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PRESHEAVES ON $\mathcal{V}\mathcal{I}$, *nil*-CLOSED UNSTABLE ALGEBRAS AND THEIR CENTRES

OURIEL BLOEDE

Abstract

A *nil*-closed, noetherian, unstable algebra K over the Steenrod Algebra is determined, up to isomorphism, by the functor $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$, which is a presheaf on the category $\mathcal{V}\mathcal{I}$ of finite dimensional vector spaces and injections, by the theory of Henn-Lannes-Schwartz. In this article, we use this theory to study the centre, in the sense of Heard, of a *nil*-closed noetherian unstable algebra.

For F a presheaf on $\mathcal{V}\mathcal{I}$, we construct a groupoid \mathcal{G}_F which encodes F . Then, taking $F := \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$, we show how the centre of K is determined by the associated groupoid. We also give a generalisation of the second theorem of Adams-Wilkerson, defining sub-algebras $H^*(W)^{\mathcal{G}}$ of $H^*(W)$ for appropriate groupoids \mathcal{G} .

There is a $H^*(C)$ -comodule structure on K that is associated with the centre. For K integral, we explain how the algebra of primitive elements of this $H^*(C)$ -comodule structure is also determined by the groupoid associated with $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$. Along the way, we prove that this algebra of primitive elements is also noetherian.

1 Introduction

1.1 The two theorems of Adams-Wilkerson

For p a prime number, W a finite dimensional \mathbb{F}_p -vector space, and for BW the classifying space of W , $W \mapsto H^*(W) := H^*(BW; \mathbb{F}_p)$ defines a functor from \mathcal{V}^f , the category of finite dimensional vector spaces, to \mathcal{K} , the category of unstable algebra over the Steenrod algebra over \mathbb{F}_p . Then, for G a sub-group of $\text{Gl}(W)$, G acts on $H^*(W)$ and we can define $H^*(W)^G \in \mathcal{K}$, the unstable algebra of invariant elements of $H^*(W)$ under the action of G . For K an integral, noetherian, unstable algebra of transcendence degree $\dim(W)$, the first theorem of Adams-Wilkerson states that there always exists an injection ϕ from K to $H^*(W)$ and that this injection induces a structure of finitely generated K -module on $H^*(W)$. Then, for $\text{Gal}(\phi)$ the sub group of $\text{Gl}(W)$ whose elements are the automorphisms α such that $\alpha^*\phi = \phi$ with $\alpha^*\phi$ the composition of ϕ with the isomorphism induced by α on $H^*(W)$, the image of ϕ is a sub-algebra of $H^*(W)^{\text{Gal}(\phi)}$. $H^*(W)^{\text{Gal}(\phi)}$ is a first approximation of K , but usually ϕ does not define an isomorphism between K and $H^*(W)^{\text{Gal}(\phi)}$.

The category of unstable modules over the Steenrod algebra admits a localizing subcategory $\mathcal{N}il$ (cf [Sch94]), whose objects are called nilpotent unstable modules. We call an unstable module *nil*-closed if its localization away from $\mathcal{N}il$ is an isomorphism. The algebras $H^*(W)^G$ are *nil*-closed, hence ϕ cannot be an isomorphism when K is not *nil*-closed. In this case, and when $p = 2$ the

second theorem of Adams-Wilkerson gives a condition for ϕ to be an isomorphism, when K is *nil*-closed. Namely, if K is

1. noetherian,
2. integral,
3. *nil*-closed,
4. integrally closed in its field of fractions,

the morphism ϕ of the first theorem of Adams-Wilkerson is an isomorphism between K and $H^*(W)^{\text{Gal}(\phi)}$. (There is a similar statement when p is odd.)

The first objective of this article is the following, for \mathcal{G} a groupoid whose objects are the subspaces of W and whose morphisms satisfy a restriction property, we define an object $H^*(W)^{\mathcal{G}}$ that generalises the algebra of invariants $H^*(W)^G$. We will show that the $H^*(W)^{\mathcal{G}}$ give a complete list of the *nil*-closed, noetherian, unstable sub-algebras of $H^*(W)$.

1.2 The centre of an unstable algebra

In [DW92b], Dwyer and Wilkerson introduced the notion of a central element of an unstable algebra, this notion allowed them to exhibit the only exotic finite loop space at prime 2 in [DW92a]. In the case where K is noetherian and connected, the set of central elements of K coincides with the set of pairs (V, ϕ) such that

1. $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$,
2. K admits a structure κ of $H^*(V)$ -comodule in \mathcal{K} , such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ & \searrow \phi & \downarrow \epsilon_K \otimes \text{id} \\ & & H^*(V), \end{array}$$

where ϵ_K denotes the augmentation of K (which is uniquely defined because of the connectedness of K).

In [Hea21], Heard showed that for K noetherian, K admits a unique (up to isomorphism) central element (C, γ) such that γ induces a structure of finitely generated K -module on $H^*(C)$ and $\dim(C)$ is maximal among such central elements. Heard called this central element the centre of K . The centre of an unstable algebra have been shown to be an important invariant. In [Kuh07] and [Kuh13], Kuhn used it to approximate the depth of K as well as invariants $d_0(K)$ and $d_1(K)$ introduced by Henn, Lannes and Schwartz in [HLS95], in the case where K is the cohomology of a group. Heard generalised those results for K noetherian in [Hea20] and [Hea21].

For K noetherian, since the centre of K is associated with a $H^*(C)$ -comodule structure on K , it gives rise to a second invariant: the sub-algebra of primitive elements of K under this $H^*(C)$ -comodule structure. The second objective of this article is to explain how the centre of $H^*(W)^{\mathcal{G}}$ and its sub-algebra of primitive elements are determined by \mathcal{G} . We will then be able to classify *nil*-closed, noetherian sub-algebras of $H^*(W)$ with a given centre and algebra of primitive elements.

1.3 The category $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$

We consider $\mathcal{K}/\mathcal{N}il$ the localization of \mathcal{K} with respect to morphisms whose kernels and cokernels are nilpotent unstable modules. In [HLS93], Henn, Lannes and Schwartz proved that the functor which sends an unstable algebra K to $\text{Hom}_{\mathcal{K}}(K, H^*(-))$, the functor which maps $V \in \mathcal{V}^f$ to $\text{Hom}_{\mathcal{K}}(K, H^*(V))$, induces an equivalence of category between $\mathcal{K}/\mathcal{N}il$ and a certain sub-category of $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$ the category of contravariant functors from \mathcal{V}^f to $\mathcal{S}et$. Furthermore, they used abundantly the fact that, when K is noetherian, the functor $\text{Hom}_{\mathcal{K}}(K, H^*(-))$ is fully determined by the $\text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(V))$ for V running through \mathcal{V}^f , where $\text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(V))$ is the subset of $\text{Hom}_{\mathcal{K}}(K, H^*(V))$ whose objects are the morphisms from K to $H^*(V)$ which turn $H^*(V)$ into a finitely generated K -module.

When K is noetherian, $\text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(-))$ defines a contravariant functor from $\mathcal{V}\mathcal{I}$ to $\mathcal{S}et$, where $\mathcal{V}\mathcal{I}$ is the wide sub-category of \mathcal{V}^f whose morphisms are the injective morphisms. The category $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ is a lot easier to understand than the category $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$, and that is going to be our main tool.

The two first sections of this article consist of recollections about the equivalences of categories constructed in [HLS93] and the definition of central elements of an unstable algebra. In the third section, we will introduce the category $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and its connections with noetherian unstable algebras. Furthermore, we will define a notion of central elements for functors in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. For $F = \text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(-))$, with K a noetherian nil -closed unstable algebra, the central elements of F will coincide with the central elements (V, ϕ) of K , such that ϕ turns $H^*(V)$ into a finitely generated K -module.

Theorem 4.51. *For K a noetherian unstable algebra, $F = \text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(-))$ and $\phi \in F(V)$, (V, ϕ) is central for K if and only if it is central for F .*

1.4 The groupoid \mathcal{G}_F

An injection ϕ from a noetherian unstable algebra K to some $\prod_{i \in I}^{\wedge} H^*(W_i)$, where $\prod_{i \in I}^{\wedge}$ denotes the product in the category of connected unstable algebras and such that ϕ turns each $H^*(W_i)$ into a finitely generated K -module, induces a surjection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to $\text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(-))$. For I a set, we consider ${}_{(W_i)_{i \in I}}\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, the category whose objects are pairs (F, q_F) with $F \in \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and q_F a natural surjection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F .

In the fourth section, we define an application which sends an object $(F, q_F) \in {}_{(W_i)_{i \in I}}\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ to a groupoid $\mathcal{G}_{(F, q_F)}$ whose set of objects is the disjoint union of the sub-spaces of the W_i . This groupoid satisfies a property called the restriction property. The first main result of this article is that the isomorphism classes of objects in ${}_{(W_i)_{i \in I}}\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ are in one-to-one correspondence with such groupoids. This is stated in the following theorem, where $\sim_{\mathcal{G}}$ is an equivalence relation on $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ characterised by the groupoid \mathcal{G} .

Theorem 5.17. *1. For \mathcal{G} a groupoid whose objects are the sub-spaces of the W_i and whose*

morphisms are isomorphisms of vector spaces such that \mathcal{G} has the restriction property,

$$\mathcal{G}(\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}, q}) = \mathcal{G},$$

for q the canonical surjection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}$.

2. Conversely, let $F \in {}_{(W_i)_{i \in I}} \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$. Then, F is isomorphic to $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}_F}$.

We also show, in Theorem 5.30, how the central elements of $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}$ are determined by the groupoid \mathcal{G} .

Now, for K a noetherian, *nil*-closed, unstable sub algebra of $H^*(W)$ with transcendence degree $\dim(W)$, the inclusion of K in $H^*(W)$ turns $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ into an object of ${}_W \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$. This implies the following:

Theorem 6.7. *For all $W \in \mathcal{V}^f$, there is a one-to-one correspondence between the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ and the set of groupoids with the restriction property, whose objects are the sub-vector spaces of W .*

Then, for \mathcal{G} a groupoid with the restriction property whose objects are sub-spaces of W , we will denote by $H^*(W)^{\mathcal{G}}$ the noetherian, *nil*-closed sub-algebra of $H^*(W)$ such that the groupoid associated with the surjection $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) \rightarrow \text{Hom}_{\mathcal{K}\text{f.g.}}(H^*(W)^{\mathcal{G}}, H^*(-))$ is \mathcal{G} .

By Theorems 4.51 and 5.30, the central elements of $H^*(W)^{\mathcal{G}}$ are determined by \mathcal{G} . In the fifth section we will prove the following, where $P(K, x)$ denotes the algebra of primitive elements of K for the $H^*(V)$ -comodule structure induced by a central element (V, x) .

Theorem 6.25. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi)$ is central, for ϕ the inclusion of K in $H^*(W)$ and δ some morphism from $H^*(W)$ to $H^*(V)$. Then, $P(K, \delta^* \phi)$ is *nil*-closed and noetherian.*

In this context, we will identify naturally $P(K, \delta^* \phi)$ with a sub algebra of $H^*(W/\text{Im}(\delta))$ of transcendence degree $\dim(W/\text{Im}(\delta))$. By Theorem 6.7, $P(K, \delta^* \phi)$ identifies with some $H^*(W/\text{Im}(\delta))^{\mathcal{G}'}$. In the case where $K = H^*(W)^{\mathcal{G}}$, we explain in Theorem 6.28 how to determine \mathcal{G}' from \mathcal{G} .

We conclude this article by giving examples on how to use those constructions to determine *nil*-closed, noetherian, integral, unstable algebras whose transcendence degree is fixed, with a $H^*(V)$ -comodule structure whose primitive elements are isomorphic to some $H^*(U)^{\mathcal{G}'}$ with U in \mathcal{V}^f and \mathcal{G}' a groupoid whith the restriction property and whose objects are the sub spaces of U . For example:

Theorem 6.30. *Let K be a noetherian, *nil*-closed, integral, unstable algebra of transcendence degree d . We assume that the centre of K is of dimension $d - 1$. Then, there exists G , a sub-group of $Gl(W)$, such that K is isomorphic to the algebra of invariant elements $H^*(W)^G$ with $\dim(W) = d$. Furthermore, G satisfies that the set of element $x \in W$ such that $g(x) = x$ for all $g \in G$ is a sub-vector space of W of dimension $d - 1$.*

This Theorem implies that for K noetherian, *nil*-closed, integral of transcendence degree d whose centre is of dimension $d - 1$ and for ϕ the injection of Adams-Wilkerson, ϕ is an isomorphism from K to $H^*(W)^{\text{Gal}(\phi)}$.

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2 Unstable algebras over the Steenrod algebra and the functor f

In this section, we recall some known facts about Lannes' T functor as well as results from [HLS93] about the localization of the categories of unstable modules and algebras modulo away from nilpotent objects. Recollections about unstable algebras, unstable modules and nilpotent objects can be found in [Sch94]. In the following, \mathcal{A} denotes the Steenrod algebra over \mathbb{F}_p with p a prime number, \mathcal{U} and \mathcal{K} denote the category of unstable modules and unstable algebras over \mathcal{A} and \mathcal{Nil} denotes the class of nilpotent objects in \mathcal{U} .

2.1 The T functor

Let us recall the definition of Lannes' T functor.

Theorem 2.1. [Lan87, Proposition 2.1] *For V , a finite dimensional vector space the functor $- \otimes H^*(V)$ has a left adjoint T_V .*

Proposition 2.2. [Sch94, Proposition 3.8.4]

1. *Let K be an unstable algebra, then $T_V(K)$ is given in a natural way a structure of unstable algebra.*
2. *Then, T_V defines a functor from \mathcal{K} to \mathcal{K} which is left adjoint to the tensor product with $H^*(V)$ in \mathcal{K} .*

Example 2.3. [Sch94, 3.9.1] For V and W two finite dimensional \mathbb{F}_p -vector spaces, there is an isomorphism of unstable algebras, natural in both V and W , $T_V(H^*(W)) \cong H^*(W) \otimes \mathbb{F}_p^{\text{Hom}(V,W)}$. By the adjunction property, we get that $(\mathbb{F}_p^{\text{Hom}(V,W)})^\# \cong \text{Hom}_{\mathcal{U}}(H^*(W), H^*(V))$. In other words, $\mathbb{F}_p[\text{Hom}(V, W)] \cong \text{Hom}_{\mathcal{U}}(H^*(W), H^*(V))$ which is a theorem first proved by Adams, Gunawardena and Miller.

2.2 \mathcal{Nil} -localisation of unstable modules

The class of nilpotent modules is a Serre class in \mathcal{U} , we recall the existence of an equivalence of categories between $\mathcal{U}/\mathcal{Nil}$ (defined as in [Gab62]) and a category of functors. The proofs can be found in [HLS93].

Theorem 2.4. [HLS93, Part I.4] *There is an adjunction of functors:*

$$r_1 : \mathcal{U} \rightleftarrows \mathcal{U}/\mathcal{Nil} : s_1,$$

such that, for ϕ a morphism of unstable modules, $r_1(\phi)$ is an isomorphism if and only if $\ker(\phi)$ and $\text{coker}(\phi)$ are objects in $\mathcal{N}il$.

Then, $\mathcal{U}/\mathcal{N}il$ satisfies the following universal property: for A an abelian category and $F : \mathcal{U} \rightarrow A$ an exact functor such that for all $M \in \mathcal{N}il$, $F(M) = 0$, there exists a unique $G : \mathcal{U}/\mathcal{N}il \rightarrow A$ such that $F = G \circ r_1$.

Definition 2.5. For M an unstable module, $l_1(M) := s_1 \circ r_1(M)$ is the *nil*-localisation of M , and we say that M is *nil*-closed if the unit of the adjunction $M \rightarrow l_1(M)$ is an isomorphism.

For \mathcal{F} the category of functors from the category \mathcal{V}^f of finite dimensional \mathbb{F}_p -vector spaces to the category \mathcal{V} of \mathbb{F}_p -vector spaces, we consider $f : \mathcal{U} \rightarrow \mathcal{F}$ the functor which assigns to M in \mathcal{U} ,

$$f(M) : V \mapsto T_V(M)^0.$$

The class $\mathcal{N}il$ satisfies that $f(M) = 0$ if and only if $M \in \mathcal{N}il$. Then, f induces a functor f' from $\mathcal{U}/\mathcal{N}il$ to \mathcal{F} such that $f = f' \circ r_1$.

We will denote by \mathcal{F}_ω the essential image of f in \mathcal{F} .

Theorem 2.6. [Sch94, Theorem 5.2.6] *The functor f' induces an equivalence of categories between $\mathcal{U}/\mathcal{N}il$ and the category \mathcal{F}_ω .*

In [HLS93], the authors exhibit a right adjoint to f' , m' such that the restriction of m' to \mathcal{F}_ω is an inverse of the equivalence of categories induced by f' .

Definition 2.7. Let $m : \mathcal{F} \rightarrow \mathcal{U}$ be the composition of m' with s_1 .

Definition 2.8. For $F \in \mathcal{F}$ and $V \in \mathcal{V}^f$, let $\Delta_V F$ denote the object of \mathcal{F} such that $\Delta_V F(W) = F(V \oplus W)$ and $\Delta_V F(\alpha) = F(\text{id}_V \oplus \alpha)$ for all $W \in \mathcal{V}^f$ and for all morphism α in \mathcal{V}^f .

We recall the following results from [Kuh94]

Proposition 2.9. *For $M \in \mathcal{U}$ and $V \in \mathcal{V}^f$, $f(T_V(M)) = \Delta_V(f(M))$.*

For F an object of \mathcal{F} , $T_V(m(F)) \cong m(\Delta_V(F))$.

Corollary 2.10. *For $M \in \mathcal{U}$ nil-closed, $T_V(M)$ is also nil-closed.*

2.3 *Nil*-localisation of unstable algebras

Since \mathcal{K} is not abelian, one cannot define a localized category of K in the sense of [Gab62]. In [HLS93], Henn, Lannes and Schwartz constructed a localized category $\mathcal{K}/\mathcal{N}il$ with respect to the morphisms whose kernels and cokernels are in $\mathcal{N}il$, in the sense of [KS05]. Then, the functor f restricted to \mathcal{K} factorises through a functor from $\mathcal{K}/\mathcal{N}il$ to \mathcal{F} . The authors of [HLS93] identified the essential image of f restricted to \mathcal{K} and they deduced an equivalence of category between $\mathcal{K}/\mathcal{N}il$ and a category of contravariant functors from \mathcal{V}^f to the category of profinite sets.

Definition 2.11. A p -boolean algebra, is an algebra B over \mathbb{F}_p , such that, for all $x \in B$, $x^p = x$.

Since $T_V(K)$ is an unstable algebra, $T_V(K)^0$ is a p -boolean algebra. We can then use standard results on p -boolean algebras to study $f(K)$.

For \mathcal{B} the category of p -boolean algebras and B a p -boolean algebra, we consider $\text{Hom}_{\mathcal{B}}(B, \mathbb{F}_p)$ the set of morphisms of \mathbb{F}_p -algebras from B to \mathbb{F}_p . Then B is the direct limit of its finite dimensional subalgebras B_α . Therefore, $\text{Hom}_{\mathcal{B}}(B, \mathbb{F}_p)$ is the inverse limit of the $\text{Hom}_{\mathcal{B}}(B_\alpha, \mathbb{F}_p)$ which are finite. $\text{Hom}_{\mathcal{B}}(B, \mathbb{F}_p)$ inherits a structure of profinite set.

Proposition 2.12. [HLS93] For $\mathcal{P}fin$ the category of profinite sets, the functor $\text{spec} : \mathcal{B}^{op} \rightarrow \mathcal{P}fin$, where $\text{spec}(B) := \text{Hom}_{\mathcal{B}}(B, \mathbb{F}_p)$, is an equivalence of categories whose inverse is the functor which sends S to the algebra of continuous maps from S to \mathbb{F}_p , \mathbb{F}_p^S .

In particular, for $K \in \mathcal{K}$, by adjunction $\text{Hom}_{\mathcal{B}}(T_V(K)^0, \mathbb{F}_p) \cong \text{Hom}_{\mathcal{K}}(K, H^*(V))$ and this isomorphism is an isomorphism of profinite sets, where the structure of profinite set on $\text{Hom}_{\mathcal{K}}(K, H^*(V))$ comes from the fact that K is the direct limit of the unstable sub-algebra of K which are finitely generated as \mathcal{A} -algebras. Then, $T_V(K)^0$ is isomorphic as a p -boolean algebra to $\mathbb{F}_p^{\text{Hom}_{\mathcal{K}}(K, H^*(V))}$.

- Definition 2.13.**
1. Let $\mathcal{P}fin^{(\mathcal{V}^f)^{op}}$ be the category of functors from $(\mathcal{V}^f)^{op}$ to $\mathcal{P}fin$,
 2. let \mathcal{L} be Lannes' linearization functor from $(\mathcal{P}fin^{(\mathcal{V}^f)^{op}})^{op}$ to \mathcal{F} defined by $\mathcal{L}(F)(V) := \mathbb{F}_p^{F(V)}$,
 3. let $g : \mathcal{K} \rightarrow (\mathcal{P}fin^{(\mathcal{V}^f)^{op}})^{op}$ be the functor which sends K to the functor $g(K) : V \mapsto \text{Hom}_{\mathcal{K}}(K, H^*(V))$.

We have a commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{g} & (\mathcal{P}fin^{(\mathcal{V}^f)^{op}})^{op} \\ \downarrow & & \downarrow \mathcal{L} \\ \mathcal{U} & \xrightarrow{f} & \mathcal{F}, \end{array}$$

where the functor from \mathcal{K} to \mathcal{U} is the forgetful functor. We denote by $\mathcal{P}fin_{\omega}^{(\mathcal{V}^f)^{op}}$ the full subcategory of $\mathcal{P}fin^{(\mathcal{V}^f)^{op}}$, whose objects are those whose image under \mathcal{L} are in \mathcal{F}_{ω} .

The functor g has a unique factorisation of the following form:

$$\mathcal{K} \rightarrow \mathcal{K}/\mathcal{N}il \rightarrow \mathcal{P}fin_{\omega}^{(\mathcal{V}^f)^{op}} \rightarrow \mathcal{P}fin^{(\mathcal{V}^f)^{op}}.$$

Theorem 2.14. [HLS93, Theorem 1.5 of Part II] The functor from $\mathcal{K}/\mathcal{N}il$ to $\mathcal{P}fin_{\omega}^{(\mathcal{V}^f)^{op}}$ induced by g is an equivalence of categories.

The following lemma will be of importance in the following.

Lemma 2.15. The functor g turns injections into surjections and finite inverse limits into direct limits.

Proof. Since f is exact, f sends injections into injections and commutes with finite inverse limits. The result is then a consequence of the isomorphism $f(K) \cong \mathbb{F}_p^{g(K)}$. \square

3 Connected components of $T_V(K)$ and central elements of an unstable algebra

In this section, we recall the definition of central elements of an unstable algebra first introduced by Dwyer and Wilkerson in [DW92b].

3.1 Connected components of $T_V(K)$

For K an unstable algebra, we recall the definition of the connected components of $T_V(K)$ which is exposed in [Hea20] and [Hea21]. Such a decomposition exists for any graded algebra over a p -boolean algebra.

Lemma 3.1. *For K an unstable algebra which is finitely generated as an algebra over \mathcal{A} , $\text{Hom}_{\mathcal{K}}(K, H^*(V))$ is finite.*

Definition 3.2. For $V \in \mathcal{V}^f$ and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, let $T_{(V,\phi)}(K) := T_V(K) \otimes_{T_V(K)^0} \mathbb{F}_p(\phi)$, where the structure of $T_V(K)^0$ -module over $\mathbb{F}_p(\phi)$ is induced by the morphism from $T_V(K)^0$ to \mathbb{F}_p adjoint to ϕ .

Proposition 3.3. [Hea20, Equation (2.6)] *For K an unstable algebra finitely generated as an algebra over \mathcal{A} and $V \in \mathcal{V}^f$, we have the following natural isomorphism of unstable algebra*

$$T_V(K) \cong \prod_{\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))} T_{(V,\phi)}(K).$$

Lemma 3.4. *For K a nil-closed unstable algebra, $V \in \mathcal{V}^f$ and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, $T_{(V,\phi)}(K)$ is nil-closed.*

Proof. By corollary 2.10 $T_V(K)$ is nil-closed. By the isomorphism of proposition 3.3, $T_{(V,\phi)}(K)$ is the kernel of the morphism from $T_V(K)$ to $\bigoplus_{\phi \neq \psi} T_V(K)$ which sends x to the direct sum of the components of x in each $T_{(V,\psi)}(K)$ with $\phi \neq \psi$. Since $\bigoplus_{\phi \neq \psi} T_V(K)$ is nil-closed and l_1 is left exact, $l_1(T_{(V,\phi)}(K))$ is the kernel of the same morphism, thus, it is isomorphic to $T_{(V,\phi)}(K)$. \square

3.2 Central elements of an unstable algebra

The notion of a central element of an unstable algebra K is defined by Dwyer and Wilkerson in [DW92b] and they used it in [DW92a] to exhibit the only exotic finite loop space at the prime 2. The centre of K has been studied in details in [Hea20] and [Hea21].

