}} \in \mathbb{Q}[z]$ .  $\square$

**Example 3.9.** If  $\mathbb{R}$  is of type  $B_n$ ,  $C_n$  or  $G_2$ , then  $\mathcal{T} = \widehat{\mathcal{T}}$  and so it is not necessary to distinguish.

The same holds for  $D_n$ , if  $n$  is even. If  $n$  is odd, then  $T_{k e_i}(z) = \widehat{T}_{k e_i}(\widehat{z})$  with  $\frac{z_{n-1}+z_n}{2} = \widehat{z}_{n-1}$ ,  $\frac{z_{n-1}-z_n}{2i} = \widehat{z}_n$ .

For  $A_{n-1}$ , we have  $\frac{T_{k e_i}(z) + T_{-k e_i}(z)}{2} = \frac{T_{k e_i}(z) + T_{k e_{n-i}}(z)}{2} = \widehat{T}_{k e_i}(\widehat{z})$  with  $\frac{z_{n-i}+z_i}{2} = \widehat{z}_i$ ,  $\frac{z_{n-i}-z_i}{2i} = \widehat{z}_{n-i}$ .

Specifically, for type  $A_2$  with Weyl group  $\mathcal{W} \cong \mathfrak{S}_3$ , we have  $-e_1 \in \mathcal{W} e_2 \subseteq \mathbb{Z}^2$  and the generalized Chebyshev polynomials of degree 2 are

$$T_{02} = 3z_2^2 - 2z_1, \quad T_{11} = 3/2 z_1 z_2 - 1/2, \quad T_{20} = 3z_1^2 - 2z_2 \in \mathbb{Q}[z].$$

After substitution  $z_1 \mapsto \widehat{z}_1 + i \widehat{z}_2$ ,  $z_2 \mapsto \widehat{z}_1 - i \widehat{z}_2$ , we obtain

$$T_{20}, T_{02} = (3\widehat{z}_1^2 - 3\widehat{z}_2^2 - 2\widehat{z}_1) \pm i(6\widehat{z}_1 \widehat{z}_2 + 2\widehat{z}_2), \quad T_{11} = (3/2 \widehat{z}_1^2 + 3/2 \widehat{z}_2^2 - 1/2),$$

and thus the new polynomials from [Proposition 3.8](#) are

$$\widehat{T}_{02} = 6\widehat{z}_1 \widehat{z}_2 + 2\widehat{z}_2, \quad \widehat{T}_{11} = 3/2 \widehat{z}_1^2 + 3/2 \widehat{z}_2^2 - 1/2, \quad \widehat{T}_{20} = 3\widehat{z}_1^2 - 3\widehat{z}_2^2 - 2\widehat{z}_1.$$

Those assume real values on  $\widehat{\mathcal{T}}$ , which is the ‘‘deltoid’’ depicted in [Figure 5](#) as a subset of  $\mathbb{R}^2$ .

## 4 Symmetric polynomial systems

In order to give a polynomial description for the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of a Weyl group  $\mathcal{W}$ , our first step is to characterize when a symmetric polynomial system has a solution in the compact torus. The present section should be regarded as a toolbox for the proofs of the main results.

For  $1 \leq i \leq n$ , we denote by  $\sigma_i(y_1, \dots, y_n) = \sum_{|K|=i} \prod_{k \in K} y_k$  the  $i$ -th elementary symmetric function in  $n$  indeterminates  $y_i$ , where  $K$  ranges over all subsets of  $\{1, \dots, n\}$  with cardinality  $i$ . We shall be confronted with the following two types of polynomial systems.

$$\begin{aligned} \text{(I)} \quad & \sigma_i(y_1, \dots, y_n) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{C} \\ \text{(II)} \quad & \sigma_i\left(\frac{y_1+y_1^{-1}}{2}, \dots, \frac{y_n+y_n^{-1}}{2}\right) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n \quad \text{with } c_1, \dots, c_n \in \mathbb{R} \end{aligned}$$

The goal is to determine, whether all complex solutions  $y = (y_1, \dots, y_n)$  of system (I), respectively (II), are contained in the compact torus  $\mathbb{T}^n$ .

We pursue this goal, because the above situation arises in [Sections 5 to 10](#). Later, the  $y_i$  will assume the role of multivariate monomials in  $\mathbb{C}[x^\pm]$ , such that  $y(x) \in \mathbb{T}^n$  if and only if  $z = \vartheta(x) \in \mathcal{T}$ .

## 4.1 Solutions in the compact torus

For now,  $x$  is a univariate indeterminate. Recall Vieta's formula  $\prod_{k=1}^n (x-y_k) = x^n + \sum_{i=1}^n (-1)^i \sigma_i(y_1, \dots, y_n) x^{n-i}$ .

**Lemma 4.1.** 1. *The map*

$$\begin{aligned} \mathbb{C}^* &\rightarrow \mathbb{C}, \\ x &\mapsto \frac{x+x^{-1}}{2} \end{aligned}$$

*is surjective and the preimage of the real interval  $[-1, 1] \subseteq \mathbb{R}$  is  $\mathbb{T}$ .*

2. *System (I) has a unique solution  $y \in \mathbb{C}^n$  up to permutation of its coordinates  $y_i$ .*

3. *System (II) has a unique solution  $y \in (\mathbb{C}^*)^n$  up to permutation and inversion of its coordinates  $y_i$ .*

*Proof.* 1. For  $r \in \mathbb{C}$ , consider the univariate polynomial  $p := x^2 - 2rx + 1$ . Then 0 is not a root of  $p$  and we have  $p(x) = 0$  if and only if  $r = (x + x^{-1})/2$ , that is,  $r$  is in the image of the map. If  $r \in [-1, 1]$ , then the discriminant of  $p$  is  $r^2 - 1 \leq 0$  and the two roots are  $x, \bar{x} = x, x^{-1} = r \pm i\sqrt{1-r^2} \in \mathbb{T}$ . On the other hand, for  $x \in \mathbb{T}$ , we have  $(x + x^{-1})/2 = (x + \bar{x})/2 = \Re(x) \in [-1, 1]$ .

2. By Vieta's formula, a solution  $y$  of system (I) is the vector of roots of the polynomial  $x^n + c_1 x^{n-1} + \dots + c_n$ . Since  $\mathbb{C}$  is algebraically closed, the statement follows.

3. By 1., we can write the roots  $r_1, \dots, r_n$  of the polynomial  $x^n + c_1 x^{n-1} + \dots + c_n$  as  $r_i = (y_i + y_i^{-1})/2$  for some  $y \in (\mathbb{C}^*)^n$ . Then  $y$  is a unique solution of (II) up to permutation and inversion.  $\square$

From now on, we speak of “the” solution of system (I), respectively (II).

**Proposition 4.2.** *For  $c_1, \dots, c_n \in \mathbb{R}$ , the corresponding solution of system (II) is contained in  $\mathbb{T}^n$  if and only if all the roots of the univariate polynomial*

$$x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$$

*are contained in  $[-1, 1]$ .*

*Proof.* If  $y \in (\mathbb{C}^*)^n$  is the solution of system (II), then the roots of  $x^n + c_1 x^{n-1} + \dots + c_n$  are  $r_i := (y_i + y_i^{-1})/2$ . According to Lemma 4.1, we have  $y \in \mathbb{T}^n$  if and only if  $r_1, \dots, r_n \in [-1, 1]$ .  $\square$

Similarly, we can characterize solutions of system (I). Recall that the univariate Chebyshev polynomial of the first kind associated to  $\ell \in \mathbb{N}$  is the unique  $T_\ell$  with  $T_\ell((x + x^{-1})/2) = (x^\ell + x^{-\ell})/2$ .

**Proposition 4.3.** *For  $c_1, \dots, c_{n-1} \in \mathbb{C}$  with  $\bar{c}_i = (-1)^n c_{n-i}$  and  $c_0 := (-1)^n c_n := 1$ , the corresponding solution of system (I) is contained in  $\mathbb{T}^n$  if and only if all the roots of the univariate polynomial*

$$T_n(x) + d_1 T_{n-1}(x) + \dots + d_{n-1} T_1(x) + \frac{d_n}{2} T_0(x) \quad \text{with} \quad d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

*are contained in  $[-1, 1]$ .*

*Proof.* The solution of system (I) is contained in  $\mathbb{T}^n$  if and only if all the roots of the univariate polynomial  $p := x^n + c_1 x^{n-1} + \dots + c_n$  are contained in  $\mathbb{T}$ .

The roots of  $p$  are nonzero, because  $p(0) = c_n = (-1)^n \neq 0$ . We define another univariate polynomial  $\tilde{p} := x^n + \bar{c}_1 x^{n-1} + \dots + \bar{c}_n$ . By the hypothesis, we have  $\tilde{p}(x) = (-x)^n p(x^{-1})$ . Hence, the set of all roots

of  $p\tilde{p}$  consists of the roots of  $p$  and their inverses. In particular, all the roots of  $p$  are contained in  $\mathbb{T}$  if and only if this holds for all the roots of  $p\tilde{p}$ .

We now show that  $p\tilde{p}$  is the polynomial from the statement: For  $0 \leq \ell \leq n$ , the coefficients of  $p\tilde{p}$  satisfy

$$\text{Coeff}(x^\ell, p\tilde{p}) = \sum_{i=0}^{\ell} c_{n-i} \overline{c_{n-\ell+i}} = \sum_{i=0}^{\ell} \overline{c_i} c_{\ell-i} = \sum_{i=0}^{\ell} \text{Coeff}(x^{n-i}, \tilde{p}) \text{Coeff}(x^{n-\ell+i}, p) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}).$$

The coefficients are real, because

$$\sum_{i=0}^{\ell} \overline{c_i} c_{\ell-i} = \sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \underbrace{(\overline{c_i} c_{\ell-i} + \overline{c_{\ell-i}} c_i)}_{\in \mathbb{R}} + \begin{cases} c_{\ell/2} \overline{c_{\ell/2}}, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases} \in \mathbb{R}.$$

Thus,  $\text{Coeff}(x^\ell, p\tilde{p}) = \text{Coeff}(x^{2n-\ell}, p\tilde{p}) = d_\ell \in \mathbb{R}$  and we can write

$$p\tilde{p} = \sum_{\ell=1}^n d_{n-\ell} (x^{n+\ell} + x^{n-\ell}) + d_n = 2x^n \left( \sum_{\ell=1}^n d_{n-\ell} T_\ell \left( \frac{x+x^{-1}}{2} \right) + \frac{d_n}{2} T_0 \left( \frac{x+x^{-1}}{2} \right) \right) =: 2x^n g \left( \frac{x+x^{-1}}{2} \right).$$

With [Lemma 4.1](#), we see that  $x \in \mathbb{T}$  is a root of  $p\tilde{p}$  if and only if  $(x+x^{-1})/2 \in [-1, 1]$  is a root of  $g$ .  $\square$

The following example illustrates the proof and reappears in the example in [Section 5](#).

**Example 4.4.** For  $c \in \mathbb{C}$ , set  $p = x^3 - cx^2 + \bar{c}x - 1$  and consider the palindromic polynomial  $p\tilde{p} \in \mathbb{R}[x]$  with

$$\begin{aligned} & \frac{1}{2x^3} p(x)\tilde{p}(x) \\ &= \frac{1}{2x^3} ((x^6 + 1) - (c + \bar{c})(x^5 + x) + (c\bar{c} + c + \bar{c})(x^4 + x^2) - (c^2 + \bar{c}^2 + 2)x^3) \\ &= T_3 \left( \frac{x+x^{-1}}{2} \right) - (c + \bar{c}) T_2 \left( \frac{x+x^{-1}}{2} \right) + (c\bar{c} + c + \bar{c}) T_1 \left( \frac{x+x^{-1}}{2} \right) - \frac{c^2 + \bar{c}^2 + 2}{2} T_0 \left( \frac{x+x^{-1}}{2} \right) \\ &=: g \left( \frac{x+x^{-1}}{2} \right). \end{aligned}$$

All the roots of  $p$  are contained in  $\mathbb{T}$  if and only if all the roots of  $g$  are contained in  $[-1, 1]$ , that is, if and only if the solution of system (I) with right hand side  $c_1 = -c, c_2 = \bar{c}, c_3 = -1$  is in  $\mathbb{T}^3$ .

## 4.2 Characterization via Hermite quadratic forms

Let  $p, q \in \mathbb{R}[x]$  be univariate polynomials with  $p \neq 0$ . The multiplication by  $q$  in the  $\mathbb{R}$ -algebra  $\mathbb{R}[x]/\langle p \rangle$  is

$$\begin{aligned} m_q : \mathbb{R}[x]/\langle p \rangle &\rightarrow \mathbb{R}[x]/\langle p \rangle, \\ f + \langle p \rangle &\mapsto qf + \langle p \rangle. \end{aligned}$$

A classical result, usually referred to as Sylvester's version of Sturm's theorem, allows us to effectively characterize when a univariate polynomial has all of its roots in  $[-1, 1]$ .

**Theorem 4.5.** All the roots of  $p$  are contained in  $S(q) := \{x \in \mathbb{R} \mid q(x) \geq 0\}$  if and only if the Hermite quadratic form

$$\begin{aligned} H(p, q) : \mathbb{R}[x]/\langle p \rangle &\rightarrow \mathbb{R}, \\ f + \langle p \rangle &\mapsto \text{Tr}(m_q f^2) \end{aligned}$$

is positive semi-definite.

