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OPTIMAL CONTROL OF MIXED LOCAL-NONLOCAL PARABOLIC PDE WITH SINGULAR BOUNDARY-EXTERIOR DATA

JEAN-DANIEL DJIDA, GISELE MOPHOU, AND MAHAMADI WARMA

ABSTRACT. We consider parabolic equations on bounded smooth open sets $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) with mixed Dirichlet type boundary-exterior conditions associated with the elliptic operator $\mathcal{L} := -\Delta + (-\Delta)^s$ ($0 < s < 1$). Firstly, we prove several well-posedness and regularity results of the associated elliptic and parabolic problems with smooth, and then with singular boundary-exterior data. Secondly, we show the existence of optimal solutions of associated optimal control problems, and we characterize the optimality conditions. This is the first time that such topics have been presented and studied in a unified fashion for mixed local-nonlocal PDEs with singular data.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with a smooth boundary $\partial\Omega$. We consider the minimization problem:

$$\min_{(u_1, u_2) \in \mathcal{Z}_D} J(\psi(u_1, u_2)), \quad (1.1a)$$

subject to the constraints that the state $\psi := \psi(u_1, u_2)$ solves the following initial-boundary-exterior value problem:

$$\begin{cases} \psi_t + \mathcal{L}\psi = 0 & \text{in } Q := \Omega \times (0, T), \\ \psi = u_1 & \text{on } \Gamma := \partial\Omega \times (0, T), \\ \psi = u_2 & \text{in } \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \psi(\cdot, 0) = 0, & \text{in } \Omega. \end{cases} \quad (1.1b)$$

Here, the operator \mathcal{L} is given by

$$\mathcal{L} := -\Delta + (-\Delta)^s, \quad 0 < s < 1, \quad (1.1c)$$

the functional $J : \mathcal{Z}_D \rightarrow [0, \infty]$ is weakly lower-semicontinuous (we shall give the precise expression of J later), the control $(u_1, u_2) \in \mathcal{Z}_{ad}$ with $\mathcal{Z}_{ad} \subset \mathcal{Z}_D$ being a closed and convex subset, where

$$\mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma). \quad (1.1d)$$

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In (1.1c), Δ is the classical Laplacian and $(-\Delta)^s$ ($0 < s < 1$) denotes the fractional Laplace operator given formally by the following singular integral:

$$(-\Delta)^s \psi = \text{P.V.} \int_{\mathbb{R}^N} C_{N,s} \frac{\psi(x) - \psi(y)}{|x - y|^{N+2s}} dy,$$

where $C_{N,s}$ is a normalization constant depending only on N and s . We refer to Section 2 for more details.

Let us clarify how we interpret the boundary-exterior conditions in (1.1b).

- If the function ψ has a well-defined trace on Γ , then $u_1 = \psi|_{\Gamma}$. In that case, the condition on Γ can be dropped and the problem will still be well-posed as we shall see later. But if the condition in Σ is removed, then the system will be ill-posed.
- If ψ does not have a well-defined trace on Γ , then the condition on Γ will be seen in a very-weak sense that we shall explain later. In that case, none of the two conditions (boundary and exterior) can be removed, otherwise the system will be ill-posed.

These important facts will be clarified in Section 3.

The first main concern of the present paper is to prove several well-posedness results of the parabolic problem (1.1b) and the associated elliptic (time independent) equation:

$$\begin{cases} \mathcal{L}\phi = f & \text{in } \Omega, \\ \phi = u_1 & \text{on } \partial\Omega, \\ \phi = u_2 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.2)$$

Notice that throughout the paper, since we are considering smooth open sets Ω , it follows that a.e. in $\mathbb{R}^N \setminus \Omega$ is the same as a.e. in $\mathbb{R}^N \setminus \bar{\Omega}$. The system (1.2), with $f \in L^2(\Omega)$ and u_1, u_2 are zero, has been very recently studied in [10] where the authors have proved some well-posedness, local and boundary regularity, and some maximum principle results. Here, we shall show that in this case, the associated self-adjoint operator on $L^2(\Omega)$ is a generator of a strongly continuous submarkovian semigroup $(T(t))_{t \geq 0}$ which is also ultracontractive in the sense that the operator $T(t)$ maps $L^1(\Omega)$ into $L^\infty(\Omega)$ for every $t > 0$. This will be used to have some fine regularity results of the dual problem associated with the system (1.1b) which will be crucial in the study of our optimal control problems.

In [10], the authors also considered briefly the case where u_1 and u_2 in (1.2) are smooth functions. In that case, as we mentioned above, the condition on $\partial\Omega$ can be dropped and the associated system will still be well-posed. In the first part of the present article, for non-smooth boundary-exterior data, we shall introduce the notion of solutions by transposition (or very-weak solutions) of (1.2), study their existence and regularity. Our main result in this direction reads that if $u_1 \in L^2(\partial\Omega)$, $u_2 \in L^2(\mathbb{R}^N \setminus \Omega)$ and $0 < s \leq 3/4$, then the associated very-weak solution ψ of (1.2) belongs to $H^{1/2}(\Omega) \cap L^2(\mathbb{R}^N)$ (see Theorem 3.7 and Remark 3.8). For the associated parabolic problem (1.1b), if $u_1 \in L^2(\Gamma)$, $u_2 \in L^2(\Sigma)$ and $0 < s \leq 3/4$, then $\psi \in L^2((0, T) \times \mathbb{R}^N) \cap C([0, T]; H^{-1}(\Omega))$ (see Theorem 3.18).

Notice that the eigenvalues problem associated with nonlocal Neumann exterior conditions, that is, when $\phi = 0$ on $\partial\Omega$ and $\phi = 0$ in $\mathbb{R}^N \setminus \Omega$ are replaced with $\partial_\nu \phi = 0$ on $\partial\Omega$ and $\mathcal{N}_s \phi = 0$ in $\mathbb{R}^N \setminus \bar{\Omega}$, respectively (see (2.5) below for the definition of \mathcal{N}_s) has been recently investigated in [25], where the authors have

shown that the associated operator has a compact resolvent, hence, has a discrete spectrum formed with eigenvalues. We mention that the case of the fractional Laplace operator with the nonlocal exterior condition $\mathcal{N}_s \phi = 0$ in $\mathbb{R}^N \setminus \bar{\Omega}$ has been introduced and investigated in [26]. Here, we are not interested in the Neumann type boundary-exterior conditions.

Our second main concern is to study the existence of optimal solutions to optimal control problems involving the mixed operator \mathcal{L} with singular Dirichlet boundary-exterior data, and to characterize the associated optimality conditions. More precisely, we shall consider the following two different optimal control problems:

$$\min_{(u_1, u_2) \in \mathcal{Z}_{ad}} J_i((u_1, u_2)), \quad i = 1, 2, \quad (1.3)$$

subject to the constraint that the state $\psi := \psi(u_1, u_2)$ solves the parabolic system (1.1b). We recall that the control $(u_1, u_2) \in \mathcal{Z}_{ad}$ with \mathcal{Z}_{ad} being a closed and convex subset of $\mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma)$, which is endowed with the norm given by

$$\|(u_1, u_2)\|_{\mathcal{Z}_D} = \left(\|u_1\|_{L^2(\Gamma)}^2 + \|u_2\|_{L^2(\Sigma)}^2 \right)^{\frac{1}{2}}.$$

The functionals J_1 and J_2 are given by

$$J_1(u_1, u_2) := \frac{1}{2} \|\psi((u_1, u_2)) - z_d^1\|_{L^2(Q)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2 \quad (1.4)$$

and

$$J_2(u_1, u_2) := \frac{1}{2} \|\psi(T; (u_1, u_2)) - z_d^2\|_{H^{-1}(\Omega)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2, \quad (1.5)$$

where $\beta > 0$ is a real number, $z_d^1 \in L^2(Q)$, $z_d^2 \in H^{-1}(\Omega)$, and

$$\|\phi\|_{H^{-1}(\Omega)}^2 := \langle (-\Delta_D)^{-1} \phi, \phi \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.$$

Here, $-\Delta_D$ is the realization in $L^2(\Omega)$ of the Laplace operator $-\Delta$ with the zero Dirichlet boundary condition. The functionals J_1 and J_2 can be replaced with more general functionals satisfying suitable conditions, without any substantial modification of the proofs.

The novelties and difficulties of the present paper can be summarized as follows.

- (a) For the first time, elliptic and parabolic equations associated with a mixed local-nonlocal operator and singular boundary-exterior data have been studied.
- (b) Since we are considering singular data, the right definition of solutions, their existence, and their fine regularity, are more challenging than the case of the single local, or the single nonlocal operator. In particular, the regularity of solutions is crucial in the study of the optimal control problems.
- (c) For singular boundary-exterior data, we have introduced the notion of solutions by transposition to the system (1.1b). This definition requires that solutions ϕ of the dual system associated with (1.1b) satisfy $\partial_\nu \phi \in L^2(\Gamma)$ and $\mathcal{N}_s \phi \in L^2(\Sigma)$, where $\partial_\nu \phi$ is the classical normal derivative of ϕ , and $\mathcal{N}_s \phi$ denotes the nonlocal normal derivative of ϕ (see (2.5)). We have been able to show this regularity only in the range of exponents $0 < s \leq 3/4$. The case $3/4 < s < 1$ remains an open problem. In the classical local case $s = 1$ or $s = 0$ (in the situation of the present paper), classical elliptic regularity results show that ϕ belongs to $L^2((0, T); H^2(\Omega))$ so that its normal derivative exists and belongs to $L^2((0, T); L^2(\partial\Omega))$. This seems not to be

the case for the nonlocal case investigated here. We shall give more details in Section 3.

- (d) Let us notice that the operator \mathcal{L} is a sum of the local operator $-\Delta$ and the nonlocal one $(-\Delta)^s$. On the one hand, for this operator, regarding well-posedness and regularity of associated elliptic and parabolic systems, the nonlocal operator seems to be dominant as we shall see in Section 3. This shows that the operator \mathcal{L} cannot be just seen as a simple perturbation of $-\Delta$ with a lower order operator. To see that, for example, $\mathcal{L}\psi \in L^2(\Omega)$ does not mean that $\Delta\psi \in L^2(\Omega)$ and $(-\Delta)^s\psi \in L^2(\Omega)$. This is the case only for certain values of s , that is, when $0 < s \leq 3/4$. On the other hand, regarding the ultracontractivity property of the semigroup $(T(t))_{t \geq 0}$ mentioned above, the Laplace operator $-\Delta$ seems to be dominant.
- (e) For the first time, optimal control problems associated with a mixed local-nonlocal operator and singular boundary-exterior data have been investigated. The existence and uniqueness of minimizers and the characterization of the associated optimality conditions have been obtained under the assumption that $0 < s \leq 3/4$. This is a substantial extension of the nowadays well-known local case of the Laplace operator (see e.g. the monograph [44] and the references therein), and the nonlocal case recently investigated in [4, 5, 6, 7] and the references therein.

It is a great idea from the authors in [10] for having introduced the operator \mathcal{L} and initiated the study of its qualitative properties.

We mention that fractional order operators have recently received a great deal of attention due to the fact that they model many diverse real-life phenomena which could not be adequately modeled by using classical local or non-fractional operators. These type of operators belong to the broad class of nonlocal operators and have merged as a modeling alternative in various branches of science. The theory of these operators has discovered many applications, for example in fluid dynamics [20], diffusion of biological species [51], transitions across an interface, multiscale behavior in cardiac tissue [16], and phase field models [3]. Despite the numerous applications, still, there is a lot of undiscovered potential lying in the connection between local (induced by Brownian motion) and nonlocal (induced by Lévy process), since they can both be understood from the viewpoint of mathematical analysis and then combined in a natural way. However, there is no agreement on exactly how to define mixed local and nonlocal operators as a combination process of the Brownian and Lévy processes. For this reason, attempts have been made recently in [10], to unify local and nonlocal operators in the single operator \mathcal{L} given in (1.1c). Operators of the kind (1.1c) arise naturally from the superposition of two stochastic processes with different scales (namely, a classical random walk and a Lévy flight): roughly speaking, when a particle can follow either of these two processes according to a certain probability, the associated limit diffusion equation is described by an operator of the form (1.1c). See also [19] which discuss about the advection-mediated coexistence of competing species.

Optimal control problems of partial differential equations involving classical local operators with boundary control through Dirichlet and Neumann conditions have been widely investigated this last five decades. For instance, J. L. Lions [44] considered an optimal control problem with Neuman boundary observation subjected to an elliptic equation with Dirichlet boundary control. Using a transposition method, the author proved the existence and uniqueness of solutions when the control is in

$L^2(\partial\Omega)$ and gave a sense to the normal derivative of the considered observation in $H^{-1}(\partial\Omega)$. Then, the author showed the existence of the optimal control and gave its characterization. These results have been extended by the same author to an optimal control problem with Neuman boundary observation subjected to a parabolic equation with Dirichlet boundary control in [44]. Note that for both elliptic and parabolic equations, if the set of admissible controls are respectively $L^2(\partial\Omega)$ and $L^2((0, T) \times \partial\Omega)$, and the domain Ω is smooth enough, then the regularity of the control and the normal derivative of the adjoint state are the same.

We observe that there is a fundamental difference between the regularity of solutions of parabolic problems with Dirichlet L^2 -boundary data and the regularity of optimal solutions of the associated control problems. In the classical local case ($s = 1$ or $s = 0$ in the setting of the present paper), it turns out that the latter is always higher. In fact, Lions and Magenes [43, Section 2.2] were first to prove that solutions of the open loop system with $L^2(\Gamma)$ -Dirichlet data belong to $L^2((0, T); H^{1/4}(\Omega)) \cap H^{1/2}((0, T); L^2(\Omega))$. In [38], Lasiecka used a semigroup approach to solve a Dirichlet boundary optimal control and a distributed observation subjected to parabolic systems. The author proved that if Ω is smooth enough or Ω is a parallelepiped, then the optimal control belongs to $L^2((0, T); H^{1/2}(\partial\Omega))$ which is more regular than the allowed control space, $L^2(\Gamma)$, and the optimal trajectory is in $L^2((0, T); H^1(\Omega))$. These results were significantly improved by Lasiecka and Triggiani [39, 40] (see also [23]) for smooth domains. Actually, considering a more general parabolic problem with a control in $L^2(\Gamma)$ acting on a Dirichlet boundary, and a distributed observation, using a transposition method, the authors proved that the optimal control belongs to $L^2((0, T); H^{1/2}(\partial\Omega)) \cap H^{1/4}((0, T); L^2(\partial\Omega))$ and the optimal trajectory is in $L^2((0, T); H^1(\Omega)) \cap H^{1/2}((0, T); L^2(\partial\Omega))$. An extension to non-autonomous systems is contained in [1]. Whether such a result can be obtained in the fractional case considered in the present paper is still an interesting open problem. Notice that for the fractional case, we have an exterior control that is no longer a trace of the state on $\Gamma := \partial\Omega \times (0, T)$, but a restriction of the state in $(\mathbb{R}^N \setminus \Omega) \times (0, T)$. For more information on boundary optimal control problems, we refer to [42, 43, 44, 50] and the references therein.

