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# A comment on “Robust stabilization of delayed neural fields with partial measurement and actuation” [Automatica 83 (2017) 262-274]

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**Abstract.** In [2], the authors study the stabilization of a class of delayed neural fields through output proportional feedback. They provide a condition under which the resulting closed-loop system is input-to-state stable (ISS). However, a key assumption in that paper is the existence of an equilibrium for the closed-loop system. We show here that such an equilibrium does exist if the activation functions are bounded.

## 1 Introduction

In [2, Section 4], the following delayed neural fields are considered:

$$\tau_1(r) \frac{\partial z_1}{\partial t}(r, t) = -z_1(r, t) + S_1 \left( I_1^*(r) + \alpha(r)u(r, t) + \sum_{j=1}^2 \int_{\Omega} w_{1j}(r, r') z_j(r', t - d_j(r, r')) dr' \right), \quad (1a)$$

$$\tau_2(r) \frac{\partial z_2}{\partial t}(r, t) = -z_2(r, t) + S_2 \left( I_2^*(r) + \sum_{j=1}^2 \int_{\Omega} w_{2j}(r, r') z_j(r', t - d_j(r, r')) dr' \right), \quad (1b)$$

where  $\Omega \subset \mathbb{R}^q$  is a compact set,  $z_i(r, t)$  is the neural activity at position  $r \in \Omega$  and time  $t \in \mathbb{R}_+$  of population  $i \in \{1, 2\}$ ,  $\tau_i : \Omega \rightarrow \mathbb{R}_{>0}$  is a time constant distribution,  $S_i : \mathbb{R} \rightarrow \mathbb{R}$  is a non-decreasing continuous function,  $w_{ij} \in L^2(\Omega^2; \mathbb{R})$ ,  $d_j : \Omega^2 \rightarrow [0, \bar{d}]$  for some  $\bar{d} \geq 0$ ,  $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is the controlled input,  $\alpha : \Omega \rightarrow \mathbb{R}_+$  is a bounded function reflecting the in-homogeneity of the received input, and  $I_i^* \in L^2(\Omega; \mathbb{R})$  is a constant uncontrolled input. The system is controlled in closed loop with a partial proportional feedback:

$$u(r, t) = -k(z_1(r, t) - z_{ref}(r)). \quad (2)$$

where  $k \in \mathbb{R}_+$  denotes the proportional gain and  $z_{ref} : \Omega \rightarrow \mathbb{R}$  is a target distribution. In order to investigate the robust stability of the closed-loop system, the authors assume<sup>1</sup> *a priori* the

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<sup>1</sup>More precisely, on page 266 of [2]: “For now on, we simply assumed that such an equilibrium exists.”

existence of an equilibrium point  $(z_1^*, z_2^*) \in L^2(\Omega; \mathbb{R})^2$  for (1)-(2), at which they aim to stabilize the system. A similar assumption was made in [3].

The existence of such an equilibrium in the absence of proportional control can be established by invoking [4, Theorem 3.6], which exploits compactness arguments. As stressed in [2], this result cannot readily be invoked for (1)-(2). As a matter of fact, the control law (2) does not define a compact operator, thus making the use of Schaefer's fixed point theorem more delicate.

In this note, we provide mild conditions under which such an equilibrium exists. We show in particular that boundedness of the activation functions  $S_i$  are enough to guarantee the existence of an equilibrium.

## 2 Main result

Our main result is the following.

**Theorem 2.1** *Let  $\Omega$  be a compact set of  $\mathbb{R}^q$ ,  $q \in \mathbb{N}$ . Given any  $i, j \in \{1, 2\}$ , let  $I_i^* \in L^2(\Omega, \mathbb{R})$ ,  $\tau_i : \Omega \rightarrow \mathbb{R}_{>0}$ ,  $d_i : \Omega^2 \rightarrow [0, \bar{d}]$  for some  $\bar{d} \geq 0$ ,  $w_{ij} \in L^2(\Omega^2, \mathbb{R})$ ,  $\alpha : \Omega \rightarrow \mathbb{R}_+$  be a bounded function, and  $z_{ref} \in L^2(\Omega, \mathbb{R})$ . If  $k \geq 0$  and  $S_i : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous bounded function for each  $i \in \{1, 2\}$ , then the closed-loop system (1)-(2) admits at least one an equilibrium in  $L^2(\Omega; \mathbb{R})^2$ .*