*Proof.* Let  $H \in \mathbb{R}^{n \times n}$  be the symmetric matrix associated to  $H(p, q)$  for a fixed basis of  $\mathbb{R}[x]/\langle p \rangle$ , where  $n = \deg(p)$ . Denote by  $N_+$ , respectively  $N_-$ , the number of strictly positive, respectively negative, eigenvalues of  $H$ , including their multiplicities. By [CLO05, Ch. 2, Thm. 5.2], the rank and the signature of  $H(p, q)$  are

$$\begin{aligned} N_+ + N_- &= \text{Rank}(H(p, q)) = |\{x \in \mathbb{C} \mid p(x) = 0, q(x) \neq 0\}|, \\ N_+ - N_- &= \text{Sign}(H(p, q)) = \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) > 0\}|}_{=: n_+} - \underbrace{|\{x \in \mathbb{R} \mid p(x) = 0, q(x) < 0\}|}_{=: n_-}. \end{aligned}$$

If all the roots of  $p$  are contained in  $S(q)$ , then  $n_- = 0$ , and consequently

$$N_+ + N_- = \text{Rank}(H(p, q)) = \text{Sign}(H(p, q)) = n_+ = N_+ - N_-.$$

This means that  $N_- = 0$  and all eigenvalues of  $H$  are nonnegative and so  $H(p, q)$  is positive semi-definite.

Conversely, assume that  $H(p, q)$  is positive semi-definite. Then  $N_- = 0$  and so

$$N_+ = \text{Sign}(H(p, q)) = n_+ - n_- \leq \text{Rank}(H(p, q)) = N_+,$$

Hence, we have  $n_+ - n_- = \text{Rank}(H(p, q))$ . On the other hand, if not all the roots of  $p$  were contained in  $S(q)$ , then  $n_+ - n_-$  would be strictly smaller than  $\text{Rank}(H(p, q))$ .  $\square$

Finally, we compute the matrix of the Hermite quadratic form. If  $p = x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$ , then the matrix of the multiplication  $m_x$  in  $\mathbb{R}[x]/\langle p \rangle$  in the basis  $\{1, x, \dots, x^{n-1}\}$  is the companion matrix

$$\begin{bmatrix} 0 & 0 & -c_n \\ 1 & \ddots & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & 1 & -c_1 \end{bmatrix} \quad (4.1)$$

of  $p$ , because  $x x^i = x^{i+1}$  whenever  $0 \leq i \leq n-2$  and  $x x^{n-1} = x^n \equiv -c_1 x^{n-1} - \dots - c_{n-1} x - c_n \pmod{\langle p \rangle}$ .

**Corollary 4.6.** *For  $c_1, \dots, c_n \in \mathbb{R}$ , denote by  $C \in \mathbb{R}^{n \times n}$  the matrix from Equation (4.1). Then the solution of system (II) is contained in  $\mathbb{T}^n$  if and only if  $H \succeq 0$ , where  $H \in \mathbb{R}^{n \times n}$  has entries  $H_{ij} = \text{Tr}(C^{i+j-2} - C^{i+j})$ .*

*Proof.*  $H$  is the matrix associated to the Hermite quadratic form  $H(p, q)$  from Theorem 4.5 with  $q = 1 - x^2$  in the basis  $\{1, x, x^2, \dots, x^{n-1}\}$ . Indeed, by [CLO05, Ch. 2, Prop. 4.2], the matrix entries are

$$H_{ij} = \text{Tr}(m_q x^{i-1} x^{j-1}) = \text{Tr}(m_x x^{i+j-2} - x^{i+j}) = \text{Tr}(m_x^{i+j-2} - m_x^{i+j})$$

for  $1 \leq i, j \leq n$ . Since the trace is basis independent, we can replace  $m_x$  with its matrix  $C$  given by Equation (4.1). The statement now follows from Proposition 4.2, because  $\{x \in \mathbb{R} \mid q(x) \geq 0\} = [-1, 1]$ .  $\square$

On the other hand, the univariate Chebyshev polynomials also form a basis  $\{T_0, T_1, \dots, T_{n-1}\}$  of  $\mathbb{R}[x]/\langle p \rangle$ . If  $n \geq 3$  and  $p = T_n + d_1 T_{n-1} + \dots + d_{n-1} T_1 + d_n/2 T_0 \in \mathbb{R}[x]$ , then the matrix of  $m_x$  is

$$\begin{bmatrix} 0 & 1/2 & & 0 & -d_n/4 \\ 1 & 0 & \ddots & & -d_{n-1}/2 \\ & 1/2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3/2 \\ & & & \ddots & 0 & (1-d_2)/2 \\ 0 & & & & 1/2 & -d_1/2 \end{bmatrix}. \quad (4.2)$$

The entries originate from the recurrence formula  $2xT_j = T_{j+1} + T_{j-1}$ . In particular, the last column is obtained through

$$2xT_{n-1} = T_n + T_{n-2} \equiv -d_1T_{n-1} + (1-d_2)T_{n-2} - d_3T_{n-3} - \dots - d_{n-1}T_1 - d_n/2T_0 \pmod{\langle p \rangle}.$$

**Corollary 4.7.** *Let  $n \geq 3$ . For  $c_1, \dots, c_{n-1} \in \mathbb{C}$  with  $\bar{c}_i = (-1)^n c_{n-i}$  and  $c_0 := (-1)^n c_n := 1$ , set*

$$d_\ell = \sum_{i=0}^{\ell} \bar{c}_i c_{\ell-i} \in \mathbb{R}$$

for  $1 \leq \ell \leq n$  and denote by  $C \in \mathbb{R}^{n \times n}$  the matrix from Equation (4.2). Then the solution of system (I) is contained in  $\mathbb{T}^n$  if and only if  $H \geq 0$ , where  $H \in \mathbb{R}^{n \times n}$  has entries  $H_{ij} = \text{Tr}(C^{i+j-2} - C^{i+j})$ .

*Proof.* Using Proposition 4.3, the proof is analogous to the one of Corollary 4.6.  $\square$

## 5 Type $A_{n-1}$

The group  $\mathfrak{S}_n$  acts on  $\mathbb{R}^n$  by permutation of coordinates and leaves the subspace  $V := \{u \in \mathbb{R}^n \mid u_1 + \dots + u_n = 0\}$  invariant. By [Bou68, Planche I],  $A_{n-1}$  is a root system in  $V$  with fundamental weights

$$\omega_i = \sum_{j=1}^i \varepsilon_j - \frac{i}{n} \sum_{j=1}^n \varepsilon_j = \frac{1}{n} \underbrace{(n-i, \dots, n-i)}_{i \text{ times}} \underbrace{(-i, \dots, -i)}_{n-i \text{ times}} \quad (1 \leq i \leq n-1). \quad (5.1)$$

Here, the  $\varepsilon_i$  are the standard basis vectors in  $\mathbb{R}^n$ . The weight lattice is  $\Omega := \{u/n \mid u \in V \cap \mathbb{Z}^n\}$  and the Weyl group is  $\mathcal{W} \cong \mathfrak{S}_n$ . For  $1 \leq i \leq n-1$ , we have  $-\omega_i \in \mathcal{W}\omega_{n-i}$  and the orbit  $\mathcal{W}\omega_i$  has cardinality  $\binom{n}{i}$ .

Since the rank is  $n-1$ , the ring of Laurent polynomials is  $\mathbb{C}[x^\pm] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}]$  and the polynomial ring is  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_{n-1}]$ .

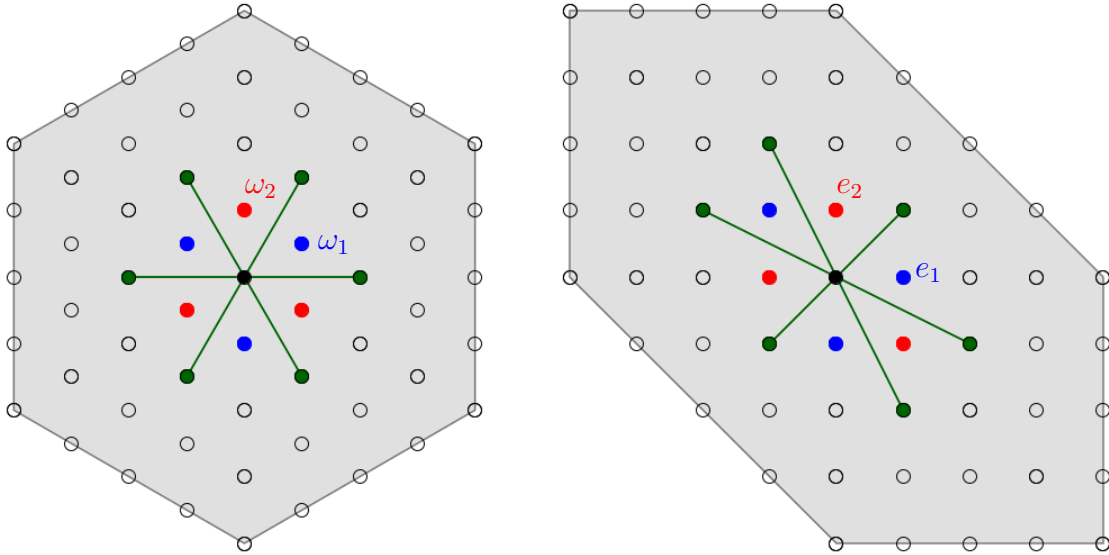


Figure 4: The root system  $A_2$  and its weight lattice. The orbits of the fundamental weights are the blue and red lattice elements.



## 5.1 Orbit polynomials

With  $\mathcal{W} \cong \mathfrak{S}_n$  and [Equation \(5.1\)](#), one observes that a weight  $\mu \in \Omega$  is contained in the  $\mathcal{W}$ -orbit of the first fundamental weight  $\omega_1$  if and only if  $\mu \in \{\omega_1, \omega_2 - \omega_1, \dots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\}$ . Hence, the orbit  $\mathcal{W} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{W}\}$  consists of  $n$  distinct monomials in  $\mathbb{C}[x^\pm]$ , namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-1} := x_{n-1} x_{n-2}^{-1}, \quad y_n := x_{n-1}^{-1}. \quad (5.2)$$

For  $1 \leq i \leq n$ , let  $\sigma_i$  be the  $i$ -th elementary symmetric function in  $n$  indeterminates. Recall that, for  $1 \leq i \leq n-1$ , the  $\mathcal{W}$ -invariant orbit polynomial associated to  $e_i \in \mathbb{Z}^{n-1}$  from [Equation \(3.3\)](#) is denoted by  $\theta_i = \Theta_{e_i}$ . We show that  $\sigma_i(y_1, \dots, y_n) \in \mathbb{C}[x^\pm]^\mathcal{W}$  by expressing it explicitly in terms of the set of fundamental invariants  $\{\theta_1, \dots, \theta_{n-1}\}$ .

**Proposition 5.1.** *In  $\mathbb{Q}[x^\pm]$ , we have*

$$\sigma_i(y_1, \dots, y_n) = \binom{n}{i} \theta_i \quad (1 \leq i \leq n-1) \quad \text{and} \quad \sigma_n(y_1, \dots, y_n) = 1.$$

*Proof.* It follows from [Equation \(5.2\)](#) that  $x_i = y_1 \dots y_i$  and  $\mathcal{W}$  acts on the  $y_i$  by permutation. Hence,

$$\theta_i = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B e_i} = \frac{|\text{Stab}_\mathcal{W}(e_i)|}{|\mathcal{W}|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} y_j = \frac{1}{|\mathcal{W} \cdot x_i|} \sigma_i(y_1, \dots, y_n).$$

With  $|\mathcal{W} \cdot x_i| = \binom{n}{i}$  and  $y_1 \dots y_n = 1$ , we obtain the statement. □

The next statement follows from [Equation \(5.2\)](#) and the fact that  $x_i = y_1 \dots y_i$  for  $1 \leq i \leq n-1$ .

**Lemma 5.2.** *Let  $\mathbb{T}_1^n := \{y \in \mathbb{T}^n \mid y_1 \dots y_n = 1\}$ . The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^{n-1} &\rightarrow \mathbb{T}_1^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)) \end{aligned}$$

*is bijective.*

## 5.2 Hermite characterization with standard monomials

We now characterize, whether a given point  $z$  is contained in the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of  $\mathcal{W}$ , that is, if the equation  $\theta_i(x) = z_i$  has a solution  $x \in \mathbb{T}^{n-1}$ . As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (I) has its solution in  $\mathbb{T}^n$ . We state our main result for  $\mathbb{A}_{n-1}$  in the standard monomial basis.

**Theorem 5.3.** For  $n = 2$ , we have  $\mathcal{T} = \widehat{\mathcal{T}} = [-1, 1]$  (type  $A_1$ ). Let  $n \geq 3$ . Define the  $(n - 1)$ -dimensional  $\mathbb{R}$ -vector space  $\mathcal{Z} := \{z \in \mathbb{C}^{n-1} \mid \forall 1 \leq i \leq n - 1 : \bar{z}_i = z_{n-i}\}$  and the matrix polynomial  $H \in \mathbb{Q}[z]^{n \times n}$  by

$$H(z)_{ij} = \text{Tr}((C(z))^{i+j-2} - (C(z))^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & 1/2 & & 0 & -d_n(z)/4 \\ 1 & 0 & \ddots & & -d_{n-1}(z)/2 \\ & 1/2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 & -d_3(z)/2 \\ & & & \ddots & 0 & (1 - d_2(z))/2 \\ 0 & & & & 1/2 & -d_1(z)/2 \end{bmatrix},$$

$$d_\ell(z) = (-1)^\ell \sum_{i=0}^{\ell} \binom{n}{i} \binom{n}{\ell-i} z_i z_{n-\ell+i} \Big|_{z_0=z_n=1} \quad \text{for } 1 \leq \ell \leq n.$$

For  $z \in \mathcal{Z}$ , we have  $H(z) \in \mathbb{R}^{n \times n}$  and  $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$ .