Recently, partial differential equations with time fractional derivatives and/or the Fractional Laplace operator have emerged as excellent tools to described phenomenon with memory effects. Hence, the control of such equations are of great interest. However, for time fractional diffusion equations involving classical second order elliptic operators, boundary conditions are the same as for the classical diffusion equations. Instead, for diffusion equations involving the fractional Laplace operator, they may not have boundary conditions but external conditions. These lead to boundary control problems for time fractional diffusion equations involving second order elliptic operators, and external control problems for diffusion equations involving the fractional Laplace operator. Compared to the classical diffusion equations, the literature on quadratic boundary control associated to such equations is scarce. In [47], R. Dorville et al. used a transposition method to study a Dirichlet boundary control problem associated to a time fractional diffusion equation involving Riemann-Liouville fractional derivatives of order $\alpha \in (0, 1)$ and a final time observation of Riemann-Liouville integrals of order $\alpha \in (0, 1)$. They succeeded in proving the existence and the uniqueness of the optimal control that they characterized by an optimality system. They showed that the optimal control

is in $L^2((0, T); H^{1/2}(\partial\Omega))$ when the domain is smooth enough and the set of the admissible controls is $L^2((0, T) \times \partial\Omega)$.

Using variational methods, the authors in [4] investigated an external optimal control problem of a nonlocal elliptic equation involving the fractional Laplace operator with distributed observation. They proved the existence and uniqueness of the optimal control and gave the optimality system that characterized this optimal control. They also noticed that even if the domain Ω is smooth enough, contrary to the results obtained for the classical Laplace operator, the external optimal control remains in $L^2((0, T) \times \mathbb{R}^N \setminus (\Omega))$, the allowed control space. This result was extended in [5] to the external optimal control problem of parabolic equations involving the fractional Laplace operator with distributed observation.

In this paper, we consider two boundary quadratic optimal control problems subject to a diffusion equation involving the classical and the fractional Laplace operators. We have two controls: an external control and a control acting through a Dirichlet condition. We first consider distributed observations and prove the existence and uniqueness of the optimal control. Then, observing as in [39] that for an allowed control in $L^2((0, T) \times \partial\Omega)$, it may happen that the response ψ is such that $\psi(\cdot, T) \notin L^2(\Omega)$, we consider for the second control problem a final time observation in $H^{-1}(\Omega)$ and prove the existence and uniqueness of the optimal control. In both optimal control problems we observe that if the set of admissible controls is $L^2(\Gamma) \times L^2(\Sigma)$, then even though, we deal with a mixed local and nonlocal diffusion operator, optimal control belongs to $L^2((0, T); H^{1/2}(\partial\Omega)) \times L^2((0, T); H_{\text{loc}}^1(\mathbb{R}^N \setminus \Omega))$ and the optimal trajectory remains in $L^2((0, T) \times \mathbb{R}^N)$ whenever $0 < s \leq 3/4$. We have not been successful in the case $0 < s < 3/4$. Of course the cases $s = 0$ and $s = 1$ correspond to the classical Laplace operator mentioned above.

The rest of the paper is structured as follows. In Section 2 we fix some notations, give a rigorous definition of the fractional Laplace operator, and introduce the function spaces needed to study our problems. The results of well-posedness and regularity of solutions to the elliptic problem (1.2), and the parabolic problem (1.1b), are contained in Sections 3.1 and 3.2, respectively. In Section 4.1 we show the existence and uniqueness of optimal solutions to the control problem (1.3)-(1.4), and we characterize the associated optimality conditions. The same study for the control problem (1.3)-(1.5) is contained in Section 4.2.

2. NOTATIONS AND PRELIMINARIES

In this section we fix some notations and recall some known results as they are needed throughout the paper. These results can be found for example in [5, 7, 10, 13, 18, 24, 28, 29, 30, 36, 52, 53] and the references therein.

Let us first give a rigorous definition of the fractional Laplacian. Given $0 < s < 1$, we let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ w : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|w(x)|}{(1 + |x|)^{N+2s}} dx < \infty \right\}.$$

For $w \in \mathcal{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$, we set

$$(-\Delta)_\varepsilon^s w(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant given by

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}. \quad (2.1)$$

The fractional Laplacian $(-\Delta)^s$ is defined by the following singular integral:

$$(-\Delta)^s w(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s w(x), \quad x \in \mathbb{R}^N, \quad (2.2)$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$. We refer to [24] and their references regarding the class of functions for which the limit in (2.2) exists for a.e. $x \in \mathbb{R}^N$.

Next, we introduce the function spaces needed to study our problems. We start with fractional order Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. Given $0 < s < 1$ a real number, we let

$$H^s(\Omega) := \left\{ w \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy < \infty \right\},$$

and we endow it with the norm defined by

$$\|w\|_{H^s(\Omega)} := \left(\int_{\Omega} |w(x)|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

We set

$$H_0^s(\Omega) := \left\{ w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

Then, $H_0^s(\Omega)$ endowed with the norm

$$\|w\|_{H_0^s(\Omega)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2}, \quad (2.3)$$

is a Hilbert space (see e.g. [49, Lemma 7]). We let $H^{-s}(\Omega) := (H_0^s(\Omega))^*$ be the dual space of $H_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$, so that we have the following continuous and dense embeddings (see e.g. [9]):

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega). \quad (2.4)$$

Under the assumption that Ω is bounded and has a Lipschitz continuous boundary, we have that $\mathcal{D}(\Omega)$ is dense in $H_0^s(\Omega)$ (for every $0 < s < 1$, see e.g. [27]), and by [36, Chapter 1], if $0 < s \neq 1/2 < 1$, then

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)},$$

with equivalent norms, where $\mathcal{D}(\Omega)$ denotes the space of all continuously infinitely differentiable functions with compact support in Ω . But if $s = 1/2$, then $H_0^s(\Omega)$ is a proper subspace of $\overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$.

For more information on fractional order Sobolev spaces, we refer to [24, 36, 52] and their references.

Next, for $\varphi \in H^s(\mathbb{R}^N)$ we introduce the *nonlocal normal derivative* \mathcal{N}_s given by

$$\mathcal{N}_s \varphi(x) := C_{N,s} \int_{\Omega} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \quad (2.5)$$

where $C_{N,s}$ is the constant given in (2.1). We notice that the nonlocal normal derivative \mathcal{N}_s has been first introduced in [26].

Remark 2.1. It has been shown in [32, 33] that the operator \mathcal{N}_s maps $H^s(\mathbb{R}^N)$ into $H_{\text{loc}}^s(\mathbb{R}^N \setminus \bar{\Omega})$. Furthermore, if $u \in H_0^1(\Omega)$ and $(-\Delta)^s u \in L^2(\Omega)$, then $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$, and there is a constant $C > 0$ such that

$$\|\mathcal{N}_s u\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \leq C \|u\|_{H_0^1(\Omega)}. \quad (2.6)$$

The following integration by parts formula is contained in [26, 54]. Let $\varphi \in H^s(\mathbb{R}^N)$ be such that $(-\Delta)^s \varphi \in L^2(\Omega)$ and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$. Then for every $\psi \in H^s(\mathbb{R}^N)$, the identity

$$\begin{aligned} \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy \\ = \int_{\Omega} \psi (-\Delta)^s \varphi dx + \int_{\mathbb{R}^N \setminus \Omega} \psi \mathcal{N}_s \varphi dx \end{aligned} \quad (2.7)$$

holds.

Observing that

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega),$$

we have that if $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$ or $\psi = 0$ in $\mathbb{R}^N \setminus \Omega$, then

$$\begin{aligned} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy \\ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Throughout the remainder of the paper, we shall let the bilinear form $\mathcal{F} : H_0^s(\Omega) \times H_0^s(\Omega) \rightarrow \mathbb{R}$ be given by

$$\mathcal{F}(\varphi, \psi) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy. \quad (2.8)$$

Next, we introduce the classical first order Sobolev space

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} |\nabla u|^2 dx < \infty \right\}$$

which is endowed with the norm defined by

$$\|u\|_{H^1(\Omega)} = \left(\int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

In order to study the solvability of (1.1b), we shall also need the following function space

$$H_0^1(\Omega) := \left\{ w \in H^1(\mathbb{R}^N) : w \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}, \quad (2.9)$$

which is a (real) Hilbert space endowed with the scalar product

$$\int_{\Omega} \nabla w \cdot \nabla \varphi dx,$$

and associated norm

$$\|\varphi\|_{H_0^1(\Omega)} := \|\nabla \varphi\|_{L^2(\Omega)}. \quad (2.10)$$

Furthermore, the classical Poincaré inequality holds in $H_0^1(\Omega)$. That is, there is a constant $C > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{H_0^1(\Omega)} \quad \text{for all } \varphi \in H_0^1(\Omega). \quad (2.11)$$

We shall denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$ so that we have the following continuous and dense embeddings:

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

Here also, if Ω is bounded and has a Lipschitz continuous boundary, then by [36, Chapter 1]

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}.$$

In addition, under the same assumption on Ω , every function $u \in H^1(\Omega)$ has a trace $u|_{\partial\Omega}$ that belongs to $H^{1/2}(\partial\Omega)$, and the mapping trace

$$H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega), \quad u \mapsto u|_{\partial\Omega} \quad (2.12)$$

is continuous and surjective.

Now assume that Ω is a bounded open set with a Lipschitz continuous boundary. Then every function $u \in H^2(\Omega)$ has a normal derivative $\partial_\nu u := \nabla u \cdot \nu$ that belongs to $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \sigma)$. Here, σ denotes the restriction to $\partial\Omega$ of the $(N-1)$ -dimensional Hausdorff measure which coincides with the Lebesgue surface measure since Ω has a Lipschitz continuous boundary, and ν is the outer normal vector at the boundary. More precisely, there is a constant $C > 0$ such that for every $u \in H^2(\Omega)$,

$$\|\partial_\nu u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^2(\Omega)}. \quad (2.13)$$

In addition, for every $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, the integration by parts formula

$$-\int_{\Omega} v \Delta u \, dx = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \partial_\nu u \, d\sigma \quad (2.14)$$

holds. We refer to [36, Lemma 1.5.3.7] for the proof.

Assume that Ω is a bounded open set of class $C^{1,1}$ and let $E(\Delta, L^2(\Omega)) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. Then $C^\infty(\overline{\Omega})$ is dense in $E(\Delta, L^2(\Omega))$ and every $u \in E(\Delta, L^2(\Omega))$ has a normal derivative $\partial_\nu u := \nabla u \cdot \nu$ that belongs to $H^{-1/2}(\partial\Omega) := (H^{1/2}(\partial\Omega, \sigma))^*$. More precisely, there is a constant $C > 0$ such that for every $u \in E(\Delta, L^2(\Omega))$,

$$\|\partial_\nu u\|_{H^{-1/2}(\partial\Omega)} \leq C (\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}). \quad (2.15)$$

In addition, for every $u \in E(\Delta, L^2(\Omega))$ and $v \in H^1(\Omega)$, the integration by parts formula

$$-\int_{\Omega} v \Delta u \, dx = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \langle \partial_\nu u, v \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \quad (2.16)$$

holds (see e.g. [36, Formula 1.5.3.30, pp 62]). We refer to [2, 17, 34, 36, 45] and the references therein for more details on this topic.

Throughout the remainder of the paper, without any mention, we shall assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. Under this assumption, we have the following continuous and dense embedding for every $0 < s < 1$ (see e.g. [36, 24]):

$$H_0^1(\Omega) \hookrightarrow H_0^s(\Omega). \quad (2.17)$$

In view of (2.10) and (2.17), we can deduce that

$$(\varphi, \psi)_{H_0^1(\Omega)} := \mathcal{F}(\varphi, \psi) + \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx \quad (2.18)$$

defines a scalar product on $H_0^1(\Omega)$ with associated norm

$$\|\varphi\|_{H_0^1(\Omega)} := \left(\mathcal{F}(\varphi, \varphi) + \int_{\Omega} |\nabla\varphi|^2 dx \right)^{\frac{1}{2}}. \quad (2.19)$$

The norm given in (2.19) is equivalent to the one given in (2.10).

Throughout the remainder of the article, otherwise stated, $H_0^1(\Omega)$ will be endowed with the scalar product and norm given in (2.18) and (2.19), respectively.

We conclude this section by giving the following result. Let $T > 0$ be a real number and set

$$W(0, T) := \{ \zeta \in L^2((0, T); H_0^1(\Omega)) : \zeta_t \in L^2((0, T); H^{-1}(\Omega)) \}. \quad (2.20)$$

Since $H_0^1(\Omega)$ is a real Hilbert space, it follows from Lions–Magenes [15, Theorem II.5.12] and [41] that $W(0, T)$ endowed with the norm

$$\|\zeta\|_{W(0, T)} := \left(\|\zeta\|_{L^2((0, T); H_0^1(\Omega))}^2 + \|\zeta_t\|_{L^2((0, T); H^{-1}(\Omega))}^2 \right)^{\frac{1}{2}}, \quad (2.21)$$

is a Hilbert space. Moreover, we have the following continuous embedding:

$$W(0, T) \hookrightarrow C([0, T], L^2(\Omega)). \quad (2.22)$$

Furthermore, from (2.11) and the Lions–Aubin Lemma [15, Theorem II.5.16] the following embedding is compact

$$W(0, T) \hookrightarrow L^2((0, T); L^2(\Omega)). \quad (2.23)$$

3. WELL-POSEDNESS OF THE STATE EQUATION

In this section we are interested in establishing some existence, uniqueness and regularity results of the state equation (1.1b) that will be needed in the proof of the existence of minimizers to the optimal control problem (1.1). We recall that without any mention, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$.

