Periodic geodesics for contact sub-Riemannian 3D manifolds

Yves Colin de Verdière

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The goal of this paper is to study periodic geodesics for sub-Riemannain metrics on a contact 3D-manifold. We develop two rather independent subjects:

1. The existence of closed geodesics spiraling around periodic Reeb orbits for a generic metric.

2. The precise study of the periodic geodesics for a right invariant metric on a quotient $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$ for which the Reeb flow is the geodesic flow of the corresponding hyperbolic surface $\Gamma\backslash\mathbb{H}$.

In the first part (Section 2 to Section 7), we prove the following result which was conjectured in [C-H-T-21]:

Theorem 0.1 *Let (M, D, g) be a contact 3D sub-Riemannian manifold and Γ a periodic orbit of the canonically associated Reeb flow with period $T_0 > 0$. Let us assume that Γ is non degenerate, meaning that 1 is not an eigenvalue of the linearized Poincaré map of this orbit. Then, there exists a sequence γ_k, $k \geq k_0$, of periodic sub-Riemannian geodesics of (M, D, g) with $\lim_{k\to+\infty} \gamma_k = \Gamma$ as closed sets with the Hausdorff topology and the length l_k of γ_k is equivalent as $k \to +\infty$ to $2\sqrt{\pi k T_0}$; more precisely, l_k admits a full asymptotic expansion in powers of $k^{-\frac{1}{2}}$.*

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In fact, as we will see, the γ_k's are spiraling around Γ. The assumptions are generic for closed manifolds: it is known that there exists closed Reeb orbits in any closed 3-manifold [Ta-07] and that they are generically non degenerate [Bo-03]. The precise definitions of the terms in Theorem 0.1 will be given in Section 2.

In the second part (Section 8 to Section 14), we give a description of the closed geodesics in the case of the *Liouville* contact structure on the quotient $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$, the unit co-tangent bundle of an hyperbolic Riemann surface. Roughly speaking we show the existence of continuous families of 2D-tori on which the geodesics spiral linearly and are periodic for a dense set of parameters. This involves the Casimir Hamiltonian and the Euler equations in the dual of the Lie algebra of $\mathrm{PSL}_2(\mathbb{R})$.

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1 Motivations

The study of periodic geodesics is a classical part of Riemannian geometry (see [Be-03], chap. 10). There are not many works on closed geodesics on sub-Riemannian manifolds, see however [K-V-19, D-K-P-S-V-18] (for explicit calculations on spheres and Lie groups) and [Sh-21] and references therein for the Heisenberg Kepler problem. Starting from the work [C-H-T-21], it is natural to ask about closed geodesics spiraling around closed orbits of the Reeb flow. A related motivation comes from inverse spectral problems: roughly speaking, is the set of periods of the Reeb flow a spectral invariant of the sub-Riemannian Laplacian? Our main theorem shows that it could be true; indeed in [Me-84], Richard Melrose proved an extension of the Chazarain-Duistermaat-Guillemin wave trace formula [D-G-75] (see also [CdV-07]) for sR contact Laplacian showing that the set of lengths of closed geodesics (called the "lengths spectrum") is, generically, a spectral invariant. One can then hope to recover the Reeb periods from the lengths of closed geodesics or at least to prove a rigidity result.

2 The setup

A nice introduction to sub-Riemannian geometry can be found in the book [Mo-02], see also [C-H-T-18] for what follows. Let us recall a few things: M is a smooth manifold of dimension 3, $D \subset TM$ is a smooth distribution of dimension 2 defined globally by $D = \ker \alpha$ where α is a non vanishing real 1-form, so that $\alpha \wedge d\alpha$ is a non vanishing volume form. It follows that M is oriented and D is orientable. We choose some orientation of D. The metric g is a smooth metric defined on D. We define the so-called co-metric $g^\star : T^\star M \to \mathbb{R}^+$ by

$$g^\star(q,p) = \|p_{|D_q}\|^2_{g(q)}$$

where the norm is the norm on the dual of the Euclidean space D_q. The geodesic flow is the Hamiltonian flow of $\frac{1}{2}g^\star$ restricted to $g^\star = 1$. The projections of the integral curves of the geodesic flow onto M are the geodesics of the sub-Riemannian manifold with speed 1.

To g and the orientation of D is associated a choice of a 1- form defining D as follows: we define α_g so that $\ker \alpha_g = D$ and $d\alpha_g$ restricts to D as the oriented volume defined by g.

To α_g is associated the Reeb vector field $\vec{R}$ on M characterized by

$$\alpha_g(\vec{R}) = 1,\ d\alpha_g(\vec{R}, .) = 0$$

A periodic Reeb orbit is said to be *non degenerate* if the linearized Poincaré map does not admit 1 as an eigenvalue.

The Reeb vector field has the following Hamiltonian interpretation: the cone $\Sigma = \{(q,p) \in T^\star M | p_{|D_q} = 0\}$, generated by α_g, is a symplectic submanifold of $T^\star M$. We define the function $\rho(\alpha) = \alpha/\alpha_g$ on Σ. The function ρ is homogeneous of degree 1 and hence the Hamiltonian vector field $\vec{\rho}$ of ρ is homogeneous of degree 0. Let us denote $\Sigma^+ := \Sigma \cap \{\rho > 0\}$ and by π_Σ the projection of Σ^+ onto M. The projection $(\pi_\Sigma)_\star(\vec{\rho})$ on M is well defined and is the Reeb vector field $\vec{R}$ ([C-H-T-18], sec. 2.4).

Let us denote by π the canonical projection of $T^\star M$ onto M. If Γ is a periodic Reeb orbit, there exist a neighbourhood Ω of Γ and a conical neighbourhood U of $\pi_\Sigma^{-1}(\Omega)$ in $T^\star M \setminus 0$ so that a "Birkhoff normal form" holds in U. This Birkhoff normal form is defined as follows (see Section 5.1.4 of [C-H-T-18]): let us consider the conic symplectic manifold $\Sigma_\sigma \times \mathbb{R}^2_{u,v}$ with the symplectic form

$$\omega_\Sigma + dv \wedge du$$

and the positive dilations $\lambda(\sigma, u, v) = (\lambda\sigma, \sqrt{\lambda}u, \sqrt{\lambda}v)$. There exists an homogeneous symplectic diffeomorphism χ of U onto an open cone $V \subset \Sigma^+ \times \mathbb{R}^2$ so that, $\forall \sigma \in \pi_\Sigma^{-1}(\Omega), \chi(\sigma) = (\sigma, 0)$ and

$$F(\sigma, u, v) := g^\star \circ \chi^{-1}(\sigma, u, v) = \rho I + \rho_2 I^2 + \cdots + O\left(I^2(I/\rho)^\infty\right)$$

where $I = u^2 + v^2$, the ρ_j's are functions homogeneous of degree $2 - j$ on $\pi_\Sigma^{-1}(\Omega)$ and the remainder depends in general of u and v and not only of I and σ. Note that the remainder is natural: it is the expected estimate for a remainder which is flat along Σ and homogeneous of degree 2.