*Proof.* For  $z \in \mathbb{C}^{n-1}$ , define  $c_0 := 1$ ,  $c_n := (-1)^n$ ,  $c_i := (-1)^i \binom{n}{i} z_i \in \mathbb{C}$  for  $1 \leq i \leq n - 1$  as well as  $d_\ell := d_\ell(z)$  for  $1 \leq \ell \leq n$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$  and fix  $x \in \mathbb{T}^{n-1}$ , such that  $\theta_i(x) = z_i$ . By [Proposition 5.1](#) and [Lemma 5.2](#), the unique solution of

$$(I) \quad \sigma_i(y_1, \dots, y_n) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is  $y = \Upsilon(x) \in \mathbb{T}_1^n$ . Note that  $\theta_j(x)$  and  $\theta_{n-j}(x)$  are complex conjugates, because  $-\omega_j \in \mathcal{W}\omega_{n-j}$ . Therefore,  $z \in \mathcal{Z}$  and  $d_\ell = \sum_{i=0}^{\ell} c_i \bar{c}_{\ell-i} \in \mathbb{R}$  yields the last column of  $C(z)$ . [Corollary 4.7](#) gives us  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $z \in \mathcal{Z}$  with  $H(z) \succeq 0$ . By [Corollary 4.7](#), the solution  $y$  of system (I) is contained in  $\mathbb{T}_1^n$ . Let  $x \in \mathbb{T}^{n-1}$  be the unique preimage of  $y$  under  $\Upsilon$ . Then by [Proposition 5.1](#),

$$z_i = (-1)^i \binom{n}{i}^{-1} c_i = \binom{n}{i}^{-1} \sigma_i(y_1, \dots, y_n) = \theta_i(x)$$

for  $1 \leq i \leq n - 1$  and so  $z = \vartheta(x) \in \mathcal{T}$ . □

### Example: $A_2$

We investigate the root system from [Figure 4](#) with Weyl group  $\mathcal{W} \cong \mathfrak{S}_3$ . Since  $\mathcal{W}\omega_1 = \{\omega_1, \omega_2 - \omega_1, -\omega_2\}$  and  $\mathcal{W}\omega_2 = \{\omega_2, \omega_1 - \omega_2, -\omega_1\}$ , the integral representation of  $\mathcal{W}$  is

$$\mathcal{W} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\} \subseteq \text{GL}_2(\mathbb{Z}).$$

Let  $H \in \mathbb{Q}[z]^{3 \times 3}$  be the matrix polynomial from [Theorem 5.3](#). Then  $z \in \mathcal{Z}$  is contained in  $\mathcal{T}$  if and only if  $H(z) \in \mathbb{R}^{3 \times 3}$  is positive semi-definite. Let

$$\text{Det}(xI_3 - H(z)) = x^3 - h_1(z)x^2 + h_2(z)x - h_3(z) \tag{5.3}$$

be the characteristic polynomial of  $H(z)$ . We write  $z = (z_1 + iz_2, z_1 - iz_2)$  to simplify the notation. Then  $z \in \mathcal{T} \subseteq \mathcal{Z}$  is equivalent to  $(z_1, z_2) \in \widehat{\mathcal{T}} \subseteq \mathbb{R}^2$  and we have

$$\begin{aligned}
h_3(z) &= -\text{Coeff}(x^0, \text{Det}(x I_3 - H(z))) \\
&= 2187/64 z_2^4 (3 z_1 + 1)^2 (-3 z_1^4 - 6 z_1^2 z_2^2 - 3 z_2^4 + 8 z_1^3 - 24 z_1 z_2^2 - 6 z_1^2 - 6 z_2^2 + 1), \\
h_2(z) &= \text{Coeff}(x^1, \text{Det}(x I_3 - H(z))) \\
&= 243/256 z_2^2 (-243 z_1^8 - 972 z_1^6 z_2^2 - 1458 z_1^4 z_2^4 - 972 z_1^2 z_2^6 - 243 z_2^8 + 324 z_1^7 - 1620 z_1^5 z_2^2 \\
&\quad - 4212 z_1^3 z_2^4 - 2268 z_1 z_2^6 - 432 z_1^6 - 2052 z_1^4 z_2^2 - 5400 z_1^2 z_2^4 - 324 z_2^6 + 180 z_1^5 - 3384 z_1^3 z_2^2 \\
&\quad - 684 z_1 z_2^4 + 18 z_1^4 - 804 z_1^2 z_2^2 + 42 z_2^4 + 76 z_1^3 + 404 z_1 z_2^2 - 8 z_1^2 - 108 z_2^2 + 60 z_1 + 25), \\
h_1(z) &= -\text{Coeff}(x^2, \text{Det}(x I_3 - H(z))) \\
&= 1/32 (-729 z_1^6 + 1458 z_1^5 + (10935 z_2^2 - 1215) z_1^4 + (-2916 z_2^2 + 540) z_1^3 + 351 z_2^2 + 63 \\
&\quad + (-10935 z_2^4 + 1458 z_2^2 - 135) z_1^2 + (-4374 z_2^4 + 972 z_2^2 + 18) z_1 + 729 z_2^6 - 1215 z_2^4).
\end{aligned}$$

The matrix  $H(z)$  is positive semi-definite if and only if  $h_i(z) \geq 0$  for  $1 \leq i \leq 3$ . In [Figure 5](#), a solid red line, blue dots and green dashes indicate the varieties of these three polynomials. From those plots, we suspect that the real  $\mathbb{T}$ -orbit space is invariant under rotation by  $2\pi/3$  and horizontal reflection, that is, under the dihedral group  $\mathfrak{D}_3$  of order 6. The ring of  $\mathfrak{D}_3$ -invariants is  $\mathbb{Q}[z_1, z_2]^{\mathfrak{D}_3} = \mathbb{Q}[g_1, g_2]$  with  $g_1 := z_1^2 + z_2^2$  and  $g_2 := z_1(z_1^2 - 3z_2^2)$ . One can check that

$$h_3(z) = \underbrace{2187/64 z_2^4 (3 z_1 + 1)^2}_{\geq 0} \underbrace{(-6 g_1(z_1, z_2) - 3 g_1(z_1, z_2)^2 + 8 g_2(z_1, z_2) + 1)}_{\mathfrak{D}_3\text{-invariant}} \quad (5.4)$$

and the variety of the  $\mathfrak{D}_3$ -invariant factor is the boundary of  $\widehat{\mathcal{T}}$ .

We now identify the vertices of  $\widehat{\mathcal{T}}$ , which correspond to the fundamental weights and the origin. With  $\widehat{\vartheta} = \left(\frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 - \theta_2}{2i}\right)$ , those are

$$\text{Vertex}_0 := \widehat{\vartheta}(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \widehat{\vartheta}(1, 1) = (1, 0),$$

$$\text{Vertex}_1 := \widehat{\vartheta}(\exp(-2\pi i \langle \omega_1, \omega_1 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1 \rangle)) = \widehat{\vartheta}\left(\exp\left(-\frac{4}{3}\pi i\right), \exp\left(-\frac{2}{3}\pi i\right)\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\text{Vertex}_2 := \widehat{\vartheta}(\exp(-2\pi i \langle \omega_1, \omega_2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2 \rangle)) = \widehat{\vartheta}\left(\exp\left(-\frac{2}{3}\pi i\right), \exp\left(-\frac{4}{3}\pi i\right)\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

For a generic point  $(z_1, z_2) \in \mathbb{R}^2$ ,  $H(z)$  has rank 3. If  $h_1(z) = h_2(z) = 0$ , then  $h_3(z) = 0$  as well and  $H(z)$  has rank 0. This holds for two points, namely  $\text{Vertex}_0$  and

$$\widehat{\vartheta}(\exp(-2\pi i \langle \omega_1, (\omega_1 + \omega_2)/2 \rangle), \exp(-2\pi i \langle \omega_2, (\omega_1 + \omega_2)/2 \rangle)) = \widehat{\vartheta}(\exp(-\pi i), \exp(\pi i)) = \left(-\frac{1}{3}, 0\right).$$

If  $h_1(z) = h_3(z) = 0$  or  $h_2(z) = h_3(z) = 0$ , then  $H(z)$  has rank 1. This holds for four points, namely  $\text{Vertex}_1$ ,  $\text{Vertex}_2$  and  $(-1/3, \pm 2/3)$ . For every other point on the boundary of  $\widehat{\mathcal{T}}$ ,  $H(z)$  has rank 2.

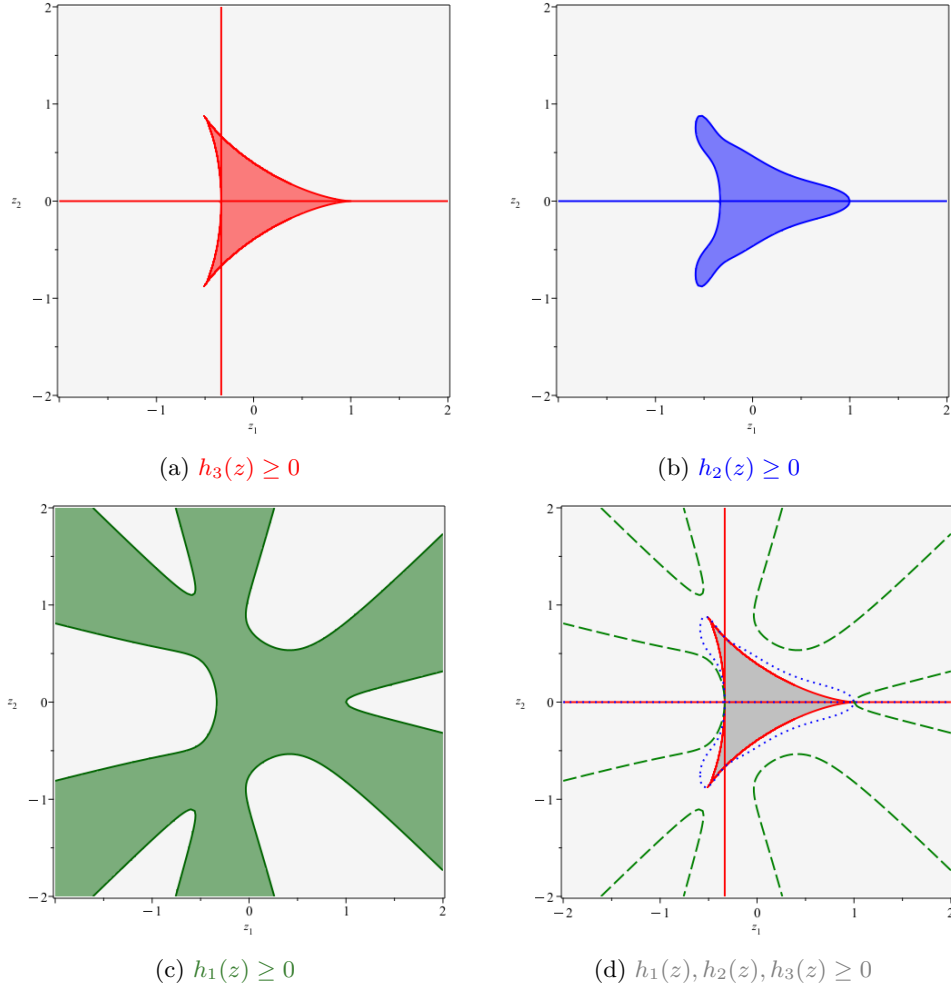


Figure 5: The intersection of the semi-algebraic sets defined by the coefficients of the characteristic polynomial of  $H(z)$  is the  $\mathbb{T}$ -orbit space associated to  $A_2$ .

## 6 Type $C_n$

The group  $\{\pm 1\}^n$  acts on  $\mathbb{R}^n$  by multiplication of coordinates with  $\pm 1$ . By [Bou68, Planche III],  $C_n$  is a root system in  $\mathbb{R}^n$  with fundamental weights

$$\omega_i = \varepsilon_1 + \dots + \varepsilon_i \quad (1 \leq i \leq n). \quad (6.1)$$

Here, the  $\varepsilon_i$  are the standard basis vectors in  $\mathbb{R}^n$ . The Weyl group is  $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}^n$ . For  $1 \leq i \leq n$ , we have  $-\omega_i \in \mathcal{W}\omega_i$  (because  $-I_n \in \mathcal{W}$ ) and the orbit  $\mathcal{W}\omega_i$  has cardinality  $2^i \binom{n}{i}$ .

Since the rank is  $n$ , the ring of Laurent polynomials is  $\mathbb{C}[x^\pm] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$  and the polynomial ring is  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$ .