The aim of this subsection is to recall some known facts about central elements of an unstable algebra.

Notation 3.5. Let M be an unstable module, V a finite dimensional vector space, K an unstable algebra and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$. Denote by:

- $\eta_{M,V} : M \rightarrow T_V(M) \otimes H^*(V)$ the unit of the adjunction between T_V and $- \otimes H^*(V)$;

- $\rho_{M,V}$ the following composition

$$\rho_{M,V} : M \xrightarrow{\eta_{M,V}} T_V(M) \otimes H^*(V) \xrightarrow{id \otimes \epsilon_V} T_V(M),$$

where ϵ_V denote the augmentation of $H^*(V)$;

- $\rho_{K,(V,\phi)}$ the composition of $\rho_{K,V}$ with the projection onto $T_{(V,\phi)}(K)$.

Remark 3.6. The morphism $\rho_{M,V}$ identifies with $T_{\iota_0^V}(M) : M \cong T_0(M) \rightarrow T_V(M)$, the morphism induced by naturality of $T_V(M)$ with respect to V by ι_0^V , the injection from 0 to V .

Definition 3.7. Let K be an unstable algebra and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$. Then, the pair (V, ϕ) is said to be central if $\rho_{K,(V,\phi)} : K \rightarrow T_{(V,\phi)}(K)$ is an isomorphism.

Let $\mathbf{C}(K)$ be the set of central elements of K .

The classical example and first motivation for studying the centre of an unstable algebra is the example of $H^*(G)$, the cohomology of a group G . The details can be found in [Hen01].

Example 3.8. For G a discrete group or a compact Lie group, $\text{Hom}_{\mathcal{K}}(H^*(G), H^*(V)) \cong \mathbb{F}_p[\text{Rep}(V, G)]$, where $\text{Rep}(V, G)$ denote the conjugacy classes of morphisms from V to G .

Let ρ represent a conjugacy class in $\text{Rep}(V, G)$. We consider the morphism $V \times C_G(\rho) \rightarrow G$, where $C_G(\rho)$ denote the centraliser in G of the image of ρ , which sends (v, g) to $\rho(v) \cdot g$. It induces a morphism from $H^*(G) \rightarrow H^*(V) \otimes H^*(C_G(\rho))$. By adjunction, it gives us a morphism $T_V(H^*(G)) \rightarrow H^*(C_G(\rho))$ which depends only on the conjugacy class of ρ . This morphism induce an isomorphism between $T_{(V,\rho)}(H^*(G))$ and $H^*(C_G(\rho))$, and

$$\rho_{H^*(G),(V,\rho)} : H^*(G) \rightarrow T_{(V,\rho)}(H^*(G)) \cong H^*(C_G(\rho))$$

is the morphism induced by the injection $C_G(\rho) \hookrightarrow G$.

Hence, (V, ρ) is central if and only if the injection $C_G(\rho) \hookrightarrow G$ induces an isomorphism in cohomology.

Definition 3.9. Let K be an unstable algebra, K is connected if K has an augmentation $\epsilon_K : K \rightarrow \mathbb{F}_p$ which induces an isomorphism $K^0 \xrightarrow{\cong} \mathbb{F}_p$.

Remark 3.10. The connected components $T_{(V,\phi)}(K)$ are connected, hence, if K is not connected, $\mathbf{C}(K) = \emptyset$.

Example 3.11. The functor T_0 is the identity, hence, if K is connected, $T_{(0,\epsilon_K)}(K) \cong K$, for $\epsilon_K : K \rightarrow \mathbb{F}_p$ the unique morphism of unstable algebra from K to \mathbb{F}_p . Hence $(0, \epsilon_K)$ is central.

Notation 3.12. Let $\epsilon_{K,V}$, be the composition of ϵ_K with the injection from \mathbb{F}_p to $H^*(V)$.

For K a connected unstable algebra, $I(K)$ denotes the augmentation ideal of K . Then the module of indecomposable elements of K is defined by $\mathcal{Q}(K) := I(K)/I(K)^2$. An unstable module M is said to be locally finite if, for all $x \in M$, $\mathcal{A}x$ is finite.

In [DW90], Dwyer and Wilkerson exhibit how, when $\mathcal{Q}(K)$ is locally finite, central elements of K are related to $H^*(V)$ -comodule structures on K .

Proposition 3.13. [DW90, Proof of Theorem 3.2]

Let K be a connected unstable algebra such that $\mathcal{Q}(K)$ is locally finite as an unstable module, then $(V, \epsilon_{K,V})$ is central for all $V \in \mathcal{V}^f$.

In particular, if K is a connected, noetherian, unstable algebra, then $(V, \epsilon_{K,V})$ is central for all vector space V .

We recall the following results of [DW92b].

Proposition 3.14. [DW92b, Proposition 3.4]

Let K be a connected unstable algebra such that $\mathcal{Q}(K)$ is locally finite. Then, for $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, (V, ϕ) is central if and only if there exists a morphism from K to $K \otimes H^*(V)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & & K \\
 & \nearrow \text{id} & \uparrow \text{id} \otimes \epsilon_{H^*(V)} \\
 K & \longrightarrow & K \otimes H^*(V) \\
 & \searrow \phi & \downarrow \epsilon_K \otimes \text{id} \\
 & & H^*(V).
 \end{array}$$

Corollary 3.15. [DW92b] Let K be a connected unstable algebra such that $\mathcal{Q}(K)$ is locally finite. For $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, (V, ϕ) is central if and only if K has a structure of $H^*(V)$ -comodule κ in \mathcal{K} , such that the following diagram commutes:

$$\begin{array}{ccc}
 K & \xrightarrow{\kappa} & K \otimes H^*(V) \\
 & \searrow \phi & \downarrow \epsilon_K \otimes \text{id} \\
 & & H^*(V).
 \end{array}$$

In particular, this implies:

Proposition 3.16. Let K be an unstable algebra such that $\mathcal{Q}(K)$ is locally finite, then for $\phi \in \mathcal{C}(K)$ and $\alpha : V \rightarrow E$ a morphism in \mathcal{V}^f , $(V, \alpha^* \circ \phi) \in \mathcal{C}(K)$.

Example 3.17. For $W \in \mathcal{V}^f$, the addition in W , ∇_W , induces on $H^*(W)$ a coalgebra structure in \mathcal{K} . Then, for every morphism of unstable modules ϕ from $H^*(W)$ to $H^*(V)$, one can take the composition of ∇_W^* with $\text{id}_{H^*(W)} \otimes f$ to define a $H^*(V)$ -comodule structure on $H^*(W)$ satisfying the hypothesis of corollary 3.15. Therefore (V, ϕ) is central.

4 The category $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$

In [Rec84], Rector used the fact that for a noetherian unstable algebra K , the functor which maps V to $\text{Hom}_{\mathcal{K}}(K, H^*(V))$ is fully determined by the functor which maps V to $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(V))$, where $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(V))$ is the set of morphism from K to $H^*(V)$ which makes $H^*(V)$ a finitely generated K -module. The functor $V \mapsto \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(V))$ is defined on the category $\mathcal{V}\mathcal{I}$, whose objects

are finite dimensional vector spaces on \mathbb{F}_p and whose morphisms are injective morphisms. In this section, we start by recalling the notion of a regular element of a functor in $\mathcal{P}\text{fin}^{(\mathcal{V}^f)^{\text{op}}}$, introduced in [HLS93]. This allows us to understand the passage from $\text{Hom}_{\mathcal{K}}(K, H^*(-))$ to $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ as a construction on functors in $\mathcal{P}\text{fin}^{(\mathcal{V}^f)^{\text{op}}}$.

Then, we define a shift functor for contravariant functors on the category $\mathcal{V}\mathcal{I}$. This will allow us to define a notion of centrality for objects of $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, which coincides for noetherian unstable algebras, with the notion of centrality "away from $\mathcal{N}il$ " for pairs (V, ϕ) such that $\phi \in \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(V))$.

4.1 The functor $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$

Definition 4.1. Let $\mathcal{V}\mathcal{I}$ be the wide sub-category of \mathcal{V}^f with only injective morphisms.

In the following, we will navigate between different presheaf categories from \mathcal{V}^f or $\mathcal{V}\mathcal{I}$ to variants of Set . Let us define all those categories.

- Definition 4.2.**
1. Let $\text{Set}^{(\mathcal{V}^f)^{\text{op}}}$ and $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ be the categories of contravariant functors from \mathcal{V}^f and $\mathcal{V}\mathcal{I}$ to the category of sets.
 2. Let $\mathcal{F}\text{in}^{(\mathcal{V}^f)^{\text{op}}}$ and $\mathcal{F}\text{in}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ be the categories of contravariant functors from \mathcal{V}^f and $\mathcal{V}\mathcal{I}$ to the category of finite sets.
 3. Let $\mathcal{P}\text{fin}^{(\mathcal{V}^f)^{\text{op}}}$ and $\mathcal{P}\text{fin}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ be the categories of contravariant functors from \mathcal{V}^f and $\mathcal{V}\mathcal{I}$ to the category of profinite sets.

Definition 4.3. For K in \mathcal{K} , we define $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ which maps $V \in \mathcal{V}\mathcal{I}$ to the set of morphisms ϕ from K to $H^*(V)$ such that ϕ makes $H^*(V)$ a finitely generated K -module.

Remark 4.4. If K is finitely generated as an algebra over \mathcal{A} , then, for all finite dimensional vector spaces V , $\text{Hom}_{\mathcal{K}}(K, H^*(V))$ is finite. Thus $\text{Hom}_{\mathcal{K}}(K, H^*(-))$ and $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ are respectively in $\mathcal{F}\text{in}^{(\mathcal{V}^f)^{\text{op}}}$ and $\mathcal{F}\text{in}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$.

In particular, this is the case for K noetherian.

4.2 Regular elements of an object in $\text{Set}^{(\mathcal{V}^f)^{\text{op}}}$

In this sub-section we recall the notion of a regular element of an object of $\text{Set}^{(\mathcal{V}^f)^{\text{op}}}$. The interest of this notion is that, for F in $\text{Set}^{(\mathcal{V}^f)^{\text{op}}}$ satisfying a noetherianity condition introduced in [HLS93], the regular elements of F define an object in the much simpler category $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. Moreover, if $F \cong \text{Hom}_{\mathcal{K}}(K, H^*(-))$ for K a noetherian unstable algebra, the regular elements of F are given by $\text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ and we can show that F is fully determined by its regular elements. This will allow us to make the connection between the study of the $\mathcal{N}il$ -localisation of noetherian unstable algebras, and the study of functors in $\mathcal{F}\text{in}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$.

Proposition 4.5. [HLS93, Proposition-Definition 5.1] *Let $G \in \text{Set}^{(\mathcal{V}^f)^{\text{op}}}$, $V \in \mathcal{V}^f$ and $s \in G(V)$. Then, there exists a unique sub-vector space U of V , denoted by $\ker(s)$, such that:*

1. *For all $t \in G(W)$ and all morphism $\alpha : V \rightarrow W$ such that $s = G(\alpha)(t)$, $\ker(\alpha) \subset U$.*

2. There exists W_0 in $(\mathcal{V}^f)^{op}$, $t_0 \in G(W_0)$ and $\alpha_0 : V \rightarrow W_0$ such that $s = G(\alpha_0)(t_0)$ and $\ker(\alpha_0) = U$.

3. There exists $t_0 \in G(V/U)$ such that $s = G(\pi)(t_0)$, where π is the projection of V onto V/U .

Definition 4.6. Let $G \in \text{Set}^{(\mathcal{V}^f)^{op}}$, $V \in \mathcal{V}^f$ and $s \in G(V)$. We say that s is regular if $\ker(s) = 0$. Let $\text{reg}(G)(V) := \{x \in G(V) ; \ker(x) = 0\}$.

We recall the definition of a noetherian functor from [HLS93].

Definition 4.7. Let F be in $\mathcal{P}\text{fin}^{(\mathcal{V}^f)^{op}}$, we say that F is noetherian if it satisfies the following:

1. $F \in \mathcal{F}\text{in}^{(\mathcal{V}^f)^{op}}$,
2. there exists an integer d such that $F(V) = \emptyset$ for $\dim(V) > d$,
3. for all $V \in \mathcal{V}^f$ and $s \in F(V)$ and for all morphisms α which takes values in V , $\ker(F(\alpha)s) = \alpha^{-1}(\ker(s))$.

Proposition 4.8. [HLS93, Theorem 7.1]

1. If $K \in \mathcal{K}$ is noetherian, $\text{Hom}_{\mathcal{K}}(K, H^*(\cdot))$ is noetherian.
2. If $F \in \text{Set}^{(\mathcal{V}^f)^{op}}$ is noetherian, then $m \circ \mathcal{L}(F) \in \mathcal{K}$ is noetherian.

Lemma 4.9. Let F be a noetherian functor, then, $\text{reg}(F)$ is an object in $\mathcal{F}\text{in}^{(\mathcal{V}^{\mathcal{I}})^{op}}$.

Proof. We only have to prove that if $x \in \text{reg}(F)(V)$, and if α from E to V is an injection, α^*x is regular. But, since F is supposed noetherian, $\ker(\alpha^*x) = \alpha^{-1}(\ker(x))$ which is equal to $\{0\}$ since x is regular and α is an injection. \square

Remark 4.10. reg is not a functor, since the image under a natural transformation of a regular element is not necessarily regular. For example, fix V and U where $U \neq 0$ is a sub-vector space of V , and consider the natural transformation from $\text{Hom}_{\mathbb{F}_2}(\cdot, V) \xrightarrow{\pi} \text{Hom}_{\mathbb{F}_2}(\cdot, V/U)$ induced by the projection from V to V/U . Then, id_V is regular in $\text{Hom}_{\mathbb{F}_2}(V, V)$ but $\ker(\pi \circ \text{id}_V) = U$ in $\text{Hom}_{\mathbb{F}_2}(V, V/U)$.

Proposition 4.11. [DW92b, Proposition 4.8] For K a noetherian unstable algebra, we have a natural isomorphism $\text{reg}(\text{Hom}_{\mathcal{K}}(K, H^*(\cdot))) \cong \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(\cdot))$.

Let us denote by \mathcal{O} the forgetful functor from $\text{Set}^{(\mathcal{V}^f)^{op}}$ to $\text{Set}^{(\mathcal{V}^{\mathcal{I}})^{op}}$ induced by restriction to $(\mathcal{V}^{\mathcal{I}})^{op}$. We define the left Kan extension of \mathcal{O} along the identity.

Proposition 4.12. The functor \mathcal{O} has a left adjoint, which maps a functor $F \in \text{Set}^{(\mathcal{V}^{\mathcal{I}})^{op}}$ to \tilde{F} , which is defined by $\tilde{F}(V) := \bigsqcup_{U \in \mathcal{S}(V)} F(V/U)$, where $\mathcal{S}(V)$ is the set of sub-vector spaces of V .

Proof. Let us first prove that \tilde{F} defines an object in $\text{Set}^{(\mathcal{V}^f)^{op}}$. To a morphism $\alpha : V \rightarrow W$ we associate $\tilde{F}(\alpha)$ in the following way: for $x \in F(W/U)$ we consider $\pi \circ \alpha$ where π is the projection from W to W/U , we factorise $\pi \circ \alpha$ as $\tilde{\alpha} \circ \psi$ with ψ the projection from V to $V/\ker(\pi \circ \alpha)$, then $\tilde{\alpha}$ is injective and we define $\tilde{F}(\alpha)(x) = F(\tilde{\alpha})(x) \in F(V/\ker(\pi \circ \alpha))$.

Let ϕ be a natural transformation from \tilde{F} to A , where A is an object in $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$. For all $V \in \mathcal{V}^f$, we have a morphism in $\mathcal{S}et$,

$$\phi_V : \bigsqcup_{U \in \mathcal{S}(V)} F(V/U) \rightarrow A(V).$$

Then,

$$\phi_V|_{F(V)}$$

induces a morphism $\iota(\phi)_V$ from $F(V)$ to $\mathcal{O}(A)(V)$. $\iota(\phi)$ is a natural transformation in $\mathcal{S}et^{(\mathcal{V}^{\mathcal{I}})^{op}}$. Conversely, for ϕ a natural transformation from F to $\mathcal{O}(A)$, we define

$$\gamma(\phi)_V : \tilde{F}(V) \rightarrow A(V)$$

in the following way: for $x \in F(V/U)$ we define $\gamma(\phi)_V(x) := A(\pi)(\phi_{V/U}(x))$, for π the projection from V to V/U . Then, $\gamma(\phi)$ is a natural transformation in $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$, and γ and ι are mutually inverse. \square

Remark 4.13. For $F \in \mathcal{S}et^{(\mathcal{V}^{\mathcal{I}})^{op}}$, \tilde{F} always satisfies the second condition in the definition of a noetherian functor. Moreover, \tilde{F} is in $\mathcal{F}in^{(\mathcal{V}^f)^{op}}$ if and only if $F \in \mathcal{F}in^{(\mathcal{V}^{\mathcal{I}})^{op}}$ and there is an integer d such that $F(V) = \emptyset$ for $\dim(V) \geq d$, hence \tilde{F} is noetherian if and only if $F \in \mathcal{F}in^{(\mathcal{V}^{\mathcal{I}})^{op}}$ and there exists $d \in \mathbb{N}$ such that $F(V)$ is empty for $\dim(V)$ greater than d .

Proposition 4.14. For F a noetherian functor, $\widetilde{reg(F)} \cong F$.

Proof. Let F be a noetherian functor. For V a finite dimensional vector space, we define the following morphism from $\widetilde{reg(F)}(V)$ to $F(V)$:
$$\begin{array}{ccc} \widetilde{reg(F)}(V) & \rightarrow & F(V) \\ x \in \widetilde{reg(F)}(V/U) & \mapsto & F(\pi)(x), \end{array}$$
 where π is the projection from V to V/U . Then, Proposition 4.5 implies that this morphism is an isomorphism, and the fact that F is noetherian implies that it is natural in V . \square

Corollary 4.15. For K a noetherian unstable algebra, there is a natural bijection with respect to V ,

$$\text{Hom}_{\mathcal{K}}(K, H^*(V)) \cong \bigsqcup_{U \in \mathcal{S}(V)} \text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(V/U)).$$

4.3 Central elements of a noetherian algebra

In the last subsection, we explained why, for K a noetherian unstable algebra, $\text{Hom}_{\mathcal{K}}(K, H^*(\cdot))$ is fully determined by its regular elements, given by $\text{Hom}_{\mathcal{K}_{f.g.}}(K, H^*(\cdot))$. In this section, we prove that for K noetherian, one can deduce $\mathbf{C}(K)$ from the central elements (V, ϕ) of K such that ϕ is regular.

We recall the definition of the centre of a noetherian unstable algebra from [Hea21] which is in some sense the maximal regular central element of K .

We state the following results for noetherian algebras, since it is in this case that we will use those, but all of them are true if we only suppose that $\mathcal{Q}(K)$ is locally finite.

Proposition 4.16. *Let K be a noetherian unstable algebra and let ϕ be in $\text{Hom}_{\mathcal{K}}(K, H^*(E))$, then if (E, ϕ) is central, let ϕ_0 be the only element in $\text{Hom}_{\mathcal{K}}(K, H^*(E/\ker(\phi)))$ such that $\phi = \pi^*\phi_0$. Then, ϕ_0 is regular and $(E/\ker(\phi), \phi_0)$ is central.*

Conversely, if $(E/U, \phi)$ is central with ϕ regular, $(E, \pi^ \circ \phi)$ is central. Thus, (E, ϕ) is central if and only if $(E/\ker(\phi), \phi_0)$ is central.*

Proof. We consider s a section of π , the projection from E to $E/\ker(\phi)$, then $\phi_0 = s^* \circ \phi$. Then, by Proposition 3.16, if (E, ϕ) is central, so is $(E/\ker(\phi), \phi_0)$. The converse is a direct consequence of Proposition 3.16. □

Lemma 4.17. *[DW92b, Lemma 4.6] Let K be a noetherian unstable algebra, let f be in $\text{Hom}_{\mathcal{K}}(K, H^*(E))$ and $(C, g) \in \mathbf{C}(K)$, then there exists a unique pair $(E \oplus C, f \boxplus g)$ with $f \boxplus g \in \text{Hom}_{\mathcal{K}}(K, H^*(E \oplus C))$ such that $f \boxplus g$ composed respectively with the projections on $H^*(E)$ and $H^*(C)$ gives f and g .*

$$\begin{array}{ccc}
 & & H^*(C) \\
 & \nearrow g & \uparrow \text{id} \otimes \epsilon_{H^*(E)} \\
 K & \xrightarrow{f \boxplus g} & H^*(C) \otimes H^*(E) \\
 & \searrow f & \downarrow \epsilon_{H^*(C)} \otimes \text{id} \\
 & & H^*(E)
 \end{array}$$

Remark 4.18. In Lemma 4.17, even if f and g are regular, $f \boxplus g$ is not regular in general.

We recall now the definition of the centre of a noetherian algebra.

Definition 4.19. [Hea21, Definition 3.9] Let K be a noetherian unstable algebra, let f be in $\text{Hom}_{\mathcal{K}}(K, H^*(E))$ and $(C, g) \in \mathbf{C}(K)$, we define $(E \circ C, \sigma(f, g))$ by $E \circ C = (E \oplus C)/\ker(f \boxplus g)$, and $\sigma(f, g) : K \rightarrow E \circ C$ such that the composition of $\sigma(f, g)$ by $H^*(\pi_{(E \oplus C)/\ker(f \boxplus g)})$ is equal to $f \boxplus g$, for $\pi_{(E \oplus C)/\ker(f \boxplus g)}$ the projection on $(E \oplus C)/\ker(f \boxplus g)$.

Proposition 4.20. [Hea21, Corollary 3.11] *Let K be a noetherian unstable algebra and (C, g) and (E, f) be two central elements of K . Then $(E \oplus C, f \boxplus g)$ and $(E \circ C, \sigma(f, g))$ are central.*

Theorem 4.21. [Hea21, Theorem 3.13] *Let K be a connected, noetherian, unstable algebra. Then, up to isomorphism, there is a unique couple $(C, \gamma) \in \mathbf{C}(K)$ which is regular and satisfy the following: for all central element (E, f) with f regular, there exists an injection ι from E to C , such that $f = \iota^*\gamma$. We will call (C, γ) the centre of K .*

The idea of the proof is to consider (C, γ) a regular central element of maximal dimension. Then, show that for (E, f) regular and central, either $(E, f) = (E, \iota^*\gamma)$ for ι an injection from E to C , or the dimension of $E \circ C$ is greater than that of C which contradicts the assumption.

Corollary 4.22. *For $E \in \mathcal{V}^f$ and $f \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, (E, f) is central if and only if there is a morphism α from E to C , with (C, γ) the centre of K , such that $f = \alpha^*\gamma$.*

Proof. It is a direct consequence of Proposition 3.16 and Theorem 4.21. □

4.4 The shift functor

We want to be able to discuss centrality away from $\mathcal{N}il$, and define accordingly a notion of central elements for objects in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. The aim of this subsection is to define, for $F \in \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, $V \in \mathcal{V}^f$ and $\phi \in F(V)$ a functor $\sigma_{(V,\phi)}F$, in such a way that for $F = \mathit{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$, $\sigma_{(V,\phi)}F \cong \mathit{Hom}_{\mathcal{K}f.g.}(T_{(V,\phi)}(K), H^*(-))$.

We start by defining functors Σ_V from $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$ to $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ such that, for all unstable algebras K , $\Sigma_V \mathit{Hom}_{\mathcal{K}}(K, H^*(-)) \cong \mathit{Hom}_{\mathcal{K}}(T_V(K), H^*(-))$, and such that $\Sigma_V \mathit{Hom}_{\mathcal{K}}(K, H^*(-))$ naturally decomposes in $\Sigma_{(V,\phi)} \mathit{Hom}_{\mathcal{K}}(K, H^*(-)) \cong \mathit{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(-))$, with ϕ running through $\mathit{Hom}_{\mathcal{K}}(K, H^*(V))$.

Then, we define a shift functor σ_V in the category $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and we identify the desired functors $\sigma_{(V,\phi)}F$ as the connected components of $\sigma_V F$. It is worth pointing out that since the decomposition of $\mathit{Hom}_{\mathcal{K}f.g.}(T_V(K), H^*(-))$ is induced by elements (V, ϕ) with $\phi \in \mathit{Hom}_{\mathcal{K}}(K, H^*(V))$ (and not only elements in $\mathit{Hom}_{\mathcal{K}f.g.}(K, H^*(V))$), $\sigma_V \mathit{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$ is only a sub functor of $\mathit{Hom}_{\mathcal{K}f.g.}(T_V(K), H^*(-))$.