In what follows, we will always study the geodesic flow in the Birkhoff coordinates. Note that there exists some $I_0 > 0$ so that the energy shell $\{F = 1,\ I < I_0\}$ is properly included in the cone V of $\Sigma \times \mathbb{R}^2$. Hence, for I small enough and $q \in \Omega$, we stay in the domain of the Birkhoff normal form.

3 The "integrable" case

In this section, we will assume that the Birkhoff normal form is convergent: it means that $g^\star$ is symplectically equivalent in the cone U to some smooth function $(\sigma, I) \to F(\sigma, I)$ where F is a smooth homogeneous function of degree 2, defined in the cone V. Moreover, F has an asymptotic expansion as before

$$F(., I) = \rho I + \rho_2 I^2 + \cdots$$

but the remainder depends only of σ and I. Clearly, in this case, the function I is a first integral of the flow. Integrable here does not imply Liouville integrability in general, because we have only two integrals of the flow.

Note that we will consider the geodesic flow (identified to the flow of $\frac{1}{2}F$) in the energy shell $\{F = 1,\ I < I_0\}$. For an Hamiltonian H on a symplectic manifold, we denote by $\vec{H}$ the associated Hamiltonian vector field.

3.1 Closed geodesics

Theorem 3.1 *Let us assume that there exists a periodic non degenerate Reeb orbit of period $T_0 > 0$ and that the Birkhoff normal form is convergent. Then, there exists a sequence of periodic geodesics γ_k, $k \geq k_0$, of the sub-*

Riemannian manifold (M, D, g) *accumulating on* Γ *with lengths*

$$l_k = 2\sqrt{\pi k T_0} + \sum_{j=0}^{\infty} a_j k^{-j/2} + O\left(k^{-\infty}\right) \tag{1}$$

Proof.– Let us denote by H_I the Hamiltonian on Σ defined by $\frac{1}{2}F(.,I) = \frac{1}{2}\rho I + O(I^2)$. For the Hamiltonian $H = \frac{1}{2}g^\star$ of the geodesic flow expressed in the "Birkhoff coordinates", we have

$$\frac{1}{2}\vec{F} = \overrightarrow{H_I} + \frac{\partial F}{\partial I}\partial_\theta$$

where

$$(u, v) = (\sqrt{I}\cos\theta, \sqrt{I}\sin\theta). \tag{2}$$

Let us start with the

Lemma 3.1 *Let us consider the map* π_I *which is the restriction of* π_Σ *to* $\{\sigma | F(\sigma, I) = 1\}$. *Then, for* I *small enough,* π_I *is a diffeomorphism over a fixed neighbourhood of* Γ.

Proof.– For I small enough, there exists a smooth function $\lambda_I : \{\rho = 1\} \to \mathbb{R}^+$ so that $F(\lambda_I(\sigma)\sigma, I) = 1$. The function λ_I admits an expansion $\lambda_I = 1/I + O(1)$. We define $\Lambda_I : \{\rho = 1\} \to \{F(.,I) = 1\}$ by $\Lambda_I(\sigma) = \lambda_I(\sigma)\sigma$. The map Λ_I is a diffeomorphism. We can hence consider the map $\pi_I \circ \Lambda_I$ from $\{\rho = 1\}$ on M. We have $\pi_I \circ \Lambda_I(\sigma) = \pi_I(\lambda_I(\sigma)\sigma) = \pi_\Sigma(\sigma)$ which is clearly a diffeomorphism. □

We consider then the geodesic flow. For each value of I, we project it on Σ, i.e. we consider the flow of the Hamiltonian $\frac{1}{2}F(.,I)$ on Σ restricted to $F = 1$. We have the

Lemma 3.2 *The previous flow, projected by* π_I *and with a change of time* $s = 2t/I$, *is a smooth perturbation of the Reeb flow on* M.

Proof.– The flow projected on Σ is $\frac{1}{2}I\vec{\rho} + O(I^2)$. Hence the change of times reduces to $\vec{\rho} + O(I)$. This last vector field projects on M as $\vec{R} + O(I)$. □

From this and the fact that Γ is non degenerate (see Appendix D), we obtain a periodic orbit of the Hamiltonian $F(.,I)$ of period $T(I)$ with

$$T(I) \sim \frac{2T_0}{I} + \sum_{j\geq 0} c_j I^j \tag{3}$$

We have now to close the angular part of the dynamics given by $\partial F/\partial I \partial_\theta$.

$$\theta(T(I)) - \theta(0) = \int_0^{T(I)} \frac{\partial F}{\partial I}(\Gamma_I(t), I)dt$$

The righthandside of this equation admits a full expansion

$$\frac{2T_0}{I^2}\left(1 + \sum_{j=1}^{\infty} b_j I^j\right)$$

which has to be equal to $2k\pi$ in order to close the geodesic. Hence, we get an asymptotic expansion of I in terms of powers of $k^{-\frac{1}{2}}$ which gives the asymptotic of the lengths by inserting into the Equation (3). □

3.2 Poincaré section

Let us now describe a *Poincaré section* S_0 and the corresponding *Poincaré map* P_0 for the geodesic flow assuming for simplicity that $F = \rho I$. For the definitions and properties of the Poincaré maps, see Appendix D.