3.1. The elliptic problem. We start with the stationary problem. That is, we consider the following non-homogeneous Dirichlet problem associated with the operator \mathcal{L} , as defined in (1.1c). That is,

$$\begin{cases} \mathcal{L}w = f & \text{in } \Omega, \\ w = g_1 & \text{on } \partial\Omega, \\ w = g_2 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.1)$$

Even if we are not considering control problems for elliptic equations, the complete analysis of (3.1) will be crucial in the study of the associated time dependent problem, and is also interesting in its own, independently of the applications given in the present paper.

To introduce our notion of solutions to the system (3.1), we start with the simple case $g_1 = 0$ on $\partial\Omega$ and $g_2 = 0$ in $\mathbb{R}^N \setminus \Omega$.

Definition 3.1. Let $f \in H^{-1}(\Omega)$, $g_2 = 0$ in $\mathbb{R}^N \setminus \Omega$ and $g_1 = 0$ on $\partial\Omega$. A function $w \in H_0^1(\Omega)$ is said to be a *weak solution* of (3.1) if for every function $\varphi \in H_0^1(\Omega)$, the identity

$$\int_{\Omega} \nabla w \cdot \nabla \varphi dx + \mathcal{F}(w, \varphi) = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad (3.2)$$

holds, where we recall that the bilinear form \mathcal{F} has been defined in (2.8).

The following existence result can be established by using the classical Lax-Milgram Lemma. We refer to [10, Theorem 1.1] for more details.

Proposition 3.2. *Let $g_2 = 0$ in $\mathbb{R}^N \setminus \Omega$ and $g_1 = 0$ on $\partial\Omega$. Then for every $f \in H^{-1}(\Omega)$, there is a unique function $w \in H_0^1(\Omega)$ satisfying (3.1) in the sense of Definition 3.1. In addition, there is a constant $C > 0$ such that*

$$\|w\|_{H_0^1(\Omega)} \leq C \|f\|_{H^{-1}(\Omega)}.$$

Next, we consider the case of smooth non-zero boundary-exterior data.

Definition 3.3. Let $f \in H^{-1}(\Omega)$. Let $g_2 \in H^1(\mathbb{R}^N \setminus \bar{\Omega})$ and $g_1 \in H^{1/2}(\partial\Omega)$ be such that $g_1 = g_2|_{\partial\Omega}$. Let $G \in H^1(\mathbb{R}^N)$ be such that $G|_{\mathbb{R}^N \setminus \bar{\Omega}} = g_2$. A function $w \in H^1(\mathbb{R}^N)$ is said to be a weak solution of (3.1) if $w - G \in H_0^1(\Omega)$ and the identity

$$\int_{\Omega} \nabla w \cdot \nabla \varphi \, dx + \mathcal{F}(w, \varphi) = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad (3.3)$$

holds for every $\varphi \in H_0^1(\Omega)$.

The following existence result can be easily established. We also refer to [10, Corollary 2.6] where the same result has been proved under the assumption that the function g_2 is smooth enough. The version given here can be obtained by using [10, Corollary 2.6] and an approximation argument.

Proposition 3.4. *Let g_1 and g_2 be as in Definition 3.3 and $f \in H^{-1}(\Omega)$. Then, there is a unique $w \in H^1(\mathbb{R}^N)$ satisfying (3.1) in the sense of Definition 3.3. In addition, there is a constant $C > 0$ such that*

$$\|w\|_{H^1(\mathbb{R}^N)} \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|g_2\|_{H^1(\mathbb{R}^N \setminus \bar{\Omega})} \right).$$

Remark 3.5. We notice that in the situation of Definitions 3.1- 3.3, and Propositions 3.2-3.4, the system (3.1) becomes

$$\mathcal{L}w = f \quad \text{in } \Omega, \quad w = g_2 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

that is, the condition on $\partial\Omega$ is not needed in order to have a well-posed problem.

Finally, we consider the case of non-smooth boundary-exterior data. This case has not been discussed in [10] and we need a new notion of solutions that we introduce next.

Definition 3.6. Let $f \in H^{-1}(\Omega)$, $g_1 \in L^2(\partial\Omega)$, and $g_2 \in L^2(\mathbb{R}^N \setminus \Omega)$. A function $w \in L^2(\mathbb{R}^N)$ is called a very-weak solution (or a solution by transposition) of (3.1), if the identity

$$\int_{\Omega} w \mathcal{L}\varphi \, dx = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\partial\Omega} g_1 \partial_\nu \varphi \, d\sigma - \int_{\mathbb{R}^N \setminus \bar{\Omega}} g_2 \mathcal{N}_s \varphi \, dx \quad (3.4)$$

holds, for every $\varphi \in \mathbb{V} := \left\{ \varphi \in H_0^1(\Omega) : \mathcal{L}\varphi \in L^2(\Omega) \right\}$.

We notice that Definition 3.6 of very-weak solutions makes sense if every function $\varphi \in \mathbb{V}$ satisfies $\partial_\nu \varphi \in L^2(\partial\Omega)$, and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$.

We have the following existence theorem which is the main result of this section.

Theorem 3.7. *Let $0 < s \leq 3/4$. Then for every $f \in H^{-1}(\Omega)$, $g_1 \in L^2(\partial\Omega)$ and $g_2 \in L^2(\mathbb{R}^N \setminus \Omega)$, the system (3.1) has a unique very-weak solution $w \in L^2(\mathbb{R}^N)$ in the sense of Definition 3.6, and there is a constant $C > 0$ such that*

$$\|w\|_{L^2(\mathbb{R}^N)} \leq C (\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \Omega)}). \quad (3.5)$$

In addition, if g_1 and g_2 are as in Definition 3.3, then the following assertions hold.

- (a) Every weak solution of (3.1) is also a very-weak solution.
- (b) Every very-weak solution of (3.1) that belongs to $H^1(\mathbb{R}^N)$ is also a weak solution.

Proof. The proof follows similarly as the case of the single fractional Laplace operator (see e.g. [4]) with the exception of the restriction on s which is an important and delicate step. We include the full proof for the sake of completeness. We proceed in several steps.

Step 1: Let \mathbb{A} be the realization of \mathcal{L} in $L^2(\Omega)$ with zero Dirichlet exterior condition, that is,

$$D(\mathbb{A}) = \mathbb{V} := \{u \in H_0^1(\Omega) : (\mathcal{L}u)|_\Omega \in L^2(\Omega)\}, \quad \mathbb{A}u = (\mathcal{L}u)|_\Omega \text{ in } \Omega. \quad (3.6)$$

We shall give a detailed description of this operator later. Notice that $\|v\|_{\mathbb{V}} := \|\mathcal{L}v\|_{L^2(\Omega)}$ defines an equivalent norm on \mathbb{V} . This follows from the fact that the operator \mathbb{A} is invertible, has a compact resolvent, and its first eigenvalue is strictly positive (see the proof of Theorem 3.11 below for more details). We claim that

$$\mathbb{V} = \left\{ \varphi \in H_0^1(\Omega) : \mathcal{L}\varphi \in L^2(\Omega), \partial_\nu \varphi \in L^2(\partial\Omega) \text{ and } \mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega}) \right\}.$$

It suffices to show that $\partial_\nu \varphi \in L^2(\partial\Omega)$ and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$ for every $\varphi \in \mathbb{V}$. Indeed, let $\varphi \in \mathbb{V}$. We have two cases.

- Case $0 < s < 1/2$. Since $\varphi \in H_0^1(\Omega) \hookrightarrow H_0^s(\Omega)$, it follows from the regularity result contained in [37, Theorem 4.1] that $(-\Delta)^s \varphi \in L^2(\Omega)$. Thus, $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$ by Remark 2.1. As $\mathcal{L}u \in L^2(\Omega)$, this also implies that $\Delta \varphi \in L^2(\Omega)$. Since Ω is assumed to be smooth, using the well-known elliptic regularity results for the Laplace operator, (see e.g. [34]), we have that $\varphi \in H^2(\Omega)$. Thus, $\partial_\nu \varphi \in H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.
- Case $1/2 \leq s < 1$. Since $\varphi \in H_0^1(\Omega)$, it follows from [37, Theorem 4.1] again that $(-\Delta)^s \varphi \in H^{1-2s}(\Omega)$. Hence, $\Delta \varphi \in H^{1-2s}(\Omega)$ and this implies that $\varphi \in H^{3-2s}(\Omega)$. Thus, $\partial_\nu \varphi \in H^{3/2-2s}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ if $3/2 - 2s \geq 0$. Since $\varphi \in H^{3-2s}(\Omega)$, it follows from [37] again that if $3 - 4s \geq 0$, then $(-\Delta)^s \varphi \in L^2(\Omega)$. Since $\mathcal{L}\varphi \in L^2(\Omega)$, we can deduce that $\Delta \varphi \in L^2(\Omega)$. From elliptic regularity for the Laplace operator we can conclude that $\varphi \in H^2(\Omega)$. Thus, $\partial_\nu \varphi \in H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

Since we have assumed that $0 < s \leq 3/4$, it follows that $3/2 - 2s \geq 0$ and $3 - 4s \geq 0$. Thus, using Remark 2.1, we can conclude that $\partial_\nu \varphi \in L^2(\partial\Omega)$ and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \bar{\Omega})$. The claim is proved.

Step 2: We apply the Babuška-Lax-Milgram theorem. Let $\mathbb{F} : L^2(\Omega) \times \mathbb{V} \rightarrow \mathbb{R}$ be the bilinear form defined by

$$\mathbb{F}(u, v) := \int_{\Omega} u \mathcal{L}v \, dx.$$

It is clear that \mathbb{F} is bounded on $L^2(\Omega) \times \mathbb{V}$. That is,

$$|\mathbb{F}(u, v)| \leq \|u\|_{L^2(\Omega)} \|\mathcal{L}v\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)} \|v\|_{\mathbb{V}}, \quad \forall (u, v) \in L^2(\Omega) \times \mathbb{V}.$$

Next, we show the inf-sup condition. Letting

$$u := \frac{\mathcal{L}v}{\|\mathcal{L}v\|_{L^2(\Omega)}} \in L^2(\Omega),$$

we obtain that

$$\sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)}=1} |(u, \mathcal{L}v)_{L^2(\Omega)}| \geq \frac{|(\mathcal{L}v, \mathcal{L}v)_{L^2(\Omega)}|}{\|\mathcal{L}v\|_{L^2(\Omega)}} = \|\mathcal{L}v\|_{L^2(\Omega)} = \|v\|_{\mathbb{V}}.$$

Next, we choose $v \in \mathbb{V}$ as the unique weak solution of the Dirichlet problem

$$\mathcal{L}v = \frac{h}{\|h\|_{L^2(\Omega)}} \quad \text{in } \Omega \quad \text{for some } 0 \neq h \in L^2(\Omega).$$

Then we readily obtain that

$$\sup_{v \in \mathbb{V}, \|v\|_{\mathbb{V}}=1} |(h, \mathcal{L}v)_{L^2(\Omega)}| \geq \frac{|(h, h)_{L^2(\Omega)}|}{\|h\|_{L^2(\Omega)}} = \|h\|_{L^2(\Omega)} > 0 \quad \text{for all } 0 \neq h \in L^2(\Omega).$$

Finally, we have to show that the right-hand-side in (3.4) defines a linear continuous functional on \mathbb{V} . Indeed, applying the Hölder inequality, the fact that ∂_ν maps \mathbb{V} continuously into $L^2(\partial\Omega)$, and \mathcal{N}_s maps \mathbb{V} continuously into $L^2(\mathbb{R}^N \setminus \bar{\Omega})$ (by Step 1), we obtain that there is a constant $C > 0$ such that

$$\begin{aligned} & \left| \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\partial\Omega} g_1 \partial_\nu v \, d\sigma - \int_{\mathbb{R}^N \setminus \bar{\Omega}} g_2 \mathcal{N}_s v \, dx \right| \\ & \leq \|f\|_{H^{-1}(\bar{\Omega})} \|v\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} \|\partial_\nu v\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \|\mathcal{N}_s v\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \\ & \leq C \left(\|f\|_{H^{-1}(\bar{\Omega})} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \right) \|v\|_{\mathbb{V}}. \end{aligned} \quad (3.7)$$

In view of (3.7), we can deduce that the right-hand-side in (3.4) defines a linear continuous functional on the Hilbert space \mathbb{V} . Therefore, all the requirements of the Babuška–Lax–Milgram theorem hold. Thus, we can conclude that there exists a unique $w \in L^2(\Omega)$ satisfying (3.4). Letting $w := g_2$ in $\mathbb{R}^N \setminus \bar{\Omega}$, we have that $w \in L^2(\mathbb{R}^N)$ and satisfies (3.4). We have shown the existence and uniqueness of a very-weak solution to the system (3.1).

Step 3: Next, we show the estimate (3.5). Let $w \in L^2(\mathbb{R}^N)$ be a very-weak solution of (3.1). Let $v \in \mathbb{V}$ and set $w := \mathbb{A}v = \mathcal{L}v$ in Ω . Taking v as a test function in (3.4), using (3.7), (2.6), (2.17), and (2.13), we can deduce that there is a constant $C > 0$ such that

$$\begin{aligned} \|w\|_{L^2(\Omega)}^2 & \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \right) \|v\|_{\mathbb{V}} \\ & = C \left(\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \right) \|w\|_{L^2(\Omega)}. \end{aligned}$$

Thus,

$$\|w\|_{L^2(\Omega)} \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \right).$$

Since $w = g_2$ in $\mathbb{R}^N \setminus \bar{\Omega}$, it follows from the preceding estimate that

$$\|w\|_{L^2(\mathbb{R}^N)} \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N \setminus \bar{\Omega})} \right).$$

We have shown the estimate (3.5) and this completes the proof of the first part.

Step 4: We prove the assertions (a) and (b) of the theorem. Assume that $g := g_2 \in H^1(\mathbb{R}^N \setminus \overline{\Omega})$, $g_1 \in H^{1/2}(\partial\Omega)$, and that $g_2|_{\partial\Omega} = g_1$.