Definition 4.23. For $V \in \mathcal{V}^f$, define the functor Σ_V from $\mathcal{S}et^{(\mathcal{V}^f)^{op}}$ to $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ by

$$\Sigma_V F(W) = F(V \oplus W),$$

and $\Sigma_V F(\alpha) = F(\text{id}_V \oplus \alpha)$, for $F \in \mathcal{S}et^{(\mathcal{V}^f)^{op}}$, $W \in \mathcal{V}^f$ and α a morphism in \mathcal{V}^f .

Lemma 4.24. For $K \in \mathcal{K}$ and $V \in \mathcal{V}^f$, there is an isomorphism natural both in K and V , $\mathit{Hom}_{\mathcal{K}}(T_V(K), H^*(-)) \cong \Sigma_V \mathit{Hom}_{\mathcal{K}}(K, H^*(-))$.

Proof. This is a direct consequence of the definition of T_V as the adjoint of the tensor product by $H^*(V)$ and of the natural isomorphism $H^*(V) \otimes H^*(W) \cong H^*(V \oplus W)$. \square

Definition 4.25. For $F \in \mathcal{S}et^{(\mathcal{V}^f)^{op}}$, $V \in \mathcal{V}^f$ and $\phi \in F(V)$, we consider $\Sigma_{(V,\phi)} F(W)$ to be the fibre over $\{\phi\}$ of the morphism $\Sigma_V F(W) \rightarrow \Sigma_V F(0) \cong F(V)$ induced by the injection from 0 to W .

Lemma 4.26. Let K be an unstable algebra, $V \in \mathcal{V}^f$ and $\phi \in \mathit{Hom}_{\mathcal{K}}(K, H^*(V))$. There is a natural isomorphism $\Sigma_{(V,\phi)} \mathit{Hom}_{\mathcal{K}}(K, H^*(W)) \cong \mathit{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(W))$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \Sigma_V \mathit{Hom}_{\mathcal{K}}(K, H^*(W)) & \xrightarrow{\cong} & \mathit{Hom}_{\mathcal{K}}(T_V(K), H^*(W)) \\ \downarrow & & \downarrow \\ \mathit{Hom}_{\mathcal{K}}(K, H^*(V)) & \xrightarrow{\cong} & \mathit{Hom}_{\mathcal{K}}(T_V(K), \mathbb{F}_p), \end{array}$$

where the vertical maps are induced by the injection from $\{0\}$ to W , and the horizontal ones are given by the natural isomorphism of Lemma 4.24. By definition, $\Sigma_{(V,\phi)} \mathit{Hom}_{\mathcal{K}}(K, H^*(W))$ is the fibre over ϕ of the left map, and by construction $\mathit{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(W))$ is the fibre over the adjoint of ϕ of the right one. This concludes the proof. \square

Proposition 4.27. *If F is noetherian, for x an element in $F(V)$, $\Sigma_V F$ and $\Sigma_{(V,x)} F$ are also noetherian.*

Proof. If F is finite, $\Sigma_V F$ and $\Sigma_{(V,x)} F$ are obviously finite, we only have to prove the second condition. Let a be an element in $\sigma_V F(W)$, then $a \in F(V \oplus W)$. There is an ambiguity in considering $\ker(a)$, since a can be viewed either as an element in F or $\Sigma_V F$. Let $\ker^1(a)$ denote the kernel of a considered as an element in F , and $\ker^2(a)$ its kernel in $\Sigma_V F$. We notice that $\ker^1(a)$ is a sub-vector space of $V \oplus W$, whereas $\ker^2(a)$ is a sub-space of W . Then, $\ker^2(a) = \ker^1(a) \cap W$. Indeed, for t regular such that $a = \pi^* t$ for π the projection from $V \oplus W$ to $(V \oplus W)/\ker^1(a)$, π factorises as $\pi_2 \circ \pi_1$, for π_1 the projection from $V \oplus W$ to $V \oplus (W/(\ker^1(a) \cap W))$ and π_2 the projection from $V \oplus (W/(\ker^1(a) \cap W))$ to $(V \oplus W)/\ker^1(a)$, then $a = \Sigma_V F(\pi')(\pi_2^* t)$, for π' the projection from W to $W/(\ker^1(a) \cap W)$. Therefore, $\ker^1(a) \cap W \subset \ker^2(a)$.

Conversely, if π is now the projection from W to $W/\ker^2(a)$, let $t \in F(V \oplus (W/\ker^2(a)))$ such that $a = \Sigma_V F(\pi)(t)$. Then, $a = (\text{id}_V \oplus \pi)^* t$, hence $\ker^2(a) \subset \ker^1(a)$. Since, we also have $\ker^2(a) \subset W$, $\ker^2(a) = \ker^1(a) \cap W$. Then, for α a morphism from a finite dimensional vector space U to W , $\ker^2(\Sigma_V F(\alpha)(a)) = \ker^1((\text{id}_V \oplus \alpha)^* a) \cap W$ which is equal to $(\text{id}_V \oplus \alpha)^{-1}(\ker^1(a) \cap W)$, since F is noetherian. Hence, it is equal to $\alpha^{-1}(\ker^1(a) \cap W) = \alpha^{-1}(\ker^2(a))$. This proves that $\Sigma_V F$ is noetherian. The proof for $\Sigma_{(V,x)} F$ is similar. \square

Corollary 4.28. *If K is noetherian and nil-closed, $T_V(K)$ and $T_{(V,\phi)} K$ are noetherian, for $V \in \mathcal{V}^f$ and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$.*

Proof. By Proposition 4.27, $g(T_V(K))$ and $g(T_{(V,\phi)}(K))$ are noetherian, then by Proposition 4.8 $l_1(T_V(K)) \cong (m \circ \mathcal{L} \circ g)(T_V(K))$ and $l_1(T_{(V,\phi)}(K))$ are noetherian. Therefore, since by Corollary 2.10 and Lemma 3.4 $T_V(K)$ and $T_{(V,\phi)}(K)$ are nil-closed, they are noetherian. \square

We want to identify a shift functor σ_V from $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ to $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, in so that $\sigma_V \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-))$ captures the behaviour of $T_V(K)$ away from nilpotent objects for K noetherian, and such that for F in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, $\sigma_V F$ comes with a decomposition in $\sigma_{(V,\phi)} F$ with ϕ running through $F(V)$.

In order to define this shift functor, we have to discuss pushouts in \mathcal{V}^f and $\mathcal{V}\mathcal{I}$. It is worth noticing that the pull-back in \mathcal{V}^f of a diagram whose morphisms are in $\mathcal{V}\mathcal{I}$ is also a pull-back in $\mathcal{V}\mathcal{I}$.

Remark 4.29. Pushouts usually don't exist in $\mathcal{V}\mathcal{I}$, for example, if one consider the following diagram

$$\begin{array}{ccc} 0 & \longrightarrow & V \\ \downarrow & & \\ V & & \end{array},$$

the pushout “should be” $V \oplus V$, but since non injective morphisms are not in $\mathcal{V}\mathcal{I}$ the commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & V \\ \downarrow & & \downarrow \\ V & \longrightarrow & V \end{array}$$

does not give rise to a morphism from $V \oplus V$ to V satisfying the universal property of the pushout. So, in the following, we will consider pushouts in \mathcal{V}^f of diagrams in $\mathcal{V}\mathcal{I}$.

The following simple lemma deals with this.

Lemma 4.30. *Consider a pushout square in \mathcal{V}^f :*

$$\begin{array}{ccc} M & \xrightarrow{\omega} & W \\ \nu \downarrow & & \downarrow \\ V & \longrightarrow & P. \end{array}$$

Suppose that ν and ω are injections. Then, for all pullback squares

$$\begin{array}{ccc} M & \xrightarrow{\omega} & W \\ \nu \downarrow & & \downarrow \\ V & \longrightarrow & N, \end{array}$$

such that the morphisms from V and W to N are injections, the morphism from P to N induced by the universal property of the pushout in \mathcal{V}^f , is a morphism in $\mathcal{V}\mathcal{I}$.

Let us also recall the pasting law for pullbacks:

Lemma 4.31. [AHS90, Proposition 11.10] *We consider a diagram of the following shape in a category \mathcal{C} :*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F. \end{array}$$

If the right square is a pullback square, then the outer square is a pullback square if and only if the left square is.

Definition 4.32. For V and W some fixed objects of \mathcal{V}^f , let $B(V, W)$ be the set of triples (M, ν, ω) , where $M \in \mathcal{V}^f$ and ν and ω are morphisms from M to V and W in $\mathcal{V}\mathcal{I}$. Let also \equiv be the relation on $B(V, W)$ defined by $(M, \nu, \omega) \equiv (M', \nu', \omega')$ if there exists an isomorphism μ from M' to M such that $\nu' = \nu \circ \mu$ and $\omega' = \omega \circ \mu$.

The following lemma is obvious.

Lemma 4.33. \equiv is an equivalence relation on $B(V, W)$.

Definition 4.34. For V and W two objects of $\mathcal{V}\mathcal{I}$, let $\mathbf{B}(V, W)$ be the set of equivalence classes for \equiv in $B(V, W)$. For $[M, \nu, \omega] \in \mathbf{B}(V, W)$, let $V \oplus_{\nu, \omega} W$ denote the pushout of the following diagram in \mathcal{V}^f :

$$\begin{array}{ccc} M & \xrightarrow{\omega} & W \\ \nu \downarrow & & \\ V & & \end{array},$$

let also denote by $\iota_V^{V \oplus_{\nu, \omega} W}$ and $\iota_W^{V \oplus_{\nu, \omega} W}$ the induced injections from V and W to $V \oplus_{\nu, \omega} W$. When there is no ambiguity, we will denote them by ι_V and ι_W .

Remark 4.35. $V \oplus_{\nu, \omega} W$ does not depend on the choice of $(M, \nu, \omega) \in [M, \nu, \omega]$.

Proposition 4.36. $\mathbf{B}(-, -)$ is a bifunctor on \mathcal{VT}^{op} .

Proof. Let $\alpha : V' \hookrightarrow V$ and $\beta : W' \hookrightarrow W$ be morphisms in \mathcal{VT} , for $[M, \nu, \omega] \in \mathbf{B}(V, W)$ we define $\mathbf{B}(\alpha, \beta)([M, \nu, \omega]) \in \mathbf{B}(V', W')$ in the following way. We consider the following diagram, where M_V, M_W and M' are defined by pullback.

$$\begin{array}{ccccc}
 M' & \longrightarrow & M_W & \longrightarrow & W' \\
 \downarrow & & \downarrow & & \downarrow \beta \\
 M_V & \longrightarrow & M & \xrightarrow{\omega} & W \\
 \downarrow & & \downarrow \nu & & \\
 V' & \xrightarrow{\alpha} & V & & .
 \end{array}$$

Then, ν' and ω' are given by the compositions $M' \rightarrow M_V \rightarrow V'$ and $M' \rightarrow M_W \rightarrow W'$. \square

Remark 4.37. By the pasting law of pullbacks (lemma 4.31, that we used two times) we can show that the outer square of the following diagram is a pullback square:

$$\begin{array}{ccccc}
 M' & \longrightarrow & M_W & \longrightarrow & W' \\
 \downarrow & & \downarrow & & \downarrow \\
 M_V & \longrightarrow & M & \xrightarrow{\omega} & W \\
 \downarrow & & \downarrow \nu & & \downarrow \\
 V' & \longrightarrow & V & \longrightarrow & V \oplus_{\nu, \omega} W.
 \end{array}$$

Definition 4.38. Let $\alpha : V' \hookrightarrow V$ and $\beta : W' \hookrightarrow W$ be injections in \mathcal{VT} , for $[M, \nu, \omega] \in \mathbf{B}(V, W)$ and for $[M', \nu', \omega'] := \mathbf{B}(\alpha, \beta)([M, \nu, \omega]) \in \mathbf{B}(V', W')$ we define

$$\alpha \oplus_{\nu, \omega} \beta : V' \oplus_{\nu', \omega'} W' \hookrightarrow V \oplus_{\nu, \omega} W,$$

the injective (cf lemma 4.30) morphism induced by the universal property of the pushout in \mathcal{V}^f of the following pullback square:

$$\begin{array}{ccc}
 M' & \xrightarrow{\nu'} & W' \\
 \omega' \downarrow & & \downarrow \omega \circ \beta \\
 V' & \xrightarrow{\nu \circ \alpha} & V \oplus_{\nu, \omega} W.
 \end{array}$$

Definition 4.39. For $F \in \text{Set}^{(\mathcal{VT})^{op}}$ and V and W two objects in \mathcal{VT} , we define

$$\sigma_V F(W) := \bigsqcup_{[M, \nu, \omega] \in \mathbf{B}(V, W)} F(V \oplus_{\nu, \omega} W).$$

Even though the roles of V and W are symmetric in the definition of $\sigma_V F(W)$, we use an asymmetric notation to reflect that of $T_V(K)$.

Proposition 4.40. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$, σ_-F is a bifunctor on $\mathcal{V}\mathcal{I}^{op}$.

Proof. For $\alpha : V' \hookrightarrow V$ and $\beta : W' \hookrightarrow W$ two morphisms in $\mathcal{V}\mathcal{I}$ we have to define $\sigma_\alpha(\beta)$ from $\sigma_V F(W)$ to $\sigma_{V'} F(W')$. For $x \in F(V \oplus_{\nu, \omega} W)$ let $\sigma_\alpha F(\beta)(x) = F(\alpha \oplus_{\nu, \omega} \beta)(x) \in F(V' \oplus_{\nu', \omega'} W')$. Then, the fact that σ_-F is a bifunctor comes from the fact that $\text{id}_V \oplus_{\nu, \omega} \text{id}_W = \text{id}_{V \oplus_{\nu, \omega} W}$ and that for $\alpha' : V'' \rightarrow V'$ and $\beta' : W'' \rightarrow W'$, $(\alpha \oplus_{\nu, \omega} \beta) \circ (\alpha' \oplus_{\nu', \omega'} \beta') = (\alpha \circ \alpha') \oplus_{\nu, \omega} (\beta \circ \beta')$. \square

Corollary 4.41. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$ and V and W some fixed objects in $\mathcal{V}\mathcal{I}$, $\sigma_V F$ and $\sigma_-F(W)$ are objects in $\text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$.

Let us now define the analogue of $T_{(V, \phi)}$ in the category $\text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$.

Lemma 4.42. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$ and V and W in $\mathcal{V}\mathcal{I}$, we have natural isomorphisms $\sigma_0 F(W) \cong F(W)$ and $\sigma_V F(0) \cong F(V)$.

Definition 4.43. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$ and V and W two objects of $\mathcal{V}\mathcal{I}$, the morphism from 0 to W induces a morphism $\sigma_V F(W) \rightarrow \sigma_V F(0) \cong F(V)$. For $x \in F(V)$, let $\sigma_{(V, x)} F(W)$ be the inverse image of $\{x\}$ under this morphism.

Lemma 4.44. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$, V and W objects in $\mathcal{V}\mathcal{I}$ and for $x \in F(V)$, $\sigma_{(V, x)} F$ is a subfunctor of $\sigma_V F$.

Proof. We only have to prove that for $\alpha : U \rightarrow W$ a morphism in $\mathcal{V}\mathcal{I}$ and for $y \in \sigma_{(V, x)} F(W)$, $\sigma_V F(\alpha)(y) \in \sigma_{(V, x)} F(U)$. This follows directly from the fact that the morphism $0 \rightarrow W$ factorizes through $0 \rightarrow U \rightarrow W$. \square

As we stated in the beginning of this sub-section,

$$\sigma_V \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-)) \cong \text{Hom}_{\mathcal{K}\text{f.g.}}(T_V(K), H^*(-)).$$

Nonetheless, as we will see in corollary 4.46, for $\phi \in \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(V))$, we have that

$$\sigma_{(V, \phi)} \text{Hom}_{\mathcal{K}\text{f.g.}}(K, H^*(-)) \cong \text{Hom}_{\mathcal{K}\text{f.g.}}(T_{(V, \phi)}(K), H^*(-)).$$

Proposition 4.45. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{op}}$, $V \in \mathcal{V}^f$ and $x \in F(V)$, $\widetilde{\sigma_{(V, x)} F} \cong \Sigma_{(V, x)} \tilde{F}$.

Proof. Let W be a finite dimensional vector space. Let us first notice that both $\widetilde{\sigma_{(V, x)} F}(W)$ and $\Sigma_{(V, x)} \tilde{F}(W)$ can be seen as sub-sets of $\tilde{F}(V \oplus W) = \bigsqcup_{H \in \mathbf{S}(V \oplus W)} F((V \oplus W)/H)$. It is obvious for

$\Sigma_{(V, x)} \tilde{F}(W)$. For $\widetilde{\sigma_{(V, x)} F}(W) \subset \widetilde{\sigma_V F}(W)$, for every $V \oplus_{\nu, \omega} (W/U)$ the universal property of the product induces a projection $\pi_{\nu, \omega}$ from $V \oplus W$ to $V \oplus_{\nu, \omega} (W/U)$. It induces an isomorphism from $(V \oplus W)/\ker(\pi_{\nu, \omega})$ to $V \oplus_{\nu, \omega} (W/U)$, we identify $F(V \oplus_{\nu, \omega} (W/U))$ with $F((V \oplus W)/\ker(\pi_{\nu, \omega})) \subset \tilde{F}(V \oplus W)$ through this isomorphism. Then we only have to prove that those sub-sets are equals.

$\Sigma_{(V, x)} \tilde{F}(W)$ identifies with the set of elements γ in $\tilde{F}(V \oplus W)$ such that $\tilde{F}(\iota_V^{V \oplus W})(\gamma) = x \in F(V)$. Since \tilde{F} is noetherian and x is regular, $\ker(\gamma) \cap V = \{0\}$. Then, there exists U a subspace of W and a class $[M, \nu, \omega] \in \mathbf{B}(V, W/U)$ such that the canonical projection from $V \oplus W$

to $V \oplus_{\nu, \omega} (W/U)$ induces an isomorphism $(V \oplus W)/\ker(\gamma) \cong V \oplus_{\nu, \omega} (W/U)$. Then, up to this isomorphism $\gamma \in F(V \oplus_{\nu, \omega} (W/U))$ and $(\iota_V^{V \oplus_{\nu, \omega} (W/U)})^* \gamma = x$, so $\gamma \in \sigma_{(V, x)} F(W/U) \subset \widetilde{\sigma_{(V, x)} F(W)}$.

Conversely, for $\gamma \in \sigma_{(V, x)} F(W/U) \subset \widetilde{\sigma_{(V, x)} F(W)}$, there is $[M, \nu, \omega] \in \mathbf{B}(V, W/U)$ such that $\gamma \in F(V \oplus_{\nu, \omega} (W/U))$ and $(\iota_V^{V \oplus_{\nu, \omega} (W/U)})^* \gamma = x$, then for H the kernel of the projection from $V \oplus W$ to $V \oplus_{\nu, \omega} (W/U)$, we have the following isomorphism induced by the first isomorphism theorem $(V \oplus W)/H \cong V \oplus_{\nu, \omega} (W/U)$. Up to this isomorphism, $\gamma \in F((V \oplus W)/H) \subset \tilde{F}(V \oplus W)$. Then, by construction and up to the isomorphism from $(V \oplus W)/H$ to $V \oplus_{\nu, \omega} (W/U)$, $\tilde{F}(\iota_V^{V \oplus W})(\gamma) = (\iota_V^{V \oplus_{\nu, \omega} (W/U)})^* \gamma = x$. Then, $\gamma \in \Sigma_{(V, x)} \tilde{F}(W)$. □

Corollary 4.46. For K a noetherian algebra, $V \in \mathcal{V}^f$ and $\phi \in \text{Hom}_{\mathcal{K}f.g.}(K, H^*(V))$,

$$\sigma_{(V, \phi)} \text{Hom}_{\mathcal{K}f.g.}(K, H^*(-)) \cong \text{Hom}_{\mathcal{K}f.g.}(T_{(V, \phi)}(K), H^*(-)).$$

Proof. This is a direct consequence of Lemma 4.26, of Proposition 4.45 and of the fact that, if K is noetherian, $T_{(V, \phi)}(K)$ is also noetherian. □

Remark 4.47. Since the morphism $\sigma_V F(W) \rightarrow F(V)$ is a surjection, $\sigma_V F(W)$ is the disjoint union of the fibres over singletons in $F(V)$, we then have an isomorphism which is natural in both K and W , $\sigma_V \text{Hom}_{\mathcal{K}f.g.}(K, H^*(W)) \cong \bigsqcup_{\phi \in \text{Hom}_{\mathcal{K}f.g.}(K, H^*(V))} \text{Hom}_{\mathcal{K}f.g.}(T_{(V, \phi)}(K), H^*(W))$.

4.5 Central elements of an object of $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$

In Definition 3.7, we defined the notion of a central elements of an unstable algebra K . In the following, we define a notion of centrality "away from $\mathcal{N}il$ ".

Definition 4.48. For $K \in \mathcal{K}$, $V \in \mathcal{V}^f$ and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$, we will say that (V, ϕ) is central away from $\mathcal{N}il$, if $g(\rho_{K, (V, \phi)})$ (or equivalently $f(\rho_{K, (V, \phi)})$) is an isomorphism.

Remark 4.49. Since, by lemma 3.4, for K $\mathcal{N}il$ -closed, $T_{(V, \phi)}(K)$ is also $\mathcal{N}il$ -closed, if K is $\mathcal{N}il$ -closed (V, ϕ) is central away from $\mathcal{N}il$ if and only if it is central.

The centrality away from $\mathcal{N}il$ can be characterised by properties of the functor $\text{Hom}_{\mathcal{K}f.g.}(K, H^*(-)) \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, $V \in \mathcal{V}^f$ and $\phi \in F(V)$, we want to define $\rho_{F, (V, \phi)} : \sigma_{(V, \phi)} F \rightarrow F$ in such a way that, when $F \cong \text{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$ where K is a noetherian unstable algebra, $\rho_{F, (V, \phi)}$ is an isomorphism if and only if $g(\rho_{K, (V, \phi)})$ is.

Definition 4.50. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, V and W two objects of $\mathcal{V}\mathcal{I}$, let $\rho_{F, V} : \sigma_V F \rightarrow F$ be the natural transformation induced by the morphism $0 \rightarrow V$ and the natural isomorphism $\sigma_0 F \cong F$.

For $x \in F(V)$, we also define

$$\rho_{F, (V, x)} : \sigma_{(V, x)} F \rightarrow F$$

as the restriction of $\rho_{F, V}$ to $\sigma_{(V, x)} F$.

If $\rho_{F, (V, x)}$ is an isomorphism we will say that (V, x) is a central element of F . We will denote by $\mathbf{C}(F)$ the set of central elements of F .

Theorem 4.51. For K a noetherian unstable algebra and $\phi \in \text{Hom}_{\mathcal{K}f.g.}(K, H^*(V))$, (V, ϕ) is central away from $\mathcal{N}il$ for K if and only if it is central for $\text{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$.

Proof. By construction,

$$g(\rho_{K,(V,\phi)}) : \text{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(-)) \cong \Sigma_{(V,\phi)}g(K) \rightarrow g(K),$$

is the morphism which sends $\delta \in \Sigma_{(V,\phi)}\text{Hom}_{\mathcal{K}}(K, H^*(W))$ to $(\iota_W^{V \oplus W})^*\delta \in \text{Hom}_{\mathcal{K}}(K, H^*(W))$. Then, for $F_K = \text{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$ and for ϕ regular, by construction the isomorphisms $\sigma_{(V,\phi)}\widetilde{F}_K(W) \cong \Sigma_{(V,\phi)}\text{Hom}_{\mathcal{K}}(K, H^*(W))$ and $\widetilde{F}_K(W) \cong \text{Hom}_{\mathcal{K}}(K, H^*(W))$ (Propositions 4.14 and 4.11), fit into the following commutative diagram, which is natural with respect to K :

$$\begin{array}{ccc} \Sigma_{(V,\phi)}\text{Hom}_{\mathcal{K}}(K, H^*(-)) & \xrightarrow{g(\rho_{K,(V,\phi)})} & \text{Hom}_{\mathcal{K}}(K, H^*(-)) \\ \cong \uparrow & & \cong \uparrow \\ \sigma_{(V,\phi)}\widetilde{F}_K & \xrightarrow{\rho_{F,(V,\phi)}} & \widetilde{F}_K. \end{array}$$

This concludes the proof. \square

Definition 4.52. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, we will say that F is connected if $F(0)$ is reduced to one element.

Remark 4.53. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, $V \in \mathcal{V}\mathcal{I}$ and $x \in F(V)$, $\sigma_{(V,x)}F(0) = \{x\}$, therefore it is connected. For F not connected, $\mathbf{C}(F) = \emptyset$.