Let X_R be a Poincaré section of the Reeb flow in the energy shell $\rho = R$ so that all X_R's project on a fixed Poincaré section of the periodic orbit of the Reeb flow in M. Let us use polar coordinates given by Equation (2) identifying $\mathbb{R}^2_{u,v} \setminus 0$ to $T^\star_{\theta,I}(\mathbb{R}/2\pi\mathbb{Z})$. The latter manifold will be denoted by $T^\star$ in what follows. The manifold

$$S_0 := \{(\sigma, \theta, I) | \sigma \in X_R,\ RI = 1,\ I < I_0\} \tag{4}$$

is a Poincaré section of the geodesic flow. But the Reeb flow projects onto M; we can hence identify any Poincaré section X_R with a Poincaré section of the Reeb flow in M denoted by X, which can be assumed to be independent of R. The Poincaré S_0 section is then parametrized by $\mathbf{S}_0$ given by $\mathbf{S}_0 = X_q \times T^\star_{\theta,I}$. The Poincaré map is given in these coordinates by

$$P_0(q, \theta, I) = (\Pi(q), \theta + 2T(q)/I^2, I)$$

where $T(q)$ is the return time of the Reeb flow when starting from q and Π is the Poincaré map of the Reeb orbit. Note that the cylinder $C_0 := \{q_0\} \times T^\star$ is invariant by P_0 as well as the symplectic form $d\theta \wedge dI$ on it. Note also that the latter symplectic form is the restriction of the symplectic form on $\Sigma \times T^\star$ to C_0. We will show latter that the Poincaré map is weakly perturbed when I is small in the non integrable case.

4 Summary of the proof

In what follows, we will assume, for technical simplicity, that the co-metric admits the simple normal form $g^\star = \rho I + O\left(I^2(I/\rho)^\infty\right)$ in some conical neighbourhood of $\pi^{-1}(\Gamma) \cap \Sigma^+$. The general case would be $g^\star = F(\sigma, I) + O\left(I^2(I/\rho)^\infty\right)$ with F a sum of the BNF as in the previous section. The same strategy will work in the latter case. So we will see the flow of $\frac{1}{2}g^\star$ as a perturbation of the flow of $\frac{1}{2}\rho I$.

The scheme of the proof is as follows.

1. Using the structural stability of the non degenerate closed Reeb orbit, we get a symplectic cylinder C of dimension 2 close to C_0 invariant by the Poincaré map.
2. We show the existence of invariant circles c_k inside that cylinder.
3. We apply the Poincaré-Birkhoff fixed point theorem to the annuli between c_{k-1} and c_{k+1}.

5 The invariant cylinder C

Let us first build a Poincaré section for the geodesic flow. Let $S_0 \subset \Sigma \times T^\star$ be the Poincaré section for $\frac{1}{2}\rho I$ given in Equation (4). We choose a Poincaré section which is a dilation of S_0 (respecting the fibers) of the form λS_0 with $\lambda : \Sigma \times T^\star \to \mathbb{R}^+$ a germ along $\pi^{-1}(\Gamma)$. We have to satisfy $g^\star = 1$ with $g^\star = \rho I + O((\lambda I)^\infty)$ and $\rho I = 1$ on S_0 , this gives

$$\lambda^2 + O((\lambda I)^\infty) = 1$$

and hence $\lambda = 1 + O(I^\infty)$. This is clearly a Poincaré section because it is still transversal to the flow. Using the projection onto M, we can still identify this Poincaré section with $\mathbf{S}_0$. We need the

Lemma 5.1 *On the energy shell $F = 1$ and for times $O(1/I)$ the geodesic flow differs from the unperturbed one by $O(I(0)^\infty)$.*

Proof.– We put $I_0 := I(0)$. First, from $dI/dt = O(I^\infty)$, we get $I(t) = I_0 + O(I_0^\infty)$ for times $t = O(1/I)$. We get $\rho = 1/I_0 + O(I_0^\infty)$. The Lemma follows then by looking at

$$\frac{1}{2}\overrightarrow{g^\star} = \frac{I}{2}\vec{\rho} + \rho\partial_\theta + O(I^\infty)$$

where we can replace I by I_0 and ρ by $1/I_0$ modulo $O(I_0^\infty)$. We use then Equation (6) in Appendix C. Recall that the flow of $\frac{1}{2}\rho\vec{I}$ is given by $G_t^0(\sigma,\theta,I) = \left(R_{It/2}(\sigma), \theta + \frac{t}{I}, I\right)$. The differential of that flow is hence

$$DG_t^0(\sigma,\theta,I) = \begin{pmatrix} DR_{It/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{t}{I^2} & 1 \end{pmatrix}$$

We get the following estimate

$$\| \left(DG_t^0(\sigma,\theta,I)\right)^{-1} \| \leq \| \left(DR_{It/2}(\sigma)\right)^{-1} \| + |t/I^2|$$

Hence, for times $O(1/I)$, we have

$$\| \left(DG_t^0(\sigma,\theta,I)\right)^{-1} \| = O\left(I^{-3}\right)$$

Using Equation (6) in Appendix C and the notations there, we get $\| \frac{d}{dt}"w(t,x)" \| = O\left(I_0^\infty\right)$ for times $O(1/I_0)$ and $w(t,x) = x + O\left(I_0^\infty\right)$. The result follows. $\square$

We get

Proposition 5.1 *For $a > 0$, let us denote $T_a^\star := T^\star \cap \{|I| < a\}$. There exists $a > 0$ and smooth functions $q : T_a^\star \to X$ and $T : T_a^\star \to \mathbb{R}$ so that*

1. $$q(\theta, I) = q_0 + O\left(I^\infty\right)$$

2. $$T(\theta, I) = \frac{2T_0}{I} + O\left(I^\infty\right)$$

3. *The flow G_t of $\frac{1}{2}g^\star$ restricted to $g^\star = 1$ satisfies*

$$G_{T(\theta,I)}(q(\theta,I),\theta,I) = \left(q(\theta,I), \theta + \frac{2T_0}{I^2} + O\left(I^\infty\right), I + O\left(I^\infty\right)\right)$$

Hence the cylinder

$$C := \{(q(\theta,I),\theta,I) | (\theta,I) \in T^\star\}$$

is invariant by $G_{T(\theta,I)}$ which is the restriction of the Poincaré map to C. Moreover the symplectic form restricts to C as a symplectic form $dI \wedge d\theta + O\left(I^\infty\right)$.

Proof.– Let us consider the return map to X of the q-component for (θ, I) fixed with I small. It follows from Lemma 5.1 that the return map is $O\left(I^{\infty}\right)$ close to the unperturbed Poincaré map of the Reeb flow with a return time $O(1/I)$. The conclusion follows then from the non degeneracy of the Reeb orbit. □

6 The invariant circles c_k

Applying Theorem A.1 to the restriction P_C of P to the cylinder C and the circles $c_k^0 := \{(\theta, I) | I = \sqrt{T_0/k\pi}\}$, we will get circles c_k globally, but not pointwise, invariant by the map P_C and close to c_k^0.