In order to establish this result, we first observe that  $(z_1^*, z_2^*)$  is an equilibrium point of (1)-(2) if and only if it is a fixed point of the nonlinear map  $T : L^2(\Omega; \mathbb{R})^2 \rightarrow L^2(\Omega; \mathbb{R})^2$  defined by  $T(z_1, z_2) := (T_1(z_1, z_2), T_2(z_1, z_2))$ , where

$$T_1(z_1, z_2) := S_1 \left( I_1^*(r) - k\alpha(r)(z_1(r) - z_{ref}(r)) + \sum_{j=1}^2 \int_{\Omega} w_{1j}(r, r') z_j(r') dr' \right),$$

$$T_2(z_1, z_2) := S_2 \left( I_2^*(r) + \sum_{j=1}^2 \int_{\Omega} w_{2j}(r, r') z_j(r') dr' \right).$$

We rely on the change of coordinates  $z_i(r) = S_i(x_i(r))$  for each  $i \in \{1, 2\}$ . More precisely, we consider the map  $\mathcal{T} : L^2(\Omega; \mathbb{R})^2 \rightarrow L^2(\Omega; \mathbb{R})^2$  defined by  $\mathcal{T}(x_1, x_2) := (\mathcal{T}_1(x_1, x_2), \mathcal{T}_2(x_1, x_2))$ , where

$$\mathcal{T}_1(x_1, x_2) := I_1^*(r) - k\alpha(r)(S_1(x_1(r)) - z_{ref}(r)) + \sum_{j=1}^2 \int_{\Omega} w_{1j}(r, r') S_1(x_j(r')) dr',$$

$$\mathcal{T}_2(x_1, x_2) := I_2^*(r) + \sum_{j=1}^2 \int_{\Omega} w_{2j}(r, r') S_2(x_j(r')) dr'.$$

Then  $(z_1^*, z_2^*)$  is a fixed point of  $T$  if and only if there exists a fixed point  $(x_1^*, x_2^*)$  of  $\mathcal{T}$  such that  $(z_1^*, z_2^*) = (S_1(x_1^*), S_2(x_2^*))$ . Hence, it is sufficient to find a fixed point of  $\mathcal{T}$  in  $L^2(\Omega, \mathbb{R})^2$  in order to prove Theorem 2.1. The existence of  $(x_1^*, x_2^*)$  follows from the next lemma.

**Lemma 2.2** *Let  $X$  be a Hilbert space,  $f \in X$ ,  $W : X \rightarrow X$  be a continuous nonlinear compact operator,  $\rho : X \rightarrow X$  be a continuous nonlinear uniformly bounded operator and  $\sigma : X \rightarrow X$  be a continuous nonlinear monotone operator that maps bounded sets to bounded sets. Then the map  $G : X \rightarrow X$  defined by*

$$G(x) := W(\rho(x)) - \sigma(x) + f$$

*admits at least one fixed point in  $X$ .*

*Proof.* The proof is an adaptation of [4, Theorem 3.6], that dealt with the uncontrolled case (i.e.,  $k = 0$ ). It is based on Schaefer's fixed point theorem. Since  $\sigma$  is continuous, monotone, and maps bounded sets to bounded sets, the map  $x \mapsto x/2 + \sigma(x)$  is a maximal monotone operator on  $X$  according to [1, Chapter 2, Corollary 1.1]. Hence the nonlinear map  $H : X \rightarrow X$  defined

by  $H(x) := x + \sigma(x)$  has a continuous inverse  $H^{-1}$  on  $X$ . Consider the map  $\pi : X \rightarrow X$  defined by  $\pi(x) := H^{-1}(W(\rho(x)) + f)$ . Then  $\pi$  is continuous and compact, since  $H^{-1}$ ,  $\rho$  and  $W$  are continuous and  $W$  is compact. Set  $E := \{x \in X \mid \exists \lambda \in (0, 1), x = \lambda\pi(x)\}$ . Since  $\rho$  is uniformly bounded, there exists a bounded set  $B \subset X$  such that  $\rho(E) \subset B$ . Since  $W$  is compact and  $H^{-1}$  is continuous,  $H^{-1}(W(B) + f)$  is a relatively compact set, hence  $\pi(E)$  is bounded and so is  $E$ . Thus, according to Schaefer's fixed point theorem,  $\pi$  admits at least one fixed point  $x^*$  in  $X$ . Then  $H(x^*) = W(x^*) + f$ , *i.e.*  $x^*$  is a fixed point of  $G$ . ■