We give an alternative criterion for the centrality of (V, x) . This alternative criterion will often prove to be easier to check, when we have to prove the centrality of a given element.

Lemma 4.54. Let $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, $V \in \mathcal{V}^f$ and $x \in F(V)$. Then, (V, x) is central if for all $W \in \mathcal{V}^f$ and $y \in F(W)$ there is a unique class $[M, \nu, \omega] \in \mathbf{B}(V, W)$ and a unique $b \in F(V \oplus_{\nu, \omega} W)$ such that $\iota_V^*b = x$ and $\iota_W^*b = y$.

Proof. By construction, $\rho_{F,V}$ is the morphism which maps $b \in F(V \oplus_{\nu, \omega} W)$ to ι_V^*b , then $b \in \sigma_{F,(V,x)}F(W)$ if and only if $\iota_V^*b = x$. So, y has a unique element in its inverse image under $\rho_{F,(V,x)}$ if there is a unique $[M, \nu, \omega] \in \mathbf{B}(V, W)$ and a single $b \in F(V \oplus_{\nu, \omega} W)$ such that $\iota_V^*b = x$ and $\iota_W^*b = y$. \square

In the rest of this subsection, we want to prove an analogue of Theorem 4.21 for objects in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. Namely, we want to prove that, for $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ such that $F(V) = \emptyset$ when $\dim(V)$ is greater than some integer d , there is, up to isomorphism, a unique maximal central element (C, c) in the following sense, for all central element (V, x) there is an injection ι from V to C such that $x = \iota^*c$.

Lemma 4.55. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, V an object in \mathcal{V}^f , $x \in F(V)$ and $\alpha : T \rightarrow V$ a morphism in $\mathcal{V}\mathcal{I}$, α^*x is always in the image of $\rho_{F,(V,x)}$. Furthermore, the morphism $\iota_V^{T \oplus_{id_T, \alpha} V}$ is invertible and $((\iota_V^{T \oplus_{id_T, \alpha} V})^{-1})^*x$ is an element in the inverse image of α^*x .

Proof. The morphism $\iota_V^{T \oplus_{\text{id}_T, \alpha} V} : V \rightarrow T \oplus_{\text{id}_T, \alpha} V$ is an isomorphism. Indeed, the following commutative diagram is a pushout in \mathcal{V}^f :

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & V \\ \text{id}_T \downarrow & & \downarrow \text{id}_V \\ T & \xrightarrow{\alpha} & V. \end{array}$$

Then, $((\iota_V^{T \oplus_{\text{id}_T, \alpha} V})^{-1})^* x$ satisfies the two following conditions, $\iota_V^{T \oplus_{\text{id}_T, \alpha} V} * ((\iota_V^{T \oplus_{\text{id}_T, \alpha} V})^{-1})^* x = x$ and $\iota_T^{T \oplus_{\text{id}_T, \alpha} V} * ((\iota_V^{T \oplus_{\text{id}_T, \alpha} V})^{-1})^* x = \alpha^* x$, where the second is a consequence of the following commutative diagram:

$$\begin{array}{ccccc} T & \xrightarrow{\alpha} & V & & \\ \text{id}_T \downarrow & & \downarrow \iota_V & \searrow \text{id}_V & \\ T & \xrightarrow{\iota_T} & T \oplus_{\text{id}_T, \alpha} V & \xrightarrow{\iota_V^{-1}} & V. \\ & \searrow \alpha & & & \end{array}$$

Therefore it is in the inverse image of $\alpha^* x$ under $\rho_{F, (V, x)}$, if (V, x) is central it is the only element in the inverse image. □

In the following, for V, W and T finite dimensional vector spaces, we think of spaces of the form $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$ as $V + W + T$, where we identified V, W and T with their images in $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$, in order to use the associativity $(V + W) + T = V + (W + T)$. The following technical construction aims to identify a canonical isomorphism $\zeta_{\nu, \omega, g, \tau}$ from $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$ to the appropriate $V \oplus_{\nu', g'} (W \oplus_{\omega', \tau'} T)$.

Notation 4.56. For V, W and T in \mathcal{V}^f , we denote

$$T_1(V, W; T) := \bigsqcup_{[M, \nu, \omega] \in \mathbf{B}(V, W)} \mathbf{B}(V \oplus_{\nu, \omega} W, T),$$

and

$$T_2(V; W, T) := \bigsqcup_{[M', \omega', \tau'] \in \mathbf{B}(W, T)} \mathbf{B}(V, W \oplus_{\omega', \tau'} T).$$

Lemma 4.57. 1. $T_1(-, -, -)$ and $T_2(-, -, -)$ are trifunctors on $\mathcal{V}\mathcal{I}^{op}$.

2. There is a natural transformation ζ from $T_1(-, -, -)$ and $T_2(-, -, -)$,

3. for $[M, g, \tau] \in \mathbf{B}(V \oplus_{\nu, \omega} W, T) \subset T_1(V, W; T)$ and for $(\nu', g', \omega', \tau')$ such that $\zeta([M, g, \tau]) = [M', \nu', g'] \in \mathbf{B}(V, W \oplus_{\omega', \tau'} T)$, there is a natural isomorphism

$$\zeta_{\nu, \omega, g, \tau} : (V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T \rightarrow V \oplus_{\nu', g'} (W \oplus_{\omega', \tau'} T),$$

such that the canonical injections from V, W and T to $V \oplus_{\nu', g'} (W \oplus_{\omega', \tau'} T)$ factorizes through $\zeta_{\nu, \omega, g, \tau}$ and their injections in $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$.

Proof. The fact that $T_1(-, -, -)$ and $T_2(-, -, -)$ are trifunctors is a direct consequence of the fact that $\mathbf{B}(-, -)$ is a bifunctor. Let us construct the natural transformation ζ .

For $[M, g, \tau] \in \mathbf{B}(V \oplus_{\nu, \omega} W, T) \subset T_1(V, W; T)$, we consider $\iota_W + \iota_T$ from $W \oplus T$ to $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$. Since ι_W and ι_T are injections from W and T to $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$, $\ker(\iota_W + \iota_T) \cap W = \ker(\iota_W + \iota_T) \cap T = \{0\}$. Then, for all $x \in \ker(\iota_W + \iota_T)$, x has a non trivial component both in W and T , hence there exists $\omega'(x) \in W \setminus \{0\}$ and $\tau'(x) \in T \setminus \{0\}$ such that $x = \omega'(x) - \tau'(x)$. Then, by the first isomorphism theorem, $\iota_W + \iota_T$ factorises through an injection $\iota_{W \oplus_{\omega', \tau'} T}$ from $W \oplus_{\omega', \tau'} T$ to $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$.

We do the same construction a second time. We consider $\iota_V + \iota_{W \oplus_{\omega', \tau'} T}$ from $V \oplus (W \oplus_{\omega', \tau'} T)$ to $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$. Since the two are injections, there are injections ν' and g' from $\ker(\iota_V + \iota_{W \oplus_{\omega', \tau'} T})$ to V and $W \oplus_{\omega', \tau'} T$ such that, for all $x \in \ker(\iota_V + \iota_{W \oplus_{\omega', \tau'} T})$, $x = \nu'(x) - g'(x)$. Then, by the first isomorphism theorem, $\iota_V + \iota_{W \oplus_{\omega', \tau'} T}$ factorises through an injection from $V \oplus_{\nu', g'} (W \oplus_{\omega', \tau'} T)$ to $(V \oplus_{\nu, \omega} W) \oplus_{g, \tau} T$. But, by construction, $\iota_V + \iota_{W \oplus_{\omega', \tau'} T}$ is surjective, therefore this injection is an isomorphism that we denote $\zeta_{\nu, \omega, g, \tau}$. We define $\zeta([M, g, \tau]) := [\ker(\iota_V + \iota_{W \oplus_{\omega', \tau'} T}), \nu', g']$.

By construction, $\zeta_{\nu, \omega, g, \tau}$ satisfies the desired factorizations. \square

Lemma 4.58. *Let F be an object in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, V be an object of $\mathcal{V}\mathcal{I}$ and x be an element of $F(V)$ such that (V, x) is central. For $\alpha : T \rightarrow V$ in $\mathcal{V}\mathcal{I}$, (T, α^*x) is central.*

Proof. By Lemma 4.54, we have to prove that for every $y \in F(W)$ there is a unique $b \in F(W \oplus_{\omega, \tau} T)$ for some $[M, \omega, \tau] \in \mathbf{B}(W, T)$, such that $\iota_W^* b = y$ and $\iota_T^* b = \alpha^* x$.

We will prove the existence first. For $a \in F(V \oplus_{\nu, \omega} W)$ satisfying $\iota_V^* a = x$, we have that

$$\sigma_\alpha F(W)(a) = (\alpha \oplus_{\nu, \omega} \text{id}_W)^* a$$

is in some $F(T \oplus_{\tau, \epsilon} W)$ and satisfies $\iota_T^* (\alpha \oplus_{\nu, \omega} \text{id}_W)^* a = \alpha^* \iota_V^* a = \alpha^* x$, so $\sigma_\alpha F(W)(a) \in \Sigma_{(T, \alpha^* x)} F(W)$, then if a is the unique element such that $\iota_W^* a = y$ (which exists since (V, x) is central), then $b := \sigma_\alpha F(W)(a)$ satisfies $\iota_T^* b = \alpha^* x$ and $\iota_W^* b = \iota_W^* a = y$.

Let us now prove the uniqueness. Let b be in the preimage of y in $\sigma_{F, (T, \alpha^* x)} F(W)$, with

$$b \in F(W \oplus_{\omega, \tau} T).$$

Then, since (V, x) is central, there is a unique $[M, g, \nu] \in \mathbf{B}(W \oplus_{\omega, \tau} T, V)$ and a

$$c \in F((W \oplus_{\omega, \tau} T) \oplus_{g, \nu} V)$$

such that $\iota_V^* c = x$ and $\iota_{W \oplus_{\omega, \tau} T}^* c = b$. We consider the isomorphism $\zeta_{\omega, \tau, g, \nu}$ given by Lemma 4.57, which take values in some space $W \oplus_{\omega', g'} (T \oplus_{\tau', \nu'} V)$. Then, by construction, $\iota_{T \oplus_{\tau', \nu'} V}^* \zeta_{\omega, \tau, g, \nu}^* c$ satisfies $\iota_T^* \iota_{T \oplus_{\tau', \nu'} V}^* \zeta_{\omega, \tau, g, \nu}^* c = \alpha^* x$ and $\iota_V^* \iota_{T \oplus_{\tau', \nu'} V}^* \zeta_{\omega, \tau, g, \nu}^* c = x$. Therefore, since (V, x) is central, Lemma 4.55 implies that $\iota_V : V \rightarrow T \oplus_{\tau', \nu'} V$ is an isomorphism and $(\iota_V^{-1})^* \iota_{T \oplus_{\tau', \nu'} V}^* \zeta_{\omega, \tau, g, \nu}^* c = x$. Then, identifying $W \oplus_{\omega', g'} (T \oplus_{\tau', \nu'} V)$ with the appropriate $W \oplus_{\omega'', \nu''} V$, we get that $\zeta_{\omega, \tau, g, \nu}^* c \in F(W \oplus_{\omega'', \nu''} V)$ satisfies $\iota_W^* \zeta_{\omega, \tau, g, \nu}^* c = y$ and $\iota_V^* \zeta_{\omega, \tau, g, \nu}^* c = x$. $\zeta_{\omega, \tau, g, \nu}^* c$ is then the unique element in the inverse image of y by $\rho_{F, (V, x)}$ and $b = \sigma_\alpha F(W)(\zeta_{\omega, \tau, g, \nu}^* c)$, which proves the uniqueness of b . \square

Lemma 4.59. *Let F be in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and (V, x) and (T, y) be central elements of F , then there exists, up to isomorphism, a unique pair (R, z) , with $R = V \oplus_{\nu, \tau} T$ for some $[M, \nu, \tau] \in \mathcal{B}(V, T)$ and with $z \in F(R)$, such that $x = \iota_V^* z$ and $y = \iota_T^* z$. Moreover, (R, z) is central.*

Proof. By centrality of (V, x) and by Lemma 4.54, z is necessarily the only element in the inverse image of y under $\rho_{F, (V, x)}$, which prove the existence and the uniqueness of (R, z) .

Let us prove that it is central. Let e be in $F(W)$, we take e' in $F(W \oplus_{\omega, \tau} T)$ the only element in the inverse image of e under $\rho_{F, (T, y)}$ and $e'' \in F((W \oplus_{\omega, \tau} T) \oplus_{g, \nu} V)$ the only element in the inverse image of e' under $\rho_{F, (V, x)}$. We consider $\zeta_{\omega, \tau, g, \nu}$ as in lemma 4.57, an isomorphism from $(W \oplus_{\omega, \tau} T) \oplus_{g, \nu} V$ to some $W \oplus_{\omega', g'} (T \oplus_{\tau', \nu'} V)$. Then, $\iota_{T \oplus_{\tau', \nu'} V}^* (\zeta_{\omega, \tau, g, \nu}^{-1})^* e''$ is in the inverse image of y under $\rho_{F, (V, x)}$, since (V, x) is central $\iota_{T \oplus_{\tau', \nu'} V}^* (\zeta_{\omega, \tau, g, \nu}^{-1})^* e'' = z$. Moreover, $\iota_W^* (\zeta_{\omega, \tau, g, \nu}^{-1})^* e'' = e$, this imply that $(\zeta_{\omega, \tau, g, \nu}^{-1})^* e''$ is in the inverse image of e under $\rho_{F, (R, z)}$.

Let us now show the uniqueness of the element in the inverse image of e . We take R to be of the form $V \oplus_{\nu, \tau} T$, and we take $a \in F(W \oplus_{\omega, g} (V \oplus_{\nu, \tau} T))$ and $b \in F(W \oplus_{\epsilon, \gamma} (V \oplus_{\nu, \tau} T))$ two elements in the inverse image of e under $\rho_{F, (R, z)}$. We consider the isomorphisms $\zeta_{\omega', \nu', g', \tau'}$ and $\zeta_{\epsilon', \nu'', \gamma', \tau''}$ given by Lemma 4.57, from some vector spaces $(W \oplus_{\omega', \nu'} V) \oplus_{g', \tau'} T$ and $(W \oplus_{\epsilon', \nu''} V) \oplus_{\gamma', \tau''} T$ to $W \oplus_{\omega, g} (V \oplus_{\nu, \tau} T)$ and $W \oplus_{\epsilon, \gamma} (V \oplus_{\nu, \tau} T)$. Set $a' = \iota_{W \oplus_{\omega', \nu'} V}^* \zeta_{\omega', \nu', g', \tau'}^* a \in F(W \oplus_{\omega', \nu'} V)$ and $b' = \iota_{W \oplus_{\epsilon', \nu''} V}^* \zeta_{\epsilon', \nu'', \gamma', \tau''}^* b \in F(W \oplus_{\epsilon', \nu''} V)$. They both satisfy that their image by ι_W^* is e and by ι_V^* is x . Then, by the centrality of (V, x) we have that $a' = b'$. Moreover, applying ι_T^* to $\zeta_{\omega', \nu', g', \tau'}^* a$ and $\zeta_{\epsilon', \nu'', \gamma', \tau''}^* b$, we obtain y . Therefore, $\zeta_{\omega', \nu', g', \tau'}^* a$ and $\zeta_{\epsilon', \nu'', \gamma', \tau''}^* b$ are both in the inverse image of $a' = b'$ under $\rho_{F, (T, y)}$. Since (T, y) is also central, $\zeta_{\omega', \nu', g', \tau'}^* a = \zeta_{\epsilon', \nu'', \gamma', \tau''}^* b$, hence $a = b$. \square

Theorem 4.60. *Let F be an object in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ such that $\mathcal{C}(F)$ is not empty and such that $F(V) = \emptyset$ for $\dim(V)$ greater than some integer d , then there is a unique central element (C, c) up to isomorphism satisfying that (V, x) is central if and only if there is an injective morphism α from V to C such that $x = \alpha^* c$. We call (C, c) the centre of F .*

Proof. It is a direct consequence of lemma 4.58 and 4.59. Indeed, for (C, c) a central element with $\dim(C)$ maximal, if (V, x) is a central element such that there is no injection from V to C satisfying $\alpha^* c = x$, then the dimension of (R, z) the unique pair satisfying the assumptions of lemma 4.59 for $(T, y) = (C, c)$ is greater than $\dim(C)$ which is absurd, since $\dim(C)$ is supposed to be maximal among central elements. \square

Corollary 4.61. *For K a noetherian unstable algebra, if K is nil-closed, the centre of K is equal to the centre of $\text{Hom}_{\mathcal{K}f.g.}(K, H^*(-))$.*

5 Definition of the groupoid \mathcal{G}_F and application to the computation of the centre of F

For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, Yoneda's Lemma implies the existence of surjections from functors of the form $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F . This statement is not specific to $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, we could make a similar one for $\text{Set}^{(\mathcal{V}^f)^{\text{op}}}$. The interest we have about such surjections in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ is that the simplicity of the category $\mathcal{V}\mathcal{I}$ makes it easy to classify functors in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ with a given surjections from

$\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$. We start this section by explaining how to associate to objects F in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ with a given surjection from some $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F , a groupoid \mathcal{G}_F with objects the sub-vector spaces of the W_i . We will prove that the isomorphism class of F in the coslice category with respect to $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ is determined by \mathcal{G}_F .

In the second sub-section, we will prove that the centre of F is explicitly determined by the groupoid \mathcal{G}_F of the first sub-section.

5.1 Definition of the groupoid

By Yoneda's lemma, $F(W)$ is naturally isomorphic to $\text{Hom}_{\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}(\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W), F)$ under the morphism which sends a natural transformation η to $\eta(\text{id}_W)$. Then, we can always exhibit a surjection from some $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F , where the cardinal of I may be very big. The most obvious one being defined by

$I = \{x \in F(\mathbb{F}_p^d) ; d \in \mathbb{N}\}$, $W_x = \mathbb{F}_p^d$ for $x \in F(\mathbb{F}_p^d)$, and q is the only natural transformation which sends id_{W_x} to x for all $x \in I$.

So F is always isomorphic to some functor of the form $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_F$ with \sim_F an equivalence relation on $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$. In this part, we will show how the equivalence relation \sim_F associated with a surjection

$$\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) \twoheadrightarrow F,$$

is encoded by a groupoid whose objects are the sub-spaces of the W_i .

Definition 5.1. Let F be an object in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, a generating family of F is a family $(W_i)_{i \in I}$ of objects of $\mathcal{V}\mathcal{I}$, together with a surjection $q : \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) \twoheadrightarrow F$.

Even though the results of this sub-section do not require any finiteness condition on the cardinal of I , for $((W_i)_{i \in I}, q)$ a generating family, the constructions might not be exploitable in the case where I is not finite.

Definition 5.2. An object F in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ is finitely generated if F admits a generating family $((W_i)_{i \in I}, q)$ with $|I|$ finite.

Let F be an object in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, and let $((W_i)_{i \in I}, q)$ be a generating family for F . We denote by F_i the image of $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ under q . Then, we have $F = \bigcup_{i \in I} F_i$. We also denote by ϕ_i^F (or just ϕ_i when there is no ambiguity) the image of id_{W_i} under q .

Let us now define the main ingredient of this section.

Definition 5.3. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and $((W_i)_{i \in I}, q)$ a generating family of F , let $\mathcal{G}_F^{((W_i)_{i \in I}, q)}$ be the groupoid whose set of objects is the disjoint union of the sets of sub-spaces of the W_i and whose morphisms from $U \subset W_i$ to $U' \subset W_j$ are the isomorphisms α from U to U' such that $\alpha^* \iota_{U'}^* \phi_j = \iota_U^* \phi_i$.

The groupoid $\mathcal{G}_F^{((W_i)_{i \in I}, q)}$ depends heavily on the choice of a generating family $((W_i)_{i \in I}, q)$, so cannot be made functorial on the category $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. It can nonetheless be made functorial on the category whose objects are provided with a surjection from a fixed $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$.

Definition 5.4. Let $(W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ be the slice category whose objects are pairs (F, q_F) with $F \in \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and q_F a natural surjection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F , and whose morphisms from (F, q_F) to (G, q_G) are natural transformations η from F to G such that the following diagram commutes:

$$\begin{array}{ccc}
 & & F \\
 & \nearrow^{q_F} & \downarrow \eta \\
 \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) & & G \\
 & \searrow_{q_G} &
 \end{array}$$

Proposition 5.5. $(F, q_F) \mapsto \mathcal{G}_{(F, q_F)} := \mathcal{G}_F^{((W_i)_{i \in I}, q_F)}$ is a functor from the category $(W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ to the category of groupoids.

Proof. For η a morphism from (F, q_F) to (G, q_G) two objects in $(W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, we have to define a morphism \mathcal{G}_η from $\mathcal{G}_{(F, q_F)}$ to $\mathcal{G}_{(G, q_G)}$. $\mathcal{G}_{(F, q_F)}$ has the same objects as $\mathcal{G}_{(G, q_G)}$, furthermore for all $i \in I$, $\eta(\phi_i^F) = \phi_i^G$, otherwise the diagram of Definition 5.4 would not commute. Then, for all $\alpha \in \mathcal{G}_{(F, q_F)}(U \subset W_i, U' \subset W_j)$, the commutativity of the preceding diagram implies that

$$\alpha^* \iota_{U'}^* \phi_j^G = \eta(\alpha^* \iota_{U'} \phi_j^F) = \iota_U^* \phi_i^G,$$

therefore $\alpha \in \mathcal{G}_{(G, q_G)}$. Hence, $\mathcal{G}_{(F, q_F)}$ is a sub-groupoid of $\mathcal{G}_{(G, q_G)}$ and we define \mathcal{G}_η to be the inclusion of this sub-groupoid. \square

In the following, we fix a family $(W_i)_{i \in I}$. When there can be no ambiguity, for $(F, q_F) \in (W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, we will use the notation \mathcal{G}_F instead of $\mathcal{G}_{(F, q_F)}$. We want to prove that the isomorphism class of (F, q_F) in $(W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ is determined by the groupoid $\mathcal{G}_{(F, q_F)}$.

Lemma 5.6. Let (F, q_F) be an object in $(W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. Then, \mathcal{G}_F satisfies the following property. For $\alpha \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$, for M a sub-space of U , and for $\alpha_M : M \rightarrow \alpha(M)$ the restriction of α to M corestricted to $\alpha(M)$, $\alpha_M \in \mathcal{G}_F(M \subset W_i, \alpha(M) \subset W_j)$.

Proof. For ι_M^U the inclusion of M in U (we reserve the notation ι_M for the inclusion of M in W_i) we have, $\alpha \circ \iota_M^U = \alpha_M$. Then, $\alpha^* \iota_{U'}^* \phi_j = \iota_U^* \phi_i$ implies that $\alpha_M^* \iota_{\alpha(M)}^* \phi_j = (\iota_M^U)^* \alpha^* \iota_{U'}^* \phi_j = (\iota_M^U)^* \iota_U^* \phi_i = \iota_M^* \phi_i$. Then, $\alpha_M \in \mathcal{G}_F(M \subset W_i, \alpha(M) \subset W_j)$. \square

Example 5.7. Let W be a finite dimensional vector-space and G be a sub-group of $\text{Gl}(W)$. We consider $F(V) := \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W)/G$ where G acts on $\text{Hom}_{\mathcal{V}\mathcal{I}}(V, W)$ by composition. F is an object in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. Then, for q the canonical projection $\text{Hom}_{\mathcal{V}\mathcal{I}}(V, W) \rightarrow \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W)/G$, $(F, q) \in {}_W \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. \mathcal{G}_F is the groupoid whose objects are the sub-vector spaces of W , and whose morphisms from U to U' are the isomorphisms α from U to U' such that $\alpha^* q(\iota_{U'}) = q(\iota_U)$. But, $\alpha^* q(\iota_{U'}) =$

$q(\iota_U)$ implies that $q(\iota_{U'} \circ \alpha) = q(\iota_U)$, by definition of q , it implies that there is $g \in G$ such that $\iota_{U'} \circ \alpha = g \circ \iota_U$, in other terms $\alpha = g_U$. So $\mathcal{G}_F(U, U')$ is the set of isomorphisms of the form g_U , for $g \in G$ such that $g(U) = U'$.

Definition 5.8. For \mathcal{G} a groupoid whose objects are the sub-vector spaces of the W_i with $i \in I$, and whose morphisms are isomorphisms of vector spaces, we say that \mathcal{G} has the restriction property if, for all $U \subset W_i$, for all $U' \subset W_j$ and for all $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$, α_M is in $\mathcal{G}(M \subset W_i, \alpha(M) \subset W_j)$.

Remark 5.9. Thus, Lemma 5.6 asserts that, for $F \in \coprod_{(W_i)_{i \in I}} \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, \mathcal{G}_F has the restriction property.