More precisely, the restriction P_C of the Poincaré map to the cylinder C writes

$$P_C(\theta, I) = (\theta + 2T_0/I^2, I) + O\left(I^{\infty}\right)$$

using the coordinates (θ, I) in C. Putting $J = 1/I^2$ and $\theta' = \theta + 2k\pi$, we get the map:

$$S(\theta', J) = (\theta' + (2T_0 J - 2k\pi), J) + O\left(J^{-\infty}\right)$$

Near $J = k\pi/T_0$, we get

$$S(\theta', J) = (\theta' + (2T_0 J - 2k\pi), J) + O\left(k^{-\infty}\right) \tag{5}$$

This map is a perturbation of the map $(\theta', J) \to (\theta' + (2T_0 J - 2k\pi), J)$ to which we can apply Appendix A with $X = T^{\star}$ and $Y_0 = \{J = k\pi/T_0\}$ and get the curves c_k which are $O\left(I^{\infty}\right)$ close to c_k^0.

7 Applying the Poincaré-Birkhoff theorem

We apply the Poincaré-Birkhoff Theorem B.1. to the annuli between c_{k-1} and c_{k+1} and the lift to the universal cover of the annulus moving the lift C_{k+1} of c_{k+1} by a map close to $\theta \to \theta + 2\pi$ and the lift C_{k-1} of c_{k-1} by a map close to $\theta \to \theta - 2\pi$. This way we get a fixed point in the annulus for the lift. This point is $O\left(I^{\infty}\right)$ close to c_k^0, because the other points are moved at some speed like I^m for some $m \geq 0$: it is because of the estimate (5) of the map S. This finishes the proof of Theorem 0.1.

8 Periodic geodesics on $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$

The goal of this second part, which is quite independent of the previous ones, is to describe the periodic geodesics of a specific right invariant sub-Riemannian contact structure on compact quotients

$$\Gamma\backslash \mathrm{PSL}_2(\mathbb{R}) := \{\Gamma.g | g \in \mathrm{PSL}_2(\mathbb{R})\}$$

We assume for simplicity that Γ has no elliptic elements, so that all elements of $\Gamma \setminus \mathrm{Id}$ are hyperbolic. Because we took the quotient $\mathrm{PSL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ by $\pm\mathrm{Id}$, we can represent any hyperbolic transform by a matrix with eigenvalues $\lambda,\ 1/\lambda$ with $\lambda > 1$.

Our analysis can be extended to some Riemannian case (see [Sa-98]) or to another sub-Riemannian structure like the magnetic one (see [Ch-20]), or even to any right invariant Hamiltonian. Note also that the Quantum version of this study, namely the spectral theory of the associated sub-Riemannian Laplacian, follows some parallel path (see [C-H-W-22?]).

9 Lie-Poisson bracket

Let G be a Lie group. We identify its Lie algebra $\mathcal{G}$ to the space of right invariant vector fields equipped with the bracket of vector fields. We consider also the algebra of right invariant differential operators, called the envelopping algebra, generated by the Lie algebra. The principal symbols of these right invariant operators are determined by their values on the dual $\mathcal{G}^\star = T^\star_{\mathrm{Id}}G$ of the Lie algebra. This gives a Poisson bracket on $\mathcal{G}^\star$. We need only to compute it for coordinates functions: if $X \in \mathcal{G}$, the symbol of X at $g = \mathrm{Id}$ is $\sigma(X)(p) = ip(X)$. If $X, Y \in \mathcal{G}$, the symbol of $[X, Y] = Z$ is $-i\{\sigma(X), \sigma(Y)\}$. Hence the formula

$$\{a, b\}(p) = -[da(p), db(p)](p)$$

can be checked for operators of the Lie algebra: if $p \in \mathcal{G}^\star$,

$$\sigma([X, Y])(p) = -i\{\sigma(X), \sigma(Y)\}(p) = ip([X, Y]|)$$

or, if the functions ξ and η on $\mathcal{G}^\star$ are defined by $\xi(p) = p(X),\ \eta(p) = p(Y)$,

$$\{\xi, \eta\}(p) = -p([X, Y])$$

This bracket is called the *Lie-Poisson bracket*. The associated Hamiltonian dynamics are given by *Euler equations*.

This gives the *Euler equations* using for a in the previous equation a coordinates system on $\mathcal{G}^\star$. All of this is explained in the book [M-R-98], Sec. 13.1.

Let us compute the Poisson bracket in the case of $G = \mathrm{PSL}_2(\mathbb{R})$. The Lie algebra is the 3D space of trace free real 2×2 matrices

$$M(x, y, z) := \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$$

We write $M = xX + yY + zZ$. We have

$$[X, Y] = -Z,\ [X, Z] = 2X,\ [Y, Z] = -2Y$$

There is a minus sign w.r. to the matrix bracket because of the right invariance!

Hence, if (ξ, η, ζ) are the coordinates on $\mathcal{G}^\star$ dual to (x, y, z), we have

$$\{\xi, \eta\} = \zeta,\ \{\xi, \zeta\} = -2\xi,\ \{\eta, \zeta\} = 2\eta$$

The symplectic leaves of the Poisson bracket are the connected components of the level sets of the *Casimir Hamiltonian* defined by $\mathrm{Cas}(\xi, \eta, \zeta) := \frac{1}{2}\zeta^2 + 2\xi\eta$. The Hamiltonian Cas Poisson commutes with all functions on $\mathcal{G}^\star$.

We consider the right invariant contact distribution generated by X and Y and the sub-Riemannian metric g for which (X, Y) is an orthonormal basis. The right invariant Hamiltonian of the geodesic flow is $\frac{1}{2}g^\star$ with $g^\star = \xi^2 + \eta^2$.

10 Invariant 2-tori, the frequency ω and the period $\tau(C)$

The Hamiltonian $g^\star$ Poisson commutes with Cas. Both flows are complete because the momentum map $(\mathrm{Cas}, g^\star) : T^\star M \to \mathbb{R}^2$ is proper. We get hence an $\mathbb{R}^2$-action Φ on $M \times \mathcal{G}^\star \equiv T^\star M$ preserving the values of the momentum map. We will see that for most values of C, there exists orbits of Φ which are tori $\mathbb{T}_C$ on which the action Φ is linear and such that the orbits of $\overrightarrow{\mathrm{Cas}}$ are periodic of period $T^{\mathrm{Cas}}(C)$. The geodesic dynamics on $\mathbb{T}_C$ induces a Poincaré map on an orbit of the Hamiltonian vector field $\overrightarrow{\mathrm{Cas}}$ of Cas. This

map is a translation of $t \to t + \omega(C)/T^{\mathrm{Cas}}(C)$ of the choosen Casimir orbit. For $\omega = p/q \bmod \mathbb{Z}$ with $p, q,\ q \geq 1$, which are coprime, we get periodic geodesics. These geodesics are crossing q times the Casimir periodic orbit.