*Proof of Theorem 2.1.* To prove Theorem 2.1 from Lemma 2.2, we set  $X = L^2(\Omega; \mathbb{R})^2$ ,  $f = (I_1^* + k\alpha z_{ref}, I_2^*)$ ,  $W(x_1, x_2)(r) = \sum_{j=1}^2 (\langle w_{1j}(r, \cdot), x_j \rangle_{L^2}, \langle w_{2j}(r, \cdot), x_j \rangle)$ ,  $\rho(x_1, x_2) = (S_1(x_1), S_2(x_2))$ ,  $\sigma(x_1, x_2) = (k\alpha S_1(x_1), 0)$ . Then  $\mathcal{T} = G$ . The operator  $W$  is compact as a Hilbert–Schmidt integral operator (since the maps  $w_{ij}$  are in  $L^2(\Omega^2, \mathbb{R})$ ). Moreover,  $\rho$  is continuous and uniformly bounded since each  $S_i$  is continuous and bounded by assumption. Finally,  $\sigma$  is continuous, monotone and maps bounded sets to bounded sets since  $S_1$  is continuous, non-decreasing, and bounded, and  $k\alpha \geq 0$ . Therefore, all the assumptions of Lemma 2.2 are satisfied, which ends the proof of Theorem 2.1. ■

### 3 Remarks

In [2], the maps  $S_i$  are not assumed to be bounded. However, in most neural fields models, these activation functions are supposed bounded (as in [4] for example). This boundedness reflects the fact that the activity of a given neuronal population cannot exceed a certain value due to biological considerations. Consequently, the boundedness of the activation functions does not induce a too demanding additional requirement in practice. In particular, this boundedness requirement holds naturally for the modeling of the neuronal populations involved in the generation of pathological oscillations related to Parkinson's disease, which is the main scope of [2].

We stress that this boundedness requirement can be removed in the case when the maps  $S_i$  are linear. In that case,  $\sigma$ ,  $W$  and  $\rho$  in Lemma 2.2 are also linear. Hence  $\mathcal{T}$  admits a fixed point if and only if  $f$  lies in the range of the linear operator  $x \mapsto x + \sigma(x) - W(\rho(x))$ .

Sufficient conditions are given in [2] for the ISS of (1)-(2) at some equilibrium point under the assumption of the existence of an equilibrium. Naturally, this implies the uniqueness of the equilibrium point, hence of the fixed point of  $T$ .

Under the additional assumption that  $I_i$ ,  $S_i$ ,  $\alpha$  and  $z_{ref}$  are continuous maps, it can be proved that  $T$  defines a mapping from  $C(\Omega; \mathbb{R})^2$  into itself, and admits a fixed point in  $C(\Omega; \mathbb{R})^2$ . Indeed, following the proof of Lemma 2.2, the only missing assumptions are that  $X = C(\Omega; \mathbb{R})^2$  is not a Hilbert space but a Banach space, and  $\sigma$  is not monotone. However, the map  $H : C(\Omega; \mathbb{R})^2 \rightarrow C(\Omega; \mathbb{R})^2$  defined by  $H(x) := x + \sigma(x)$  still admits a continuous inverse. Therefore, the conclusion of Lemma 2.2 remains valid. In particular, if the fixed point given in Theorem 2.1 (*a priori* lying in  $L^2(\Omega; \mathbb{R})^2$ ) is unique due to the ISS property shown in [2], then it actually lies  $C(\Omega; \mathbb{R})^2$ .

Note that Lemma 2.2 allows to take into account more general neural fields than (1) and more general feedback laws than (2). In particular, higher dimensional models (with state  $(z_i)_{1 \leq i \leq N}$ ,  $N \in \mathbb{N}$ ) as well as nonlinear feedback laws can be considered. The only assumption to check is that  $\sigma$  remains a continuous monotone operator, mapping bounded sets to bounded sets, or more generally that  $H : x \mapsto x + \sigma(x)$  has a continuous inverse.

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