Remark 5.10. The set of groupoids whose set of objects is the disjoint union of the sets of sub-vector spaces of the W_i and that satisfies the restriction property, is ordered by inclusion, where $\mathcal{G} \subset \mathcal{H}$ if, for each pair $(U \subset W_i, U' \subset W_j)$, $\mathcal{G}(U \subset W_i, U' \subset W_j) \subset \mathcal{H}(U \subset W_i, U' \subset W_j)$.

Notation 5.11. We denote by $\text{Groupoid}((W_i)_{i \in I})$ the poset of groupoids whose objects are the sub-spaces of the W_i and that satisfies the restriction property.

For W a finite dimensional vector space, let us also denote by $\text{Group}(W)$ the poset of sub-groups of $\text{Gl}(W)$.

Definition 5.12. Let \mathfrak{g} be the poset preserving map from $\text{Group}(W)$ to $\text{Groupoid}(W)$ which sends G a subgroup of $\text{Gl}(W)$ to the groupoid $\mathfrak{g}(G)$ such that, for U and U' subspaces of W and α from U to U' , $\alpha \in \mathfrak{g}(G)(U, U')$ if and only if there is $g \in G$ such that $\alpha = g_U$.

Remark 5.13. We have seen in Example 5.7 that $\mathcal{G}_{(\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)/G, q_G)} = \mathfrak{g}(G)$, for q_G the canonical projection from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)$ to $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)/G$.

In the following, we want to show that for \mathcal{G} a groupoid whose objects are the sub-vector spaces of the W_i and whose morphisms are isomorphisms of vector spaces, there exists $F \in \coprod_{(W_i)_{i \in I}} \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ such that $\mathcal{G}_F = \mathcal{G}$ if and only if \mathcal{G} has the restriction property. We will then show that those groupoids are in on-to-one correspondence with isomorphism classes of objects in $\coprod_{(W_i)_{i \in I}} \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$.

Definition 5.14. For $\mathcal{G} \in \text{Groupoid}((W_i)_{i \in I})$, let $\sim_{\mathcal{G}}$ be the relation on $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W_i)$ defined by $\zeta \sim_{\mathcal{G}} \epsilon$ if there is $\alpha \in \mathcal{G}(\text{Im}(\zeta) \subset W_i, \text{Im}(\epsilon) \subset W_j)$ such that $\tilde{\epsilon} = \alpha \circ \tilde{\zeta}$, where $\tilde{\zeta}$ and $\tilde{\epsilon}$ denote the corestriction of ζ and ϵ to their images.

Lemma 5.15. For $\mathcal{G} \in \text{Groupoid}((W_i)_{i \in I})$, $\sim_{\mathcal{G}}$ is an equivalence relation.

We will denote by $[\epsilon]_{\mathcal{G}}$ the equivalence class of ϵ for this equivalence relation.

Proof. Since for all $U \subset W_i$, $\text{id}_U \in \mathcal{G}(U \subset W_i, U \subset W_i)$, the relation is reflexive. The transitivity comes from the composition of morphisms in \mathcal{G} and the symmetry from the fact that \mathcal{G} is a groupoid and hence that every morphism in \mathcal{G} has an inverse. \square

Proposition 5.16. For $\mathcal{G} \in \text{Groupoid}((W_i)_{i \in I})$, $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)/\sim_{\mathcal{G}}$ defines an object of $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$.

Moreover, $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)/\sim_{\mathcal{G}}$ is connected if and only if, for all $(i, j) \in I^2$, $\mathcal{G}(0 \subset W_i, 0 \subset W_j)$ contains the only morphism from 0 to 0.

Proof. To prove that $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}$ is an object in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, let us prove that if $\rho \sim_{\mathcal{G}} \zeta$, for $\rho \in \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W_i)$ and $\zeta \in \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W_j)$ with $V \in \mathcal{V}\mathcal{I}$, and if $\beta : U \rightarrow V$ is a morphism in $\mathcal{V}\mathcal{I}$, then $\beta^* \rho \sim_{\mathcal{G}} \beta^* \zeta$. Let α be in $\mathcal{G}(\text{Im}(\rho) \subset W_i, \text{Im}(\zeta) \subset W_j)$ such that $\tilde{\zeta} = \alpha \circ \tilde{\rho}$. Then since \mathcal{G} has the restriction property, $\alpha_{\rho \circ \beta(U)} : \rho \circ \beta(U) \rightarrow \zeta \circ \beta(U)$ is in $\mathcal{G}(\text{Im}(\rho \circ \beta) \subset W_i, \text{Im}(\zeta \circ \beta) \subset W_j)$ and $\widetilde{\zeta \circ \beta} = \alpha_{\rho \circ \beta(U)} \circ \widetilde{\rho \circ \beta}$ therefore $\beta^* \rho \sim_{\mathcal{G}} \beta^* \zeta$. \square

Theorem 5.17. 1. For $\mathcal{G} \in \text{Groupoid}((W_i)_{i \in I})$,

$$\mathcal{G}_{(\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}, q)} = \mathcal{G},$$

for q the canonical surjection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}$.

2. Conversely, let $F \in {}_{(W_i)_{i \in I}} \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. Then, F is isomorphic to $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}_F}$.

Proof. Let us prove the first point. Let α be an isomorphism from $U \subset W_i$ to $U' \subset W_j$. $\alpha \in \mathcal{G}_{\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i) / \sim_{\mathcal{G}}}(U \subset W_i, U' \subset W_j)$ if and only if $\alpha^* [\iota_{U'}]_{\mathcal{G}} = [\iota_U]_{\mathcal{G}}$. This the case if and only if $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$.

By construction, for α an isomorphism from U to U' with U and U' sub-spaces of W_i and W_j , $\alpha^* \iota_{U'} [\text{id}_{W_j}]_{\mathcal{G}} = \iota_U [\text{id}_{W_i}]_{\mathcal{G}}$ if and only if $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$. We only have to prove that $[\epsilon]_{\mathcal{G}_F} \mapsto \epsilon^* \phi_i$, for $\epsilon \in \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W_i)$, is well defined and defines a bijection from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(V, W_i) / \sim_{\mathcal{G}_F}$ to $F(V)$ for $V \in \mathcal{V}\mathcal{I}$.

For the second point, if $\epsilon \sim_{\mathcal{G}_F} \zeta$, there exists $\alpha : \epsilon(V) \rightarrow \zeta(V)$ such that $\alpha \in \mathcal{G}_F(\text{Im}(\epsilon) \subset W_i, \text{Im}(\zeta) \subset W_j)$ and $\tilde{\zeta} = \alpha \circ \tilde{\epsilon}$. Then,

$$\begin{aligned} \zeta^* \phi_j &= \tilde{\zeta}^* \iota_{\zeta(V)}^* \phi_j, \\ &= \tilde{\epsilon}^* \alpha^* \iota_{\zeta(V)}^* \phi_j, \\ &= \tilde{\epsilon}^* \iota_{\epsilon(V)}^* \phi_i, \\ &= \epsilon^* \phi_i. \end{aligned}$$

The map which maps $[\epsilon]_{\mathcal{G}_F}$ to $\epsilon^* \phi_i$ is then well defined. It is obviously surjective.

Let us prove that it is injective. Let ϵ and ζ be morphisms from V to W_i and W_j such that $\epsilon^* \phi_i = \zeta^* \phi_j$. Then, $\tilde{\epsilon}^* \iota_{\epsilon(V)}^* \phi_i = \tilde{\zeta}^* \iota_{\zeta(V)}^* \phi_j$. Since $\tilde{\epsilon}$ is an isomorphism from V to the image of ϵ , we have $\iota_{\epsilon(V)}^* \phi_i = (\tilde{\zeta} \circ \tilde{\epsilon}^{-1})^* \iota_{\zeta(V)}^* \phi_j$. Therefore $\tilde{\zeta} \circ \tilde{\epsilon}^{-1} \in \mathcal{G}_F(\text{Im}(\epsilon) \subset W_i, \text{Im}(\zeta) \subset W_j)$ and $\epsilon \sim_{\mathcal{G}_F} \zeta$. \square

The following is a ‘‘converse’’ of example 5.7.

Example 5.18. Let F be an object in ${}_W \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and consider the group $G = \mathcal{G}_F(W, W)$. We know, since \mathcal{G}_F has the restriction property, that $\mathcal{G}_F \subset \mathfrak{g}(G)$ (see Definition 5.12). Suppose that this inclusion is an equality. Then, $\sim_{\mathcal{G}}$ is the equivalence relation defined by $\rho \sim_{\mathcal{G}} \zeta$ if and only if there is $g \in G$ such that $g \circ \rho = \zeta$. By Theorem 5.17, $F \cong \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / G$.

5.2 Computation of $C(F)$ using \mathcal{G}_F

In this sub-section, we take $F \in {}_{(W_i)_{i \in I}}\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. We want to prove that, under some assumptions on the generating family of F , the central elements of F are determined by the groupoid \mathcal{G}_F .

We start by proving that, if F has a generating family with one element, $\rho_{F,(V,x)}$ is surjective for all $V \in \mathcal{V}\mathcal{I}$ and for all $x \in F(V)$.

Notation 5.19. For $F \in \text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and for (W, q) a generating family with one element, we denote by $\phi^F \in F(W)$ the image of id_W under q .

On one hand, for $F \in {}_W\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, the functor F is determined up to isomorphism by the groupoid \mathcal{G}_F . On the other hand, Lemma 4.54 gives a criterion for the centrality of a pair (V, x) through the sets $F(V \oplus_{\nu, \mu} U)$ with the $V \oplus_{\nu, \mu} U \in \mathbf{B}(V, U)$ which are not sub-vector spaces of W . To characterise the centre of F using \mathcal{G}_F , we need to reformulate the centrality condition by comparing the $V \oplus_{\nu, \mu} U$ with sub-vector spaces of W .

Remark 5.20. Consider $\delta : V \rightarrow W$ and $\epsilon : U \rightarrow W$ two morphisms in $\mathcal{V}\mathcal{I}$. For $\delta + \epsilon$ the morphism from $V \oplus U$ to W which sends v to $\delta(v) + \epsilon(v)$, since δ and ϵ are injective, there exists ν and μ from $\ker(\delta + \epsilon)$ to V and U such that for all $v \in \ker(\delta + \epsilon)$, $v = \nu(v) - \mu(v)$. Then, by the universal property of the pushout, $(V \oplus U)/\ker(\delta + \epsilon)$ is isomorphic to $V \oplus_{\nu, \mu} U$. The first isomorphism theorem implies that $\delta + \epsilon$ induces an isomorphism between $V \oplus_{\nu, \mu} U$ and $\delta(V) + \epsilon(U)$ (which is the image of $\delta + \epsilon$).

Notation 5.21. For $\delta : V \rightarrow W$ and $\epsilon : U \rightarrow W$ two morphisms in $\mathcal{V}\mathcal{I}$, let ν and μ be as in Remark 5.20. Then, let $\epsilon \uparrow \delta$ denote the isomorphism from $V \oplus_{\nu, \mu} U$ to $\delta(V) + \epsilon(U)$ induced by the first isomorphism theorem. It satisfies $\delta + \epsilon = \iota_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta) \circ \pi$, for π the canonical projection from $V \oplus U$ to $V \oplus_{\nu, \mu} U$ and $\iota_{\delta(V) + \epsilon(U)}$ the inclusion from $\delta(V) + \epsilon(U)$ in W .

Lemma 5.22. For $F \in {}_W\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, δ from V to W and ϵ from U to W , $(\epsilon \uparrow \delta)^* \iota_{\delta(V) + \epsilon(U)}^* \phi^F \in F(V \oplus_{\nu, \mu} U)$ is in the inverse image of $\epsilon^* \phi^F$ under $\rho_{F,(V, \delta^* \phi^F)}$.

Proof. By definition of $\epsilon \uparrow \delta$, we have $\iota_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta) \circ \iota_V = \delta$ and $\iota_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta) \circ \iota_U = \epsilon$, for ι_V and ι_U the inclusions of U and V in $V \oplus_{\nu, \mu} U$. Therefore, $\iota_U^* ((\epsilon \uparrow \delta)^* \iota_{\delta(V) + \epsilon(U)}^* \phi^F) = \epsilon^* \phi^F$ and $\iota_V^* ((\epsilon \uparrow \delta)^* \iota_{\delta(V) + \epsilon(U)}^* \phi^F) = \delta^* \phi^F$. \square

This lemma implies two things, firstly that $\rho_{F,(V, \delta^* \phi^F)}$ is always surjective, for $F \in {}_W\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and δ from V to W . Secondly, that $\delta^* \phi^F \in F(V)$ is central if and only if, for all morphisms $\epsilon : U \rightarrow W$ in $\mathcal{V}\mathcal{I}$, $(\epsilon \uparrow \delta)^* \iota_{\delta(V) + \epsilon(U)}^* \phi^F \in F(V \oplus_{\nu, \mu} U)$ is the only element in the inverse image of $\epsilon^* \phi^F$ under $\rho_{F,(V, \delta^* \phi^F)}$.

In the case $F \in {}_{(W_i)_{i \in I}}\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ with $|I|$ greater than one, we recall that F_i denotes the image of $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ in F . We want to use the surjectivity of each $\rho_{F_i,(V,x)}$ with $i \in I$ and $x \in F_i(V)$. We start by proving that, under some condition on the surjection q_F from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W_i)$ to F , if (V, x) is central, then $x \in F_i(V)$ for all $i \in I$.

Definition 5.23. Let F be an object in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, with generating family $((W_i)_{i \in I}, q)$. We say that $((W_i)_{i \in I}, q)$ is a minimal generating family, if for all $i \neq j$, $F_i \not\subseteq F_j$.

Remark 5.24. We can always extract a minimal sub-generating family from a given generating family.

Lemma 5.25. *Let F be in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and $((W_i)_{i \in I}, q)$ a minimal generating family of F . Let also (V, x) be a central element of F . Then, $x \in \bigcap_{i \in I} F_i(V)$ and (V, x) is a central element for each F_i .*

Proof. For each i , let $x_i \in F(W_i \oplus_{\omega_i, \nu_i} V)$ be in the inverse image of ϕ_i under $\rho_{F, (V, x)}$. If $x_i \in F_j(W_i \oplus_{\omega_i, \nu_i} V)$, $\phi_i = \iota_{W_i}^* x_i$ hence F_i is a sub-functor of F_j . Since we supposed the generating family to be minimal, this implies that $i = j$. Then, $x = \iota_V^* x_i$ and therefore $x \in F_i(V)$.

By Lemma 5.22, we get that an element in $F_i(E)$, for $E \in \mathcal{V}\mathcal{I}$, has at least one element in its inverse image under $\rho_{F_i, (V, x)}$. If (V, x) is central for F , this element is unique and (V, x) is also central for F_i . \square

Now, let us consider $x \in \bigcap_{i \in I} F_i(V)$. We want to find a criterion on \mathcal{G}_F to determine whether (V, x) is central. Since, x is in every $F_i(V)$ and that each F_i are generated by one element, Lemma 5.22 implies that every $\rho_{F_i, (V, x)}$ is surjective, and so $\rho_{F, (V, x)}$ is surjective. So the only way that (V, x) can fail to be central is if a given element has two elements in its inverse image under $\rho_{F, (V, x)}$.

Since $x \in \bigcap_{i \in I} F_i(V)$, for all $i \in I$, there is some inclusion δ_i from V to W_i such that $x = \delta_i^* \phi_i$ and we have the following lemma.

Lemma 5.26. *For $F \in_{(W_i)_{i \in I}} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and (V, x) a central element of F , we consider for all $i \in I$, δ_i from V to W_i such that $x = \delta_i^* \phi_i$. Then, for $\epsilon_k : U \rightarrow W_k$ and $\epsilon_j : U \rightarrow W_j$ such that $\epsilon_k^* \phi_k = \epsilon_j^* \phi_j$, we have $(\epsilon_k \uparrow \delta_k)^* \iota_{\delta_k(V) + \epsilon_k(U)}^* \phi_k = (\epsilon_j \uparrow \delta_j)^* \iota_{\delta_j(V) + \epsilon_j(U)}^* \phi_j$.*

Proof. Lemma 5.22 states that $(\epsilon_k \uparrow \delta_k)^* \iota_{\delta_k(V) + \epsilon_k(U)}^* \phi_k$ and $(\epsilon_j \uparrow \delta_j)^* \iota_{\delta_j(V) + \epsilon_j(U)}^* \phi_j$ are in the inverse image of $\epsilon_k^* \phi_k = \epsilon_j^* \phi_j$, respectively under $\rho_{F_j, (V, x)}$ and $\rho_{F_k, (V, x)}$. Hence, each of them are in its inverse image under $\rho_{F, (V, x)}$. Thus, if (V, x) is central, we necessarily have that $(\epsilon_k \uparrow \delta_k)^* \iota_{\delta_k(V) + \epsilon_k(U)}^* \phi_k = (\epsilon_j \uparrow \delta_j)^* \iota_{\delta_j(V) + \epsilon_j(U)}^* \phi_j$. \square

This gives us a necessary condition on \mathcal{G}_F , for (V, x) to be central. This condition will also be sufficient.

For $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$, $\alpha^* \iota_{U'}^* \phi_j = \iota_U^* \phi_i$ has two, possibly equal, elements in its inverse image under $\rho_{F, (V, x)}$, where $x = \delta_i^* \phi_i$ for all i . Namely $((\iota_{U'} \circ \alpha) \uparrow \delta_j)^* \iota_{U' + \delta_j(V)}^* \phi_j$ and $(\iota_U \uparrow \delta_i)^* \iota_{U + \delta_i(V)}^* \phi_i$. If (V, x) is central, those two have to be equal, in particular they have to live in the same $F(V \oplus_{\nu, \mu} U)$, which gives us a first condition on \mathcal{G}_F .

Lemma 5.27. *Let F in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and $((W_i)_{i \in I}, q)$ a minimal generating family of F . Let (V, x) be a central element of F and for all $i \in I$, let*

$$\delta_i : V \rightarrow W_i,$$

such that $x = \delta_i^ \phi_i$. Then, for $\alpha \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$, if $v \in U \cap V_i$, $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$, for V_i the image of V under δ_i .*

Proof. We consider

$$(\iota_U \uparrow \delta_i)^* (\iota_{U+V_i}^{W_i})^* \phi_i \in F_i(V \oplus_{\nu, \mu} U)$$

and

$$((\iota_{U'} \circ \alpha) \uparrow \delta_j)^* (\iota_{U'+V_j}^{W_j})^* \phi_j \in F_j(V \oplus_{\nu', \mu'} U),$$

where μ is the injection of $U \cap V_i$ into U , ν is the restriction to $U \cap V_i$ of $(\delta_i|^{V_i})^{-1}$, and ν' and μ' are the restriction to $U' \cap V_j$ of α^{-1} and $(\delta_j|^{V_j})^{-1}$. They are equal by Lemma 5.26. In particular they need to live in the same set. We get that $V \oplus_{\nu, \mu} U = V \oplus_{\nu', \mu'} U$, which implies that, for $v \in U \cap V_i$, $\alpha(v) \in V_j$ and $(\delta_j|^{V_j})^{-1}(\alpha(v)) = (\delta_i|^{V_i})^{-1}(v)$, which means that $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$. \square

If the condition of Lemma 5.27 is satisfied, both $((\iota_{U'} \circ \alpha) \uparrow \delta_j)^* \iota_{U'+\delta_j(V)}^* \phi_j$ and $(\iota_U \uparrow \delta_i)^* \iota_{U+\delta_i(V)}^* \phi_i$ live in $F(V \oplus_{\nu, \mu} U)$ where μ is the inclusion from $U \cap V_i$ in U and ν is the restriction to $U \cap V_i$ of $(\delta_i|^{V_i})^{-1}$. Since by definition $\iota_U \uparrow \delta_i$ and $(\iota_{U'} \circ \alpha) \uparrow \delta_j$ are isomorphisms from $V \oplus_{\nu, \mu} U$ respectively to $V_i + U$ and $V_j + U'$, $((\iota_{U'} \circ \alpha) \uparrow \delta_j) \circ (\iota_U \uparrow \delta_i)^{-1}$ is an isomorphism from $U + V_i$ to $U' + V_j$ that we can compute.

Lemma 5.28. *Let α be an isomorphism from $U \subset W_i$ to $U' \subset W_j$ and let δ_i and δ_j be two injections from V respectively to W_i and W_j whose images are V_i and V_j . Then, if for every $v \in U \cap V_i$, $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$, $\iota_U \uparrow \delta_i$ and $(\iota_{U'} \circ \alpha) \uparrow \delta_j$ are isomorphism from $V \oplus_{\nu, \mu} U$ respectively to $V_i + U$ and $V_j + U'$, where ν is the inclusion of $U \cap V_i$ in U and μ is the restriction to $U \cap V_i$ of $(\delta_i|^{V_i})^{-1}$.*

Furthermore, $((\iota_{U'} \circ \alpha) \uparrow \delta_j) \circ (\iota_U \uparrow \delta_i)^{-1}$ is the isomorphism from $U + V_i$ to $U' + V_j$ which sends $u \in U$ to $\alpha(u)$ and $v \in V_i$ to $\delta_j((\delta_i|^{V_i})^{-1}(v))$.

Proof. The first part of the statement is a direct consequence of the definition of $\iota_U \uparrow \delta_i$ and $(\iota_{U'} \circ \alpha) \uparrow \delta_j$.

For $u \in U$, $(\iota_U \uparrow \delta_i)^{-1}(u) = \iota_U^{V \oplus_{\nu, \mu} U}(u)$, then $((\iota_{U'} \circ \alpha) \uparrow \delta_j)(\iota_U^{V \oplus_{\nu, \mu} U}(u)) = \alpha(u)$, and for $v \in V_i$, $(\iota_U \uparrow \delta_i)^{-1}(v) = \iota_U^{V \oplus_{\nu, \mu} U}((\delta_i|^{V_i})^{-1}(v))$ and $((\iota_{U'} \circ \alpha) \uparrow \delta_j)(\iota_U^{V \oplus_{\nu, \mu} U}((\delta_i|^{V_i})^{-1}(v))) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$. \square

Notation 5.29. Let α be an isomorphism from $U \subset W_i$ to $U' \subset W_j$ and let δ_i and δ_j be two injections from V respectively to W_i and W_j whose images are V_i and V_j and such that for every $v \in U \cap V_i$, $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$. We denote by $\bar{\alpha}$ the morphism from $U + V_i$ to $U' + V_j$ which sends $u \in U$ to $\alpha(u)$ and $v \in V_i$ to $\delta_j((\delta_i|^{V_i})^{-1}(v))$.

If (V, x) is central, we have $((\iota_{U'} \circ \alpha) \uparrow \delta_j)^* \iota_{U'+\delta_j(V)}^* \phi_j = (\iota_U \uparrow \delta_i)^* \iota_{U+\delta_i(V)}^* \phi_i$, which implies that $\bar{\alpha}$ must be in $\mathcal{G}_F(U + V_i \subset W_i, U' + V_j \subset W_j)$. We state the principal theorem of this sub-section.

Theorem 5.30. *Let F be in $\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and $((W_i)_{i \in I}, q)$ a minimal generating family of F . Let also $x \in F(V)$ and $(\delta_i : V \rightarrow W_i)_{i \in I}$ a family of injective morphisms from V to the W_i , such that, for all $i \in I$, $x = \delta_i^* \phi_i$. Then, (V, x) is central if and only if the two following conditions are satisfied:*

1. for all $\alpha \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$, and for all $v \in U \cap V_i$, $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$,
2. for all sub-spaces U of W_i and U' of W_j and for all isomorphism α from U to U' satisfying $\alpha(v) = \delta_j \circ (\delta_i|^{V_i})^{-1}(v)$, for $v \in U \cap V_i$, $\alpha \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$ if and only if $\bar{\alpha} \in \mathcal{G}_F(U + V_i \subset W_i, U' + V_j \subset W_j)$.

Proof. Let us show first necessity. We suppose that (V, x) is central, then by Lemma 5.27, the first condition is satisfied. We consider $\alpha \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$, then by Lemma 5.26, $(\iota_U \uparrow \delta_i)^*(\iota_{U+V_i}^{W_i})^*\phi_i = ((\iota_{U'} \circ \alpha) \uparrow \delta_j)^*(\iota_{\alpha(U)+V_j}^{W_j})^*\phi_j$. By Lemma 5.28, this implies that $\bar{\alpha}^*\iota_{U'+V_j}^*\phi_j = \iota_{U+V_i}^*\phi_i$. Then, $\bar{\alpha} \in \mathcal{G}_F(U+V_i \subset W_i, U'+V_j \subset W_j)$. Conversely, if $\bar{\alpha} \in \mathcal{G}_F(U+V_i \subset W_i, U'+V_j \subset W_j)$, since \mathcal{G}_F has the restriction property, $\alpha = \bar{\alpha}_U \in \mathcal{G}_F(U \subset W_i, U' \subset W_j)$.