For $C \in]-1, +\infty[\backslash\{0, 1\}$, the geodesic Hamiltonian $\frac{1}{2}g^\star$ admits two periodic orbits on the symplectic leaf $\mathrm{Cas}^{-1}(C)$. These orbits are supported by the two connected components of $(g^\star)^{-1}(1) \cap \mathrm{Cas}^{-1}(C)$. We denote by $\tau(C) > 0$ the period of these orbits.

Note also that the antipodal map $p \to -p$ changes the orientation of both dynamics.

11 Main result

To each closed sub-Riemannian geodesic c of period $T > 0$, we associate the following invariants:

- The *free homotopy class* of the projection $\pi(c)$ onto $\Gamma\backslash\mathbb{H}$ which is given by a conjugacy class $[\gamma]$ in Γ. We denote it by $F_{\mathbb{H}}(c)$.
- The *spiraling integer* of any horizontal smooth closed curve in M of period T is defined as follows: the right invariant distribution D spanned by X, Y is a trivialized bundle. We look at the map $\theta : \mathbb{R}/T\mathbb{Z} \to \{\xi^2+\eta^2 = 1\}$ which associates to t the vector (ξ, η) so that $\xi(t)X(c(t))+\eta(t)Y(c(t))$ is the tangent vector to the curve at the point of parameter t. The spiraling integer is the degree of that mapping. Note that this integer is invariant by C^1 homotopies of horizontal periodic curves. We denote it by $\mathrm{Sp}(c)$.
- The *Morse index* $\mu(c)$ which is the Morse index of the energy functional restricted to horizontal curves.
- The *free homotopy class* of c as a conjugacy class in $\pi_1(M)$ denoted by $F_M(c)$.

We want to prove the following

Theorem 11.1 *For each value of $C_0 \in [-1, +\infty[\backslash\{0\}$ there exists closed sub-Riemannian geodesics whose momentum satisfies $\mathrm{Cas}(p) = C_0$, $g^\star = 1$. More precisely:*

- *If $C_0 > 1$, for each primitive hyperbolic $\gamma \in \Gamma$, there exists a 2D-torus $\mathbb{T}^{C_o}_\gamma$ invariant by the geodesic flow. The dynamics of the geodesic flow on $\mathbb{T}^{C_o}_\gamma$ is linear and periodic for the dense set of values of C_0 for which $\omega(C_0) = p/q$ is rational. For such closed geodesics, we have $\mathrm{Sp}(c) = q$ and the length $l(c) = q\tau(C_0)$.*

- *If $C_0 = 1$, for each primitive hyperbolic $\gamma \in \Gamma$, there exists one periodic geodesic of length $\sqrt{2}\log\lambda$ with $F_{\mathbb{H}}(c) = [\gamma]$ and $\mathrm{Sp}(c) = 0$.*

- *If $0 < C_0 < 1$, for each primitive hyperbolic $\gamma \in \Gamma$, there exists a 2D-torus $\mathbb{T}^{C_o}_\gamma$ invariant by the geodesic flow. The dynamics of the geodesic flow on $\mathbb{T}^{C_o}_\gamma$ is linear and periodic for a dense set of values of C_0 with $\omega(C_0) = p/q$. For such closed geodesics, we have $\mathrm{Sp}(c) = 0$ and the length $l(c) = q\tau(C_0)$.*

- *If $-1 < C_0 < 0$, there exists a 2D-torus $\mathbb{T}^{C_o}_{\mathrm{Id}}$ invariant by the geodesic flow. The dynamics of the geodesic flow on $\mathbb{T}^{C_0}_{\mathrm{Id}}$ is linear and periodic for a dense set of values of C_0 for which $\omega(C_0) = p/q$ is rational. We have $F_{\mathbb{H}}(c) = \{\mathrm{Id}\}$, $\mathrm{Sp}(c) = 0$ and the length $l(c) = q\tau(C_0)$.*

- *If $C_0 = -1$, there exists one periodic orbit of length $2\sqrt{2}\pi$ with $F_{\mathbb{H}}(c) = \{\mathrm{Id}\}$ and $\mathrm{Sp}(c) = 0$.*

It is not clear for us how to compute the other invariants...

12 Casimir periodic orbits

Recall that Γ is a co-compact lattice with no elliptic elements in G and $M = \Gamma\backslash G$. The compact 3-manifold M can be identified with the unit cotangent bundle of the compact Riemann surface $\Gamma\backslash\mathbb{H}$ where $\mathbb{H}$ is the Poincaré half-plane (see Appendix E). We consider the Hamiltonian Cas on $T^\star M \equiv M \times \mathcal{G}^\star$. We have

$$\overrightarrow{\mathrm{Cas}} = \zeta\vec{\zeta} + 2\xi\vec{\eta} + 2\eta\vec{\xi}$$

On the other hand (ξ, η, ζ) are first integrals, hence for $(\Gamma.g, p) \in M \times \mathcal{G}^\star$ with $p = (\xi, \eta, \zeta)$, if Cas_t is the Casimir flow, we have

$$\mathrm{Cas}_t(\Gamma.g, p) = (\Gamma.ge^{tA(p)}, p)$$

with

$$A(p) := \zeta Z + 2\xi Y + 2\eta X$$

and $\mathrm{Cas} = -\frac{1}{2}\det(A(p))$.

We get a periodic orbit $t \to (\Gamma.ge^{tA(p)}, p)$ of period T if and only if $\Gamma.ge^{TA(p)} = \Gamma.g$, i.e.

$$ge^{TA(p)}g^{-1} = \gamma$$

for some $\gamma \in \Gamma$ and $g \in G$. That periodic orbit is the left translate of the invariant line of γ by g^{-1}.