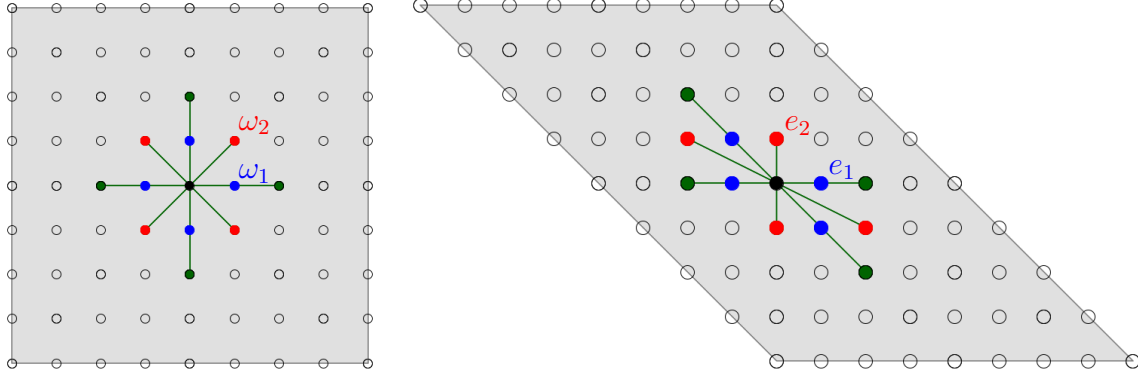


Figure 6: The root system  $C_2$  and its weight lattice. The orbits of the fundamental weights are the blue and red lattice elements.

## 6.1 Orbit polynomials

By Equation (6.1), the orbit  $\mathcal{W} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{W}\}$  consists of  $2n$  distinct monomials in  $\mathbb{C}[x^\pm]$ , namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_n := x_n x_{n-1}^{-1} \quad (6.2)$$

and their inverses. For  $1 \leq i \leq n$ , let  $\sigma_i$  be the  $i$ -th elementary symmetric function in  $n$  indeterminates and recall that  $\theta_i = \Theta_{e_i}$  is the  $\mathcal{W}$ -invariant orbit polynomial associated to  $e_i \in \mathbb{Z}^n$  from Equation (3.3).

**Proposition 6.1.** *In  $\mathbb{C}[x^\pm]$ , we have*

$$\sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = \binom{n}{i} \theta_i \quad (1 \leq i \leq n).$$

*Proof.* It follows from Equation (6.2) that  $x_i = y_1 \dots y_i$  and  $\mathcal{W}$  acts on the  $y_i^{\pm 1}$  by permutation and inversion. Hence,

$$\theta_i = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B e_i} = \frac{|\text{Stab}_{\mathcal{W}}(e_i)|}{|\mathcal{W}|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \sum_{\delta \in \{\pm 1\}^J} \prod_{j \in J} y_j^{\delta_j} = \frac{1}{|\mathcal{W} \cdot x_i|} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=i}} \prod_{j \in J} (y_j + y_j^{-1})$$

With  $|\mathcal{W} \cdot x_i| = 2^i \binom{n}{i}$ , we obtain the statement.  $\square$

The next statement follows from Equation (6.2) and the fact that  $x_i = y_1 \dots y_i$  for  $1 \leq i \leq n$ .

**Lemma 6.2.** *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)), \end{aligned}$$

*is bijective.*

## 6.2 Hermite characterization with standard monomials

We now characterize, whether a given point  $z$  is contained in the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of  $\mathcal{W}$ , that is, if the equation  $\theta_i(x) = z_i$  has a solution  $x \in \mathbb{T}^n$ . As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in  $\mathbb{T}^n$ . We state our main result for  $C_n$  in the standard monomial basis.

**Theorem 6.3.** Define the matrix polynomial  $H \in \mathbb{Q}[z]^{n \times n}$  by

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad \text{for } 1 \leq i \leq n.$$

Then  $\mathcal{T} = \widehat{\mathcal{T}} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{R}^n$  and set  $c_i := c_i(z) \in \mathbb{R}$  for  $1 \leq i \leq n$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$  for  $1 \leq i \leq n$ . By [Proposition 6.1](#) and [Lemma 6.2](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i c_i \quad (1 \leq i \leq n)$$

is  $y = \Upsilon(x) \in \mathbb{T}^n$ . Applying [Corollary 4.6](#) yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $H(z) \succeq 0$ . By [Corollary 4.6](#), the solution  $y$  of the above system (II) is contained in  $\mathbb{T}^n$ . Let  $x \in \mathbb{T}^n$  be the unique preimage of  $y$  under  $\Upsilon$ . Then  $z_i = \theta_i(x)$  and so  $z = \vartheta(x)$  is contained in  $\mathcal{T}$ .  $\square$

### Example: $C_2$

We study the root system from [Figure 6](#) with Weyl group  $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$ . Since  $\mathcal{W}\omega_1 = \{\pm\omega_1, \pm(\omega_1 - \omega_2)\}$  and  $\mathcal{W}\omega_2 = \{\pm\omega_2, \pm(2\omega_1 - \omega_2)\}$ , the integral representation of  $\mathcal{W}$  is

$$\mathcal{W} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \subseteq \text{GL}_2(\mathbb{Z}).$$

Then  $z = (z_1, z_2) \in \mathbb{R}^2$  is contained in  $\mathcal{T}$  if and only if the resulting Hermite matrix

$$H(z) = \begin{bmatrix} -4z_1^2 + 2z_2 + 2 & -8z_1^3 + 6z_1z_2 + 2z_1 \\ -8z_1^3 + 6z_1z_2 + 2z_1 & -16z_1^4 + 16z_1^2z_2 + 4z_1^2 - 2z_2^2 - 2z_2 \end{bmatrix}. \quad (6.3)$$

is positive semi-definite, or equivalently, its determinant and trace

$$\begin{aligned} \text{Det}(H(z)) &= -4(z_1^2 - z_2)(2z_1 + 1 + z_2)(2z_1 - 1 - z_2) \\ \text{Tr}(H(z)) &= -16z_1^4 + 16z_1^2z_2 - 2z_2^2 + 2 \end{aligned} \quad (6.4)$$

are nonnegative. The varieties of these two polynomials are depicted in [Figure 7](#). If  $z \in \mathbb{R}^2$  with  $\text{Det}(H(z)) = \text{Tr}(H(z)) = 0$ , then  $H(z)$  has rank 0 and  $z$  must be one of the three vertices

$$\begin{aligned} \text{Vertex}_0 &:= \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1), \\ \text{Vertex}_1 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \omega_1/2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1/2 \rangle)) = \vartheta(-1, -1) = (0, -1), \\ \text{Vertex}_2 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \omega_2/2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2/2 \rangle)) = \vartheta(-1, 1) = (-1, 1). \end{aligned}$$

The boundary of  $\mathcal{T}$  is dictated by the determinant, but from the positivity condition one can observe that the trace is also required. Alternatively, the inequality given by the trace could be replaced by the constraint

that the orbit space is contained in the square  $[-1, 1]^2$ , which is always true. Hence,  $H(z)$  has rank 1 if  $z$  is a boundary point, but not a vertex, and rank 2 if  $z$  is generic.

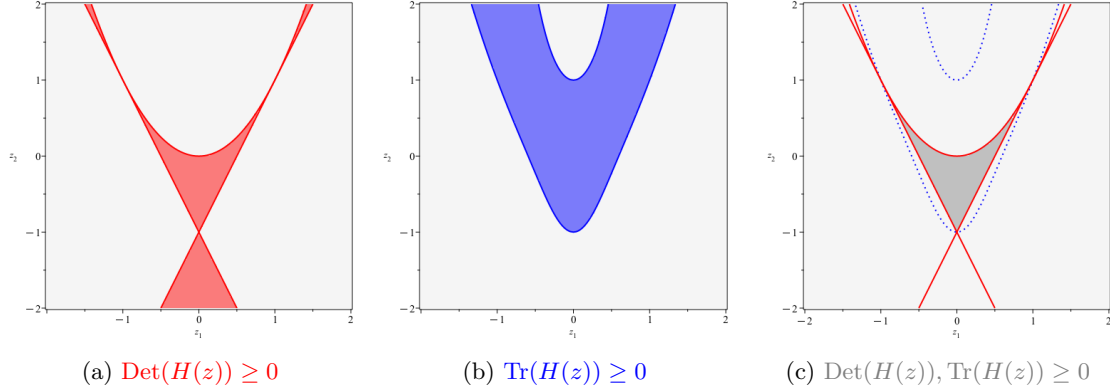


Figure 7: The semi-algebraic sets defined by the determinant and trace of  $H(z)$ .

## 7 Type $B_n$

The root system  $B_n$  from [Bou68, Planche II] is a root system in  $\mathbb{R}^n$ . Its Weyl group is the same as that of  $C_n$ , that is  $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}^n$ , but now the fundamental weights are

$$\omega_i = \varepsilon_1 + \dots + \varepsilon_i \quad (1 \leq i \leq n-1) \quad \text{and} \quad \omega_n = (\varepsilon_1 + \dots + \varepsilon_n)/2. \quad (7.1)$$

Here, the  $\varepsilon_i$  are the standard basis vectors in  $\mathbb{R}^n$ . In particular, the  $\mathbb{T}$ -orbit space is different from  $C_n$ , see Figure 1. For  $1 \leq i \leq n$ , we have  $-\omega_i \in \mathcal{W}\omega_i$  (because  $-I_n \in \mathcal{W}$ ) and the orbit  $\mathcal{W}\omega_i$  has cardinality  $2^i \binom{n}{i}$ .

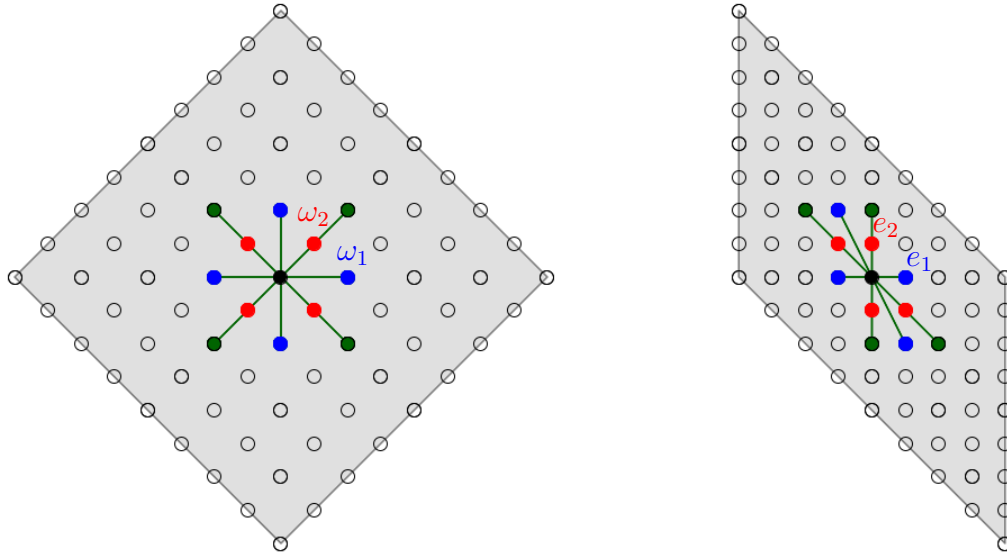


Figure 8: The root system  $B_2$  and its weight lattice. The orbits of the fundamental weights are the blue and red lattice elements.

Since the rank is  $n$ , the ring of Laurent polynomials is  $\mathbb{C}[x^\pm] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$  and the polynomial ring is  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$ .

## 7.1 Orbit polynomials

By Equation (7.1), the orbit  $\mathcal{W} \cdot x_1 = \{x^{Be_1} \mid B \in \mathcal{W}\}$  consists of  $2n$  distinct monomials in  $\mathbb{C}[x^\pm]$ , namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-1} := x_{n-1} x_{n-2}^{-1}, \quad y_n := x_n^2 x_{n-1}^{-1} \quad (7.2)$$

and their inverses. For  $1 \leq i \leq n$ , let  $\sigma_i$  be the  $i$ -th elementary symmetric function in  $n$  indeterminates and recall that  $\theta_i = \Theta_{e_i}$  is the  $\mathcal{W}$ -invariant orbit polynomial associated to  $e_i \in \mathbb{Z}^n$  from Equation (3.3).

**Proposition 7.1.** *In  $\mathbb{C}[x^\pm]$ , we have*

$$\sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = \binom{n}{i} \theta_i \quad (1 \leq i \leq n-1)$$

and  $\sigma_n \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = \Theta_{2e_n}$ .

*Proof.* By Equation (7.2),  $x_i = y_1 \dots y_i$  and  $x_n^2 = y_1 \dots y_n$ . The proof is analogous to Proposition 6.1.  $\square$

We compute the missing explicit expression for the right hand side of Proposition 7.1 in terms of the fundamental invariants  $\theta_i$ .

**Lemma 7.2.** *In  $\mathbb{C}[x^\pm]$ , we have*

$$\Theta_{2e_n} = 2^n \theta_n^2 - \sum_{j=1}^{n-1} \binom{n}{j} \theta_j - 1.$$

*Proof.* The cardinality of the orbit  $\mathcal{W}e_n \subseteq \mathbb{Z}^n$  is  $2^n$ . Let  $\alpha \in \mathcal{W}e_n$  and distinguish as follows.

1. If  $\alpha = e_n$ , then  $\Theta_{e_n + \alpha} = \Theta_{2e_n}$  is the term for which we seek an explicit formula.
2. If  $\alpha = -e_n$ , then  $\Theta_{e_n + \alpha} = \Theta_0 = 1$ .
3. Otherwise, it follows from Equation (7.1) that there exists  $1 \leq j \leq n-1$ , such that  $e_n + \alpha \in \mathcal{W}e_j$ . The number of  $\alpha$ , for which this is the case, is  $\binom{n}{j}$ .