(a) Let $w \in H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ be a weak solution of (3.1). It follows from the definition that $w = g_2$ in $\mathbb{R}^N \setminus \overline{\Omega}$, $w|_{\partial\Omega} = g_2|_{\partial\Omega} = g_1$ on $\partial\Omega$, and

$$\int_{\Omega} \nabla w \cdot \nabla v \, dx + \mathcal{F}(w, v) = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad (3.8)$$

for every $v \in \mathbb{V}$. Since $v = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\ &= \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \overline{\Omega})^2} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy. \end{aligned} \quad (3.9)$$

Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Notice that both Δv and $(-\Delta)^s v$ belong to $L^2(\Omega)$, $\partial_\nu v \in L^2(\partial\Omega)$, and $\mathcal{N}_s v \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$. Therefore, using (3.8), (3.9), the integration by parts formulas (2.7)-(2.14), we get that for every $v \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned} & \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_{\Omega} \nabla w \cdot \nabla v \, dx + \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\ &= \int_{\Omega} w(-\Delta)v + \int_{\partial\Omega} w \partial_\nu v \, d\sigma + \int_{\Omega} w(-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \overline{\Omega}} w \mathcal{N}_s v \, dx \\ &= \int_{\Omega} w \mathcal{L}v + \int_{\partial\Omega} g_1 \partial_\nu v \, d\sigma + \int_{\mathbb{R}^N \setminus \overline{\Omega}} g_2 \mathcal{N}_s v \, dx. \end{aligned} \quad (3.10)$$

Since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in \mathbb{V} , we have that (3.10) remains true for every $v \in \mathbb{V}$. Thus, w is a very-weak solution of (3.1).

(b) Finally, let w be a very-weak solution of (3.1) and assume that $w \in H^1(\mathbb{R}^N)$. Since $w = g_2$ in $\mathbb{R}^N \setminus \overline{\Omega}$, we have that $g_2 \in H^1(\mathbb{R}^N \setminus \overline{\Omega})$. Let $\tilde{g} \in H^1(\mathbb{R}^N)$ be such that $\tilde{g}|_{\mathbb{R}^N \setminus \overline{\Omega}} = g_2$. Then clearly $(w - \tilde{g}) \in H_0^1(\Omega)$. Since w is a very-weak solution of (3.1), it follows from the definition that for every $v \in \mathbb{V}$,

$$\int_{\Omega} w \mathcal{L}v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\partial\Omega} g_1 \partial_\nu v \, d\sigma - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g_2 \mathcal{N}_s v \, dx. \quad (3.11)$$

In particular, (3.11) holds for every $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Let then $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Since $v \in H^1(\mathbb{R}^N)$ and $v = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, it follows from (2.7) and (3.9) that

$$\begin{aligned} & \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\ &= \int_{\Omega} w(-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \overline{\Omega}} w \mathcal{N}_s v \, dx = \int_{\Omega} w(-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \overline{\Omega}} g_2 \mathcal{N}_s v \, dx. \end{aligned} \quad (3.12)$$

It also follows from (2.14) that

$$\begin{aligned} \int_{\Omega} \nabla w \cdot \nabla v \, dx &= - \int_{\Omega} w \Delta v \, dx + \int_{\partial\Omega} w \partial_\nu v \, d\sigma \\ &= - \int_{\Omega} w \Delta v \, dx + \int_{\partial\Omega} g_1 \partial_\nu v \, d\sigma. \end{aligned} \quad (3.13)$$

Combining (3.11)-(3.12) and (3.13), we get that for every $v \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_{\Omega} \nabla w \cdot \nabla v \, dx + \mathcal{F}(w, v) = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (3.14)$$

Since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, we have that (3.14) remains true for every $v \in H_0^1(\Omega)$. We have shown that w is a weak solution of (3.1) and the proof is finished. \square

We conclude this section with the following remark.

Remark 3.8. We observe the following facts.

- (a) We notice that in Definition 3.6 of very-weak solutions, we do not require that the function w has a well-defined trace on $\partial\Omega$ and that $w|_{\partial\Omega} = g_1$, for that reason the regularity of w cannot be improved.
- (b) But if w has a well-defined trace on $\partial\Omega$ and $w|_{\partial\Omega} = g_1 \in L^2(\partial\Omega)$, then the regularity of w can be improved. Indeed, using well-known trace theorems (see e.g. [31]) we can deduce that $w \in L^2(\mathbb{R}^N) \cap H^{1/2}(\Omega)$.
- (c) Let \mathbb{V} be the space defined in (3.6). If $0 < s \leq 3/4$, then $\mathbb{V} \subset H^2(\Omega) \cap H_0^1(\Omega)$. Indeed, let $\varphi \in \mathbb{V}$. If $0 < s < 1/2$, it follows from the proof of Theorem 3.7 Step 1 that $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. If $1/2 \leq s \leq 3/4$, then the proof of Theorem 3.7 Step 1 shows again that $\varphi \in H^{3-2s} \cap H_0^1(\Omega)$. Using [37], we get that, in fact $(-\Delta)^s \varphi \in L^2(\Omega)$. This implies that $\Delta \varphi \in L^2(\Omega)$. Thus, $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ by using elliptic regularity results for the Laplace operator.
- (d) Consider the following Dirichlet problem: Find $u \in H_0^1(\Omega)$ satisfying

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

Due to the presence of the fractional Laplace operator $(-\Delta)^s$, even if f is smooth, classical bootstrap argument cannot be used to improve the regularity of the solution u . This follows from the fact that even if f is smooth enough, if $1/2 \leq s < 1$, then a function $v \in H_0^s(\Omega)$ satisfying $(-\Delta)^s v = f$ in Ω only belongs to $\cap_{\varepsilon > 0} H^{2s-\varepsilon}(\Omega)$ and does not belong to $H^{2s}(\Omega)$. We refer to the papers [14, 48] for more details on this topic. This suggests that for functions in the space \mathbb{V} given in (3.6), the regularity discussed in Step 1 in the proof of Theorem 3.7 cannot be improved. At least, we do not know how to improve the regularity of functions belonging to \mathbb{V} .

- (e) In the case $3/4 < s < 1$, if the function g_1 is smooth, says, $g_1 \in H^{2s-3/2}(\partial\Omega)$, then we may replace (3.4) in the definition of very-weak solutions by the expression:

$$\begin{aligned} \int_{\Omega} w \mathcal{L}\varphi \, dx &= \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \langle g_1, \partial_{\nu}\varphi \rangle_{H^{2s-3/2}(\partial\Omega), H^{3/2-2s}(\partial\Omega)} \\ &\quad - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g_2 \mathcal{N}_s \varphi \, dx \end{aligned}$$

holds, for every $\varphi \in \mathbb{V}$. In that case, Theorem 3.7 will be valid for every $0 < s < 1$. But recall that the main objective of the paper is to study the minimization problem (1.1a) and our control function u_1 does not enjoy such a regularity.

3.2. The parabolic problem. First, we consider the following auxiliary problem:

$$\begin{cases} \phi_t + \mathcal{L}\phi = f & \text{in } Q, \\ \phi = 0 & \text{in } \Sigma, \\ \phi(\cdot, 0) = \phi_0, & \text{in } \Omega. \end{cases} \quad (3.15)$$

We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Here is our notion of weak solutions to the system (3.15).

Definition 3.9. Let $f \in L^2(Q)$ and $\phi_0 \in L^2(\Omega)$. We shall say that a function $\phi \in \mathbb{U} := L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$ is a weak solution to (3.15), if $\phi(\cdot, 0) = \phi_0$ a.e. in Ω and the equality

$$\langle \phi_t, \zeta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \nabla \phi \cdot \nabla \zeta \, dx + \mathcal{F}(\phi, \zeta) = \int_{\Omega} f \zeta \, dx \quad (3.16)$$

holds, for every $\zeta \in H_0^1(\Omega)$ and almost every $t \in (0, T)$.

Remark 3.10. It is worthwhile noticing that if $\phi \in \mathbb{U}$ is a weak solution of (3.15) with $f \in L^2(Q)$, then $\phi \in W(0, T)$. Thus, by (2.22) $\phi \in C([0, T]; L^2(\Omega))$, so that $\phi(\cdot, 0) = \phi_0$ a.e. in Ω makes sense.

Throughout the remainder of the article, we shall let

$$\mathbb{U} := L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega)).$$

Now, we are in position to state the well-posedness of (3.15).

Theorem 3.11. *Let $f \in L^2(Q)$ and $\phi_0 \in L^2(\Omega)$. Then, there exists a unique weak solution $\phi \in \mathbb{U}$ to (3.15) in the sense of Definition 3.9. In addition, there is a constant $C > 0$ such that*

$$\|\phi\|_{\mathbb{U}} \leq C (\|\phi_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}). \quad (3.17)$$

Proof. We prove the result in several steps. We shall use semigroups theory.

Step 1: Consider the bilinear form $\mathbb{E} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ on $L^2(\Omega)$ given by

$$\mathbb{E}(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \mathcal{F}(u, v). \quad (3.18)$$

Using the embedding (2.17), it is easy to see that the form \mathbb{E} is continuous in the sense that there is a constant $C > 0$ such that for every $u, v \in H_0^1(\Omega)$ we have,

$$|\mathbb{E}(u, v)| \leq C \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

We claim that the form \mathbb{E} is closed. Indeed, let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $H_0^1(\Omega)$ such that

$$\lim_{n, m \rightarrow \infty} \left(\mathbb{E}(u_n - u_m, u_n - u_m) + \|u_n - u_m\|_{L^2(\Omega)}^2 \right) = 0. \quad (3.19)$$

It follows from (3.19) that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Hilbert space $H_0^1(\Omega)$. Therefore, there is an $u \in H_0^1(\Omega)$ such that $u_n \rightarrow u$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$. Hence, $u_n \rightarrow u$ in $H_0^s(\Omega)$ as $n \rightarrow \infty$, by using the continuous embedding (2.17). This implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}(u_n - u, u_n - u) = 0.$$

Thus, the form \mathbb{E} is closed and we have proved the claim.

It is easy to see that the form \mathbb{E} is coercive in the sense that there is a constant $C > 0$ such that

$$\mathbb{E}(u, u) \geq C \|u\|_{H_0^1(\Omega)}^2, \quad \forall u \in H_0^1(\Omega).$$

Step 2: Let \mathbb{A} be the selfadjoint operator on $L^2(\Omega)$ associated with \mathbb{E} in the sense that

$$\begin{cases} D(\mathbb{A}) := \left\{ u \in H_0^1(\Omega) : \exists f \in L^2(\Omega), \mathbb{E}(u, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \right\}, \\ \mathbb{A}u = f. \end{cases} \quad (3.20)$$

Using an integration by parts argument and the results obtained in [10, 21], we can show that

$$D(\mathbb{A}) := \left\{ u \in H_0^1(\Omega) : (\mathcal{L}u)|_\Omega \in L^2(\Omega) \right\}, \quad \mathbb{A}u = (\mathcal{L}u)|_\Omega \text{ in } \Omega. \quad (3.21)$$

We have shown that the system (3.15) can be rewritten as the following abstract Cauchy problem

$$\begin{cases} \phi_t + \mathbb{A}\phi = f & \text{in } Q, \\ \phi(\cdot, 0) = \phi_0 & \text{in } \Omega. \end{cases} \quad (3.22)$$

Since the form \mathbb{E} is non-negative, continuous, closed and $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, it follows that the operator $-\mathbb{A}$ generates a strongly continuous semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ on $L^2(\Omega)$. This implies that for every $f \in L^2(Q)$ the Cauchy problem (3.22), hence (3.15), has a unique strong solution $\phi \in \mathbb{U}$ given for a.e. $x \in \Omega$ and a.e. $t \in (0, T)$ by

$$\phi(x, t) = e^{-t\mathbb{A}}\phi_0(x) + \int_0^t e^{-(t-\tau)\mathbb{A}}f(x, \tau) \, d\tau.$$

Step 3: It remains to prove (3.17). First, taking $\zeta = \phi$ as a test function in (3.16), integrating over $(0, T)$, and using Young's inequality, we get that for every $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{2} \|\phi(\cdot, T)\|_{L^2(\Omega)}^2 + \int_0^T \int_\Omega |\nabla \phi|^2 \, dx \, dt + \int_0^T \mathcal{F}(\phi, \phi) \, dt \\ &= \frac{1}{2} \|\phi_0\|_{L^2(\Omega)}^2 + \int_Q f \phi \, dx \, dt \\ &\leq \frac{1}{2} \|\phi_0\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|\phi\|_{L^2(Q)}^2 \\ &\leq \frac{1}{2} \|\phi_0\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|\phi\|_{L^2((0, T); H_0^1(\Omega))}^2. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we can deduce that there is a constant $C > 0$ such that

$$\|\phi(T)\|_{L^2(\Omega)}^2 + \|\phi\|_{L^2((0, T); H_0^1(\Omega))}^2 \leq C \left(\|\phi_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right). \quad (3.23)$$

Second, it follows from (3.16) and (3.23) that there is a constant $C > 0$ such that

$$\begin{aligned} \left| \int_0^T \langle \phi_t, \zeta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt \right| &\leq C \left(\|\phi\|_{L^2((0, T); H_0^1(\Omega))} + \|f\|_{L^2(Q)} \right) \|\zeta\|_{L^2((0, T); H_0^1(\Omega))} \\ &\leq C \left(\|\phi_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} \right) \|\zeta\|_{L^2((0, T); H_0^1(\Omega))}. \end{aligned} \quad (3.24)$$

Dividing both sides of (3.24) by $\|\zeta\|_{L^2((0,T);H_0^1(\Omega))}$ and taking the supremum over all functions $\zeta \in L^2((0,T);H_0^1(\Omega))$, we get that

$$\|\phi_t\|_{L^2((0,T);H^{-1}(\Omega))} \leq C (\|\phi_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}). \quad (3.25)$$

Combining (3.23)-(3.25) we get the estimate (3.17) and the proof is finished. \square

Remark 3.12. We mention that if in (3.15) $\phi_0 \in D(\mathbb{A})$, then the strong solution ϕ becomes a classical solution, and hence, enjoys the following additional regularity: $\phi \in C([0,T];D(\mathbb{A})) \cap H^1((0,T);L^2(\Omega))$. We refer to [8, 12] and the references therein for more details on semigroups theory and abstract Cauchy problems.