We now prove sufficiency. Suppose that F and (V, x) satisfy the two conditions. We consider $\zeta^*\phi_i \in F_i(V \oplus_{\nu, \mu} U)$ in the inverse image of an element $\beta^*\phi_j \in F(U)$, where j is not necessarily equal to i . The first condition implies that, if $\eta : V \rightarrow W_i$ satisfies $\eta \sim_{\mathcal{G}_F} \delta_i$, then $\eta = \delta_i$. But $\zeta|_V^*\phi_i = \delta_i^*\phi_i$, hence $\zeta|_V = \delta_i$. Then, the equality $\zeta|_U^*\phi_i = \beta^*\phi_j$ implies the existence of a morphism $\alpha \in \mathcal{G}_F(\beta(U), \zeta(U))$ such that $\widetilde{\zeta|_U} = \alpha \circ \widetilde{\beta}$, where $\zeta|_U$ and $\widetilde{\beta}$ are the corestrictions of $\zeta|_U$ and β to their images. Therefore, $\zeta = (\iota_{\zeta(U)+V_i}^{W_i}) \circ \bar{\alpha} \circ (\beta \uparrow \delta_j)$ and by the second condition, we get that $\zeta^*\phi_i = (\beta \uparrow \delta_j)^*(\iota_{\beta(U)+V_j}^{W_j})^*\phi_j$. Hence, $(\beta \uparrow \delta_j)^*(\iota_{\beta(U)+V_j}^{W_j})^*\phi_j$ is the only element in the inverse image of $\beta^*\phi_j$ under $\rho_{F, (V, x)}$. \square

Example 5.31. When $F \in {}_W\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, the two conditions become simpler. We consider δ from V to W , such that $(V, \delta^*\phi^F)$ is central.

The first condition becomes that, for $\alpha \in \mathcal{G}(U, U')$ and for $v \in U \cap \delta(V)$, $\alpha(v) = v$. Furthermore, in this case, $\bar{\alpha}$ is the morphism from $U + \delta(V)$ to $U' + \delta(V)$ which sends $u \in U$ to $\alpha(u)$ and $v \in \delta(V)$ to v .

We return to Example 5.7.

Proposition 5.32. *For G a sub-group of $Gl(W)$, the centre of $F = \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)/G$ is given by (C, ι_C) with C the maximal sub-vector space of W such that for all $v \in C$ and $g \in G$, $g(v) = v$ and with ι_C the inclusion of C in W .*

Proof. As we have seen in Example 5.7, $\mathcal{G}_F = \mathfrak{g}(G)$ (see Definition 5.12). Let δ be an injection from some vector space V to W such that $(V, q(\delta))$ is central, for q the canonical projection from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)$ to F . We will denote by V' the sub-space $\delta(V)$ of W . The first condition of Theorem 5.30 implies that, for all $g \in G$, $g \in \mathcal{G}_F(W, W)$, therefore for all $x \in V'$, $g(x) = x$. Conversely, if, for all $g \in G$ and for $x \in V'$, $g(x) = x$, since for all sub-spaces U of W , and for all $\alpha \in \mathcal{G}_F(U, U')$, there exists $g \in G$ such that $\alpha = g_U$, $(V, q(\delta))$ satisfies the first condition of the theorem.

In this case, if $(V, q(\delta))$ satisfies the first condition of the theorem, the second condition of Theorem 5.30 is immediately satisfied. Indeed, for $\alpha \in \mathcal{G}_F(U, U')$, $\bar{\alpha}$ is the morphism from $U + V'$ to $U' + V'$ which sends $u \in U$ to $\alpha(u)$ and $v \in V'$ to v . Then, let $g \in G$ such that $\alpha = g_U$. Since for all $v \in V'$, $g(v) = v$, $g_{U+V'} = \bar{\alpha}$. Hence, $\bar{\alpha} \in \mathcal{G}_F(U + V', U' + V')$.

Therefore, $q(\delta)$ is central if and only if δ factorizes through ι_C . Thus, (C, ι_C) is the centre of F . \square

6 The algebras $H^*(W)^{\mathcal{G}}$

This section shows how to apply the groupoid \mathcal{G}_F of the last section to classification problems of *nil*-closed, integral, noetherian, unstable algebras. Before we explain the focus of this section, let us recall the theorem of Adams-Wilkerson.

Definition 6.1. [HLS93, Part II.2] For $K \in \mathcal{K}$, the transcendence degree of K is $d \in \mathbb{N} \cup \{\infty\}$, the supremum of the cardinals of finite sets of homogeneous elements in K which are algebraically independent .

Remark 6.2. If K is noetherian, the transcendence degree of K is finite.

Let us recall the theorem of Adams-Wilkerson.

Theorem 6.3. [HLS93, Theorem 3] *Let K be an integral, unstable algebra of transcendence degree lesser or equal to $\dim(W)$, then there exists an injection ϕ from K to $H^*(W)$. Furthermore, this injection is regular if and only if the transcendence degree of K equals $\dim(W)$.*

Therefore, every integral, *nil*-closed, noetherian, unstable algebra is isomorphic to a *nil*-closed, noetherian sub unstable algebra of some $H^*(W)$. In the first sub-section we define $H^*(W)^{\mathcal{G}}$ for \mathcal{G} a groupoid on sub spaces of W that satisfies the restriction property. Then, $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$ defines an explicit one-to-one correspondence between the groupoids on sub spaces of W satisfying the restriction property and the noetherian, *nil*-closed, unstable sub algebras of $H^*(W)$ of transcendence degree $\dim(W)$.

For simplicity, we restrict to the integral case. The constructions generalizes to the non integral case.

Let us now recall the definition of the primitive elements of a comodule.

Definition 6.4. For $K \in \mathcal{K}$ provided with a $H^*(V)$ -comodule structure κ in \mathcal{K} , the algebra of primitive elements of K is the sub-algebra of K whose elements are those satisfying that $\kappa(x) = x \otimes 1$, for 1 the unit of $H^*(V)$. We will denote by $P(K, \kappa)$ the algebra of primitive elements of K for the $H^*(V)$ -comodule structure κ .

Remark 6.5. By Corollary 3.15, for all $(V, \phi) \in \mathbf{C}(K)$, there is a unique structure κ_ϕ of $H^*(V)$ -comodule on K such that $(\epsilon_K \otimes \text{id}_{H^*(V)}) \circ \kappa_\phi = \phi$.

Notation 6.6. We will also denote $P(K, \kappa_\phi)$ by $P(K, \phi)$.

The problem that we are interested in is the following. If we fix V some finite dimensional vector space and P some unstable algebra, can we classify, under suitable hypothesis, the connected, noetherian, *nil*-closed unstable algebras K , satisfying that K admit a $H^*(V)$ -comodule structure κ in \mathcal{K} , whose algebra of primitive elements is isomorphic to P . Since, every *nil*-closed, noetherian, integral, unstable algebra of transcendence degree $\dim(W)$ is isomorphic to some $H^*(W)^{\mathcal{G}}$, we need to be able to identify the primitive elements associated with a regular central element (V, ϕ) of $H^*(W)^{\mathcal{G}}$.

In the second subsection, we consider $H^*(W)^{\mathcal{G}}$ and an inclusion δ from some vector space V to W , such that $(V, \delta^* \phi) \in \mathbf{C}(H^*(W)^{\mathcal{G}})$ for ϕ the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$. Then, we prove that $P(H^*(W)^{\mathcal{G}}, \delta^* \phi)$ is a *nil*-closed and noetherian sub-algebra of $\pi^*(H^*(W/\text{Im}(\delta)))$ for π the projection from W to $W/\text{Im}(\delta)$. Since π^* is injective, there exists $H^*(W/\text{Im}(\delta))^{\mathcal{G}'} \subset H^*(W/\text{Im}(\delta))$ such that $P(H^*(W)^{\mathcal{G}}, \delta^* \phi) = \pi^*(H^*(W/\text{Im}(\delta))^{\mathcal{G}'})$. We conclude this sub-section by explaining how to determine \mathcal{G}' from \mathcal{G} .

Finally, in the last sub-section, we give examples of how to answer the following question: given \mathcal{G}' a groupoid with the restriction property on sub-spaces of W/V , with W a finite dimensional

vector space and V a sub-space of W , what are the groupoids \mathcal{G} with the restriction property on sub-spaces of W such that $(V, \iota_V^* \phi) \in \mathbf{C}(H^*(W)^\mathcal{G})$, for ι_V the inclusion of V in W , and such that the primitive elements of $H^*(W)^\mathcal{G}$ for the $H^*(V)$ -comodule structure induced by $\iota_V^* \phi$ are $\pi^*(H^*(W/V)^\mathcal{G})$.

6.1 Noetherian, *nil*-closed, unstable sub-algebras of $H^*(W)$

In this sub-section, we give an explicit one-to-one correspondence between the groupoids on sub-spaces of W satisfying the restriction property and the noetherian, *nil*-closed, unstable sub algebra of $H^*(W)$ of transcendence degree $\dim(W)$.

Theorem 6.7. *For all $W \in \mathcal{V}^f$, there is a one-to-one correspondence between the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ and the set of groupoids with the restriction property, whose objects are the sub-vector spaces of W .*

Proof. By Theorem 5.17, there is a one-to-one correspondence between isomorphism classes in ${}_W\mathbf{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$ and the set of groupoids satisfying the assumption. Thus, we have to justify that the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ of transcendence degree $\dim(W)$ are in one-to-one correspondence with isomorphism classes in ${}_W\mathbf{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. Let K be a *nil*-closed, noetherian, sub-algebra of $H^*(W)$ whose transcendence degree is $\dim(W)$. Then, for ϕ_K the inclusion of K in $H^*(W)$, since the transcendence degree of K is $\dim(W)$, by the Theorem of Adams-Wilkerson, ϕ_K is regular. Since K is noetherian, this implies that we can restrict $\phi_K^* : \text{Hom}_{\mathcal{V}^f}(-, W) \rightarrow \text{Hom}_{\mathcal{K}}(K, H^*(-))$ to regular elements. For $F := \text{Hom}_{\mathcal{K}^{\text{f.g.}}}(K, H^*(-))$ and $q_F : \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) \rightarrow F$ the restriction of ϕ_K^* to regular elements, we have that $\phi_K^* = \widetilde{q}_F$ and that (F, q_F) is an element in ${}_W\mathbf{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. This defines a map h from the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ to the set of isomorphism classes in ${}_W\mathbf{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$.

Let us prove that h is injective. We consider, K and K' two *nil*-closed and noetherian sub-algebras of $H^*(W)$ such that, for $h(K) = (F, q_F)$ and $h(K') = (F', q_{F'})$ are isomorphic in ${}_W\mathbf{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$. Then, there is an isomorphism η from F to F' such that $q_{F'} = \eta \circ q_F$. By applying the functor $(\widetilde{\quad})$, we get the following commutative diagram in $\mathbf{Set}^{(\mathcal{V}^f)^{\text{op}}}$:

$$\begin{array}{ccc}
 & & \text{Hom}_{\mathcal{K}}(K, H^*(-)) \\
 & \nearrow \phi_K^* & \downarrow \widetilde{\eta} \\
 \text{Hom}_{\mathcal{K}}(H^*(W), H^*(-)) & & \text{Hom}_{\mathcal{K}}(K', H^*(-)) \\
 & \searrow \phi_{K'}^* &
 \end{array}$$

Finally, by applying the functor $m \circ \mathcal{L}$, we get the following commutative diagram:

$$\begin{array}{ccc}
l_1(K) & \xrightarrow{l_1(\phi_K)} & l_1(H^*(W)) \\
\uparrow m \circ \mathcal{L}(\tilde{\eta}) & & \uparrow l_1(\phi_{K'}) \\
l_1(K') & &
\end{array}
,$$

where, $m \circ \mathcal{L}(\tilde{\eta})$ is an isomorphism. Since K , K' and $H^*(W)$ are *nil*-closed, this implies that there is an isomorphism from K to K' such that the following diagram commutes:

$$\begin{array}{ccc}
K & \xrightarrow{\phi_K} & H^*(W) \\
\cong \downarrow & & \uparrow \phi_{K'} \\
K' & &
\end{array}
,$$

so that K and K' are the same sub-algebra of $H^*(W)$, hence h is injective.

We conclude, by exhibiting a right inverse of h , for $(F, q_F) \in {}_W\text{Set}^{(\mathcal{V}\mathcal{I})^{\text{op}}}$, we consider the injection $m \circ \mathcal{L}(\tilde{F}) \xrightarrow{m \circ \mathcal{L}(\tilde{q}_F)} H^*(W)$ and we define $j(F, q_F)$ the image of $m \circ \mathcal{L}(\tilde{q}_F)$ in $H^*(W)$, which does not depend on the choice of (F, q_F) in its isomorphism class. By construction, $j(F, q_F)$ is *nil*-closed, and $g(j(F, q_F)) \cong \tilde{F}$ is noetherian so, by Proposition 4.8, $j(F, q_F)$ is noetherian. j is obviously a right inverse of h ; since h is injective, it is a bijection. \square

Definition 6.8. For \mathcal{G} a groupoid whose objects are the sub-vector spaces of W and which has the restriction property, for $F := \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / \sim_{\mathcal{G}}$ and for q_F the canonical surjection from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)$ to F , $H^*(W)^{\mathcal{G}}$ is the image of the map

$$m \circ \mathcal{L}(\tilde{q}_F) : m \circ \mathcal{L}(\tilde{F}) \hookrightarrow H^*(W).$$

Remark 6.9. $H^*(W)^{\mathcal{G}}$ is the unique *nil*-closed, noetherian, sub-algebra of $H^*(W)$, whose transcendence degree is $\dim(W)$ and such that $\mathcal{G}_{(F, q_F)} = \mathcal{G}$, for $F := \text{Hom}_{\mathcal{K}\text{f.g.}}(H^*(W)^{\mathcal{G}}, H^*(-))$ and q_F the natural surjection from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)$ to $\text{Hom}_{\mathcal{K}\text{f.g.}}(H^*(W)^{\mathcal{G}}, H^*(-))$ induced by the inclusion from $H^*(W)^{\mathcal{G}}$ to $H^*(W)$.

Furthermore, $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$ defines a contravariant functor between $\text{Groupoid}(W)$ (see Notation 5.11) and the poset of *nil*-closed, noetherian, sub-algebras of $H^*(W)$, whose transcendence degrees are $\dim(W)$, ordered by inclusion.

Corollary 6.10. Any *nil*-closed, integral, noetherian, unstable, algebra whose transcendence degree is equal to $\dim(W)$ is isomorphic to $H^*(W)^{\mathcal{G}}$ for some \mathcal{G} .

Proof. It is a reformulation of the theorem of Adams-Wilkerson using Theorem 6.7. \square

Example 6.11. For G a sub-group of $\text{Gl}(W)$, $H^*(W)^{\mathfrak{g}(G)} = H^*(W)^G$, for $H^*(W)^G$ the algebra of invariant element of $H^*(W)$ under the action of G .

Let us identify precisely the sub-algebra $H^*(W)^{\mathcal{G}}$ of $H^*(W)$.

Proposition 6.12. *Let \mathcal{G} be a groupoid whose objects are the sub-vector spaces of W with the restriction property. Then,*

$$H^*(W)^{\mathcal{G}} = \{x \in H^*(W) ; \alpha^* \iota_{U'}^*(x) = \iota_U^*(x) \text{ for all } \alpha \in \mathcal{G}(U, U')\}.$$

Proof. Let ϕ be the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$ and let $K(\mathcal{G}) = \{x \in H^*(W) ; \alpha^* \iota_{U'}^*(x) = \iota_U^*(x) \text{ for all } \alpha \in \mathcal{G}(U, U')\}$. By definition, $\alpha^* \iota_{U'}^* \phi = \iota_U^* \phi$ for all $\alpha \in \mathcal{G}(U, U')$ and for all sub-spaces U and U' of W . Then,

$$H^*(W)^{\mathcal{G}} \subset K(\mathcal{G}).$$

Furthermore, the inclusion from $K(\mathcal{G})$ to $H^*(W)$ induces a surjection from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)$ to $g(K(\mathcal{G}))$ which factorises through an isomorphism from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / \sim_{\mathcal{G}}$ to $g(K(\mathcal{G}))$. The fact that the surjection factorises through $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / \sim_{\mathcal{G}} \rightarrow g(K(\mathcal{G}))$ is a direct consequence of the definitions of $K(\mathcal{G})$ and $\sim_{\mathcal{G}}$, and the fact that this morphism is injective, comes from the fact that the isomorphism from $\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / \sim_{\mathcal{G}}$ to $g(H^*(W)^{\mathcal{G}})$ factorises as the following diagram:

$$\text{Hom}_{\mathcal{V}\mathcal{I}}(-, W) / \sim_{\mathcal{G}} \rightarrow g(K(\mathcal{G})) \rightarrow g(H^*(W)^{\mathcal{G}}).$$

Then, by applying the functor $m \circ \mathcal{L}$ to the last diagram, we get the bottom line of the following:

$$\begin{array}{ccccc} H^*(W)^{\mathcal{G}} & \hookrightarrow & K(\mathcal{G}) & \hookrightarrow & H^*(W) \\ \downarrow \eta_{H^*(W)^{\mathcal{G}}} & & \downarrow \eta_{K(\mathcal{G})} & & \downarrow \eta_{H^*(W)} \\ l_1(H^*(W)^{\mathcal{G}}) & \xrightarrow{\cong} & l_1(K(\mathcal{G})) & \hookrightarrow & l_1(H^*(W)), \end{array}$$

where η denotes the unit of the adjunction between f and m . Then, since $H^*(W)^{\mathcal{G}}$ and $H^*(W)$ are *nil*-closed, $\eta_{H^*(W)^{\mathcal{G}}}$ and $\eta_{H^*(W)}$ are isomorphisms. Furthermore, $K(\mathcal{G})$ is a sub unstable algebra of $H^*(W)$, hence it does not contains any nilpotent sub module, and $\eta_{K(\mathcal{G})}$ is injective. Then, the commutativity of the diagram implies that $\eta_{K(\mathcal{G})}$ is an isomorphism, and therefore that $H^*(W)^{\mathcal{G}} = K(\mathcal{G})$. \square

Definition 6.13. For $g \in \text{Gl}(W)$, and $\mathcal{G} \in \text{Groupoid}(W)$, $g \cdot \mathcal{G}$ is the groupoid in $\text{Groupoid}(W)$ defined by $\beta \in g \cdot \mathcal{G}(R, R')$, for β an isomorphisms between subspaces R and R' of W , if there exist $\alpha \in \mathcal{G}(U, U')$ for $U = g^{-1}(R)$ and $U' = g^{-1}(R')$, such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{g|_U^R} & R \\ \alpha \downarrow & & \downarrow \beta \\ U' & \xrightarrow{g|_{U'}^{R'}} & R'. \end{array}$$

This defines a poset preserving action of $\text{Gl}(W)$ on $\text{Groupoid}(W)$.

Remark 6.14. This action generalises the action by conjugation on $Group(W)$. Indeed, for G a subgroup of $Gl(W)$ and $g \in Gl(W)$, $g \cdot \mathfrak{g}(G) = \mathfrak{g}(gGg^{-1})$.

Proposition 6.15. For $g \in Gl(W)$ and $\mathcal{G} \in Groupoid(W)$, $H^*(W)^{g \cdot \mathcal{G}} = (g^{-1})^*(H^*(W)^\mathcal{G})$.

Proof. This is a direct consequence of Proposition 6.12. \square

Remark 6.16. We want to notice that the $(H^*(W)^\mathcal{G})_{\mathcal{G} \in Groupoid(W)}$ does not constitute a minimal list for representing elements of isomorphism classes of *nil*-closed, integral and noetherian unstable algebras of transcendence degree $\dim(W)$. For $g \in Gl(W)$ and $\mathcal{G} \in Groupoid(W)$, $g \cdot \mathcal{G}$ needs not to be equal to \mathcal{G} , but, by Proposition 6.15, $H^*(W)^\mathcal{G} \cong H^*(W)^{g \cdot \mathcal{G}}$.

Conversely, since the inclusion of $H^*(W)^\mathcal{G}$ in $H^*(W)$ induces a surjection from $\text{Hom}_{\mathcal{K}}(H^*(W), H^*(W))$ to $\text{Hom}_{\mathcal{K}}(H^*(W)^\mathcal{G}, H^*(W))$, and since $g \mapsto g^*$ induces an isomorphism between $\text{Hom}_{\mathcal{K}}(H^*(W), H^*(W))$ and $Gl(W)$, we have that if $H^*(W)^\mathcal{G} \cong H^*(W)^\mathcal{H}$, there exists $g \in Gl(W)$ such that $H^*(W)^\mathcal{H} = (g^{-1})^*(H^*(W)^\mathcal{G})$. By Proposition 6.15, $\mathcal{H} = g \cdot \mathcal{G}$.

6.2 Centrality and primitive elements of $H^*(W)^\mathcal{G}$

Throughout this sub-section, we fix V and W two objects in \mathcal{V}^f , as well as an injection δ from V to W .

We consider K a *nil*-closed, noetherian unstable sub algebra of $H^*(W)$ of transcendence degree $\dim(W)$, such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. We start by explaining why the $H^*(V)$ -comodule structure on K induced by $\delta^*\phi$ is induced from the $H^*(V)$ -comodule structure on $H^*(W)$ given by $(\text{id}_W + \delta)^* : H^*(W) \rightarrow H^*(W) \otimes H^*(V)$.

Then, for $K = H^*(W)^\mathcal{G}$, we explain how to determine the primitive elements of this comodule structure from \mathcal{G} .

Proposition 6.17. Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. The $H^*(V)$ -comodule structure κ on K , induced by $\delta^*\phi$ and Corollary 3.15, fits into the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ \phi \downarrow & & \downarrow \phi \otimes \text{id}_{H^*(V)} \\ H^*(W) & \xrightarrow{(\text{id}_W + \delta)^*} & H^*(W) \otimes H^*(V). \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ \phi \downarrow & & \downarrow \phi \otimes \text{id}_{H^*(V)} \\ H^*(W) & & H^*(W) \otimes H^*(V). \end{array}$$

The existence of a morphism ψ^* from $H^*(W)$ to $H^*(W) \otimes H^*(V)$ which turns it into a commutative diagram is a consequence of the surjectivity of ϕ^* from $\text{Hom}_{\mathcal{K}}(H^*(W), H^*(W \oplus V))$ to

$\text{Hom}_{\mathcal{K}}(K, H^*(W \oplus V))$. We only have to justify why we can take $\psi = \text{id}_W + \delta$. We have that the composition of $(\phi \otimes \text{id}_{H^*(V)}) \circ \kappa$ with $\epsilon_K \otimes \text{id}_{H^*(V)}$ is equal to $\delta^* \phi$ and that with $\text{id}_{H^*(W)} \otimes \epsilon_{H^*(V)}$ is equal to ϕ . Hence, since $\delta^* \phi$ is central, $(\phi \otimes \text{id}_{H^*(V)}) \circ \kappa$ is the unique element in the inverse image of ϕ under $\rho_{\text{Hom}_{\mathcal{K}}(K, H^*(-)), (V, \delta^* \phi)}$. But $(\text{id}_W + \delta)^* \phi$ is also in this inverse image of ϕ , hence the diagram commutes. \square

We consider $(\text{id}_W + \delta)^* : H^*(W) \rightarrow H^*(W) \otimes H^*(V)$ which is the $H^*(V)$ -comodule structure on $H^*(W)$ associated with $(V, \delta) \in \mathbf{C}(H^*(W))$.

Proposition 6.18. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathbf{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, we have a pullback diagram of the following form:*

$$\begin{array}{ccc} P(K, \delta^* \phi) & \hookrightarrow & K \\ \downarrow & & \downarrow \phi \\ H^*(W/\text{Im}(\delta)) & \xrightarrow{\pi^*} & H^*(W). \end{array}$$

Proof. Proposition 6.17 says that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ \phi \downarrow & & \downarrow \phi \otimes \text{id}_{H^*(V)} \\ H^*(W) & \xrightarrow{(\text{id}_W + \delta)^*} & H^*(W) \otimes H^*(V). \end{array}$$

This means that the $H^*(V)$ -comodule structure on K is induced by that on $H^*(W)$. Hence, the primitive elements of K are simply the primitive elements of $H^*(W)$ that are in K . But the comodule structure on $H^*(W)$ is the morphism $(\text{id}_W + \delta)^*$ whose algebra of primitive elements is the image of $H^*(W/\text{Im}(\delta))$ under π^* , for π the projection from W to $W/\text{Im}(\delta)$. \square

Corollary 6.19. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathbf{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, the following is a pushout diagram:*

$$\begin{array}{ccc} \text{Hom}_{\mathcal{V}f}(-, W) & \twoheadrightarrow & \text{Hom}_{\mathcal{K}}(K, H^*(-)) \\ \downarrow \Downarrow & & \downarrow \Downarrow \\ \text{Hom}_{\mathcal{V}f}(-, W/\text{Im}(\delta)) & \twoheadrightarrow & \text{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-)). \end{array}$$

Proof. It is a direct consequence of Lemma 2.15 and of Proposition 6.18. \square

We can thus identify $\text{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-))$ in this context. In particular, we show that P is always noetherian.