With the recurrence formula from Equation (3.4), we conclude

$$2^n \theta_n^2 = |\mathcal{W}e_n| \Theta_{e_n} \Theta_{e_n} = \sum_{\alpha \in \mathcal{W}e_n} \Theta_{\alpha + e_n} = \Theta_{2e_n} + \sum_{j=1}^{n-1} \binom{n}{j} \Theta_{e_j} + 1 = \Theta_{2e_n} + \sum_{j=1}^{n-1} \binom{n}{j} \theta_j + 1$$

and obtain the formula for  $\Theta_{2e_n}$ .  $\square$

**Lemma 7.3.** *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)), \end{aligned}$$

*is surjective. Furthermore, every  $y \in \mathbb{T}^n$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  with*

$$\theta_i(x) = \theta_i(x') \quad (1 \leq i \leq n-1) \quad \text{and} \quad \theta_n(x) = -\theta_n(x').$$



*Proof.* For  $y \in \mathbb{T}^n$ , choose  $x \in \mathbb{T}^n$  with  $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-1} = y_{n-1} x_{n-2}$  and  $x_n^2 = y_n x_{n-1}$ . Then  $x$  is a preimage of  $y$  under  $\Upsilon$  and unique up to a sign in the last coordinate.

For  $1 \leq i \leq n-1$ , we have  $\theta_i(x) = \theta_i(x')$ , because  $y_k(x) = y_k(x')$  for all  $1 \leq k \leq n$ .

It follows from Equation (7.1) that the stabilizer of  $\omega_n$  in  $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}^n$  is isomorphic to the subgroup  $\mathfrak{S}_n$ . Hence, we have  $\mathcal{W}\omega_n = \{\pm 1\}^n \omega_n$  and, for all  $\mu \in \mathcal{W}\omega_n$ , there exist  $\epsilon_i = \pm 1$  and  $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-1}$ , such that

$$\mu = \frac{\epsilon_1}{2} \varepsilon_1 + \dots + \frac{\epsilon_n}{2} \varepsilon_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-1} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_n}{2} (2\omega_n - \omega_{n-1}) = \epsilon_n \omega_n + \nu.$$

Now let  $\alpha, \beta \in \mathbb{Z}^n$ , such that  $\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n$  and  $\nu = \beta_1 \omega_1 + \dots + \beta_n \omega_n$ . Then  $\beta_n = 0$  and the monomial in  $\theta_n$  corresponding to  $\mu$  is  $x^\alpha = x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}} x_n^{\epsilon_n}$ . Thus,  $x^\alpha$  is linear in  $x_n$ . Since every monomial in  $\theta_n$  can be written in terms of such  $\beta$  and  $\epsilon_i$ , the polynomial  $\theta_n$  is linear in  $x_n$  and, with  $x, x'$  as above, we have  $\theta_n(x) = -\theta_n(x')$ .  $\square$

## 7.2 Hermite characterization with standard monomials

We now characterize, whether a given point  $z$  is contained in the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of  $\mathcal{W}$ , that is, if the equation  $\theta_i(x) = z_i$  has a solution  $x \in \mathbb{T}^n$ . As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in  $\mathbb{T}^n$ . We state our main result for  $B_n$  in the standard monomial basis.

**Theorem 7.4.** *Define the matrix polynomial  $H \in \mathbb{Q}[z]^{n \times n}$  by*

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad (1 \leq i \leq n-1) \quad \text{and} \quad c_n(z) = (-1)^n \left( 2^n z_n^2 - \sum_{i=1}^{n-1} \binom{n}{i} z_i - 1 \right).$$

Then  $\mathcal{T} = \widehat{\mathcal{T}} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{R}^n$  and set  $c_i := c_i(z) \in \mathbb{R}$  for  $1 \leq i \leq n$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$  for  $1 \leq i \leq n$ . By Proposition 7.1 and Lemmas 7.2 and 7.3, the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i c_i \quad \text{for } 1 \leq i \leq n$$

is  $y = \Upsilon(x) \in \mathbb{T}^n$ . Applying Corollary 4.6 yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $H(z) \succeq 0$ . By Corollary 4.6, the solution  $y$  of system (II) with coefficients  $c_i$  is contained in  $\mathbb{T}^n$ . According to Lemma 7.3,  $y$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  under  $\Upsilon$  with  $x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}$  and  $x_n = -x'_n$ . We have  $z_i = \theta_i(x) = \theta_i(x')$  for  $1 \leq i \leq n-1$  and  $z_n^2 = \theta_n(x)^2 = \theta_n(x')^2$  with  $\theta_n(x) = -\theta_n(x')$ . Therefore,  $z_n = \theta_n(x)$  or  $z_n = -\theta_n(x) = \theta_n(x')$  and thus,  $z$  is contained in  $\mathcal{T}$ .  $\square$

**Example:  $B_2$** 

We study the root system from Figure 8 with Weyl group  $\mathcal{W} \cong \mathfrak{S}_2 \times \{\pm 1\}^2$ . Since  $\mathcal{W}\omega_1 = \{\omega_1, -\omega_1, \omega_1 - 2\omega_2, 2\omega_2 - \omega_1\}$  and  $\mathcal{W}\omega_2 = \{\omega_2, -\omega_2, \omega_1 - \omega_1, \omega_2 - \omega_1\}$ , the integral representation of  $\mathcal{W}$  is

$$\mathcal{W} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}. \quad (7.3)$$

Then  $z = (z_1, z_2) \in \mathbb{R}^2$  is contained in  $\mathcal{T}$  if and only if the resulting Hermite matrix

$$H(z) = 16 \begin{bmatrix} -4z_1^2 + 8z_2^2 - 4z_1 & -8z_1^3 + 24z_1z_2^2 - 12z_1^2 - 4z_1 \\ -8z_1^3 + 24z_1z_2^2 - 12z_1^2 - 4z_1 & -8z_1^3 + 24z_1z_2^2 - 12z_1^2 - 4z_1 \end{bmatrix}$$

is positive semi-definite, or equivalently, its determinant and trace

$$\begin{aligned} \text{Det}(H(z)) &= -64z_2^2(-z_2^2 + z_1)(z_1 + 1 + 2z_2)(z_1 + 1 - 2z_2) \\ \text{Tr}(H(z)) &= -16z_1^4 + 64z_1^2z_2^2 - 32z_2^4 - 32z_1^3 + 32z_1z_2^2 - 24z_1^2 + 16z_2^2 - 8z_1 \end{aligned} \quad (7.4)$$

are nonnegative. The varieties of these two polynomials are depicted below.

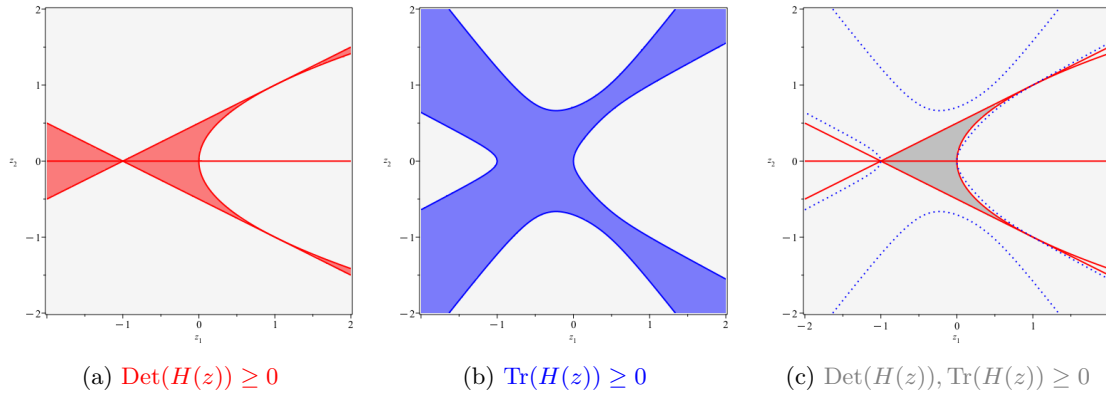


Figure 9: The semi-algebraic sets defined by the determinant and trace of  $H(z)$ .

The  $\mathbb{T}$ -orbit space in the  $B_2$ -case is obtained from the  $C_2$ -case in Figure 7 by swapping  $z_1$  and  $z_2$ . This follows immediately from the fact that their Dynkin graphs  $\bullet \Rightarrow \bullet$  and  $\bullet \Leftarrow \bullet$  become equal after reversing the arrows, see [Bou68, Planche II et III]. Let us confirm that it follows also from our semi-algebraic description: Indeed, we have  $\text{Det}(H(z))/\text{Det}(H^{C_2}(z_2, z_1)) = 16z_2^2 \geq 0$  on  $\mathbb{R}^2$  and the variety of the determinant dictates the boundary. On the other hand, the determinant of  $H(z)$  vanishes if  $z_2 = 0$  and the rank drops to 0 in  $z = (0, 0)$  although this is not a vertex. The other points which correspond to  $H$  having rank 0 are precisely the three vertices

$$\begin{aligned} \text{Vertex}_0 &:= \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1), \\ \text{Vertex}_1 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \omega_1 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1 \rangle)) = \vartheta(1, -1) = (-1, 0), \\ \text{Vertex}_2 &:= \vartheta(\exp(-2\pi i \langle \omega_1, \omega_2 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2 \rangle)) = \vartheta(-1, -1) = (1, -1). \end{aligned}$$

Hence,  $H(z)$  has rank 1 if  $z$  is a boundary point, but not a vertex or 0, and rank 2 if  $z$  is generic.

## 8 Type $D_n$

The group  $\{\pm 1\}^{n-1}$  acts on  $\mathbb{R}^n$  by scalar multiplication of coordinates with  $\pm 1$ , where only an even amount of sign changes is admissible. By [Bou68, Planche IV],  $D_n$  is a root system in  $\mathbb{R}^n$  with fundamental weights

$$\omega_i = \varepsilon_1 + \dots + \varepsilon_i \quad (1 \leq i \leq n-2) \quad \text{and} \quad \omega_{n-1} = (\varepsilon_1 + \dots + \varepsilon_{n-1} - \varepsilon_n)/2, \quad \omega_n = (\varepsilon_1 + \dots + \varepsilon_n)/2. \quad (8.1)$$

Here, the  $\varepsilon_i$  are the standard basis vectors in  $\mathbb{R}^n$ . The Weyl group of  $D_n$  is  $\mathcal{W} \cong \mathfrak{S}_n \times \{\pm 1\}^{n-1}$ . For all  $1 \leq i \leq n$ , we have  $-\omega_i \in \mathcal{W}\omega_i$ , unless  $n$  is odd. In this case,  $-\omega_{n-1} \in \mathcal{W}\omega_n$ . Furthermore, we have  $|\mathcal{W}\omega_i| = 2^i \binom{n}{i}$  for  $1 \leq i \leq n-2$  and  $|\mathcal{W}\omega_{n-1}| = |\mathcal{W}\omega_n| = 2^{n-1}$ .

Since the rank is  $n$ , the ring of Laurent polynomials is  $\mathbb{C}[x^\pm] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$  and the polynomial ring is  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_n]$ .

### 8.1 Orbit polynomials

By Equation (8.1), the orbit  $\mathcal{W} \cdot x_1 = \{x^{Be_1} \mid B \in \mathcal{W}\}$  consists of  $2n$  distinct monomials in  $\mathbb{C}[x^\pm]$ , namely

$$y_1 := x_1, \quad y_2 := x_2 x_1^{-1}, \quad \dots, \quad y_{n-2} := x_{n-2} x_{n-3}^{-1}, \quad y_{n-1} := x_n x_{n-1} x_{n-2}^{-1}, \quad y_n := x_n x_{n-1}^{-1} \quad (8.2)$$

and their inverses. For  $1 \leq i \leq n$ , let  $\sigma_i$  be the  $i$ -th elementary symmetric function in  $n$  indeterminates and recall that  $\theta_i = \Theta_{e_i}$  is the  $\mathcal{W}$ -invariant orbit polynomial associated to  $e_i \in \mathbb{Z}^n$  from Equation (3.3).

**Proposition 8.1.** *In  $\mathbb{C}[x^\pm]$ , we have*

$$\begin{aligned} \sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) &= \binom{n}{i} \theta_i \quad (1 \leq i \leq n-2) \\ \text{and} \quad \sigma_{n-1} \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) &= n \Theta_{e_{n-1} + e_n}, \\ \sigma_n \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) &= \frac{\Theta_{2e_{n-1}} + \Theta_{2e_n}}{2}. \end{aligned}$$

*Proof.* Let  $1 \leq i \leq n-2$ . It follows from Equation (8.2) that  $x_i = y_1 \dots y_i$ . Then the statement for  $\theta_i$  is proven analogously to Proposition 6.1.

With  $x_n x_{n-1} = y_1 \dots y_{n-1}$ , we obtain the equation for  $\Theta_{e_{n-1} + e_n}$  as well.