Next, we give further qualitative properties of the operator \mathbb{A} and the semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ constructed above. Even if all these results will not be used in the present paper, they are interesting on their own and deserve to be known by the mathematics community working in the field.

Proposition 3.13. *The operator \mathbb{A} has a compact resolvent. Its eigenvalues form a non-decreasing sequence of real numbers $(\lambda_n)_{n \in \mathbb{N}}$ satisfying*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = \infty. \quad (3.26)$$

The semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ is submarkovian and ultracontractive.

Proof. We prove the results in several steps.

Step 1: Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we have that the operator \mathbb{A} has a compact resolvent. Therefore, its spectrum is composed with eigenvalues satisfying $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Since the form \mathbb{E} is coercive, it follows that the first eigenvalue λ_1 is strictly positive. Thus, $(\lambda_n)_{n \in \mathbb{N}}$ satisfies (3.26).

Step 2: We claim that the semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ is positivity-preserving in the sense

$$0 \leq u \in L^2(\Omega) \quad \text{implies} \quad e^{-t\mathbb{A}}u \geq 0, \quad \forall t \geq 0. \quad (3.27)$$

The First Beurling-Deny criterion [22, Theorem 1.3.1] shows that (3.27) is equivalent to

$$u \in H_0^1(\Omega) \Rightarrow u^+ := \max\{u, 0\} \in H_0^1(\Omega) \quad \text{and} \quad \mathbb{E}(u^+, u^-) \leq 0. \quad (3.28)$$

Indeed, let $u \in H_0^1(\Omega)$ and set $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. It follows from [35, Chapter 1] that $u^+, u^- \in H_0^1(\Omega)$ and

$$\int_{\Omega} \nabla(u^+) \cdot \nabla(u^-) \, dx = \int_{\Omega} (\nabla u) \chi_{\{u \geq 0\}} (\nabla u) \chi_{\{u \leq 0\}} \, dx = 0.$$

It has been shown in [29] that $\mathcal{F}(u^+, u^-) \leq 0$. Thus,

$$\mathbb{E}(u^+, u^-) \leq 0, \quad \forall u \in H_0^1(\Omega).$$

We have shown that $(e^{-t\mathbb{A}})_{t \geq 0}$ is positivity-preserving.

Step 3: We claim that $(e^{-t\mathbb{A}})_{t \geq 0}$ is L^∞ -contractive in the sense

$$\|e^{-t\mathbb{A}}u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}, \quad \forall t \geq 0 \quad \text{and} \quad u \in L^2(\Omega) \cap L^\infty(\Omega) = L^\infty(\Omega).$$

For this, let $u \in H_0^1(\Omega)$ be such that $u \geq 0$. It follows again from [35, Chapter 1] that $u \wedge 1 \in H_0^1(\Omega)$. A simple calculation gives

$$\int_{\Omega} |\nabla(u \wedge 1)|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx.$$

By [52, Lemma 2.7], we have that $\mathcal{F}(u \wedge 1, u \wedge 1) \leq \mathcal{F}(u, u)$. We have shown that

$$\mathbb{E}(u \wedge 1, u \wedge 1) \leq \mathbb{E}(u, u), \quad \forall 0 \leq u \in H_0^1(\Omega).$$

It follows from the second Beurling–Deny criterion [22, Theorem 1.3.2] that $(e^{-t\mathbb{A}})_{t \geq 0}$ is L^∞ -contractive. We have shown that the semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ is submarkovian. As a consequence, $(e^{-t\mathbb{A}})_{t \geq 0}$ can be extended to consistent semigroups on $L^p(\Omega)$ ($1 \leq p \leq \infty$). Each semigroup is strongly continuous on $L^p(\Omega)$ if $1 \leq p < \infty$, and bounded analytic if $1 < p < \infty$ (see e.g. [22, Chapter 1] for more details).

Step 4: It remains to show that the semigroup is ultracontractive. Indeed, notice that we have the following continuous embedding:

$$H_0^1(\Omega) \hookrightarrow L^r(\Omega) \quad \text{with } r = \begin{cases} \frac{2N}{N-2} & \text{if } N > 2 \\ 1 \leq r < \infty & \text{if } N \leq 2. \end{cases}$$

That is, there is a constant $C > 0$ such that for every $u \in H_0^1(\Omega)$,

$$\|u\|_{L^r(\Omega)}^2 \leq C\mathbb{E}(u, u). \quad (3.29)$$

Using the abstract results contained in [22, 46], the estimate (3.29) is equivalent to the ultracontractivity of $(e^{-t\mathbb{A}})_{t \geq 0}$. More precisely, for every $1 \leq p \leq q \leq \infty$, the operator $e^{-t\mathbb{A}}$ maps $L^p(\Omega)$ into $L^q(\Omega)$ and there is a constant $C > 0$ such that for every $t > 0$ and $u \in L^p(\Omega)$,

$$\|e^{-t\mathbb{A}}u\|_{L^q(\Omega)} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\Omega)}.$$

The proof is finished. \square

We mention that the results obtained for the homogeneous problem (3.15) are classical and they follow from semigroups theory associated with bilinear operators studied in [22, 46] and the properties of the fractional Laplace operator investigated in [11, 24, 26, 29, 48, 52] and their references.

Next, we consider the nonhomogeneous boundary-exterior-initial value problem (1.1b), that is,

$$\begin{cases} \phi_t + \mathcal{L}\phi = 0 & \text{in } Q, \\ \phi = u_1 & \text{on } \Gamma, \\ \phi = u_2 & \text{in } \Sigma, \\ \phi(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.30)$$

First, we consider smooth boundary-exterior data.

Definition 3.14. Let $u_2 \in H^1((0, T); H^1(\mathbb{R}^N \setminus \overline{\Omega}))$ and $u_1 \in L^2((0, T); H^{1/2}(\partial\Omega))$ be such that $u_2|_{\Gamma} = u_1$. Let $\tilde{u} \in H^1((0, T); H^1(\mathbb{R}^N))$ be such that $\tilde{u} = u_2$ in Σ . A function $\phi \in L^2((0, T); H^1(\mathbb{R}^N)) \cap H^1((0, T); H^{-1}(\Omega))$ is said to be a weak solution of the system (3.30) if $\phi - \tilde{u} \in L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$ and the identity

$$\langle \phi_t, \zeta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \nabla \phi \cdot \nabla \zeta dx + \mathcal{F}(\phi, \zeta) = 0 \quad (3.31)$$

holds, for every $\zeta \in H_0^1(\Omega)$ and almost every $t \in (0, T)$.

Remark 3.15. In Definition 3.14, we have assumed that the function u_2 (hence, the solution ϕ) has a well-defined trace that coincides with $u_1 \in L^2(\Gamma)$. In that case, the condition on Γ can be dropped and the system is still well-posed. That is, (3.30) becomes

$$\begin{cases} \phi_t + \mathcal{L}\phi = 0 & \text{in } Q, \\ \phi = u_2 & \text{in } \Sigma, \\ \phi(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

We have the following existence result.

Theorem 3.16. *Let u_2 and u_1 be as in Definition 3.14. Then, the system (3.30) has a unique weak solution $\phi \in L^2((0, T); H^1(\mathbb{R}^N)) \cap H^1((0, T); H^{-1}(\Omega))$ in the sense of Definition 3.14. In addition, there is a constant $C > 0$ such that*

$$\|\phi\|_{L^2((0, T); H^1(\mathbb{R}^N)) \cap H^1((0, T); H^{-1}(\Omega))} \leq C \|u_2\|_{H^1((0, T); H^1(\mathbb{R}^N \setminus \bar{\Omega}))} \quad (3.32)$$

Proof. We prove the result in several steps.

Step 1: First, assume that u_2 does not depend on the time variable. Let $\tilde{u} \in H^1(\mathbb{R}^N)$ be the unique weak solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}\tilde{u} = 0 & \text{in } \Omega, \\ \tilde{u} = u_2 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}. \end{cases} \quad (3.33)$$

That is, $\tilde{u} \in H^1(\mathbb{R}^N)$, $\tilde{u}|_{\mathbb{R}^N \setminus \bar{\Omega}} = u_2$, and \tilde{u} solves (3.33) in the sense that

$$\int_{\Omega} \nabla \tilde{u} \cdot \nabla v \, dx + \mathcal{F}(\tilde{u}, v) = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

and there is a constant $C > 0$ such that

$$\|\tilde{u}\|_{H^1(\mathbb{R}^N)} \leq C \|u_2\|_{H^1(\mathbb{R}^N \setminus \bar{\Omega})}. \quad (3.34)$$

The proof of the existence, uniqueness of such a solution \tilde{u} and the continuous dependence on the datum u follow from Proposition 3.4 by taking $f = 0$ in (3.1).

Step 2: Second, assume that u_2 depends on both variables (x, t) and satisfies the assumption of the theorem. Let \tilde{u} be the associated solution of (3.33). It follows from the above argument that $\tilde{u} \in H^1((0, T); H^1(\mathbb{R}^N))$. Let $\Phi := \phi - \tilde{u}$. Then, it is clear that $\Phi|_{\Sigma} = 0$. In addition, a simple calculation shows that

$$\begin{cases} \Phi_t + \mathcal{L}\Phi = -\tilde{u}_t & \text{in } Q, \\ \Phi = 0 & \text{in } \Sigma, \\ \Phi(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.35)$$

Let

$$\mathbb{H} := L^2((0, T); H^1(\mathbb{R}^N)) \cap H^1((0, T); H^{-1}(\Omega)).$$

Since $\tilde{u}_t \in L^2((0, T); H^1(\mathbb{R}^N))$, using Theorem 3.11, we get that there exists a unique $\Phi \in \mathbb{H}$ solving (3.35). Thus, the unique solution ϕ of (3.30) is given by $\phi = \Phi + \tilde{u}$.

Step 3: It remains to show (3.32). Firstly, since $\Phi = 0$ in Σ and $\Phi(\cdot, 0) = 0$ in Ω , it follows from (3.17) that there is a constant $C > 0$ such that

$$\|\Phi\|_{\mathbb{H}} \leq C \|\tilde{u}_t\|_{L^2((0, T); H^1(\mathbb{R}^N))}. \quad (3.36)$$

Secondly, it follows from (3.34) that there is a constant $C > 0$ such that

$$\|\tilde{u}\|_{L^2((0,T);H^1(\mathbb{R}^N))} \leq C \|u_2\|_{L^2((0,T);H^1(\mathbb{R}^N \setminus \bar{\Omega}))}. \quad (3.37)$$

Thirdly, combining (3.36)-(3.37) and using (3.34), we get that there is a constant $C > 0$ such that

$$\begin{aligned} \|\phi\|_{\mathbb{H}} &= \|\Phi + \tilde{u}\|_{\mathbb{H}} \leq \|\Phi\|_{\mathbb{H}} + \|\tilde{u}\|_{\mathbb{H}} \\ &\leq C \left(\|\tilde{u}_t\|_{L^2((0,T);H^1(\mathbb{R}^N))} + \|u_2\|_{L^2((0,T);H^1(\mathbb{R}^N \setminus \bar{\Omega}))} \right) \\ &\leq C \left(\|u_t\|_{L^2((0,T);H^1(\mathbb{R}^N \setminus \bar{\Omega}))} + \|u_2\|_{L^2((0,T);H^1(\mathbb{R}^N \setminus \bar{\Omega}))} \right) \\ &= C \|u_2\|_{H^1((0,T);H^1(\mathbb{R}^N \setminus \bar{\Omega}))}. \end{aligned}$$

We have shown (3.32) and the proof is finished. \square

Finally, we consider singular boundary-exterior data. It is worthwhile noticing that since we are considering data $u_1 \in L^2(\Gamma)$ and $u_2 \in L^2(\Sigma)$, the system (3.30) cannot have weak solutions in the sense of Definition 3.14. For this reason, we need to introduce the notion of very-weak solutions as in the elliptic case.

Definition 3.17. Let $u_1 \in L^2(\Gamma)$ and $u_2 \in L^2(\Sigma)$. A function $\phi \in L^2((0,T) \times \mathbb{R}^N)$ is said to be a very weak-solution (or a solution by transposition) to the system (3.30) if the identity

$$\int_Q \phi \left(-\varphi_t + \mathcal{L}\varphi \right) dx dt = - \int_{\Gamma} u_1 \partial_\nu \varphi d\sigma dt - \int_{\Sigma} u_2 \mathcal{N}_s \varphi dx dt, \quad (3.38)$$

holds, for every $\varphi \in L^2((0,T), \mathbb{V}) \cap H^1((0,T); L^2(\Omega))$ with $\varphi(\cdot, T) = 0$ a.e. in Ω , where we recall that $\mathbb{V} := D(\mathbb{A})$ is the space defined in (3.6).

As in the elliptic case, Definition 3.17 of very-weak solutions makes sense if every function $\varphi \in L^2((0,T), \mathbb{V}) \cap H^1((0,T); L^2(\Omega))$ satisfies $\partial_\nu \varphi \in L^2(\Gamma)$, and $\mathcal{N}_s \varphi \in L^2(\Sigma)$.

We have the following result.

Theorem 3.18. Let $0 < s \leq 3/4$, $u_1 \in L^2(\Gamma)$, and $u_2 \in L^2(\Sigma)$. Then, there exists a unique very-weak solution $\phi \in L^2((0,T) \times \mathbb{R}^N)$ to (3.30) according to Definition 3.17, and there is a constant $C > 0$ such that

$$\|\phi\|_{L^2((0,T) \times \mathbb{R}^N)} \leq C \left(\|u_1\|_{L^2(\Gamma)} + \|u_2\|_{L^2(\Sigma)} \right). \quad (3.39)$$

Moreover, if u_1 and u_2 are as in Definition 3.14, then the following assertions hold.

- (a) Every weak solution of (3.30) is also a very-weak solution.
- (b) Every very-weak solution of (3.30) that belongs to \mathbb{H} is also a weak solution.

Proof. We prove the result in several steps.