Lemma 6.20. *For S a set, and \sim_1 and \sim_2 two equivalence relations on S , we denote by \sim the smallest equivalence relation on S (in the sense that $\{(a, b) \in S \times S ; a \sim b\} \subset S \times S$ is the smallest)*

such that, for all a and b in S such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$. Then, the following is a pushout in \mathbf{Set} :

$$\begin{array}{ccc} S & \twoheadrightarrow & S/\sim_1 \\ \downarrow & & \downarrow \\ S/\sim_2 & \twoheadrightarrow & S/\sim \end{array}$$

Proof. Let Σ denote the pushout of

$$\begin{array}{ccc} S & \twoheadrightarrow & S/\sim_1 \\ \downarrow & & \\ S/\sim_2 & & \end{array}$$

Then, for $s : S \rightarrow \Sigma$ the composition of the projection from S to S/\sim_1 with the surjective application $S/\sim_1 \rightarrow \Sigma$, s is surjective. We define \sim' the equivalence relation on S defined by $a \sim' b$ if and only if $s(a) = s(b)$. Σ is isomorphic in \mathbf{Set} with S/\sim' and we will show that $\sim' = \sim$.

By commutativity of the pushout diagram, for a and b in S such that $a \sim_1 b$ or $a \sim_2 b$, $s(a) = s(b)$. Suppose that \sim' is not the smallest such equivalence relation. Then, there exists x and y with $x \sim' y$ and an equivalence relation \sim'' , satisfying that for a and b such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim'' b$, and such that x is not equivalent to y for \sim'' . Then, the following diagram is commutative:

$$\begin{array}{ccc} S & \twoheadrightarrow & S/\sim_1 \\ \downarrow & & \downarrow \\ S/\sim_2 & \twoheadrightarrow & S/\sim'' \end{array}$$

and factorise by a morphism $S/\sim' \rightarrow S/\sim''$. This is a contradiction, so $\sim' = \sim$. \square

Remark 6.21. For \sim_1 and \sim_2 as in Lemma 6.20, and for S finite, the smallest equivalence relation \sim on S such that, for all a and b in S such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$, is the equivalence relation defined by $a \sim b$ if there is a finite family $(s_i)_{i \in [1, n]}$ of objects in S such that:

1. $s_1 = a$,
2. $s_n = b$,
3. for all $1 \leq i \leq n$, if i is odd $s_i \sim_1 s_{i+1}$ and if i is even $s_i \sim_2 s_{i+1}$.

We deduce the following proposition.

Proposition 6.22. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathbf{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, for ζ and γ in $\text{Hom}_{\mathcal{V}\mathcal{I}}(U, V)$, $\gamma^* \phi|_{P(K, \delta^* \phi)} = \zeta^* \phi|_{P(K, \delta^* \phi)} \in \text{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(U))$ if and only if there exists a family $(\epsilon_i)_{i \in [1, n]} \in \text{Hom}_{\mathcal{V}f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that:*

1. $\gamma = \epsilon_1$,

2. $\zeta = \epsilon_n$,

3. for all $1 \leq i \leq n-1$, $\epsilon_i^* \phi = \epsilon_{i+1}^* \phi$ if i is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if i is even.

Proof. By Corollary 6.19, the following is a pushout:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}f}(-, W) & \xrightarrow{k} & \mathrm{Hom}_{\mathcal{K}}(K, H^*(-)) \\ \pi_* \downarrow & & \downarrow p \\ \mathrm{Hom}_{\mathcal{V}f}(-, W/\mathrm{Im}(\delta)) & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-)), \end{array}$$

where k maps $\zeta : U \rightarrow W$ to $\zeta^* \phi : K \rightarrow H^*(U)$, p maps $\psi : K \rightarrow H^*(U)$ to $\psi|_{P(K, \delta^* \phi)}$ and π_* maps $\zeta : U \rightarrow W$ to $\pi \circ \zeta : U \rightarrow W/\mathrm{Im}(\delta)$. Then, by Lemma 6.20, $p \circ k(\zeta) = p \circ k(\gamma)$ if and only if there exists a family $(\epsilon_i)_{i \in [1, n]} \in \mathrm{Hom}_{\mathcal{V}f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $\gamma = \epsilon_1$, $\zeta = \epsilon_n$ and for all $1 \leq i \leq n-1$, $k(\epsilon_i) = k(\epsilon_{i+1})$ if i is odd and $\pi_*(\epsilon_i) = \pi_*(\epsilon_{i+1})$ if i is even. \square

Corollary 6.23. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, for $\zeta \in \mathrm{Hom}_{\mathcal{V}f}(U, W)$, $\ker(\zeta^* \phi|_{P(K, \delta^* \phi)}) = \ker(\pi \circ \zeta)$.*

Proof. Let $\zeta_0 \in \mathrm{Hom}_{\mathcal{V}f}(U/\ker(\zeta^* \phi|_{P(K, \delta^* \phi)}), W)$ such that $\zeta^* \phi|_{P(K, \delta^* \phi)} = \pi_U^* \zeta_0^* \phi|_{P(K, \delta^* \phi)}$, with π_U the projection from U to $U/\ker(\zeta^* \phi|_{P(K, \delta^* \phi)})$. Let $\epsilon_1 = \zeta_0 \circ \pi_U$, $\epsilon_n = \zeta$ and for all i $\epsilon_i^* \phi = \epsilon_{i+1}^* \phi$ if i is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if i is even. Then, since $\mathrm{Hom}_{\mathcal{K}}(K, H^*(-))$ is noetherian, $\ker(\pi \circ \epsilon_i) = \ker(\pi \circ \epsilon_{i+1})$ for all $1 \leq i \leq n-1$. Hence, $\ker(\pi \circ \zeta) = \ker(\pi \circ \zeta_0 \circ \pi_U) = \ker(\zeta^* \phi|_{P(K, \delta^* \phi)})$. \square

Corollary 6.24. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, $\mathrm{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-))$ is noetherian.*

Proof. The fact that $\mathrm{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-))$ is finite is obvious. Let $\zeta^* \phi|_{P(K, \delta^* \phi)}$ in $\mathrm{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(U))$ and let α be a morphism from a vector space Y to U . Then, by Corollary 6.23

$$\ker(\alpha^* \zeta^* \phi|_{P(K, \delta^* \phi)}) = \ker(\pi \circ \zeta \circ \alpha).$$

Since α is injective, this is equal to

$$\alpha^{-1}(\ker(\pi \circ \zeta)) = \alpha^{-1}(\ker(\zeta^* \phi|_{P(K, \delta^* \phi)})).$$

\square

Theorem 6.25. *Let K be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for ϕ the inclusion of K in $H^*(W)$. Then, $P(K, \delta^* \phi)$ is *nil-closed* and noetherian.*

Proof. Since, $P(K, \delta^* \phi)$ is the kernel of $\kappa - \mathrm{id}_K \otimes 1$ from K to $K \otimes H^*(V)$ which are *nil-closed*, for κ the comodule structure of K associated with $\delta^* \phi$, and since f is exact and m is left-exact, the following is an exact sequence:

$$0 \rightarrow l_1(P(K, \delta^* \phi)) \rightarrow l_1(K) \xrightarrow{l_1(\kappa - \mathrm{id}_K \otimes 1)} l_1(K \otimes H^*(V)).$$

Therefore, since K is *nil-closed*, $P(K, \delta^* \phi)$ is also *nil-closed*. Then, the noetherianity of $P(K, \delta^* \phi)$ is a consequence of the noetherianity of $\mathrm{Hom}_{\mathcal{K}}(P(K, \delta^* \phi), H^*(-))$ and of Proposition 4.8. \square

Remark 6.26. We have identified $P(K, \delta^* \phi)$ with a sub-algebra of $H^*(W/\text{Im}(\delta))$. Furthermore, we proved that $P(K, \delta^* \phi)$ is *nil*-closed and noetherian, and (because we took δ to be an injection) Corollary 6.23 implies that the inclusion from $P(K, \delta^* \phi)$ into $H^*(W/\text{Im}(\delta))$ is regular. Therefore, by Theorem 6.7, $P(K, \delta^* \phi)$ has the form $H^*(W/\text{Im}(\delta))^{\mathcal{G}'}$, for some $\mathcal{G}' \in \text{Groupoid}(W/\text{Im}(\delta))$.

This leads to the following question: for W and V in \mathcal{V}^f , for δ an inclusion from V to W and for \mathcal{G}' a groupoid with the restriction property and whose objects are the sub-spaces of $(W/\text{Im}(\delta))$, which are the groupoids \mathcal{G} with the restriction property and whose objects are the sub-spaces of W , such that

1. $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$ -comodule of $H^*(W)$ for the comodule structure induced by δ ,
2. the intersection of $H^*(W)^{\mathcal{G}}$ with $\pi^*(H^*(W/\text{Im}(\delta)))$ is the image under $\pi^* : H^*(W/\text{Im}(\delta)) \rightarrow H^*(W)$ of $H^*(W/\text{Im}(\delta))^{\mathcal{G}'}$.

Remark 6.27. $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$ -comodule of $H^*(W)$ for the comodule structure induced by δ if and only if \mathcal{G} satisfies the two conditions of theorem 5.30.

We recall that, from the beginning of this sub-section, V and W are fixed objects of \mathcal{V}^f and δ a fixed injective morphism from V to W .

Theorem 6.28. *Let \mathcal{G} be a groupoid with the restriction property and whose objects are the sub-vector spaces of W , such that $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$ -comodule of $(H^*(W), (id_W + \delta)^*)$. For \mathcal{G}' the only groupoid whose objects are sub spaces of $W/\text{Im}(\delta)$ which satisfies that $\pi^*(H^*(W/\text{Im}(\delta))^{\mathcal{G}'})$ is the algebra of primitive elements of $H^*(W)^{\mathcal{G}}$, the two following conditions are equivalent:*

1. $\alpha \in \mathcal{G}'(U, U')$, where U and U' are sub-vector spaces of $W/\text{Im}(\delta)$ and α is an isomorphism from U to U' ,
2. there exists N and N' sub spaces of W such that π induce isomorphisms from N and N' to U and U' , as well as an element $\beta \in \mathcal{G}(N, N')$ such that $\alpha = \pi|_{N'}^{U'} \circ \beta \circ (\pi|_N^U)^{-1}$.

Proof. We consider the pushout diagram of Corollary 6.19:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{V}^f}(-, W) & \xrightarrow{k} & \text{Hom}_{\mathcal{K}}(H^*(W)^{\mathcal{G}}, H^*(-)) \\ \pi_* \downarrow & & \downarrow q \\ \text{Hom}_{\mathcal{V}^f}(-, W/\text{Im}(\delta)) & \xrightarrow{p} & \text{Hom}_{\mathcal{K}}(H^*(W/\text{Im}(\delta))^{\mathcal{G}'}, H^*(-)), \end{array}$$

where π_* maps $\gamma : U \rightarrow W$ to $\pi \circ \gamma$, k maps γ to $\gamma^* \phi_{\mathcal{G}}$ for $\phi_{\mathcal{G}}$ the inclusion from $H^*(W)^{\mathcal{G}}$, q maps $\psi : H^*(W)^{\mathcal{G}} \rightarrow H^*(U)$ to $\psi|_{\pi^*(H^*(W/\text{Im}(\delta)))}$ the restriction of ψ to $\pi^*(H^*(W/\text{Im}(\delta)))$ and, finally, p maps ζ from U to $W/\text{Im}(\delta)$ to $\zeta^* \phi_{\mathcal{G}'}$, for $\phi_{\mathcal{G}'}$ the inclusion of $H^*(W/\text{Im}(\delta))^{\mathcal{G}'}$ into $H^*(W/\text{Im}(\delta))$.

We fix a section s from $W/\text{Im}(\delta)$ to W . Since $\pi \circ s = id_{W/\text{Im}(\delta)}$, $\pi_*(s) = id_{W/\text{Im}(\delta)}$. Then, by commutativity of the pushout diagram, we have $\phi_{\mathcal{G}'} = q(s^* \phi_{\mathcal{G}}) = s^* \phi_{\mathcal{G}}|_{\pi^*(H^*(W/\text{Im}(\delta)))}$.

By construction, there are natural isomorphisms $\text{Hom}_{\mathcal{K}f.g.}(H^*(W)^{\mathcal{G}}, H^*(-)) \cong \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W)/\sim_{\mathcal{G}}$ and $\text{Hom}_{\mathcal{K}f.g.}(H^*(W/\text{Im}(\delta))^{\mathcal{G}'}, H^*(-)) \cong \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W/\text{Im}(\delta))/\sim_{\mathcal{G}'}$. These are the isomorphisms

that map $\phi_{\mathcal{G}}$ to $[\text{id}_W]_{\mathcal{G}}$ and $\phi_{\mathcal{G}'}$ to $[\text{id}_{W/\text{Im}(\delta)}]_{\mathcal{G}'}$ respectively.

Let us first prove 2) \Rightarrow 1). We consider, $\beta \in \mathcal{G}(N, N')$ such that π induces isomorphisms $\pi|_N^U$ and $\pi|_{N'}^{U'}$ between N and U and between N' and U' . Let α be an isomorphism from U to U' such that $\alpha = \pi|_{N'}^{U'} \circ \beta \circ (\pi|_N^U)^{-1}$. Then, $(\pi|_{N'}^{U'})^{-1} \circ \alpha = \beta \circ (\pi|_N^U)^{-1}$. Therefore,

$$\begin{aligned} \alpha^*((\pi|_{N'}^{U'})^{-1})^* \iota_{N'}^* \phi_{\mathcal{G}} &= ((\pi|_N^U)^{-1})^* \beta^* \iota_{N'}^* \phi_{\mathcal{G}} \\ &= ((\pi|_N^U)^{-1})^* \iota_N^* \phi_{\mathcal{G}}. \end{aligned}$$

We can choose the section s in such a way that $s \circ \iota_{U'} = \iota_{N'} \circ ((\pi|_{N'}^{U'})^{-1})$, then

$$\alpha^* \iota_{U'}^* s^* \phi_{\mathcal{G}} = ((\pi|_N^U)^{-1})^* \iota_N^* \phi_{\mathcal{G}}.$$

This implies that $\alpha^* \iota_{U'}^* s^* q(\phi_{\mathcal{G}}) = ((\pi|_N^U)^{-1})^* \iota_N^* q(\phi_{\mathcal{G}})$. Furthermore, $\pi \circ s \circ \iota_U = \pi \circ \iota_N \circ (\pi|_N^U)^{-1}$. Hence, we also have, by Proposition 6.22, that $\iota_U^* s^* q(\phi_{\mathcal{G}}) = ((\pi|_N^U)^{-1})^* \iota_N^* q(\phi_{\mathcal{G}})$. Hence,

$$\alpha^* \iota_{U'}^* s^* q(\phi_{\mathcal{G}}) = \iota_U^* s^* q(\phi_{\mathcal{G}}).$$

Since $s^* q(\phi_{\mathcal{G}}) = [\text{id}_{W/\text{Im}(\delta)}]_{\mathcal{G}'}$, this implies that $\alpha \in \mathcal{G}'(U, U')$, as required.

Now, let us prove the far more challenging 1) \Rightarrow 2).

We consider $\alpha \in \mathcal{G}'(U, U')$ where U and U' are two sub-spaces of $W/\text{Im}(\delta)$. Then,

$$\alpha^* \iota_{U'}^* [\text{id}_{W/\text{Im}(\delta)}]_{\mathcal{G}'} = \iota_U^* [\text{id}_{W/\text{Im}(\delta)}]_{\mathcal{G}'},$$

or, equivalently,

$$[\iota_{U'} \circ \alpha]_{\mathcal{G}'} = [\iota_U]_{\mathcal{G}'}.$$

By Proposition 6.22, since $[\text{id}_{W/\text{Im}(\delta)}]_{\mathcal{G}'}$ is the image of $s^* q(\phi_{\mathcal{G}})$ under the isomorphism

$$\text{Hom}_{\mathcal{K}\text{f.g.}}(H^*(W/\text{Im}(\delta))^{\mathcal{G}'}, H^*(-)) \cong \text{Hom}_{\mathcal{V}\mathcal{I}}(-, W/\text{Im}(\delta)) / \sim_{\mathcal{G}'},$$

we have that $[\zeta]_{\mathcal{G}'} = [\gamma]_{\mathcal{G}'}$, if and only if there exists a family $(\epsilon_i)_{i \in [1, n]} \in \text{Hom}_{\mathcal{V}f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $s \circ \gamma = \epsilon_1$, $s \circ \zeta = \epsilon_n$ and, for all $1 \leq i \leq n-1$, $\epsilon_i^* \phi_{\mathcal{G}} = \epsilon_{i+1}^* \phi_{\mathcal{G}}$ if i is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if i is even.

So let $(\epsilon_i)_{i \in [1, n]} \in \text{Hom}_{\mathcal{V}f}(U, W)^n$ be such that $\epsilon_1 = s \circ \iota_U$, $\epsilon_n = s \circ \iota_{U'} \circ \alpha$ and for all $1 \leq i \leq n-1$, $\epsilon_i^* \phi_{\mathcal{G}} = \epsilon_{i+1}^* \phi_{\mathcal{G}}$ if i is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if i is even.

By induction, for all $i \in [1, n]$, $\epsilon_i^* \phi_{\mathcal{G}}$ and $\pi \circ \epsilon_i$ are regular elements respectively of $\text{Hom}_{\mathcal{K}}(H^*(W)^{\mathcal{G}}, H^*(U))$ and $\text{Hom}_{\mathcal{V}f}(U, W/\text{Im}(\delta))$. Hence, ϵ_i and $\pi \circ \epsilon_i$ are injections. For all i , let N_i denote the image of ϵ_i in W , we denote also by $\tilde{\epsilon}_i$ the corestriction of ϵ_i to N_i . Then, for i odd, $\epsilon_i^* \phi_{\mathcal{G}} = \epsilon_{i+1}^* \phi_{\mathcal{G}}$ implies that there exists β_i in $\mathcal{G}(N_i, N_{i+1})$ such that $\tilde{\epsilon}_{i+1} = \beta_i \circ \tilde{\epsilon}_i$.

We take some moment to explain subtlety in the proof. We would like, for i even, to have $\epsilon_i = \epsilon_{i+1}$. Then, the composition of the β_i with i odd would give an isomorphism β between $N_1 = s(U)$ and $N_n = s(U')$ such that $\beta \in \mathcal{G}(N_1, N_n)$, since \mathcal{G} is a groupoid and we would have $(s \circ \iota_{U'})|^{N_n} \circ \alpha = \beta \circ (s \circ \iota_U)|^{N_1}$. Since $(s \circ \iota_U)|^{N_1}$ and $(s \circ \iota_{U'})|^{N_n}$ are inverse isomorphisms of

$\pi|_{N_1}^U$ and $\pi|_{N_n}^{U'}$, we would have $\alpha = \pi|_{N_n}^{U'} \circ \beta \circ (\pi|_{N_1}^U)^{-1}$. If this were the case, we would have found a β for any N and N' such that π induces isomorphisms between U and N and between U' and N' , and we would have done so without using the assumption that $\delta^* \phi_{\mathcal{G}}$ is central. Unfortunately, this naive approach fails, and N and N' must be chosen carefully. The hypothesis on the ϵ_i for i even indicates how to modify our original N_1 and N_n to make it work, using the centrality of $\delta^* \phi_{\mathcal{G}}$.

First notice that, since $\pi \circ \epsilon_i$ is injective for all i , we always have $N_i \cap \text{Im}(\delta) = \{0\}$. Then, the assumption that, for i even, $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ implies that there exists ρ_i from U to W whose image is inside $\text{Im}(\delta)$ and such that $\epsilon_{i+1} = \epsilon_i + \rho_i$. Now, since $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$ -comodule of $(H^*(W), (\text{id}_W + \delta)^*)$ we know that $[\delta]_{\mathcal{G}}$ is a central element of $\text{Hom}_{V\mathcal{T}}(-, W) / \sim_{\mathcal{G}}$. Then, by Theorem 5.30, for i odd, we know that the isomorphisms $\bar{\beta}_i$ from $N_i \oplus \text{Im}(\delta)$ to $N_{i+1} \oplus \text{Im}(\delta)$ defined by $\bar{\beta}_i(n) = \beta_i(n)$ for $n \in N_i$ and $\bar{\beta}_i(v) = v$ for $v \in \text{Im}(\delta)$ satisfy $\bar{\beta}_i \in \mathcal{G}(N_i \oplus \text{Im}(\delta), N_{i+1} \oplus \text{Im}(\delta))$. Moreover, for i even, $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ implies that $N_i \oplus \text{Im}(\delta) = N_{i+1} \oplus \text{Im}(\delta)$. Then, at each even step i , we can “correct” ϵ_{i-1} to get $\beta_{i-1} \circ \tilde{\epsilon}_{i-1} = \tilde{\epsilon}_{i+1}$ instead of $\tilde{\epsilon}_i$.

For each $i \in [1, n]$, we define ϵ'_i (the “corrected” ϵ_i) by

$$\epsilon'_i := \iota_{N_i \oplus \text{Im}(\delta)}^W \circ (\tilde{\epsilon}_i \oplus \sum_{\{j \text{ even}; i \leq j < n\}} \beta_{i \rightarrow j}^{-1} \circ \rho_j|^{N_j \oplus \text{Im}(\delta)}),$$

where $\beta_{i \rightarrow j}$ is the composition of all the $\bar{\beta}_k$ with k odd and $i \leq k < j$. The family $(\epsilon'_i)_{i \in [1, n]}$ satisfies the following:

1. $\pi \circ \epsilon'_1 = \iota_U$, $\pi \circ \epsilon'_n = \iota_{U'} \circ \alpha$,
2. for all i , if we denote by N'_i the image of ϵ'_i , then $N'_i \oplus \text{Im}(\delta) = N_i \oplus \text{Im}(\delta)$,
3. for i odd, if we denote by β'_i the restriction of $\bar{\beta}_i$ to N'_i corestricted to N'_{i+1} , $\tilde{\epsilon}'_{i+1} = \beta'_i \circ \tilde{\epsilon}'_i$, with $\beta'_i \in \mathcal{G}(N'_i, N'_{i+1})$, since $\beta \in \mathcal{G}(N_i \oplus \text{Im}(\delta), N_{i+1} \oplus \text{Im}(\delta))$ and \mathcal{G} has the restriction property,
4. for i even, $\epsilon'_i = \epsilon'_{i+1}$.

Then, let $N = N'_1$, $N' = N'_n$ and $\beta = (\beta'_k \circ \dots \circ \beta'_3 \circ \beta'_1)$, where $k = n - 2$ if n is odd, $k = n - 1$ otherwise. Then, $\beta \in \mathcal{G}(N, N')$ and $\beta \circ \tilde{\epsilon}'_1 = \tilde{\epsilon}'_n$. Finally, $\pi \circ \epsilon'_1 = \iota_U$ implies that $\tilde{\epsilon}'_1 = (\pi|_N^U)^{-1}$ and $\pi \circ \epsilon'_n = \iota_{U'} \circ \alpha$ implies that $\tilde{\epsilon}'_n = (\pi|_{N'}^{U'})^{-1} \circ \alpha$. Hence, $\alpha = \pi|_{N'}^{U'} \circ \beta \circ (\pi|_N^U)^{-1}$. \square

6.3 Applications

We end this section by presenting some applications of Theorem 6.28. The first result was already known.

Proposition 6.29. *Let K be a noetherian, nil-closed, integral, unstable algebra of transcendence degree d . We assume that the centre of K is of dimension d . Then, $K \cong H^*(W)$ with $\dim(W) = d$.*

Proof. By Theorem 6.7 and the theorem of Adams-Wilkerson $K \cong H^*(W)^{\mathcal{G}}$ for some groupoid \mathcal{G} with the restriction property and whose objects are the sub-spaces of W . Then, since the centre of K has dimension $\dim(W)$, up to isomorphism the centre of K is given by (W, ϕ) where ϕ is the inclusion from K to $H^*(W)$ induced by the theorem of Adams-Wilkerson. Then, by Theorem 5.30, for all $\alpha \in \mathcal{G}(U, U')$, where U and U' are sub-vector spaces of W , and for all $x \in U = U \cap W$, $\alpha(x) = x$. Hence, \mathcal{G} is the groupoid in which the only morphisms are identities of sub-spaces of W . Then, $H^*(W)^{\mathcal{G}} = H^*(W)$. \square

Let us now consider the case where the centre is of dimension $d - 1$, for d the transcendence degree of K .