Finally, we have  $x_{n-1}^2 = y_1 \dots y_{n-1} y_n^{-1}$  and  $x_n^2 = y_1 \dots y_n$ . Thus,

$$\frac{\Theta_{2e_{n-1}} + \Theta_{2e_n}}{2} = \frac{1}{2^n} \left( \sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = -1}} y^\epsilon + \sum_{\substack{\epsilon \in \{\pm 1\}^n \\ \epsilon_1 \dots \epsilon_n = 1}} y^\epsilon \right) = \frac{1}{2^n} \sum_{\epsilon \in \{\pm 1\}^n} y^\epsilon = \frac{1}{2^n} \prod_{i=1}^n (y_i + y_i^{-1}),$$

where  $y^\epsilon = y_1^{\epsilon_1} \dots y_n^{\epsilon_n}$ . This proves the last equation.  $\square$

We give the explicit expression for the right hand side of Proposition 8.1 in terms of the fundamental invariants.

**Lemma 8.2.** *In  $\mathbb{C}[x^\pm]$ , the following statements hold.*

1. If  $n$  is even, then

$$\begin{aligned}\Theta_{e_{n-1}+e_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} \theta_{2j-1}, \\ \Theta_{2e_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1, \\ \Theta_{2e_n} &= 2^{n-1} \theta_n^2 - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} \theta_{2j} - 1.\end{aligned}$$

2. If  $n$  is odd, then

$$\begin{aligned}\Theta_{e_{n-1}+e_n} &= \frac{2^{n-1}}{n} \theta_{n-1} \theta_n - \frac{1}{n} \sum_{j=1}^{(n-3)/2} \binom{n}{2j} \theta_{2j} - \frac{1}{n}, \\ \Theta_{2e_{n-1}} &= 2^{n-1} \theta_{n-1}^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}, \\ \Theta_{2e_n} &= 2^{n-1} \theta_n^2 - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} \theta_{2j+1}.\end{aligned}$$

*Proof.* With Equation (8.1) and the recurrence formula from Equation (3.4), the proof is analogous to that for the case of  $B_n$  in Lemma 7.2. A rigorous computation can be found in [Met22, Lemma 1.43].  $\square$

**Lemma 8.3.** *The map*

$$\begin{aligned}\Upsilon : \mathbb{T}^n &\rightarrow \mathbb{T}^n, \\ x &\mapsto (y_1(x), \dots, y_n(x)),\end{aligned}$$

*is surjective. Furthermore, every  $y \in \mathbb{T}^n$  has exactly two distinct preimages  $x, x' \in \mathbb{T}^n$  under  $\Upsilon$  with*

$$\theta_i(x) = \begin{cases} \theta_i(x'), & \text{if } 1 \leq i \leq n-2 \\ -\theta_i(x'), & \text{if } i \in \{n-1, n\}. \end{cases}$$

*Proof.* For  $y \in \mathbb{T}^n$ , choose  $x \in \mathbb{T}^n$  with  $x_1 = y_1, x_2 = y_1 x_1, \dots, x_{n-2} = y_{n-2} x_{n-3}$  and  $x_{n-1}^2 = y_{n-1}^{-1} y_{n-1} x_{n-2}$ ,  $x_n = y_n x_{n-1}$ . Then  $x$  is a preimage of  $y$  under  $\Upsilon$  and unique up to a common sign in  $x_{n-1}$  and  $x_n$ . This second preimage is denoted by  $x'$ . By Proposition 8.1, we have  $\theta_i(x) = \theta_i(x')$  whenever  $1 \leq i \leq n-2$ .

It follows from Equation (8.1) that the stabilizers of  $\omega_{n-1}$  and  $\omega_n$  in  $\mathcal{W}$  are both  $\mathfrak{S}_n$ . Hence, we have  $\mathcal{W}\omega_{n-1} = \{\pm 1\}^{n-1} \omega_{n-1}$  and  $\mathcal{W}\omega_n = \{\pm 1\}^{n-1} \omega_n$ . Thus, for  $\mu \in \mathcal{W}\omega_n$ , there exist  $\epsilon_i = \pm 1$  with  $\epsilon_1 \dots \epsilon_n = 1$  and  $\nu \in \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_{n-2}$ , such that

$$\begin{aligned}\mu &= \frac{\epsilon_1}{2} \epsilon_1 + \dots + \frac{\epsilon_n}{2} \epsilon_n = \frac{\epsilon_1}{2} \omega_1 + \sum_{i=2}^{n-2} \frac{\epsilon_i}{2} (\omega_i - \omega_{i-1}) + \frac{\epsilon_{n-1}}{2} (\omega_n + \omega_{n-1} - \omega_{n-2}) + \frac{\epsilon_n}{2} (\omega_n - \omega_{n-1}) \\ &= \frac{\epsilon_{n-1} + \epsilon_n}{2} \omega_n + \frac{\epsilon_{n-1} - \epsilon_n}{2} \omega_{n-1} + \nu \in \Omega.\end{aligned}$$

Now let  $\alpha, \beta \in \mathbb{Z}^n$ , such that  $\mu = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n$  and  $\nu = \beta_1 \omega_1 + \dots + \beta_n \omega_n$ . Then  $\beta_{n-1} = \beta_n = 0$  and the monomial in  $\theta_n$  corresponding to  $\mu$  is

$$x^\alpha = x_1^{\beta_1} \dots x_{n-2}^{\beta_{n-2}} x_{n-1}^{(\epsilon_{n-1} - \epsilon_n)/2} x_n^{(\epsilon_{n-1} + \epsilon_n)/2}$$

with  $(\epsilon_{n-1} \pm \epsilon_n)/2 \in \{-1, 0, 1\}$ . Therefore,  $x^\alpha$  is linear in  $x_{n-1}$  and independent of  $x_n$  or vice versa. With  $x, x'$  as above we have  $x^\alpha = -(x')^\alpha$ . Since every monomial in  $\theta_n$  can be written in terms of such  $\beta$  and  $\epsilon_i$ , we obtain  $\theta_n(x) = -\theta_n(x')$ . Analogously, for  $\mu \in \mathcal{W}\omega_{n-1}$ , we have  $\epsilon_1 \dots \epsilon_n = -1$  and obtain the statement for  $\theta_{n-1}$ .  $\square$

## 8.2 Hermite characterization with standard monomials

We now characterize, whether a given point  $z$  is contained in the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  of  $\mathcal{W}$ , that is, if the equation  $\theta_i(x) = z_i$  has a solution  $x \in \mathbb{T}^n$ . As shown in previous subsections, this is equivalent to deciding whether a symmetric polynomial system of type (II) has its solution in  $\mathbb{T}^n$ . We state our main result for  $\mathbb{D}_n$  in the standard monomial basis.

**Theorem 8.4.** *Define the  $n$ -dimensional  $\mathbb{R}$ -vector space*

$$\mathcal{Z} := \begin{cases} \mathbb{R}^n, & \text{if } n \text{ is even} \\ \{z \in \mathbb{C}^n \mid z_1, \dots, z_{n-2} \in \mathbb{R}, \bar{z}_n = z_{n-1}\}, & \text{if } n \text{ is odd} \end{cases}$$

and the matrix polynomial  $H \in \mathbb{Q}[z]^{n \times n}$  by

$$H(z)_{ij} = \text{Tr}(C(z)^{i+j-2} - C(z)^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & \cdots & 0 & -c_n(z) \\ 1 & & 0 & -c_{n-1}(z) \\ & \ddots & & \vdots \\ 0 & & 1 & -c_1(z) \end{bmatrix},$$

$$c_i(z) = (-1)^i \binom{n}{i} z_i \quad (1 \leq i \leq n-2) \quad \text{and}$$

$$c_{n-1}(z) = (-1)^{n-1} \begin{cases} 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-2)/2} \binom{n}{2j-1} z_{2j-1}, & \text{if } n \text{ is even} \\ 2^{n-1} z_n z_{n-1} - \sum_{j=1}^{(n-3)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is odd} \end{cases},$$

$$c_n(z) = (-1)^n \begin{cases} 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=1}^{(n-2)/2} \binom{n}{2j} z_{2j} - 1, & \text{if } n \text{ is even} \\ 2^{n-2} (z_n^2 + z_{n-1}^2) - \sum_{j=0}^{(n-3)/2} \binom{n}{2j+1} z_{2j+1}, & \text{if } n \text{ is odd} \end{cases}.$$

For all  $z \in \mathcal{Z}$ , we have  $H(z) \in \mathbb{R}^{n \times n}$  and  $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$ .

*Proof.* Let  $z \in \mathbb{C}^n$  and set  $c_i := c_i(z)$  for  $1 \leq i \leq n$ .

To show “ $\subseteq$ ”, assume that  $z \in \mathcal{T}$ . Then there exists  $x \in \mathbb{T}^n$ , such that  $\theta_i(x) = z_i$  for  $1 \leq i \leq n$ . Furthermore, we have  $z \in \mathcal{Z}$  and  $c_i \in \mathbb{R}$ . By [Proposition 8.1](#) and [Lemmas 8.2](#) and [8.3](#), the solution of the symmetric polynomial system

$$(II) \quad \sigma_i \left( \frac{y_1 + y_1^{-1}}{2}, \dots, \frac{y_n + y_n^{-1}}{2} \right) = (-1)^i \tilde{c}_i \quad (1 \leq i \leq n)$$

is  $y = \Upsilon(x) \in \mathbb{T}^n$ . Applying [Corollary 4.6](#) yields  $H(z) \succeq 0$ .

For “ $\supseteq$ ” on the other hand, assume  $z \in \mathcal{Z}$  with  $H(z) \succeq 0$ . Hence,  $c_i \in \mathbb{R}$  and by [Corollary 4.6](#), the solution  $y$  of the above system (II) is contained in  $\mathbb{T}^n$ . According to [Lemma 8.3](#),  $y$  has two distinct preimages  $x, x' \in \mathbb{T}^n$ . We have  $z_i = \theta_i(x) = \theta_i(x')$  for  $1 \leq i \leq n-2$  and  $\theta_{n-1}(x) = -\theta_{n-1}(x')$ ,  $\theta_n(x) = -\theta_n(x')$ . Furthermore,

$z_{n-1}^2 + z_n^2 = \theta_{n-1}(x)^2 + \theta_n(x)^2 = \theta_{n-1}(x')^2 + \theta_n(x')^2$  and  $z_{n-1}z_n = \theta_{n-1}(x)\theta_n(x) = \theta_{n-1}(x')\theta_n(x')$ . Therefore,  $\{z_{n-1}, z_n\} \in \{\{\theta_{n-1}(x), \theta_n(x)\}, \{\theta_{n-1}(x'), \theta_n(x')\}\}$ . If  $z_{n-1} = \theta_{n-1}(x), z_n = \theta_n(x)$ , then  $z = \vartheta(x)$ . Otherwise, set  $\tilde{x} = (x_1, \dots, x_{n-2}, x_n, x_{n-1})$  to obtain  $z_{n-1} = \theta_{n-1}(\tilde{x}), z_n = \theta_n(\tilde{x})$  and  $z = \vartheta(\tilde{x})$ . An analogous argument applies to  $x'$  and thus,  $z$  is contained in  $\mathcal{T}$ .  $\square$

## 9 Type $G_2$

The group  $\mathfrak{S}_3 \times \{\pm 1\}$  acts on  $\mathbb{R}^3$  by permutation of coordinates and scalar multiplication with  $\pm 1$  and leaves the subspace  $V = \{u \in \mathbb{R}^3 \mid u_1 + u_2 + u_3 = 0\}$  invariant. By [Bou68, Planche IX],  $G_2$  is a root system in  $V$  with rank 2 and fundamental weights

$$\omega_1 = [1, -1, 0]^t \quad \text{and} \quad \omega_2 = [-2, 1, 1]^t. \quad (9.1)$$

The Weyl group of  $G_2$  is  $\mathcal{W} \cong \mathfrak{S}_3 \times \{\pm 1\}$ . We have  $-\omega_1 \in \mathcal{W}\omega_1$  and  $-\omega_2 \in \mathcal{W}\omega_2$ . Furthermore,  $|\mathcal{W}\omega_1| = |\mathcal{W}\omega_2| = 6$ , see Figure 10.

Since the rank is 2, the ring of Laurent polynomials is  $\mathbb{C}[x^\pm] = \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$  and the polynomial ring is  $\mathbb{Q}[z] = \mathbb{Q}[z_1, z_2]$ .

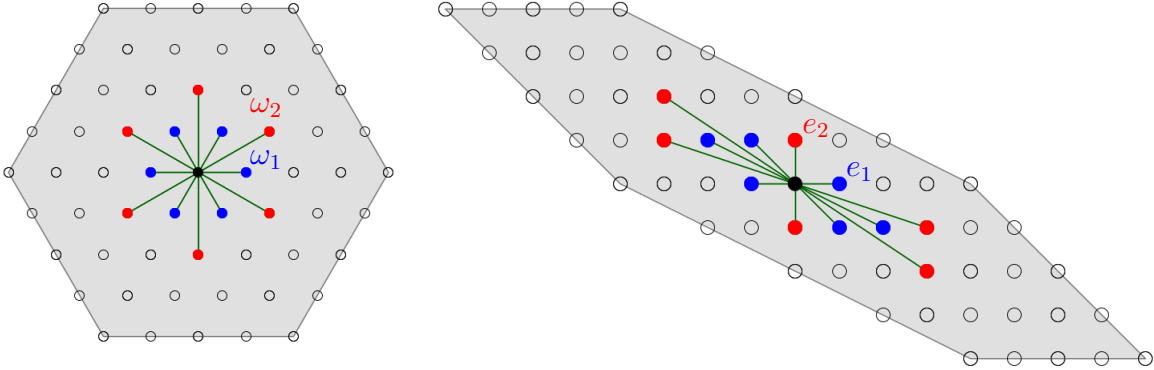


Figure 10: The root system  $G_2$  and the weight lattice. The orbits of the fundamental weights are the blue and red lattice elements.