Step 1: For a given $\eta \in L^2(Q)$, we consider the following dual problem associated to (3.30)

$$\begin{cases} -\varphi_t + \mathcal{L}\varphi = \eta & \text{in } Q, \\ \varphi = 0 & \text{in } \Gamma, \\ \varphi = 0 & \text{in } \Sigma, \\ \varphi(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.40)$$

Using semigroups theory as in the proof of Theorem 3.11, we can deduce that for the every $\eta \in L^2(Q)$ the system (3.40) has a unique weak solution $\varphi \in L^2((0, T), \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ given for a.e. $t \in (0, T)$ by

$$\varphi(\cdot, t) = \int_t^T e^{-(T-\tau)\mathbb{A}} \eta(\cdot, \tau) \, d\tau \quad \text{in } \Omega.$$

This implies that $\varphi_t \in L^2(Q)$. Since $\varphi \in L^2((0, T), \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$, we have that $\mathcal{N}_s \varphi \in L^2(\Sigma)$ and $\partial_\nu \varphi \in L^2(\Gamma)$ under the assumption $0 < s \leq 3/4$. We define the mapping

$$\Lambda : L^2(Q) \rightarrow L^2(\Gamma) \times L^2(\Sigma), \quad \eta \mapsto \Lambda \eta := (-\partial_\nu \varphi, -\mathcal{N}_s \varphi).$$

It is clear that Λ is linear. In addition, using the continuous dependence on the data of solutions to (3.40), we get that there is a constant $C > 0$ such that

$$\|\Lambda \eta\|_{L^2(\Sigma) \times L^2(\Gamma)} \leq C (\|\mathcal{N}_s \varphi\|_{L^2(\Sigma)} + \|\partial_\nu \varphi\|_{L^2(\Gamma)}) \leq C \|\varphi\|_{L^2((0, T); \mathbb{V})} \leq C \|\eta\|_{L^2(Q)}. \quad (3.41)$$

We have shown that Λ is also continuous.

Next, let $\phi := \Lambda^* u$ in Q and $\phi := u_2$ in Σ . Calculating we get the following:

$$\begin{aligned} \int_Q \phi \eta \, dx \, dt &= \int_Q \phi (-\varphi_t + \mathcal{L} \varphi) \, dx \, dt = \int_Q (\Lambda^* u) \eta \, dx \, dt \\ &= - \int_\Sigma u_2 \mathcal{N}_s \varphi \, dx \, dt - \int_\Gamma u_1 \partial_\nu \varphi \, d\sigma \, dt. \end{aligned} \quad (3.42)$$

We have constructed a function $\phi \in L^2((0, T); L^2(\mathbb{R}^N))$ that solves (3.30) in the very weak-sense.

Next, we show uniqueness. Assume that (3.30) has two very-weak solutions ϕ_1 and ϕ_2 with the same boundary datum u_1 and the same exterior datum u_2 . It follows from the definition that

$$\int_Q (\phi_1 - \phi_2) (-\varphi_t + \mathcal{L} \varphi) \, dx \, dt = 0, \quad (3.43)$$

for every $\varphi \in L^2((0, T), \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ with $\varphi(T, \cdot) = 0$ a.e. in Ω . Since for every $\eta \in L^2(Q)$ the system (3.40) has a unique weak solution φ , it follows from (3.43) and (3.40) that

$$\int_Q (\phi_1 - \phi_2) \eta \, dx \, dt = 0,$$

for every $\eta \in L^2(Q)$. It follows from the fundamental lemma of the calculus of variation that $\phi_1 = \phi_2$ a.e. in Q . Since $\phi_1 = \phi_2 = u_2$ a.e. in Σ , we can conclude that $\phi_1 = \phi_2$ a.e. in $(0, T) \times \mathbb{R}^N$, and we have shown uniqueness of very-weak solutions.

Step 2: Next, we show the estimate (3.39). Using (3.42) and (3.41), we get that there is a constant $C > 0$ such that

$$\begin{aligned} \left| \int_Q \phi \eta \, dx \, dt \right| &\leq C (\|u_2\|_{L^2(\Sigma)} \|\mathcal{N}_s \varphi\|_{L^2(\Sigma)} + \|u_1\|_{L^2(\Gamma)} \|\partial_\nu \varphi\|_{L^2(\Gamma)}) \\ &\leq C (\|u_2\|_{L^2(\Sigma)} + \|u_1\|_{L^2(\Gamma)}) (\|\partial_\nu \varphi\|_{L^2(\Gamma)} + \|\mathcal{N}_s \varphi\|_{L^2(\Sigma)}) \\ &\leq C (\|u_2\|_{L^2(\Sigma)} + \|u_1\|_{L^2(\Gamma)}) \|\eta\|_{L^2(Q)}. \end{aligned} \quad (3.44)$$

Dividing both sides of (3.44) by $\|\eta\|_{L^2(Q)}$ and taking the supremum over all $\eta \in L^2(Q)$, we get

$$\|\phi\|_{L^2(Q)} \leq C (\|u_2\|_{L^2(\Sigma)} + \|u_1\|_{L^2(\Gamma)}).$$

Since $\phi = u_2$ in Σ , it follows from the preceding estimate that

$$\|\phi\|_{L^2((0,T) \times \mathbb{R}^N)} \leq C (\|u_2\|_{L^2(\Sigma)} + \|u_1\|_{L^2(\Gamma)})$$

and we have shown the estimate (3.39).

Step 3: Next, we prove the last two assertions of the theorem. For this, we assume that u_1 and u_2 are as in Definition 3.14.

(a) Let $\phi \in \mathbb{H} := L^2(0, T); H^1(\mathbb{R}^N) \cap H^1((0, T); H^{-1}(\Omega))$ be a weak solution to (3.30). Notice that $\phi|_{\Sigma} \in L^2(\Sigma)$ and $\phi|_{\Gamma} \in L^2(\Gamma)$. It follows from the definition that $\phi = u_2$ in Σ and $\phi|_{\Gamma} = u_1$ on Γ . In particular, we have that

$$\langle \phi_t(t, \cdot), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \nabla \phi \cdot \nabla v \, dx + \mathcal{F}(\phi, v) = 0, \quad (3.45)$$

for every $v \in L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0, T); L^2(\Omega))$ with $v(\cdot, T) = 0$ a.e. in Ω , and almost every $t \in (0, T)$. Since $v(\cdot, t) = 0$ in $\mathbb{R}^N \setminus \bar{\Omega}$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(t, x) - \phi(t, y))(v(t, x) - v(t, y))}{|x - y|^{N+2s}} \, dx \, dy \\ &= \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\phi(t, x) - \phi(t, y))(v(t, x) - v(t, y))}{|x - y|^{N+2s}} \, dx \, dy. \end{aligned} \quad (3.46)$$

Using (3.45), (3.46) and the integration by parts formulas (2.7)-(2.14) (notice that the test function v is smooth enough), we get that

$$\begin{aligned} 0 &= \langle \phi_t(t, \cdot), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \nabla \phi \cdot \nabla v \, dx + \mathcal{F}(\phi, v) \\ &= \langle \phi_t(t, \cdot), v(t, \cdot) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\Omega} \phi \Delta v \, dx + \int_{\partial\Omega} \phi \partial_{\nu} v \, d\sigma \\ &\quad + \int_{\Omega} \phi (-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \Omega} \phi \mathcal{N}_s v \, dx \\ &= \langle \phi_t(t, \cdot), v(t, \cdot) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} \phi \mathcal{L} v \, dx + \int_{\partial\Omega} u_1 \partial_{\nu} v \, d\sigma + \int_{\mathbb{R}^N \setminus \Omega} u_2 \mathcal{N}_s v \, dx. \end{aligned}$$

Integrating the previous identity by parts over $(0, T)$, we get that

$$- \int_0^T (\phi(\cdot, t), v_t(\cdot, t))_{L^2(\Omega)} \, dt + \int_Q \phi \mathcal{L} v \, dx \, dt + \int_{\Gamma} u \partial_{\nu} v \, d\sigma \, dt + \int_{\Sigma} u \mathcal{N}_s v \, dx \, dt = 0.$$

Since $\phi, v \in L^2(Q)$, it follows from the preceding identity that

$$\int_Q \phi (-v_t + \mathcal{L} v) \, dx \, dt = - \int_{\Gamma} u \partial_{\nu} v \, d\sigma \, dt - \int_{\Sigma} u \mathcal{N}_s v \, dx \, dt \quad (3.47)$$

for every $v \in L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0, T); L^2(\Omega))$ with $v(\cdot, T) = 0$ a.e. in Ω . Since $L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0, T); L^2(\Omega))$ is dense in $L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$, it follows that (3.47) remains true for every $v \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ with $v(\cdot, T) = 0$ a.e. in Ω . Thus, ϕ is a very-weak solution of (3.30).

(b) Let ϕ be a very-weak solution to (3.30) and assume that $\phi \in \mathbb{H}$. Then $\phi = u_2$ in Σ and $\phi|_\Gamma = u_1$ on Γ . Let $\tilde{u} \in H^1((0, T); H^1(\mathbb{R}^N)) \cap H^1((0, T); L^2(\mathbb{R}^N))$ be such that $\tilde{u}|_\Sigma = u_2$. Then, clearly $\phi - \tilde{u} \in \mathbb{H}$. As ϕ is a very-weak solution to (3.30) and $0 < s \leq 3/4$, it follows from Definition 3.17 that for every $v \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$, we have

$$\int_Q \phi(-v_t + \mathcal{L}v) \, dx = - \int_\Gamma u_1 \partial_\nu v \, d\sigma \, dt - \int_\Sigma u_2 \mathcal{N}_s v \, dx \, dt. \quad (3.48)$$

Since $\phi \in \mathbb{H}$, $v = 0$ in Σ and $v = 0$ on Γ , using the integration by parts formulas (2.7)-(2.14) and a density argument (taking first $v \in L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0, T); L^2(\Omega))$ and then using a density argument as above), we can deduce that

$$\begin{aligned} & \int_0^T \langle \phi_t(\cdot, t), (\cdot, t)v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt + \int_Q \nabla \phi \cdot \nabla v \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} \mathcal{F}(\phi, v) \, dt \\ &= \int_Q \phi(-v_t + \mathcal{L}v) \, dx \, dt + \int_\Gamma \phi \partial_\nu v \, d\sigma \, dt + \int_\Sigma \phi \mathcal{N}_s v \, dx \, dt \\ &= \int_Q \phi(-v_t + \mathcal{L}v) \, dx \, dt + \int_\Gamma u_1 \partial_\nu v \, d\sigma \, dt + \int_\Sigma u_2 \mathcal{N}_s v \, dx \, dt. \end{aligned} \quad (3.49)$$

It follows from (3.48)-(3.49) that for every $v \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ we have

$$\int_0^T \langle \phi_t(t, \cdot), v(t, \cdot) \rangle \, dt + \int_0^T \left(\int_\Omega \nabla \phi \cdot \nabla v \, dx + \mathcal{F}(\phi, v) \right) \, dt = 0. \quad (3.50)$$

Since \mathbb{V} is dense in $H_0^1(\Omega)$ and $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$, it follows that (3.50) remains true for every $v \in L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$ with $v(\cdot, T) = 0$ a.e. in Ω . Notice that $v(\cdot, t) \in H_0^1(\Omega)$ for a.e. $t \in (0, T]$. As a result, we have that the following pointwise formulation

$$\langle \phi_t(\cdot, t), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega \nabla \phi \cdot \nabla v \, dx + \mathcal{F}(\phi, v) = 0 \quad (3.51)$$

holds for every $v \in H_0^1(\Omega)$ and a.e. $t \in (0, T)$. We have shown that ϕ is the unique weak solution of (3.30) according to Definition 3.14. The proof is finished. \square

To conclude this section we discuss the notion of very-weak solutions to the system (3.30) by taking into account the value $\phi(\cdot, T)$.

Definition 3.19. A function $\phi \in L^2((0, T) \times \mathbb{R}^N)$ is said to be a very weak-solution (or a solution by transposition) to the system (3.30) if the identity

$$\begin{aligned} \int_Q \phi(-\varphi_t + \mathcal{L}\varphi) \, dx \, dt &= \langle \phi(\cdot, T), \varphi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\quad - \int_\Gamma u_1 \partial_\nu \varphi \, d\sigma \, dt - \int_\Sigma u_2 \mathcal{N}_s \varphi \, dx \, dt, \end{aligned} \quad (3.52)$$

holds, for every $\varphi \in C([0, T], \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$.

The proof of the following result follows exactly as the proof of Theorem 3.18. We omit it for brevity.

Proposition 3.20. *Let $0 < s \leq 3/4$. Let $u_1 \in L^2(\Gamma)$ and $u_2 \in L^2(\Sigma)$. Then, there exists a unique very-weak solution $\phi \in L^2((0, T) \times \mathbb{R}^N)$ of (3.30) in the sense of Definition 3.19.*

Remark 3.21. Let us discuss the regularity of very weak solutions in the sense of Definition 3.19. Let \mathbb{V}^* denotes the dual of \mathbb{V} with respect to the pivot space $L^2(\Omega)$. Noticing that the operator \mathcal{L} maps $L^2(\Omega)$ into $\mathbb{V}^* \hookrightarrow H^{-2}(\Omega)$, and using the theory of evolution equations (see e.g. [44, Chapter III, Section 9,6]), we can show that every very-weak solution ϕ of (3.30) in the sense of Definition 3.19 belongs to $L^2((0, T) \times \mathbb{R}^N) \cap H^1((0, T), H^{-1}(\Omega))$. Thus, $\phi \in C([0, T], H^{-1}(\Omega))$ so that (3.52) makes sense for every $\varphi \in C([0, T], \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$. This fact will be used in Section 4.2 below.

4. OPTIMAL CONTROL PROBLEMS OF MIXED LOCAL-NONLOCAL PDES

The purpose of this section is to study the optimal control problem (1.1). Recall that $\mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma)$ is endowed with the norm given by

$$\|(u_1, u_2)\|_{\mathcal{Z}_D} = \left(\|u_1\|_{L^2(\Gamma)}^2 + \|u_2\|_{L^2(\Sigma)}^2 \right)^{\frac{1}{2}}. \quad (4.1)$$

We consider the following controlled equation:

$$\begin{cases} \psi_t + \mathcal{L}\psi = 0 & \text{in } Q, \\ \psi = u_1 & \text{on } \Gamma, \\ \psi = u_2 & \text{in } \Sigma, \\ \psi(\cdot, 0) = 0, & \text{in } \Omega, \end{cases} \quad (4.2)$$

where the control $(u_1, u_2) \in \mathcal{Z}_{ad}$ with \mathcal{Z}_{ad} being a closed and convex subset of \mathcal{Z}_D .