- If γ is hyperbolic with eigenvalues λ and $1/\lambda$ with $\lambda > 1$, we need to have that the eigenvalues of $A(p)$ are real and non zero, hence $C > 0$ and then

  $$e^{T\sqrt{2C}} = \lambda$$

  or

  $$T^{\mathrm{Cas}}(C) = \frac{1}{\sqrt{2C}} \log \lambda$$

  Note that this orbit is simply a translate of a lift of a periodic geodesic of $\Gamma\backslash\mathbb{H}$.

- If $-1 \leq C < 0$, then $e^{TA(p)}$ is a rotation of angle $T\sqrt{-2C}$ and hence periodic of period $T^{\mathrm{Cas}}(C) = 2\pi/\sqrt{-2C}$, independently of Γ. This orbit is homotopic to the the compact subgroup $\mathrm{SO}_2(\mathbb{R})$ of G.

- If $C = 0$, we get no periodic orbits.

13 The geodesic flow and periodic geodesics

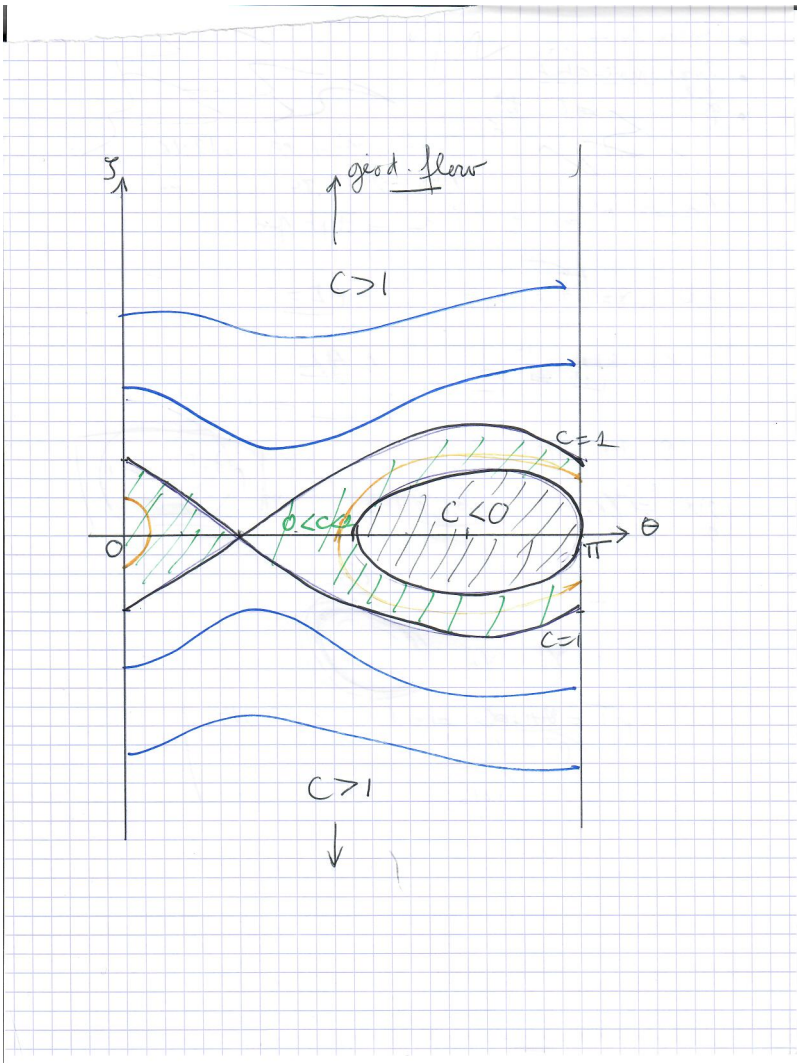
The sub-Riemannian geodesic flow G_t, which is the Hamiltonian flow of $\frac{1}{2}g^\star$, acts on the set of Casimir periodic orbits of fixed period: if $\mathrm{Cas}_T(z) = z$, we have $G_t(\mathrm{Cas}_T(z)) = \mathrm{Cas}_T(G_t(z))$, hence $\mathrm{Cas}_T(G_t(z)) = G_t(z)$. We consider the sub-Riemannian geodesic flow G_t in the energy shell $\overrightarrow{g^\star} = 1$ giving geodesics with speed 1. Recall that the periodic orbits of $\overrightarrow{\mathrm{Cas}}$ are parametrized by the value of $p(0)$. We can hence look at the Poisson action of $\frac{1}{2}g^\star$ on $\mathcal{G}^\star$ which preserves the symplectic leaves. For each values C_0 of Cas, this Poisson action is an Hamiltonian action on the 2D symplectic manifold $\mathrm{Cas} = C_0$ with Hamiltonian $\frac{1}{2}(\xi^2 + \eta^2)$. We parametrize the cylinder $\{\xi^2 +$

$\eta^2 = 1\} \subset \mathcal{G}^\star$ by $(\xi = \cos\theta,\ \eta = \sin\theta, \zeta)$ with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\zeta \in \mathbb{R}$. We get $\mathrm{Cas} = \frac{1}{2}\zeta^2 + \sin 2\theta$.

Each time this Poisson action on $\mathcal{G}^\star$ is periodic, the orbit of the $\mathbb{R}^2$-action Φ is a torus $\mathbb{T}_C$ on which Φ acts linearly. To each such a torus, we associate a rotation number $C \to \omega(C) \in \mathbb{R}/\mathbb{Z}$ as follows: a periodic orbit of the Casimir flow of period T is a Poincaré section of the geodesic flow G_t on that torus. The frequency ω is the rotation number of the Poincaré map which is conjugated to a rotation. If $\omega = p/q$ is rational with , we get a 1-parameter family of closed geodesics. We will sometimes consider a lift $\tilde{\omega}$ of ω to $\mathbb{R}$. We will in particular study below the map $\tilde{\omega} :]-1, +\infty[\backslash\{0, 1\} \to \mathbb{R}$. The lengths of these periodic geodesics is $q\tau(C)$ because they have to spiral q times before closing.

13.1 Level sets of Cas on $\xi^2 + \eta^2 = 1$

The geodesic flow preserves the level sets of Cas restricted to the unit bundle. We have hence to look at the different types of level lines. Due to right invariance, we have only to look at the restriction of these sets to $Y := \mathcal{G}^\star \cap \{g^\star = 1\}$ which is a 2D cylinder. The critical values of Cas restricted to Y are the global minimum -1 and a saddle point $+1$. We have also to look at the special value $\mathrm{Cas} = 0$ separating the elliptic from the hyperbolic case for the Casimir flow.