### 9.1 Orbit polynomials

The integral representation of  $\mathcal{W}$  is

$$\mathcal{W} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \right\} \subseteq \text{GL}_2(\mathbb{Z}).$$

Then the orbit  $\mathcal{W} \cdot x_1 = \{x^{B e_1} \mid B \in \mathcal{W}\} \subseteq \mathbb{C}[x^\pm]$  consists of 6 distinct monomials, namely

$$y_1 := x_1, \quad y_2 := x_1 x_2^{-1}, \quad y_3 := x_1^{-2} x_2 \quad (9.2)$$

and their inverses. Let  $\sigma_1, \sigma_2, \sigma_3$  be the elementary symmetric functions in 3 indeterminates and  $\theta_1 = \Theta_{e_1}, \theta_2 = \Theta_{e_2}$  be the  $\mathcal{W}$ -invariant orbit polynomials associated to  $e_1, e_2 \in \mathbb{Z}^2$  from Equation (3.3).

**Proposition 9.1.** *In  $\mathbb{C}[x^\pm]$ , we have*

$$\begin{aligned} \sigma_1 \left( \frac{y_1 + y_1^{-1}}{2}, \frac{y_2 + y_2^{-1}}{2}, \frac{y_3 + y_3^{-1}}{2} \right) &= 3\theta_1, \\ \sigma_2 \left( \frac{y_1 + y_1^{-1}}{2}, \frac{y_2 + y_2^{-1}}{2}, \frac{y_3 + y_3^{-1}}{2} \right) &= \frac{3(\theta_1 + \theta_2)}{2} \\ \text{and } \sigma_3 \left( \frac{y_1 + y_1^{-1}}{2}, \frac{y_2 + y_2^{-1}}{2}, \frac{y_3 + y_3^{-1}}{2} \right) &= \frac{9\theta_1^2 - 3\theta_1 - 3\theta_2 - 1}{2}. \end{aligned}$$

*Proof.* In principle, this can be checked by hand, but we still give a recipe to obtain the equations: It follows from Equation (9.2) that the product  $y_1 y_2 y_3$  is 1 and moreover  $x_1 = y_1$ ,  $x_1^{-1} = y_2 y_3$ ,  $x_2 = y_1^2 y_3$ ,  $x_2^{-1} = y_2^2 y_3$ . Note that we do not need the inverses of the  $y_i$ . Thus, after computing the orbits  $\mathcal{W} \cdot x_1$  and  $\mathcal{W} \cdot x_2$ , we find polynomials  $g_1, g_2 \in \mathbb{R}[y_1, y_2, y_3]$ , such that  $\theta_i = g_i(y)$ . Consider the ideal

$$\mathcal{I} := \langle \theta_1 - g_1(y), \theta_2 - g_2(y), 1 - y_1 y_2 y_3 \rangle \subseteq \mathbb{Q}[y_1, y_2, y_3, \theta_1, \theta_2].$$

Choose an elimination ordering with  $\{y_1, y_2, y_3\} \succeq \{\theta_1, \theta_2\}$  and let  $G$  be a Gröbner basis of  $\mathcal{I}$ . Note that the expression on the left-hand side of the statement is  $\mathcal{W}$ -invariant. We can write it as  $\sigma_i(y_1 + y_2 y_3, y_2 + y_1 y_3, y_3 + y_1 y_2)/2^i$ , that is, a polynomial in  $\mathbb{Q}[y_1, y_2, y_3, \theta_1, \theta_2]$ . Because of the  $\mathcal{W}$ -invariance, the normal form of this polynomial with respect to  $G$  only depends on  $\theta_1$  and  $\theta_2$ . One obtains the right-hand side.  $\square$

The next statement follows immediately from Equation (9.2).

**Lemma 9.2.** *The map*

$$\begin{aligned} \Upsilon : \mathbb{T}^2 &\rightarrow \mathbb{T}_1^3, \\ x &\mapsto (y_1(x), y_2(x), y_3(x)) \end{aligned}$$

*is bijective.*

## 9.2 Hermite characterization with standard monomials

With Proposition 9.1 and Lemma 9.2, the proof of the following is analogous to Theorem 6.3.

**Theorem 9.3.** *Define the matrix polynomial  $H \in \mathbb{Q}[z]^{3 \times 3}$  by*

$$H(z)_{ij} = \text{Tr}((C(z))^{i+j-2} - (C(z))^{i+j}), \quad \text{where } C(z) = \begin{bmatrix} 0 & 0 & (9z_1^2 - 3z_1 - 3z_2 - 1)/2 \\ 1 & 0 & -3(z_1 + z_2)/2 \\ 0 & 1 & 3z_1 \end{bmatrix}.$$

*Then  $\mathcal{T} = \widehat{\mathcal{T}} = \{z \in \mathbb{R}^2 \mid H(z) \succeq 0\}$ .*

### Example

Our result applies to the root system from Figure 10 as follows. Let  $z = (z_1, z_2) \in \mathbb{R}^2$ . The matrix polynomial from Theorem 9.3 evaluated in  $z$  is  $4/3 H(z) =$

$$\begin{bmatrix} -12z_1^2 + 4z_1 + 4z_2 + 4 & \dots & \dots \\ -36z_1^3 + 18z_1z_2 + 10z_1 + 6z_2 + 2 & \dots & \dots \\ -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots & \dots \\ \dots & -36z_1^3 + 18z_1z_2 + 10z_1 + 6z_2 + 2 & \dots \\ \dots & -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots \\ \dots & -324z_1^5 + 270z_1^3z_2 + 126z_1^3 + 45z_1^2z_2 - 45z_1z_2^2 + 15z_1^2 - 48z_1z_2 - 15z_2^2 - 11z_1 - 11z_2 - 2 & \dots \\ \dots & -108z_1^4 + 72z_1^2z_2 + 30z_1^2 + 12z_1z_2 - 6z_2^2 + 4z_1 - 4z_2 & \dots \\ \dots & -324z_1^5 + 270z_1^3z_2 + 126z_1^3 + 45z_1^2z_2 - 45z_1z_2^2 + 15z_1^2 - 48z_1z_2 - 15z_2^2 - 11z_1 - 11z_2 - 2 & \dots \\ \dots & -972z_1^6 + 972z_1^4z_2 + 432z_1^4 + 162z_1^3z_2 - 243z_1^2z_2^2 + 63z_1^3 - 207z_1^2z_2 - 81z_1z_2^2 + 9z_2^3 - 45z_1^2 - 66z_1z_2 - 3z_2^2 - 14z_1 - 6z_2 - 1 & \dots \end{bmatrix}$$

and shall have characteristic polynomial

$$\text{Det}(x I_3 - H(z)) = x^3 - h_1(z) x^2 + h_2(z) x - h_3(z). \quad (9.3)$$

Again, we have  $z \in \mathcal{T}$  if and only if  $h_i(z) \geq 0$  for  $1 \leq i \leq 3$  with

$$\begin{aligned} h_3(z) &= -\text{Coeff}(x^0, \text{Det}(x I_3 - H(z))) \\ &= 81/16 (24 z_1^3 - 12 z_1 z_2 - z_2^2 - 6 z_1 - 4 z_2 - 1) (3 z_1^2 - 2 z_2 - 1)^2 (3 z_1 + 1)^2, \\ h_2(z) &= \text{Coeff}(x^1, \text{Det}(x I_3 - H(z))) \\ &= 27/16 (3 z_1^2 - 2 z_2 - 1) (648 z_1^6 - 540 z_1^5 - 972 z_1^4 z_2 - 216 z_1^3 z_2^2 - 648 z_1^4 + 36 z_1^3 z_2 \\ &\quad + 369 z_1^2 z_2^2 + 126 z_1 z_2^3 + 9 z_2^4 - 18 z_1^3 + 480 z_1^2 z_2 + 246 z_1 z_2^2 + 36 z_2^3 + 129 z_1^2 + 202 z_1 z_2 \\ &\quad + 37 z_2^2 + 44 z_1 + 28 z_2 + 4), \\ h_1(z) &= -\text{Coeff}(x^2, \text{Det}(x I_3 - H(z))) \\ &= 9/4 - 729 z_1^6 + 243 z_1^4 + 729 z_1^4 z_2 - 729/4 z_1^2 z_2^2 + 243/2 z_1^3 z_2 + 189/4 z_1^3 - 405/4 z_1^2 z_2 \\ &\quad - 243/4 z_1 z_2^2 + 27/4 z_2^3 - 81/4 z_1^2 - 81/2 z_1 z_2 - 27/4 z_2^2 - 9/2 z_1 - 9/2 z_2. \end{aligned}$$

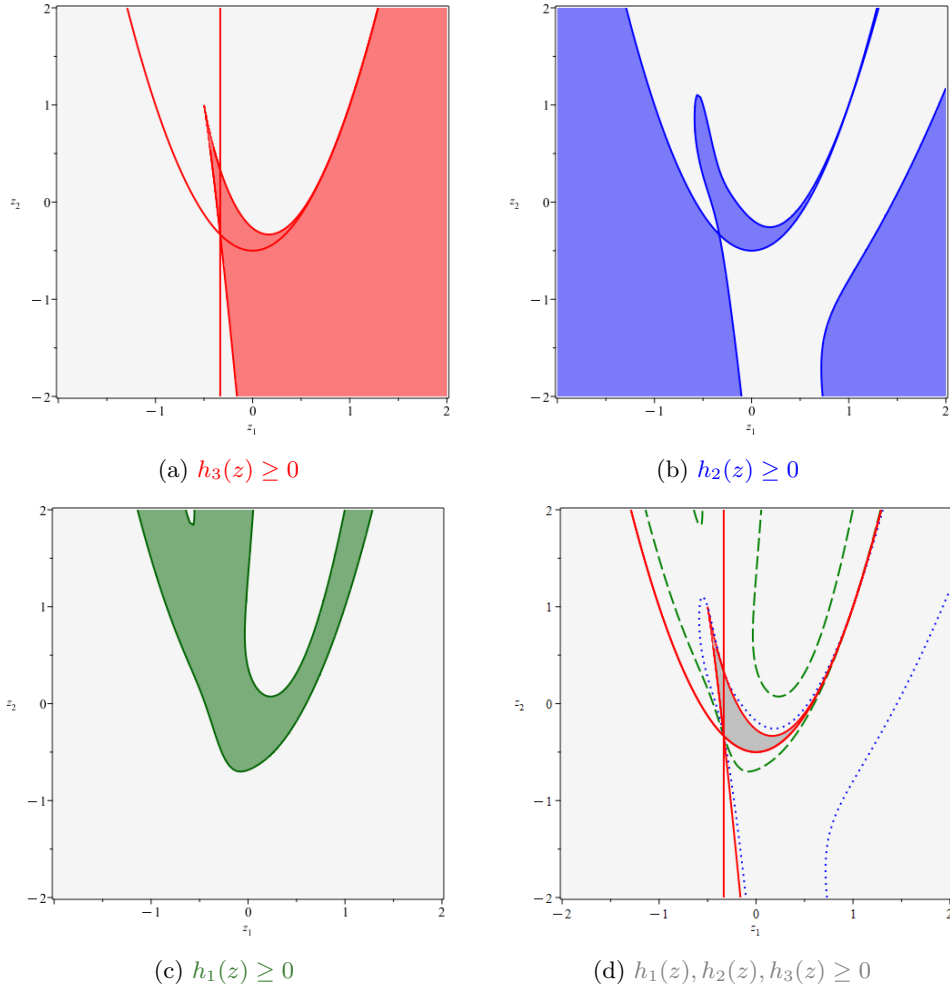


Figure 11: The semi-algebraic sets defined by the coefficients of the characteristic polynomial of  $H(z)$ .



The vertices are

$$\text{Vertex}_1 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_1/3 \rangle), \exp(-2\pi i \langle \omega_2, \omega_1/3 \rangle)) = \vartheta\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}, 1\right) = \left(-\frac{1}{2}, 1\right),$$

$$\text{Vertex}_2 := \vartheta(\exp(-2\pi i \langle \omega_1, \omega_2/6 \rangle), \exp(-2\pi i \langle \omega_2, \omega_2/6 \rangle)) = \vartheta(-1, 1) = \left(-\frac{1}{3}, -\frac{1}{3}\right),$$

$$\text{Vertex}_3 := \vartheta(\exp(-2\pi i \langle \omega_1, 0 \rangle), \exp(-2\pi i \langle \omega_2, 0 \rangle)) = \vartheta(1, 1) = (1, 1).$$

For a generic point  $(z_1, z_2) \in \mathbb{R}^2$ ,  $H(z)$  has rank 3. If  $h_1(z) = h_2(z) = 0$ , then  $h_3(z) = 0$  as well and  $H(z)$  has rank 0. This holds for two points, namely  $\text{Vertex}_0$  and

$$\vartheta_{\mathbb{R}}(\exp(-2\pi i \langle \omega_1, (\omega_1 + \omega_2)/2 \rangle), \exp(-2\pi i \langle \omega_2, (\omega_1 + \omega_2)/2 \rangle)) = \vartheta_{\mathbb{R}}(\exp(-\pi i), \exp(\pi i)) = \left(-\frac{1}{3}, 0\right).$$

If  $h_1(z) = h_3(z) = 0$  or  $h_2(z) = h_3(z) = 0$ , then  $H(z)$  has rank 1. This holds for four points, namely  $\text{Vertex}_1$ ,  $\text{Vertex}_2$  and  $(-1/3, \pm 2/3)$ . For every other point on the boundary of  $\mathcal{T}_{\mathbb{R}}$ ,  $H(z)$  has rank 2.