Under the assumption on the data, and $0 < s \leq 3/4$, we know from Theorem 3.18 that there exists a $\psi \in L^2((0, T) \times \mathbb{R}^N)$ which is the unique very-weak solution to (1.1) in the sense of Definition 3.17 or Definition 3.19. On the other hand, it follows from Remark 3.21 that $\psi(\cdot, T)$ exists and belongs to $H^{-1}(\Omega)$. Therefore, we can define on \mathcal{Z}_{ad} the following two cost functions:

$$J_1(u_1, u_2) := \frac{1}{2} \|\psi((u_1, u_2)) - z_d^1\|_{L^2(Q)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2 \quad (4.3)$$

and

$$J_2(u_1, u_2) := \frac{1}{2} \|\psi(T; (u_1, u_2)) - z_d^2\|_{H^{-1}(\Omega)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2, \quad (4.4)$$

where $\beta > 0$ is a real number, $z_d^1 \in L^2(Q)$, $z_d^2 \in H^{-1}(\Omega)$, $\psi := \psi(u_1, u_2)$ is the unique very-weak solution of (4.2), and

$$\|\phi\|_{H^{-1}(\Omega)}^2 = \langle (-\Delta_D)^{-1} \phi, \phi \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.$$

Here, $(-\Delta_D)^{-1} \phi = \varrho$ with ϱ the unique solution of the Dirichlet problem

$$-\Delta \varrho = \phi \text{ in } \Omega \text{ and } \varrho = 0 \text{ on } \partial\Omega.$$

We are interested in the following minimization problems:

$$\min_{(v_1, v_2) \in \mathcal{Z}_{ad}} J_i((v_1, v_2)), \quad i = 1, 2. \quad (4.5)$$

4.1. The first optimal control problem. In this section we consider the minimization problem (4.5)-(4.3) with the functional J_1 . We have the following existence result of optimal solutions.

Proposition 4.1. *Let $0 < s \leq 3/4$, $u_1 \in L^2(\Gamma)$, and $u_2 \in L^2(\Sigma)$. Let \mathcal{Z}_{ad} be a closed and convex subset of \mathcal{Z}_D , and let $\psi = \psi(u_1, u_2)$ satisfy (4.2) in the very-weak sense. Then there exists a unique control $(u_1^*, u_2^*) \in \mathcal{Z}_{ad}$ solution of*

$$\inf_{(v_1, v_2) \in \mathcal{Z}_D} J_1((v_1, v_2)). \quad (4.6)$$

Proof. Firstly, observe that if $(u_1, u_2) = (0, 0)$, then (4.2) has the unique solution $\psi(0, 0) = 0$.

Secondly, a simple calculation gives

$$\begin{aligned} J_1(v_1, v_2) &:= \frac{1}{2} \|\psi((v_1, v_2)) - z_d^1\|_{L^2(Q)}^2 + \frac{\beta}{2} \|(v_1, v_2)\|_{\mathcal{Z}_D}^2 \\ &= \frac{1}{2} \|\psi(v_1, v_2)\|_{L^2(Q)}^2 - \int_Q \psi(v_1, v_2) z_d^1 dx dt + \frac{1}{2} \|z_d^1\|_{L^2(Q)}^2 + \frac{\beta}{2} \|(v_1, v_2)\|_{\mathcal{Z}_D}^2 \\ &= \pi((v_1, v_2), (v_1, v_2)) - L((v_1, v_2)) + \|z_d^1\|_{L^2(Q)}^2 \end{aligned}$$

where

$$\pi((u_1, u_2), (v_1, v_2)) := \frac{1}{2} \int_Q \psi(u_1, u_2) \psi(v_1, v_2) dx dt + \frac{\beta}{2} \int_{\Gamma} v_1 u_1 dx dt + \frac{\beta}{2} \int_{\Sigma} v_2 u_2 dx dt,$$

and

$$L(v_1, v_2) := \int_Q \psi(v_1, v_2) z_d^1 dx dt.$$

It is clear that $\pi((\cdot, \cdot), (\cdot, \cdot))$ is a bilinear and symmetric functional.

(a) We claim that $\pi((\cdot, \cdot), (\cdot, \cdot))$ is continuous on \mathcal{Z}_{ad} . Indeed, let $(u_1, u_2), (v_1, v_2) \in \mathcal{Z}_{ad}$. Using (3.39), we get that there is a constant $C > 0$ such that

$$\begin{aligned} |\pi((u_1, u_2), (v_1, v_2))| &\leq \frac{1}{2} \|\psi(u_1, u_2)\|_{L^2(Q)} \|\psi(v_1, v_2)\|_{L^2(Q)} \\ &\quad + \frac{\beta}{2} \|v_1\|_{L^2(\Gamma)} \|u_1\|_{L^2(\Gamma)} + \frac{\beta}{2} \|v_2\|_{L^2(\Sigma)} \|u_2\|_{L^2(\Sigma)} \\ &\leq C \left(\|u_1\|_{L^2(\Gamma)}^2 + \|u_2\|_{L^2(\Sigma)}^2 \right)^{1/2} \left(\|v_1\|_{L^2(\Gamma)}^2 + \|v_2\|_{L^2(\Sigma)}^2 \right)^{1/2} \\ &\quad + \frac{\beta}{2} \left(\|v_1\|_{L^2(\Gamma)}^2 + \|v_2\|_{L^2(\Sigma)}^2 \right)^{1/2} \left(\|u_1\|_{L^2(\Gamma)}^2 + \|u_2\|_{L^2(\Sigma)}^2 \right)^{1/2} \\ &\leq \left(C + \frac{\beta}{2} \right) \|(v_1, v_2)\|_{\mathcal{Z}_D} \|(u_1, u_2)\|_{\mathcal{Z}_D}, \end{aligned}$$

and the claim is proved.

(b) We claim that $\pi((\cdot, \cdot), (\cdot, \cdot))$ is coercive on \mathcal{Z}_{ad} . Indeed, for all $(u_1, u_2), (v_1, v_2) \in \mathcal{Z}_{ad}$, we have

$$\pi((u_1, u_2), (u_1, u_2)) = \frac{1}{2} \|\psi(u_1, u_2)\|_{L^2(Q)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2 \geq \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2,$$

and we have shown the coercivity.

(c) Finally, using (3.39), we get that there is a constant $C > 0$ such that for all $(v_1, v_2) \in \mathcal{Z}_{ad}$,

$$|L(v_1, v_2)| \leq \|\psi(v_1, v_2)\|_{L^2(Q)} \|z_d^1\|_{L^2(Q)} \leq C \|z_d^1\|_{L^2(Q)} \|(v_1, v_2)\|_{\mathcal{Z}_D}^2.$$

We have shown that the functional L is linear and continuous on \mathcal{Z}_{ad} .

Using the abstract results in [44, Chapter II, Section 1.2], we can then deduce that there exists a unique $(u_1^*, u_2^*) \in \mathcal{Z}_{ad}$ solution to (4.6). The proof is complete. \square

Next, we characterize the optimality conditions.

Theorem 4.2. *Let $0 < s \leq 3/4$ and $\mathbb{U} = L^2((0, T); H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$. Let also \mathcal{Z}_{ad} be a closed convex subspace of \mathcal{Z}_D , and (u_1^*, u_2^*) be the minimizer (4.6) over \mathcal{Z}_{ad} . Then, there exist p^* and ψ^* such that the triplet $(\psi^*, p^*, (u_1^*, u_2^*)) \in L^2((0, T) \times \mathbb{R}^N) \times \mathbb{U} \times \mathcal{Z}_{ad}$ satisfies the following optimality systems:*

$$\begin{cases} \psi_t^* + \mathcal{L}\psi^* = 0 & \text{in } Q, \\ \psi^* = u_1^* & \text{in } \Gamma, \\ \psi^* = u_2^* & \text{in } \Sigma, \\ \psi^*(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \quad (4.7)$$

and

$$\begin{cases} -p_t^* + \mathcal{L}p^* = z_d^1 - \psi^* & \text{in } Q, \\ p^* = 0 & \text{in } \Sigma, \\ p^*(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \quad (4.8)$$

and

$$\int_{\Gamma} (\partial_\nu p^* + \beta u_1^*) (v_1 - u_1^*) \, d\sigma \, dt + \int_{\Sigma} (\mathcal{N}_s p^* + \beta u_2^*) (v_2 - u_2^*) \, dx \, dt \geq 0 \quad \forall (v_1, v_2) \in \mathcal{Z}_{ad}. \quad (4.9)$$

In addition,

$$(u_1^*, u_2^*) = \mathbb{P}(-\beta^{-1} \partial_\nu p^*, -\beta^{-1} \mathcal{N}_s p^*) \quad (4.10)$$

where \mathbb{P} is the projection onto the set \mathcal{Z}_{ad} .

Proof. Let $(u_1^*, u_2^*) \in \mathcal{Z}_{ad}$ be the unique solution of the minimization problem (4.6). We denote by $\psi^* := \psi^*(u_1^*, u_2^*)$ the associated state so that, ψ^* solves the system (4.7) in the very-weak sense.

Using classical duality arguments we have that (4.8) is the dual system associated with (4.7). Since $z_d^1 - \psi^* \in L^2(Q)$, it follows that (4.8) has a unique weak solution $p^* \in \mathbb{U}$.

To prove the last assertion (4.9), we write the Euler Lagrange first order optimality condition that characterizes the optimal control (u_1^*, u_2^*) as follows:

$$\lim_{\lambda \rightarrow 0} \frac{J_1(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) - J_1(u_1^*, u_2^*)}{\lambda} \geq 0, \quad \forall v := (v_1, v_2) \in \mathcal{Z}_{ad}. \quad (4.11)$$

Recall that

$$\begin{aligned} & J_1(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) \\ &= \frac{1}{2} \|\psi(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) - z_d^1\|_{L^2(Q)}^2 \\ & \quad + \frac{\beta}{2} \|u_1^* + \lambda(v_1 - u_1^*), (u_2^* + \lambda(v_2 - u_2^*))\|_{\mathcal{Z}_D}^2, \end{aligned}$$

where $\psi^\lambda := \psi(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*))$ is the unique very-weak solution of the system

$$\begin{cases} \psi_t^\lambda + \mathcal{L}\psi^\lambda = 0 & \text{in } Q, \\ \psi^\lambda = u_1^* + \lambda(v_1 - u_1^*) & \text{in } \Gamma, \\ \psi^\lambda = u_2^* + \lambda(v_2 - u_2^*) & \text{in } \Sigma, \\ \psi^\lambda(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (4.12)$$

Using the linearity of the system and the uniqueness of very-weak solutions, we get that

$$\psi^\lambda = \psi^*(u_1^*, u_2^*) + \lambda\psi(v_1 - u_1^*, v_2 - u_2^*) = \psi^* + \lambda\psi, \quad (4.13)$$

where ψ is the unique very-weak solution of

$$\begin{cases} \psi_t + \mathcal{L}\psi = 0 & \text{in } Q, \\ \psi = v_1 - u_1^* & \text{in } \Gamma, \\ \psi = v_2 - u_2^* & \text{in } \Sigma, \\ \psi(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (4.14)$$

It follows from (4.11) and (4.13) that

$$\begin{aligned} 0 &\leq J_1(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) \\ &= \frac{1}{2}\|\psi^*\|_{L^2(Q)}^2 + \frac{1}{2}\lambda^2\|\psi\|_{L^2(Q)}^2 + \frac{1}{2}\|z_d^1\|_{L^2(Q)}^2 \\ &\quad + \lambda \int_Q \psi^* \psi \, dx \, dt - \int_Q \psi^* z_d^1 \, dx \, dt - \lambda \int_Q \psi z_d^1 \, dx \, dt + \frac{\beta}{2}\|(u_1^*, u_2^*)\|_{\mathcal{Z}_D}^2 \\ &\quad + \frac{\lambda^2\beta}{2}\|(v_1 - u_1^*, v_2 - u_2^*)\|_{\mathcal{Z}_D}^2 + \lambda\beta \int_\Gamma u_1^*(v_1 - u_1^*) \, d\sigma dt \\ &\quad + \lambda\beta \int_\Sigma u_2^*(v_2 - u_2^*) \, dx \, dt. \end{aligned} \quad (4.15)$$

It follows from (4.15) that

$$\begin{aligned} 0 &\leq \frac{J_1(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) - J_1(u_1^*, u_2^*)}{\lambda} \\ &= \frac{1}{2}\lambda\|\psi\|_{L^2(Q)}^2 + \int_Q \psi^* \psi \, dx \, dt + \beta \int_\Sigma u_2^*(v_2 - u_2^*) \, d\sigma dt - \int_Q \psi z_d^1 \, dx \, dt \\ &\quad + \frac{\lambda\beta}{2}\|(v_1 - u_1^*, v_2 - u_2^*)\|_{\mathcal{Z}_D}^2 + \beta \int_\Gamma u_1^*(v_1 - u_1^*) \, d\sigma dt. \end{aligned} \quad (4.16)$$

Taking the limit of (4.16) as $\lambda \downarrow 0$, we obtain

$$\int_Q \psi^* \psi \, dx \, dt - \int_Q \psi z_d^1 \, dx \, dt + \beta \int_\Gamma u_1^*(v_1 - u_1^*) \, d\sigma dt + \beta \int_\Sigma u_2^*(v_2 - u_2^*) \, d\sigma dt \geq 0.$$

That is, for all $(v_1, v_2) \in \mathcal{Z}_{ad}$, we have

$$\int_Q \psi (\psi^* - z_d) \, dx \, dt + \beta \int_\Gamma u_1^*(v_1 - u_1^*) \, d\sigma dt + \beta \int_\Sigma u_2^*(v_2 - u_2^*) \, dx \, dt \geq 0. \quad (4.17)$$

Next, taking p^* (the solution of (4.8)) as a test function in the definition of very-weak solutions to (4.14) we get that

$$\int_Q \psi (z_d^1 - \psi^*) \, dx \, dt + \int_\Gamma (v_1 - u_1^*) \partial_\nu p^* \, d\sigma dt + \int_\Sigma (v_2 - u_2^*) \mathcal{N}_s p^* \, dx \, dt = 0. \quad (4.18)$$

Combining (4.17)-(4.18), we get (4.9). The justification of (4.10) is classical and the proof is finished. \square

Remark 4.3. Let $0 < s \leq 3/4$, $\mathcal{Z}_{ad} = \mathcal{Z}_D$, and (u_1^*, u_2^*) be the minimizer of (4.6) over \mathcal{Z}_{ad} . It follows from (4.10) and Step 1 in the proof of Theorem 3.7 that the regularity of u_1^* can be improved. More precisely, if $0 < s \leq 3/4$, then u_1^* belongs to $L^2((0, T); H^{1/2}(\partial\Omega))$.