Theorem 6.30. *Let K be a noetherian, nil-closed, integral, unstable algebra of transcendence degree d . We assume that the centre of K is of dimension $d - 1$. Then, there exists G a sub-group of $Gl(W)$ such that K is isomorphic to the algebra of invariant elements $H^*(W)^G$ with $\dim(W) = d$. Furthermore, G satisfies that the set of element $x \in W$ such that $g(x) = x$ for all $g \in G$ is a sub-vector space of W of dimension $d - 1$.*

Proof. By Theorem 6.7 and the theorem of Adams-Wilkerson $K \cong H^*(W)^{\mathcal{G}}$ for some groupoid \mathcal{G} with the restriction property and whose objects are the sub-spaces of W . Then, the centre of K is associated with a $H^*(V)$ -comodule structure on $H^*(W)^{\mathcal{G}}$, with $\dim(V) = d - 1$. Up to isomorphism, we can suppose that this $H^*(V)$ -comodule structure is induced by ι_V the inclusion of a sub-vector space V in W (we can assume that the comodule structure is induced by an injection, because it is associated with the centre of K which is regular by definition). By Theorem 5.30, for all $\alpha \in \mathcal{G}(U, U')$ and for all $x \in U \cap V$, $\alpha(x) = x$. Furthermore, for all sub-spaces U and U' of W , $\mathcal{G}(U, U')$ is determined from $\mathcal{G}(V \oplus U, V \oplus U')$ by the restriction property. Since we assumed $\dim(V) = \dim(W) - 1$, \mathcal{G} is uniquely determined by $\mathcal{G}(V, V)$ and $\mathcal{G}(W, W)$. Then, since for all $x \in V$ and for all $\alpha \in \mathcal{G}(V, V)$, $\alpha(x) = x$, $\mathcal{G}(V, V)$ is reduced to the identity of V . Therefore, for $g \in G := \mathcal{G}(W, W)$, $g(x) = x$ for all $x \in V$, and $\alpha \in \mathcal{G}(U, U')$ if and only if there is $g \in G$ such that $\alpha = g_U$. We get that $H^*(W)^{\mathcal{G}} = H^*(W)^G$. \square

Corollary 6.31. *Let K be a noetherian, nil-closed, integral, unstable algebra of transcendence degree $\dim(W)$, and let (C, γ) be its centre. Then if K is not isomorphic to $H^*(W)^G$ for some sub-group G of $Gl(W)$, $\dim(C) \leq d - 2$.*

In the next section, we will have examples where $\dim(C) = d - 2$ and K is not an algebra of invariant elements under some action on $H^*(W)$.

7 Two examples

In the following, we focus on the case where $p = 2$, so that $H^*(V) \cong \mathbb{F}_2[x_1, \dots, x_n]$, for (x_1, \dots, x_n) a basis of V^\sharp .

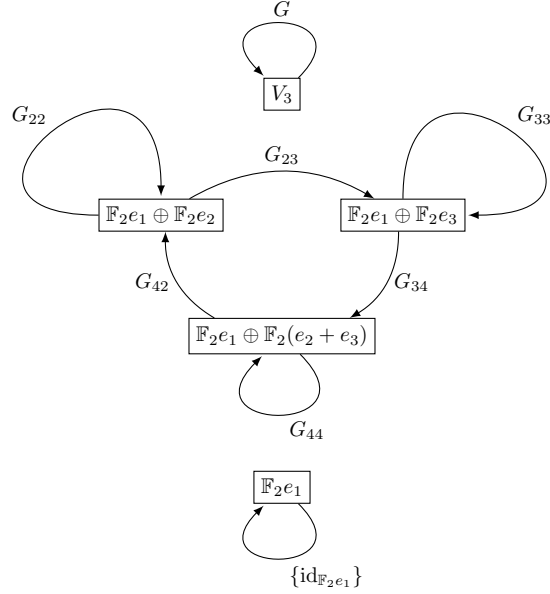
We take $V_3 = \mathbb{F}_2e_1 \oplus \mathbb{F}_2e_2 \oplus \mathbb{F}_2e_3$ and denote by (x, y, z) the dual basis of (e_1, e_2, e_3) . The morphism

$$(\text{id}_{V_3} + \iota_{\mathbb{F}_2e_1})^* : H^*(V_3) \rightarrow H^*(V_3) \otimes H^*(\mathbb{F}_2e_1),$$

which maps x to $x \otimes 1 + 1 \otimes x$, y to $y \otimes 1$ and z to $z \otimes 1$ defines an $H^*(\mathbb{F}_2e_1)$ -comodule structure on $H^*(V_3)$ in \mathcal{K} . In each of the following examples we want to find the list of noetherian, nil-closed, unstable sub-algebras of $H^*(V_3)$ of transcendence degree 3, which are sub $H^*(\mathbb{F}_2e_1)$ -comodules of $H^*(V_3)$ for the comodule structure given by $(\text{id}_{V_3} + \iota_{\mathbb{F}_2e_1})^*$ and whose algebra of primitive elements is $\pi^*(H^*(V_3/\mathbb{F}_2e_1)^{\mathcal{G}'})$ for π the projection from V_3 over V_3/\mathbb{F}_2e_1 , and for $H^*(V_3/\mathbb{F}_2e_1)^{\mathcal{G}'}$ a chosen nil-closed, noetherian, sub algebra of $H^*(V_3/\mathbb{F}_2e_1)$. By Theorem 6.28, \mathcal{G} has to satisfy that $\alpha \in \mathcal{G}'(U, U')$ if and only if there is $\beta \in \mathcal{G}(N, N')$ such that $\pi|_{N'} \circ \beta = \alpha \circ \pi|_N$.

Furthermore, by Theorem 5.30, since $[\iota_{\mathbb{F}_2e_1}]_{\mathcal{G}} \in \text{Hom}_{\mathcal{K}\text{f.g.}}(H^*(W)^{\mathcal{G}}, H^*(\mathbb{F}_2e_1))$ is central, all morphism β in $\mathcal{G}(N, N')$ is the restriction of a morphism $\tilde{\beta}$ in $\mathcal{G}(\mathbb{F}_2e_1 + N, \mathbb{F}_2e_1 + N')$ defined by

$\bar{\beta}(e_1) = e_1$ and $\bar{\beta}(n) = \beta(n)$ for $n \in N$. So \mathcal{G} can be deduced from its full sub-groupoid whose objects are V_3 , $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_2$, $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3$, $\mathbb{F}_2e_1 \oplus \mathbb{F}_2(e_2 + e_3)$ and \mathbb{F}_2e_1 . This full sub-groupoid can be represented as follows:



Where G denotes $\mathcal{G}(V_3, V_3)$ and G_{ij} denotes $\mathcal{G}(\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_i, \mathbb{F}_2e_1 \oplus \mathbb{F}_2e_j)$ with $e_4 := e_2 + e_3$, and where we omitted G_{43} , G_{24} , etc, which can be deduced by composition and inversion of the already given sets of morphisms.

We consider first the case where \mathcal{G}' contains only identities.

Proposition 7.1. *There are exactly 15 nil-closed, noetherian, unstable sub-algebras K of $H^*(V_3)$ of transcendence degree 3, which are sub $\mathbb{F}_2[x]$ -comodules of $H^*(V_3)$, for the comodule structure that maps x to $x \otimes 1 + 1 \otimes x$, y to $y \otimes 1$ and z to $z \otimes 1$, and such that the algebra of primitive elements of K is $\mathbb{F}_2[y, z]$. These are :*

1. $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)]$,
2. $\mathbb{F}_2[y, z, x(x+y)]$,
3. $\mathbb{F}_2[y, z, x(x+z)]$,
4. $\mathbb{F}_2[y, z, x(x+y+z)]$,
5. $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+y)]y$,
6. $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+z)]z$,
7. $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+y+z)](y+z)$,

8. $\mathbb{F}_2[x, y, z]$,
9. $\mathbb{F}_2[x, y, z](y+z) \oplus \mathbb{F}_2[z, x(x+z)]$,
10. $\mathbb{F}_2[x, y, z]y \oplus \mathbb{F}_2[z, x(x+z)]$,
11. $\mathbb{F}_2[x, y, z]z \oplus \mathbb{F}_2[y, x(x+y)]$,
12. $\mathbb{F}_2[z, x(x+z)] \oplus \mathbb{F}_2[x, y, z](y+z)y \oplus \mathbb{F}_2[y, x(x+y)]y$,
13. $\mathbb{F}_2[y, x(x+y)] \oplus \mathbb{F}_2[x, y, z](y+z)z \oplus \mathbb{F}_2[z, x(x+z)]z$,
14. $\mathbb{F}_2[(y+z), x(x+y+z)] \oplus \mathbb{F}_2[x, y, z]yz \oplus \mathbb{F}_2[z, x(x+z)]z$,
15. $\mathbb{F}_2[z, x(x+z)]z \oplus \mathbb{F}_2[y, x(x+y)]y \oplus \mathbb{F}_2[y, x(x+y)](y+z)y \oplus \mathbb{F}_2[x, y, z](y+z)yz$.

Proof. In this case, $\mathbb{F}_2[y, z] = \pi^*(H^*(V_3/\mathbb{F}_2e_1))$. Then, the groupoid \mathcal{G}' is the trivial groupoid, which contains only identities. Therefore, by Theorem 6.28, for all $\beta \in \mathcal{G}(N, N')$, $\pi \circ \beta = \pi$. For $\beta \in G_{ij}$ with i and j in $\{2, 3, 4\}$, $\pi \circ \beta(e_i) = \pi(e_i)$ which is not possible if $i \neq j$. Hence, $G_{ij} = \emptyset$ if $i \neq j$. We only have to determine G , G_{22} , G_{33} and G_{44} . We have two conditions on $\beta \in G$ or $\beta \in G_{ii}$, the first is that $\beta(e_1) = e_1$ (by Theorem 5.30 and the centrality of $[\iota_{\mathbb{F}_2e_1}]_{\mathcal{G}}$) and the second that $\pi \circ \beta = \pi$ (by Theorem 6.28 and the hypothesis on \mathcal{G}'). Therefore, β has a block matrix of the following form:

$$\begin{pmatrix} \text{id}_{\mathbb{F}_2e_1} & \hat{\beta} \\ 0 & \text{id}_N \end{pmatrix},$$

where $N \in \{\mathbb{F}_2e_2, \mathbb{F}_2e_3, \mathbb{F}_2(e_2 + e_3), \mathbb{F}_2e_2 \oplus \mathbb{F}_2e_3\}$ and $\hat{\beta}$ is a morphism from N to \mathbb{F}_2e_1 . Hence, the set of matrices of morphisms in G in the basis (e_1, e_2, e_3) is a sub group of

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

There are five possibilities for G , $G_1 = \{\text{id}_{V_3}\}$, $G_2 = \langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, $G_3 = \langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, $G_4 = \langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$ and G_5 the full group. The groups G_{ii} are either trivial or equal to H_{ii}

the group generated by the morphism which sends e_1 to itself and e_i to $e_1 + e_i$. Furthermore for a chosen G , the restriction property (Definition 5.8) requires that some of them are non trivial. Let us summarise the possible values of the G_{ii} for each value of G .

G	G_{22}	G_{33}	G_{44}
G_1	{id} or H_{22}	{id} or H_{33}	{id} or H_{44}
G_2	H_{22}	{id} or H_{33}	H_{44}
G_3	{id} or H_{22}	H_{33}	H_{44}
G_4	H_{22}	H_{33}	{id} or H_{44}
G_5	H_{22}	H_{33}	H_{44}

We find, 15 possible values for $(G, G_{22}, G_{33}, G_{44})$. Each one characterising precisely one $\mathcal{G} \in \text{Groupoid}(V_3)$ such that $H^*(V_3)^{\mathcal{G}}$ satisfies the required conditions.

We give details for only two computations of $H^*(V_3)^{\mathcal{G}}$. The other ones are left to the reader.

First case: If G contains all morphisms whose matrix are in H , then by the restriction property G_{ii} contains all the morphisms β that satisfy the two conditions, for $i \in \{2, 3, 4\}$. Hence, all morphisms in G_{ii} are induced by restriction from a morphism in G . Therefore, $H^*(V_3)^{\mathcal{G}}$ is equal to $H^*(V_3)^G$ the algebra of invariant elements under the action of G on $H^*(V_3)$. Let us start by using that an element of $H^*(V_3)^G$ has to be invariant under the action g_y which sends x to $x + y$, and y and z to themselves. Then, every polynomial $P(x, y, z)$ satisfies that

$$x^2P(x, y, z) = x(x + y)P(x, y, z) + yxP(x, y, z).$$

Hence, we can express every polynomial $P(x, y, z)$ as $P_1(x(x + y), y, z) + xP_2(x(x + y), y, z)$. Then, since $P_1(x(x + y), y, z)$ is invariant under g_y , $xP_2(x(x + y), y, z)$ has also to be invariant, therefore $P_2 = 0$. So every element in $H^*(V_3)^G$ can be expressed as $P(X, y, z)$ with $X = x(x + y)$.

But an element of $H^*(V_3)^G$ is also invariant under the action of g_z which send x to $x + z$ and y and z to themselves. g_z sends X to $X + y(y + z)$. We can show as before that $P(X, y, z)$ can be expressed as $P_1(X(X + y(y + z)), y, z) + XP_2(X(X + y(y + z)), y, z)$. Since $P_1(X(X + y(y + z)), y, z)$ is invariant under g_z , $P_2 = 0$. We have $X(X + y(y + z)) = x(x + y)(x + z)(x + y + z)$, so every element in $H^*(V_3)^G$ is in $\mathbb{F}_2[y, z, x(x + y)(x + z)(x + y + z)]$, but conversely, y, z and $x(x + y)(x + z)(x + y + z)$ are invariant under G . Hence, $H^*(V_3)^{\mathcal{G}} = \mathbb{F}_2[y, z, x(x + y)(x + z)(x + y + z)]$.

Second case: If $G = G_2$, by the restriction property G_{22} and G_{44} are the groups of isomorphisms of $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_i$, with $i = 2$ or 4 , generated by the morphism whose matrix in the basis (e_1, e_i) is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The only group which is not fully determined by the restriction property is G_{33} which can either be equal to $\{\text{id}_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}\}$, or to the subgroup B_2 of $\text{Gl}(\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3)$ generated by the morphism β whose matrix in the basis (e_1, e_3) is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1. If $G_{33} = \{\text{id}_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}\}$, then all the sets $\mathcal{G}(U, U')$ are determined by $G_2 = \mathcal{G}(V_3, V_3)$, by the restriction property, hence $H^*(V_3)^{\mathcal{G}} = H^*(V_3)^{G_2} = \mathbb{F}_2[y, z, x(x + y)]$.
2. If $G_{33} = B_2$, we get our first example which is not an algebra of invariant elements. By Proposition 6.12,

$$H^*(V_3)^{\mathcal{G}} = \{x \in H^*(V_3) ; \alpha^* \iota_{U'}^*(x) = \iota_U^*(x) \text{ for all } \alpha \in \mathcal{G}(U, U')\},$$

which is equal to

$$H^*(V_3)^{G_2} \cap \{x \in H^*(V_3) ; \beta^* \iota_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}^*(x) = \iota_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}^*(x)\}.$$

Hence, $H^*(V_3)^{\mathcal{G}}$ contains the element of $H^*(V_3)^{G_2}$ which are in the inverse image of $H^*(\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3)^{B_2} = \mathbb{F}_2[x(x + z), z]$ under $\iota_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}^*$ which maps $x(x + y)$ to x^2 , y to 0 and z to z . We notice that $\iota_{\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3}^*(\mathbb{F}_2[x(x + y), y, z]) \cap \mathbb{F}_2[x(x + z), z] = \mathbb{F}_2[x^2(x + z)^2, z]$ and

that $\ker(\iota_{\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_3}^*) = \mathbb{F}_2[x(x+y), y, z]y$. Since $\mathbb{F}_2[x(x+y)(x+z)(x+y+z), y, z]$ is a subalgebra of $\mathbb{F}_2[x(x+y), y, z]$ that surjects onto $\mathbb{F}_2[x^2(x+z)^2, y, z]$ under $\iota_{\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_3}^*$, we get that

$$(\iota_{\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_3}^*)^{-1}(\mathbb{F}_2[x^2(x+z)^2, z]) = \mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+y)]y.$$

Therefore,

$$H^*(V_3)^{\mathcal{G}} = \mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+y)]y.$$

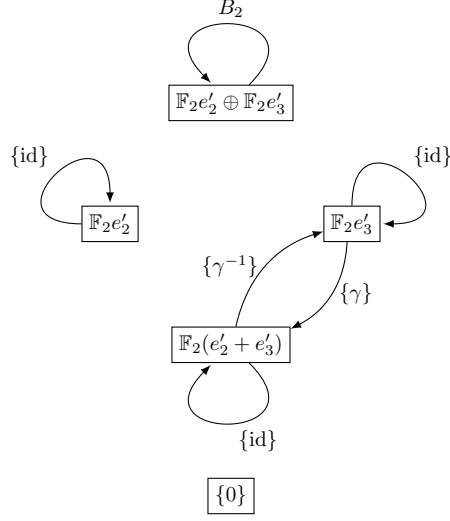
□

Remark 7.2. In Proposition 5.8, it is worth noticing that, for example, $\mathbb{F}_2[y, z, x(x+y)]$, $\mathbb{F}_2[y, z, x(x+z)]$ and $\mathbb{F}_2[y, z, x(x+y+z)]$ are conjugates. In particular, they are isomorphic. To get a minimal list of isomorphism classes of *nil*-closed, noetherian, integral, unstable algebras of transcendence degree 3, with a $\mathbb{F}_2[x]$ -comodule structure whose algebra of primitive elements is isomorphic to $\mathbb{F}_2[y, z]$, we should consider the conjugacy classes of the algebras found in Proposition 7.1. (See Remark 6.16.)

Proposition 7.3. *There are exactly 12 nil-closed, noetherian, unstable sub-algebras K of $H^*(V_3)$ of transcendence degree 3, which are sub $\mathbb{F}_2[x]$ -comodules of $H^*(V_3)$, for the comodule structure that maps x to $x \otimes 1 + 1 \otimes x$, y to $y \otimes 1$ and z to $z \otimes 1$, and such that the algebra of primitive elements of K is $\mathbb{F}_2[z, y(y+z)]$. These are :*

1. $\mathbb{F}_2[z, y(y+z), x(x+y)(x+z)(x+y+z)]$,
2. $\mathbb{F}_2[z, y(y+z), x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[z, y(y+z), x(x+y)(x+z)(x+y+z)](xz(x+z) + y^2(y+z))$,
3. $\mathbb{F}_2[z, y(y+z), x(x+z)]$,
4. $\mathbb{F}_2[y(y+z), x(x+y)(x+z)(x+y+z)] \oplus \mathbb{F}_2[z, y(y+z), x(x+z)]z$,
5. $\mathbb{F}_2[x, z, y(y+z)]$,
6. $\mathbb{F}_2[x, z, y(y+z)]z \oplus \mathbb{F}_2[y(y+z), x(x+y)(x+z)(x+y+z)]$,
7. $\mathbb{F}_2[x, z, y(y+z)]y(y+z) \oplus \mathbb{F}_2[z, x(x+z)]$,
8. $(\mathbb{F}_2[x, z, y(y+z)]y(y+z) \oplus \mathbb{F}_2[z, x(x+z)])z \oplus \mathbb{F}_2[y(y+z), x(x+y)(x+z)(x+y+z)]$,
9. $\mathbb{F}_2[z, (x+y), y(y+z)]$,
10. $\mathbb{F}_2[z, (x+y), y(y+z)]z \oplus \mathbb{F}_2[y(y+z), x(x+y)(x+z)(x+y+z)]$,
11. $\mathbb{F}_2[(x+y), z, y(y+z)]y(y+z) \oplus \mathbb{F}_2[z, (x+y)(x+y+z)]$,
12. $(\mathbb{F}_2[z, y(y+z), (x+y)]y(y+z) \oplus \mathbb{F}_2[z, (x+y)(x+y+z)])z \oplus \mathbb{F}_2[y(y+z), x(x+y)(x+z)(x+y+z)]$.

Proof. In this case, $\mathbb{F}_2[z, y(y+z)] = \pi^*(H^*(V_3/\mathbb{F}_2 e_1)^{B_2})$, for B_2 the group generated by the morphism b_2 which sends e'_2 to itself and e'_3 to $e'_2 + e'_3$, where e'_2 and e'_3 are the images of e_2 and e_3 under the canonical projection. Then, the groupoid $\mathcal{G}' = \mathfrak{g}(B_2)$ is the following:



Theorem 6.28 implies that, for $\beta \in \mathcal{G}(N, N')$, when $e_2 \in N$, $\pi \circ \beta(e_2) = e'_2$, when $e_3 \in N$, $\pi \circ \beta(e_3) = e'_3$ or e'_4 and when $e_4 \in N$, $\pi \circ \beta(e_4) = e'_3$ or e'_4 . In this case, G_{34} is not necessarily empty. We have to determine G , G_{22} , G_{33} , G_{44} and G_{34} (G_{43} being the set of inverses of morphisms in G_{34}). By the two conditions on $\beta \in G$, the matrix of β is in H , for

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Furthermore, since $b_2 \in \mathcal{G}'(\mathbb{F}_2e'_2 \oplus \mathbb{F}_2e'_3, \mathbb{F}_2e'_2 \oplus \mathbb{F}_2e'_3)$, by Theorem 6.28 there exist N and N' and $\beta \in \mathcal{G}(N, N')$ such that π induces isomorphisms from N and N' to $\mathbb{F}_2e'_2 \oplus \mathbb{F}_2e'_3$ and $b_2 \circ \pi = \pi \circ \beta$. Then, by Theorem 5.30, $\beta \in G$. Therefore G contains at least one element among

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

H is isomorphic to the dihedral group D_4 . It admits ten subgroups, among those the possible values of G are: $G_1 = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, $G_2 = \langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, $G_3 = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, $G_4 = \langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle$ and G_5 the full group.

For $\beta \in G_{ii}$, we necessarily have $\beta(e_1) = e_1$. By Theorem 6.28, β has a block matrix of the following form:

$$\begin{pmatrix} \text{id}_{\mathbb{F}_2e_1} & \hat{\beta} \\ 0 & \text{id}_N \end{pmatrix},$$

where $N \in \{\mathbb{F}_2e_2, \mathbb{F}_2e_3, \mathbb{F}_2(e_2 + e_3)\}$ and $\hat{\beta}$ is a morphism from N to \mathbb{F}_2e_1 .

Finally, G_{34} cannot be trivial, otherwise $(b_2)_{\mathbb{F}_2e_3} \notin \mathcal{G}'(\mathbb{F}_2e'_3, \mathbb{F}_2e'_4)$, and for $\beta \in G_{34}$, $\beta(e_1) = e_1$ and $\beta(e_3) = e_4$ or $e_1 + e_4$. We get that the G_{ii} with $i \in \{2, 3, 4\}$ is a subgroup of $H_{ii} := \{\text{id}, \beta_{ii}\}$, for β_{ii} the morphism which sends e_1 to itself and e_i to $e_1 + e_i$, and G_{34} is a non trivial subset of the set $H_{34} := \{\beta_1, \beta_2\}$ where β_1 and β_2 are the morphisms from $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3$ to $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_4$ whose matrix in the basis (e_1, e_3) and (e_1, e_4) are either $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

As in Proposition 7.1 (or rather its proof), some values of G imply, by restriction property, the maximality of some of the G_{ii} or the one of G_{34} . Furthermore, in this case since the groupoid admits a morphism between $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_3$ and $\mathbb{F}_2e_1 \oplus \mathbb{F}_2e_4$, for \mathcal{G} to be a groupoid we need the compositions of morphisms in G_{33} , G_{34} , G_{44} and G_{43} to be in \mathcal{G} . This is equivalent to requiring that G_{33} , G_{44} and G_{34} have the same cardinal.

G	G_{22}	G_{33}	G_{44}	G_{34}
G_1	$\{\text{id}\}$ or H_{22}	$\{\text{id}\}$ H_{33}	$\{\text{id}\}$ H_{44}	$\{\beta_1\}$ H_{34}
	$\{\text{id}\}$ or H_{22}	$\{\text{id}\}$ H_{33}	$\{\text{id}\}$ H_{44}	$\{\beta_2\}$ H_{34}
G_3	$\{\text{id}\}$ or H_{22}	H_{33}	H_{44}	H_{34}
G_4	H_{22}	H_{33}	H_{44}	H_{34}
G_5	H_{22}	H_{33}	H_{44}	H_{34}

We find 12 possible values for $(G, G_{22}, G_{33}, G_{44}, G_{34})$. Each one characterising precisely one $\mathcal{G} \in \text{Groupoid}(V_3)$ such that $H^*(V_3)^{\mathcal{G}}$ satisfies the required conditions. We leave the computations of the corresponding $H^*(V_3)^{\mathcal{G}}$ to the reader. □

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