## 10 Hermite characterization of $\mathcal{T}_{\mathbb{R}}$ with Chebyshev polynomials

Finally, we give a unifying formula for our constructed Hermite matrices in terms of the real generalized Chebyshev polynomials  $\widehat{T}_{\alpha}$  from [Proposition 3.8](#). Those depend on the root systems and so the matrices are not identical in  $\mathbb{Q}[z]$ . Instead, unifying means that the formulae follow the same pattern in the Chebyshev basis.

We have to distinguish carefully between the size of the Hermite matrix and the number of coordinates of the  $\mathbb{T}$ -orbit space: Throughout, let  $n \in \mathbb{N}$  and  $\mathbb{R}$  be a root system of type  $A_{n-1}$ ,  $B_n$ ,  $C_n$ ,  $D_n$  or  $G_2$ . In particular, the rank of  $\mathbb{R}$  is  $r := n - 1$  for  $A_{n-1}$ ,  $r := 2$  for  $G_2$  and  $r := n$  otherwise. A common property of these types is that the symmetric group  $\mathfrak{S}_n$  is a semi-direct factor of the Weyl group. This fact made the characterization via Hermite quadratic forms in the previous [Sections 5 to 9](#) possible. The  $r$ -variate polynomial ring is denoted by  $\mathbb{Q}[z] = \mathbb{Q}[z_1, \dots, z_r]$ , respectively  $\mathbb{Q}[\widehat{z}] = \mathbb{Q}[\widehat{z}_1, \dots, \widehat{z}_r]$ .

**Theorem 10.1.** *Let  $H \in \mathbb{Q}[\widehat{z}]^{n \times n}$  be the matrix polynomial with entries*

$$2^{i+j} H_{ij} = -\widehat{T}_{(i+j)e_1} + \sum_{\ell=1}^{\lceil (i+j)/2 \rceil - 1} \left( 4 \binom{i+j-2}{\ell-1} - \binom{i+j}{\ell} \right) \widehat{T}_{(i+j-2\ell)e_1} \\ + \frac{1}{2} \begin{cases} 4 \binom{i+j-2}{(i+j)/2-1} - \binom{i+j}{(i+j)/2}, & \text{if } i+j \text{ is even} \\ 0, & \text{if } i+j \text{ is odd} \end{cases}.$$

*Then the real  $\mathbb{T}$ -orbit space of the multiplicative action of  $\mathcal{W}$  on  $\mathbb{T}^n$  is  $\widehat{\mathcal{T}} = \{\widehat{z} \in \mathbb{R}^r \mid H(\widehat{z}) \succeq 0\}$ .*

We need two lemmata involving the orbit polynomials  $\Theta_{\alpha} = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$  for  $\alpha \in \mathbb{Z}^r$  from [Equation \(3.3\)](#).

**Lemma 10.2.** *The set  $\mathcal{W} \cdot x_1 \cup \mathcal{W} \cdot x_1^{-1}$  consists of  $n$  distinct monomials and their inverses, which are denoted by  $y_1^{\pm 1}, \dots, y_n^{\pm 1}$ . For  $x \in \mathbb{T}^r$  and  $k \in \mathbb{N}$ , we have  $(y_i(x)^k + y_i(x)^{-k})/2 \in [-1, 1]$  and*

$$\frac{1}{n} \sum_{i=1}^n y_i(x)^k + y_i(x)^{-k} = \Theta_{k e_1}(x) + \Theta_{-k e_1}(x).$$

*Proof.* The  $y_i$  are the monomials from [Equations \(5.2\) to \(9.2\)](#). The left hand side is  $1/n \sigma_1$  evaluated in  $y_i(x)^k + y_i(x)^{-k}$ . Then the proof is analogous to [Propositions 5.1, 6.1, 7.1, 8.1 and 9.1](#).  $\square$

**Lemma 10.3.** For  $x \in \mathbb{T}^r$  and  $y_1^{\pm 1}, \dots, y_n^{\pm 1}$  monomials as in Lemma 10.2, let  $C \in \mathbb{R}^{n \times n}$  be a matrix with eigenvalues  $(y_i(x) + y_i(x)^{-1})/2$ . Then, for  $k \in \mathbb{N}$ , the trace of the  $k$ -th power of  $C$  is

$$\mathrm{Tr}(C^k) = \frac{n}{2^k} \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left( \binom{k}{\ell} (\Theta_{(k-2\ell)e_1}(x) + \Theta_{(2\ell-k)e_1}(x)) \right) + \frac{n}{2^k} \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

*Proof.* The trace of the  $k$ -th power of  $C$  is the sum of the  $k$ -th powers of the eigenvalues. Thus,

$$\begin{aligned} \mathrm{Tr}(C^k) &= \sum_{i=1}^n \left( \frac{y_i(x) + y_i(x)^{-1}}{2} \right)^k = \frac{1}{2^k} \sum_{i=1}^n \left( \sum_{\ell=0}^k \binom{k}{\ell} y_i(x)^{k-\ell} y_i(x)^{-\ell} \right) \\ &= \frac{1}{2^k} \sum_{i=1}^n \left( \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \binom{k}{\ell} y_i(x)^{k-2\ell} + \sum_{\ell=\lfloor k/2 \rfloor + 1}^k \binom{k}{\ell} y_i(x)^{k-2\ell} + \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \right) \\ &= \frac{1}{2^k} \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left( \binom{k}{\ell} \sum_{i=1}^n (y_i(x)^{k-2\ell} + y_i(x)^{2\ell-k}) \right) + \frac{n}{2^k} \begin{cases} \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}. \end{aligned}$$

Applying Lemma 10.2 yields the formula.  $\square$

*Proof of Theorem 10.1.* The  $\mathbb{T}$ -orbit space of  $\mathcal{W}$  is by definition  $\mathcal{T} = \{\vartheta(x) \mid x \in \mathbb{T}^r\}$ . According to Theorems 5.3, 6.3, 7.4, 8.4 and 9.3, we have  $\mathcal{T} = \{z \in \mathcal{Z} \mid H(z) \succeq 0\}$ . Here,  $\mathcal{Z}$  is an  $\mathbb{R}$ -vector space with  $\dim(\mathcal{Z}) = r$  and  $H \in \mathbb{Q}[z]^{n \times n}$  is a matrix with entries  $H_{ij} = \mathrm{Tr}(C^{i+j-2}) - \mathrm{Tr}(C^{i+j})$ , where  $C \in \mathbb{Q}[z]^{n \times n}$  is defined in the respective theorem.

For  $x \in \mathbb{T}^r$  and  $z := \vartheta(x) \in \mathcal{T} \subseteq \mathcal{Z}$ , it is now essential to note that  $C(z) \in \mathbb{R}^{n \times n}$  has eigenvalues  $(y_i(x) + y_i(x)^{-1})/2 \in [-1, 1]$  with  $y_1^{\pm 1}, \dots, y_n^{\pm 1}$  as in Lemma 10.2. This follows from the construction, by which  $C$  is the matrix of a multiplication map, see Section 4.2. In particular, we obtain a formula for the entries  $H(z)_{ij}$  with Lemma 10.3. In the basis of (real) generalized Chebyshev polynomials from Definition 3.5 and Proposition 3.8, we have

$$\Theta_{(k-2\ell)e_1}(x) + \Theta_{(2\ell-k)e_1}(x) = T_{(k-2\ell)e_1}(\vartheta(x)) + T_{(2\ell-k)e_1}(\vartheta(x)) = 2\widehat{T}_{(k-2\ell)e_1}(\widehat{\vartheta}(x)).$$

We now substitute  $z \mapsto \widehat{z} := \widehat{\vartheta}(x) \in \widehat{\mathcal{T}} \subseteq \mathbb{R}^n$ . For  $1 \leq i, j \leq n$  and  $k := i + j$ , the entry  $H(z)_{ij}$  is

$$\begin{aligned} &\mathrm{Tr}(C(z)^{k-2}) - \mathrm{Tr}(C(z)^k) \\ &= \frac{2n}{2^k} \left( 4 \sum_{\ell=0}^{\lceil k/2 \rceil - 2} \left( \binom{k-2}{\ell} \widehat{T}_{(k-2(\ell+1))e_1}(\widehat{z}) \right) - \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left( \binom{k}{\ell} \widehat{T}_{(k-2\ell)e_1}(\widehat{z}) \right) \right) \\ &\quad + \frac{n}{2^k} \begin{cases} 4\binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \\ &= \frac{2n}{2^k} \left( 4 \sum_{\ell=1}^{\lceil k/2 \rceil - 1} \left( \binom{k-2}{\ell-1} \widehat{T}_{(k-2\ell)e_1}(\widehat{z}) \right) - \sum_{\ell=0}^{\lceil k/2 \rceil - 1} \left( \binom{k}{\ell} \widehat{T}_{(k-2\ell)e_1}(\widehat{z}) \right) \right) \\ &\quad + \frac{n}{2^k} \begin{cases} 4\binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \\ &= \frac{2n}{2^k} \left( -\widehat{T}_{ke_1}(\widehat{z}) + \sum_{\ell=1}^{\lceil k/2 \rceil - 1} \left( 4\binom{k-2}{\ell-1} - \binom{k}{\ell} \right) \widehat{T}_{(k-2\ell)e_1}(\widehat{z}) \right) + \frac{n}{2^k} \begin{cases} 4\binom{k-2}{k/2-1} - \binom{k}{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}. \end{aligned}$$

Dividing by  $2n$  does not change whether  $H(\widehat{z})$  is positive semi-definite and so we obtain the formula.  $\square$

**Remark 10.4.** *The matrix polynomial  $H \in \mathbb{Q}[\hat{z}]^{n \times n}$  from [Theorem 10.1](#) follows the pattern*

$$\begin{bmatrix} \frac{\hat{T}_0 - \hat{T}_{2e_1}}{4} & \frac{\hat{T}_{e_1} - \hat{T}_{3e_1}}{8} & \frac{\hat{T}_0 - \hat{T}_{4e_1}}{16} & \frac{2\hat{T}_{e_1} - \hat{T}_{3e_1} - \hat{T}_{5e_1}}{32} & \dots \\ \frac{\hat{T}_{e_1} - \hat{T}_{3e_1}}{8} & \frac{\hat{T}_0 - \hat{T}_{4e_1}}{16} & \frac{2\hat{T}_{e_1} - \hat{T}_{3e_1} - \hat{T}_{5e_1}}{32} & \frac{2\hat{T}_0 + \hat{T}_{2e_1} - 2\hat{T}_{4e_1} - \hat{T}_{6e_1}}{64} & \dots \\ \frac{\hat{T}_0 - \hat{T}_{4e_1}}{16} & \frac{2\hat{T}_{e_1} - \hat{T}_{3e_1} - \hat{T}_{5e_1}}{32} & \frac{2\hat{T}_0 + \hat{T}_{2e_1} - 2\hat{T}_{4e_1} - \hat{T}_{6e_1}}{64} & \frac{5\hat{T}_{e_1} - \hat{T}_{3e_1} - 3\hat{T}_{5e_1} - \hat{T}_{7e_1}}{128} & \dots \\ \frac{2\hat{T}_{e_1} - \hat{T}_{3e_1} - \hat{T}_{5e_1}}{32} & \frac{2\hat{T}_0 + \hat{T}_{2e_1} - 2\hat{T}_{4e_1} - \hat{T}_{6e_1}}{64} & \frac{5\hat{T}_{e_1} - \hat{T}_{3e_1} - 3\hat{T}_{5e_1} - \hat{T}_{7e_1}}{128} & \frac{5\hat{T}_0 + 4\hat{T}_{2e_1} - 4\hat{T}_{4e_1} - 4\hat{T}_{6e_1} - \hat{T}_{8e_1}}{256} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

1. *The only  $\hat{T}_\alpha$  occurring in  $H$  are those, where  $\alpha \in \mathbb{N}^r$  is a multiple of  $e_1$ . An entry is a linear combination of at most  $n + 1$  such polynomials. The largest possible degree of an entry is  $2n = \deg(\hat{T}_{2ne_1})$ .*
2.  *$H$  has Hankel structure, that is,  $H_{ij} = H_{kl}$  whenever  $i + j = k + l$ . Therefore, the number of distinct entries is at most  $2n - 1$ .*
3. *Denote by  $H^{(n)}$  the matrix  $H$  for fixed  $n$ . Then the leading principal submatrices of  $H$  have the structure of  $H^{(k)}$  for  $k \leq n$ .*

## Conclusion and future work

We have characterized the  $\mathbb{T}$ -orbit space of a Weyl group acting nonlinearly on the compact torus as a compact basic semi-algebraic set. The characterization is given through a polynomial matrix inequality and holds for the Weyl groups associated to root systems of type  $A_{n-1}$ ,  $B_n$ ,  $C_n$ ,  $D_n$  and  $G_2$ . It is the first explicit characterization for nonlinear actions in the sense that we have given a closed formula for the matrix. By expressing the entries in the basis of generalized Chebyshev polynomials, we have shown that these matrices have a common structure.

We plan to address the remaining cases  $E_6$ ,  $E_7$ ,  $E_8$  and  $F_4$  in a future work.

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