4.2. The second optimal control problem. Here we consider the minimization problem

$$\min_{(v_1, v_2) \in \mathcal{Z}_{ad}} J_2((v_1, v_2)), \quad (4.19)$$

with the functional J_2 given by

$$J_2(u_1, u_2) := \frac{1}{2} \|\psi(T; (u_1, u_2)) - z_d^2\|_{H^{-1}(\Omega)}^2 + \frac{\beta}{2} \|(u_1, u_2)\|_{\mathcal{Z}_D}^2, \quad (4.20)$$

where $\beta > 0$ is a real number, $z_d^2 \in H^{-1}(\Omega)$, the state $\psi := \psi(u_1, u_2)$ is the unique very-weak solution of (4.2), and $\psi(T; (u_1, u_2)) = \psi(\cdot, T)$.

We have the following existence result of optimal solutions.

Proposition 4.4. *Let $0 < s \leq 3/4$, \mathcal{Z}_{ad} a closed and convex subset of \mathcal{Z}_D , $u_1 \in L^2(\Gamma)$, $u_2 \in L^2(\Sigma)$, and let $\psi := \psi(u_1, u_2)$ satisfy (4.2). Then, there exists a unique solution (u_1^*, u_2^*) to the minimization problem (4.19)-(4.20).*

Proof. Here, we use minimizing sequences. Since the functional $J_2 : \mathcal{Z}_{ad} \rightarrow \mathbb{R}$ is bounded from below by zero, it is possible to construct a minimizing sequence $\{(u_{1n}, u_{2n})\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} J_2((u_{1n}, u_{2n})) = \inf_{(v_1, v_2) \in \mathcal{Z}_{ad}} J_2((v_1, v_2)). \quad (4.21)$$

We denote by $\psi_n := \psi_n((u_{1n}, u_{2n}))$ the state associated with the control (u_{1n}, u_{2n}) . Then, for each $n \in \mathbb{N}$, we have that $\psi_n(u_{1n}, u_{2n})$ is the unique very-weak solution of

$$\begin{cases} (\psi_n)_t + \mathcal{L}\psi_n = 0 & \text{in } Q, \\ \psi_n = u_{1n} & \text{on } \Gamma, \\ \psi_n = u_{2n} & \text{in } \Sigma, \\ \psi_n(\cdot, 0) = 0, & \text{in } \Omega. \end{cases} \quad (4.22)$$

It follows from (4.21), the structure of the cost function given by (4.20), and the definition of the norm on \mathcal{Z}_D given by (4.1) that, there exists a constant $C > 0$ independent of n such that,

$$\|u_{1n}\|_{L^2(\Gamma)} \leq C, \quad (4.23)$$

$$\|u_{2n}\|_{L^2(\Sigma)} \leq C, \quad (4.24)$$

$$\|\psi_n(\cdot, T)\|_{H^{-1}(\Omega)} \leq C. \quad (4.25)$$

Since $\psi_n \in L^2((0, T) \times \mathbb{R}^N)$ is the unique very-weak solution of (4.22), it follows from (3.39), (4.23) and (4.24) that,

$$\|\psi_n\|_{L^2(Q)} \leq \|\psi_n\|_{L^2((0, T) \times \mathbb{R}^N)} \leq C. \quad (4.26)$$

It follows from (4.23), (4.24), (4.25), and (4.26) that there exist $(u_1^*, u_2^*) \in L^2(\Gamma) \times L^2(\Sigma)$, $\psi_T \in H^{-1}(\Omega)$, and $\psi^* \in L^2((0, T) \times \mathbb{R}^N)$ such that, as $n \rightarrow \infty$,

$$u_{1n} \rightharpoonup u_1^* \quad \text{weakly in } L^2(\Gamma), \quad (4.27)$$

$$u_{2n} \rightharpoonup u_2^* \quad \text{weakly in } L^2(\Sigma), \quad (4.28)$$

$$\psi_n(\cdot, T) \rightharpoonup \psi_T \quad \text{weakly in } H^{-1}(\Omega), \quad (4.29)$$

$$\psi_n \rightharpoonup \psi^* \quad \text{weakly in } L^2((0, T) \times \mathbb{R}^N). \quad (4.30)$$

Note that, from (4.26) and (4.30) we have that, as $n \rightarrow \infty$,

$$\psi_n \rightharpoonup \psi^* \quad \text{weakly in } L^2(Q). \quad (4.31)$$

Using (4.27) and (4.28), we have that, as $n \rightarrow \infty$,

$$(u_{1n}, u_{2n}) \rightharpoonup (u_1^*, u_2^*) \quad \text{weakly in } \mathcal{Z}_D.$$

Since $(u_{1n}, u_{2n}) \in \mathcal{Z}_{ad}$ and \mathcal{Z}_{ad} is a closed subset of \mathcal{Z}_D , we have that

$$(u_1^*, u_2^*) \in \mathcal{Z}_{ad}. \quad (4.32)$$

It follows from the definition of very-weak solutions to the system (4.22) that

$$\begin{aligned} \int_Q \psi_n(-\phi_t + \mathcal{L}\phi) \, dx \, dt &= \langle \psi_n(\cdot, T), \phi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\quad - \int_\Gamma u_{1n} \partial_\nu \phi \, d\sigma \, dt - \int_\Sigma u_{2n} \mathcal{N}_s \phi \, dx \, dt \end{aligned} \quad (4.33)$$

for every $\phi \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ and $n \in \mathbb{N}$. Using all the above convergences, and taking the limit of (4.33) as $n \rightarrow \infty$, we get that

$$\begin{aligned} \int_Q \psi^*(-\phi_t + \mathcal{L}\phi) \, dx \, dt &= \langle \psi_T, \phi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Gamma u_1^* \partial_\nu \phi \, d\sigma \, dt \\ &\quad - \int_\Sigma u_2^* \mathcal{N}_s \phi \, dx \, dt, \end{aligned} \quad (4.34)$$

for every $\phi \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$. In particular, we have that

$$\int_Q \psi^*(-\phi_t + \mathcal{L}\phi) \, d\sigma \, dt = - \int_\Gamma u_1^* \partial_\nu \phi \, d\sigma \, dt - \int_\Sigma u_2^* \mathcal{N}_s \phi \, dx \, dt,$$

$\phi \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$ with $\phi(\cdot, T) = 0$ a.e. on Ω . Since $\psi_n \in C([0, T]; H^{-1}(\Omega))$ and $\psi_n(\cdot, 0) = 0$, we have that $\psi^*(\cdot, 0) = 0$ in Ω . We have shown that ψ^* is a very-weak solution of

$$\begin{cases} \psi_t^* + \mathcal{L}\psi^* = 0 & \text{in } Q, \\ \psi^* = u_1^* & \text{in } \Gamma, \\ \psi^* = u_2^* & \text{in } \Sigma, \\ \psi^*(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \quad (4.35)$$

in the sense of Definition 3.17. It follows from Remark 3.21 that ψ^* enjoys the following additional regularity: $\psi^* \in C([0, T]; H^{-1}(\Omega))$. This implies that

$$\begin{aligned} \int_Q \psi^*(-\phi_t + \mathcal{L}\phi) \, dx \, dt &= \langle \psi^*(\cdot, T), \phi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\quad - \int_\Gamma u_1^* \partial_\nu \phi \, d\sigma \, dt - \int_\Sigma u_2^* \mathcal{N}_s \phi \, dx \, dt, \end{aligned} \quad (4.36)$$

for every $\phi \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$. Combining (4.34)-(4.36), we get

$$\langle \psi_T, \phi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle \psi^*(\cdot, T), \phi(\cdot, T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

for every $\phi \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); L^2(\Omega))$. From which we can deduce that $\psi_T = \psi^*(\cdot, T)$.

Next, since the functional J_2 is convex and lower semi-continuous, using (4.27), (4.28), (4.32), the fact that $\psi^*(\cdot, T) = \psi_T$, and (4.29), we can deduce that

$$\begin{aligned} J_2((u_1^*, u_2^*)) &\leq \liminf_{n \rightarrow +\infty} J_2((u_{1n}, u_{2n})) = \lim_{n \rightarrow +\infty} J_2((u_{1n}^*, u_{2n}^*)) \\ &= \inf_{(u_1, u_2) \in \mathcal{Z}_{ad}} J_2((u_1, u_2)) \leq J_2((u_1^*, u_2^*)). \end{aligned}$$

We have shown that (u_1^*, u_2^*) is the optimal solution of (4.19)-(4.20). The uniqueness is straightforward and follows directly from the strict convexity of J_2 . The proof is finished. \square

The following result characterizes the optimality conditions.

Theorem 4.5. *Let $0 < s \leq 3/4$ and $\mathbb{U} := L^2(0, T; H_0^1(\Omega)) \cap H^1((0, T); H^{-1}(\Omega))$. Let \mathcal{Z}_{ad} be a closed, convex subspace of \mathcal{Z}_D , and (u_1^*, u_2^*) be the minimizer of (4.19)-(4.20) over \mathcal{Z}_{ad} . Let ψ^* be the associated unique very-weak solution of (1.1b) with boundary datum u_1^* , and exterior datum u_2^* . Then, there exists $p^* = p^*(u_1^*, u_2^*)$ such that the triplet $(\psi^*, p^*, (u_1^*, u_2^*)) \in L^2((0, T) \times \mathbb{R}^N) \times \mathbb{U} \times \mathcal{Z}_{ad}$ satisfies the following optimality systems:*

$$\begin{cases} \psi_t^* + \mathcal{L}\psi^* = 0 & \text{in } Q, \\ \psi^* = u_1^* & \text{in } \Gamma, \\ \psi^* = u_2^* & \text{in } \Sigma, \\ \psi^*(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \quad (4.37)$$

and

$$\begin{cases} -p_t^* + \mathcal{L}p^* = 0 & \text{in } Q, \\ p^* = 0 & \text{in } \Sigma, \\ p^*(\cdot, T) = (-\Delta_D)^{-1}[\psi^*(\cdot, T) - z_d^2] & \text{in } \Omega, \end{cases} \quad (4.38)$$

and for all $(v_1, v_2) \in \mathcal{Z}_{ad}$, we have

$$\int_{\Gamma} (\partial_\nu p^* - \beta u_1^*)(u_1^* - v_1) \, d\sigma \, dt + \int_{\Sigma} (\mathcal{N}_s p^* - \beta u_2^*)(u_2^* - v_2) \, dx \, dt \geq 0. \quad (4.39)$$

In addition,

$$(u_1^*, u_2^*) = \mathbb{P}(-\beta^{-1} \partial_\nu p^*, -\beta^{-1} \mathcal{N}_s p^*), \quad (4.40)$$

where \mathbb{P} denotes the projection onto the set \mathcal{Z}_{ad} .

Proof. It follows from the proof of Proposition 4.4 that ψ^* is the unique very-weak solution of (4.37) associated with the minimizer (u_1^*, u_2^*) . As above, some classical duality arguments show that (4.38) is the associated dual system. In addition, using the change of variable $t \mapsto T - t$, we have that p^* , solution of (4.38), satisfies (3.15) with $f := 0$ and $\phi_0 := (-\Delta_D)^{-1}[\psi^*(\cdot, T) - z_d^2] \in H_0^1(\Omega)$. Thus, we can deduce from Theorem 3.11 that $p^* \in \mathbb{U}$.

To prove the last assertion (4.39), we write the Euler Lagrange first order optimality conditions that characterize the optimal control (u_1^*, u_2^*) as follows:

$$\lim_{\lambda \rightarrow 0} \frac{J_2(u_1^* + \lambda(v_1 - u_1^*), u_2^* + \lambda(v_2 - u_2^*)) - J_2(u_1^*, u_2^*)}{\lambda} \geq 0, \quad \forall (v_1, v_2) \in \mathcal{Z}_{ad}.$$

After some calculations, and proceeding as in the proof of Theorem 4.2, we obtain that

$$\begin{aligned} & \langle \psi^*(\cdot, T), (-\Delta_D)^{-1}[\phi^*(\cdot, T) - z_d^2] \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ & + \beta \int_{\Gamma} u_1^*(v_1 - u_1^*) d\sigma dt + \beta \int_{\Sigma} u_2^*(v_2 - u_2^*) dx dt \geq 0 \quad \forall (v_1, v_2) \in \mathcal{Z}_{ad}, \end{aligned} \quad (4.41)$$

where $\phi^* = \phi^*(v_1 - u_1^*, v_2 - u_2^*)$ is the unique very-weak solution of the system

$$\begin{cases} \phi_t^* + \mathcal{L}\phi^* = 0 & \text{in } Q, \\ \phi^* = v_1 - u_1^* & \text{in } \Gamma, \\ \phi^* = v_2 - u_2^* & \text{in } \Sigma, \\ \phi(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (4.42)$$

Recall that under the assumption $0 < s \leq 3/4$, we have shown in Section 3.2 that $\partial_\nu p^* \in L^2(\Gamma)$ and $\mathcal{N}_s p^* \in L^2(\Sigma)$. In addition, we have that $(-\Delta_D)^{-1}[\psi^*(\cdot, T) - z_d^2] \in H_0^1(\Omega)$. So, taking p^* as a test function in the definition of very-weak solutions of (4.42), we obtain

$$\begin{aligned} 0 = & \langle \psi^*(\cdot, T), v_2 - u_2^* \rangle, (-\Delta_D)^{-1}[\phi^*(\cdot, T) - z_d^2] \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ & + \int_{\Gamma} (v_1 - u_1^*) \partial_\nu p^* d\sigma dt + \int_{\Sigma} (v_2 - u_2^*) \mathcal{N}_s p^* dx dt. \end{aligned} \quad (4.43)$$

Combining (4.41)-(4.43), we get (4.39). Here also, the justification of (4.40) is classical. The proof is finished. \square

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