We have hence many cases to consider which are the classical versions of the irreducible representations of $\mathrm{PSL}_2(\mathbb{R})$ (see [Tay-86], chap. 8).

13.1.1 The case $C_0 > 1$: "principal series"

The action of $\frac{1}{2}g^\star$ on the Poisson manifold $\mathcal{G}^\star$ restricted to $g^\star = 1$ is periodic. This implies that the corresponding orbits of the $\mathbb{R}^2$-action is a 2-torus. We get hence periodic geodesics each times ω is rational.

13.1.2 The case where $C_0 = 1$

In this case, the orbits of Φ are cylinders and circles. The circles corresponds to the critical points of Cas with critical value 1. For the critical point $\zeta = 0, \xi = \eta = 1/\sqrt{2}$, we have periodic geodesics parametrized by the hyperbolic conjugacy classes of Γ with lengths the lengths of the corresponding Riemannian periodic geodesics of $\Gamma\backslash\mathbb{H}$.

13.1.3 The case where $0 < C_0 < 1$: "complementary series"

Again we get tori as orbits of the $\mathbb{R}^2$-action and periodic geodesics each times ω is rational.

13.1.4 The case where $C_0 = 0$

No periodic orbit for the Casimir flow which is parabolic.

13.1.5 The case where $-1 < C_0 < 0$: "discrete series"

Again we get tori as orbits of the $\mathbb{R}^2$-action and periodic geodesics each times ω is rational.

13.1.6 The case where $C_0 = -1$

For the critical point $\zeta = 0, \xi = -\eta = 1/\sqrt{2}$, we get periodic geodesics which are the orbits of the SO_2 action, i.e. the fibers of $M \to \Gamma\backslash\mathbb{H}$.

14 The frequency map ω

If $\omega(C)$ is rational, we get a family of periodic geodesics. We will study the function $\tilde{\omega}$ and show that it is rational for a dense set of values of C. We have to study first two other functions of C the period T^{Cas} of the Casimir flow and the period T^{geod} of the action of the geodesic flow on the periodic orbits of the Casimir flow.

14.1 The function T^{Cas}

We know the that $T^{\mathrm{Cas}}(C) = l_\gamma/\sqrt{2C}$ for $C > 0$ and $T^{\mathrm{Cas}}(C) = \sqrt{2}\pi/\sqrt{-C}$ for $C < 0$.

14.2 The function T^{geod}

The function T^{geod} is smooth and non vanishing on $]-1,+\infty[\backslash\{1\}$. For $C \to +\infty$ is $T^{\text{geod}} \sim 1/\sqrt{C}$, the limits at $+1$ are $+\infty$, the limit at -1 is some > 0 number.

14.3 The function ω

Recall that the function $\tilde{\omega}$ is a lift to $\mathbb{R}$ of the rotation number of the Poincaré map induced on the periodic Casimir orbit by the geodesic flow. If $\omega(C)$ is rational, we get a full torus of periodic geodesics. In order to understand ω, we look at the projection of the tori on M. The projected Hamiltonian vector fields are respectively $V^{\text{Cas}} = 2\eta X + 2\xi Y + \zeta Z$ and $V^{\text{geod}} = \xi X + \eta Y$. Both are independent outside the critical points $\zeta = 0, \xi = \pm\eta = \pm 1/\sqrt{2}$. We consider the angle α between the two vectors (in the Riemannian metric on M whose (X, Y, Z) is an orthonormal basis). We have

$$|\cos\alpha| = \frac{|\xi\eta|}{\sqrt{1+\zeta^2/4}}$$

Hence, we get the following behaviour of ω:

- As $C \to +\infty$, both periods are of the same order $1/\sqrt{C}$ and the angle tends to $\pi/2$, hence we can choose the lift so that $\tilde{\omega} \to 0$.
- As $C \to 1^{\pm}$, the T^{Cas} tends to a finite limit, while $T^{\text{geod}} \to \infty$ and the angle α tends to 0 for most of the time along the closed orbit of the geodesic. Hence $\tilde{\omega} \to \infty$. .
- As $C \to 0$, T^{Cas} tends to ∞ while T^{geod} is smooth. The angle α is bounded below. Hence we can choose $\tilde{\omega} \to 0$.
- As $C \to -1$, T^{Cas} tends to $\sqrt{2}\pi$ while T^{geod} is smooth. The angle α tends to 0, hence $\tilde{\omega} \to \infty$.

From all this, we see that the function $\tilde{\omega}$ which is analytic is non constant on any interval, hence a dense set of values of C for which we get periodic geodesics.

15 Appendices

A Manifolds of fixed points

Our goal is to prove the following

Theorem A.1 *Let Y_0 a compact submanifold of a manifold X and $F_0 : X \to X$ a smooth map satisfying, $\forall y \in Y_0$, $F_0(y) = y$ and $\ker(F_0'(y) - \mathrm{Id}) = T_y Y_0$. and consider a smooth family of maps F_ε. Then, for any m, there exists $\varepsilon(m) > 0$ so that, for $|\varepsilon| < \varepsilon(m)$, there exists a C^m manifold Y_ε depending smoothly of ε globally invariant by F_ε.*

In our paper this will be used with Y_0 a circle embedded into a cylinder.

For the proof, we use the simplifying assumption that the normal bundle $TX_{|Y_0}/TY_0$ is trivial. We can then reduce to the case where $X = (Y_0)_y \times \mathbb{R}^n_z$.

We will search Y_ε as a graph of a C^m map f_ε from Y_0 into $\mathbb{R}^n$. The invariance by $F_\varepsilon(y,z) = (A_\varepsilon(y,z), B_\varepsilon(y,z))$ writes

$$B_\varepsilon(y, f_\varepsilon(y)) = f_\varepsilon(A_\varepsilon(y, f_\varepsilon(y)))$$

Differentiating with respect to ε at $\varepsilon = 0$ gives

$$\forall y \in Y_0, (\mathrm{Id} - B_0'(y))\delta f(y) = \delta B(y, 0)$$

where the δ's are the derivatives w.r. to ε and $B_0'(y)$ is the derivative of B_0 with respect to z at the point $(y, 0)$. Note that there is no derivatives of A_ε appearing because $f_0 = 0$. Hence the derivative of the righhandside w.r. to ε is δf. This can be solved with $\delta f \in C^m(Y, \mathbb{R}^n)$ by the assumption of the Theorem. Hence we can apply implicit function Theorem in the Banach space $C^m(Y, \mathbb{R}^n)$ and conclude.

B Poincaré-Birkhoff for twist maps

The goal is to give a simple proof of the Poincaré-Birkhoff theorem for twist map of the annulus (see [Go-01]). Let $A = (\mathbb{R}/\mathbb{Z})_x \times [a, b]_y$ be an annulus equipped with some area form. A smooth map $F = (X, Y) : A \to A$ preserving the boundaries of A is called a *twist map* if

1. $\partial X/\partial y > 0$

2. F is area preserving

3. There exists a choice $\tilde{F} = (\tilde{X}, \tilde{Y})$ of a lift of F to $\mathbb{R} \times [a, b]$ so that, for all $x \in \mathbb{R}$, $\tilde{X}(x, a) < x$ and $\tilde{X}(x, b) > x$.

Then

Theorem B.1 *If F is a twist map, it admits a fixed point.*

The proof is as follows: for each $x \in \mathbb{R}/\mathbb{Z}$, the twist conditions (1) and (3) implies that there exists a unique $y(x)$ so that $X(x, y(x)) = x$. Moreover $x \to y(x)$ is smooth. Let us consider the curves which are the graphs of $x \to y(x)$ and $x \to Y(x, y(x))$. The second curve is the image of the first one by F. They have to cross because F is area preserving (2), otherwise the image of the domain below the first curve will have an area smaller or larger than the area of that domain. Any intersection point is a fix point of F.

C Perturbation of flows

Let us consider a vector field of the form $\vec{V} = \vec{V_0} + \vec{R}$ and let ϕ_t^0 (resp. ϕ_t) the flows of $\overrightarrow{V_0}$ (resp. $\vec{V}$). We want to compare both flows. For that, we write $\phi_t(x)$ as $\phi_t(x) = \phi_t^0(w(t, x))$, with $w(0, x) = x$, and will write a differential equation for w. We get the two equations for the integral curve $t \to y(t) = \phi_t(x)$:

$$\frac{d}{dt} y(t) = \overrightarrow{V_0}(\phi_t^0(w(t, x))) + \vec{R}(\phi_t^0(w(t, x)))$$

and

$$\frac{d}{dt} y(t) = D\phi_t^0(w(t, x)) \frac{d}{dt} w(t, x) + \overrightarrow{V_0}(\phi_t^0(w(t, x)))$$

By identification of both equations, we get

$$D\phi_t^0(w(t, x)) \frac{d}{dt} w(t, x) = \vec{R}(\phi_t^0(w(t, x)))$$

and finally

$$\frac{d}{dt} w(t, x) = \left(D\phi_t^0(w(t, x))\right)^{-1} \vec{R}(\phi_t^0(w(t, x))) \tag{6}$$

This imply that, if we have a weak control of w and moreover we know ϕ_t^0 and the inverse of its differential, we get the closeness of both flows.

D Poincaré maps

Let γ be a periodic orbit of a vector field $\vec{V}$ in a manifold M. We choose a germ of hypersurface S transverse to γ at some point x_0. Then we can define a return map F along γ which is a germ of map from (S, x_0) into itself. That germ is independent of the choices of x_0 and S up to conjugation by a germ of diffeomorphism. The linearisation L of F at x_0 is hence also well defined up to conjugation. The orbit γ is said to be non degenerate if 1 is not an eigenvalue of L. In this case, we have the

Proposition D.1 *If γ is non degenerate and if $\vec{V}'$ is close enough to $\vec{V}$, there is a close orbit γ' of $\vec{V}'$ which depends smoothly of $\vec{V}'$.*

The non degeneracy condition allows to apply the inverse function theorem to the perturbation of the return map.

In case of an Hamiltonian vector field, the manifold M to be considered is not the full phase space, but the energy shell containing γ. The germ S is then symplectic and F is a symplectic germ of diffeomorphism.

The Hamiltonian $g^\star$ is $g^\star = \xi^2 + \eta^2$. Let us show that $g^\star$ is "integrable" in the sense of Section 3

E Geodesic flow on hyperbolic surfaces and Poincaré group of $M := \Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$

For this section, one can look at [Bu-92]. We consider, for $\Gamma \subset G = \mathrm{PSL}_2(\mathbb{R})$ a co-compact lattice with no elliptic elements, the compact smooth oriented hyperbolic surface $N := \Gamma\backslash\mathbb{H}$, where G acts on $\mathbb{H}$ by $g(z) = (az+b)/(cz+d)$ with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The group G acts also on the unit tangent bundle of N obtained by taking the derivative of the previous action. The action is transitive with a trivial isotropy group. The map $g \to Dg(v_0)$ where $v_0 = (i, \partial_y)$ identifies $\Gamma\backslash G$ to the unit tangent bundle of N. The 1-parameter group $\exp(tZ)$ acting on the right on G identifies then with the geodesic flow with speed 2. For each hyperbolic element $\gamma \in \Gamma$, there exists an unique periodic geodesic c of length $l_\gamma = \log \lambda$, whose lift to $\mathbb{H}$ is invariant by γ. We have $g\exp(\frac{1}{2} l_\gamma Z) = \gamma g$. If c is given, γ is determined up to conjugation in Γ.

Given a group H, we denote by $C(H)$ the set of conjugacy classes of H. We have the following exact homotopy sequence associated to the fibration $\Gamma \to \mathrm{PSL}_2(\mathbb{R}) \to \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$:

$$\{1\} \to \mathbb{Z} \to \pi_1\left(\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})\right) \to \Gamma \to \{1\}$$

The first arrow is an injective morphism whose image is the center of $K := \pi_1\left(\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})\right)$. The second arrow is surjective so that K is an extension of Γ by $\mathbb{Z}$. Concerning conjugacy classes in K, there is an action of $\mathbb{Z}$ on $C(K)$ whose orbits are the fibers of the projection $C(K) \to C(\ \Gamma)$. In fact there is a canonical parametrization of $C(K)$ by $C(\Gamma) \times \mathbb{Z}$ as follows: any element of $C(\Gamma)$ can be represented by a closed geodesic γ of $\Gamma\backslash\mathbb{H}$ which can be lifted as a perodic curve in M by looking at the canonical lift of γ to the unit tangent bundle. Then we look at the action of $\mathbb{Z}$ by composing this loop with a loop consisting in rotating the unit vector at a fixed point of $\Gamma\backslash\mathbb{H}$ of 2π.

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