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ϕ -FEM: an optimally convergent and easily implementable immersed boundary method for particulate flows and Stokes equations

Michel Duprez*, Vanessa Lleras[†], Alexei Lozinski[‡]

February 25, 2022

Abstract

We present an immersed boundary method to simulate the creeping motion of a solid in a fluid thanks to the Stokes equation and a FEM element strategy called ϕ -FEM using the description of the solid with a level-set function. One of the advantages of our method is the use of standard finite element spaces and classical integration tools. We obtain the optimal convergence of the method theoretically and numerically.

1 Introduction

The main goal of the present article is to demonstrate that the recently proposed ϕ -FEM methodology [12, 11, 10] is suitable for numerical simulation of incompressible viscous fluid flow past moving rigid bodies. This approach allows us to use simple (Cartesian) computational meshes, not evolving in time and not fitted to the moving rigid bodies, while achieving the optimal accuracy with classical finite element (FE) spaces of any order and performing the usual numerical integration on the whole mesh cells and facets, allowing for the use of standard FEM libraries for the implementation. We consider here only the creeping motion regime (zero Reynolds number), neglecting all the inertial terms in the equation governing both the fluid and the rigid bodies.

Numerical simulations of flows around moving rigid or elastic structures using immobile simple grids is a popular approach in, for instance, biomechanics, starting from the work of Peskin [22]. Different approaches have emerged since then, such as the Immersed Boundary method [18, 21], the Fictitious Domain method [15, 16], the penalty approximation [2], etc. All these classical methods suffer from poor accuracy due to the necessity to approximate the singularities near the fluid-solid interfaces which arise as the artefact of extending the fluid velocity field inside the solid domain. More recently, several optimally convergent fictitious domain-type methods have been proposed for the Stokes equations, which can also be used to simulate the fluid-solid motions. We cite in particular [8, 20, 17] following the CutFEM paradigm, and [1, 14] following the X-FEM paradigm. The common feature of all these methods is that they discretize the variational formulation of the Stokes equation on the physical fluid domain Ω using the FE spaces defined on the background mesh occupying a domain Ω_h , slightly larger than Ω . On

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the one hand, this permits to avoid a non-smooth extension of the solution outside its natural domain and to retrieve the optimal accuracy of the employed finite elements. On the other hand, this introduces the integrals on the cut cells into the FE scheme, i.e. the numerical integration should be performed on the portions of mesh cells, cut by the fluid-solid interface, making the methods difficult to implement.

The ϕ -FEM approach, which is the subject of the present paper, aims at combining the advantages of both classical Immersed Boundary/Fictitious Domain methods, and more recent CutFEM/X-FEM. Similarly to the former, ϕ -FEM does not need non-standard numerical integration on the cut cells; similarly to the latter, ϕ -FEM achieves the optimal accuracy of the finite elements employed. The general procedure of ϕ -FEM can be summarized as follows:

- Supposing that the physical domain Ω is given by a level set function $\Omega = \{\phi < 0\}$ and that it is embedded into a simple background mesh, introduce the *active* computational mesh \mathcal{T}_h by getting rid of the mesh cells lying completely outside Ω . The active mesh thus occupies a domain $\Omega_h \supset \Omega$, cf. Fig. 1, as in CutFEM/X-FEM.
- Extend the governing equations from Ω to Ω_h and write down a formal variational formulation on Ω_h without taking into account the boundary conditions on Γ (the relevant part of the boundary of Ω).
- Impose the boundary conditions on Γ using an appropriate ansatz or some additional variables, explicitly involving the level set ϕ which provides the link to the actual boundary. For instance, the homogeneous Dirichlet boundary conditions ($u = 0$ on Γ) can be imposed by the ansatz $u = \phi w$ thus reformulating the problem in terms of the new unknown w .
- Add appropriate stabilization, typically combining the ghost penalty [7] with a least square imposition of the governing equation on the mesh cells intersected by Γ , to guarantee coerciveness/stability on the discrete level.

This program has been successfully carried out for elliptic scalar PDEs with Dirichlet boundary conditions in [12] and for Neumann boundary conditions in [11]. Its feasibility is also demonstrated in [10] for the linear elasticity with mixed boundary conditions including the cases of internal interfaces between different materials or cracks, and for the heat equation. However, the adaptation to the equations governing the fluid flow around the moving particles is not straightforward. In particular, the following challenges are dealt with in the present article:

- The discrete inf-sup stability theory should be adapted to the case of a non-standard variational formulation of the Stokes equations posed on Ω_h rather than on Ω , and lacking the saddle-point structure. We shall show that this is possible by adapting the ghost penalty, which should be taken somewhat more complicated than in the case of scalar elliptic equations [12]. We shall do it here for Taylor-Hood finite elements of any order, but similar ideas should be also applicable to other classical inf-sup stable FE spaces.
- The motion equations for the solid particles involve the forces exerted on them by the surrounding fluid. These are defined through the integrals of some functions of fluid velocity and pressure on the particle boundary. However, the whole point of ϕ -FEM is to avoid such integrals. Indeed, the particle boundary is not resolved by the mesh, and our goal is to provide a method that necessitates the integration on the whole mesh cells or facets only. The way out of this paradox, pursued in the present paper, lies in providing a weak formulation of the governing equations, extended to Ω_h , that incorporates in an appropriate way the force balance equations, without stating them directly. This formulation is similar in spirit but different from that in [16].

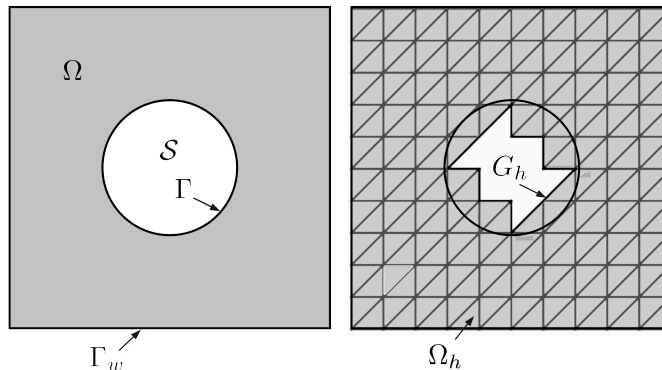


Figure 1: Left: an example of geometry for the fluid Ω with a solid \mathcal{S} inside; Right: the non-conforming active mesh \mathcal{T}_h on Ω_h with its internal boundary G_h .

We note that the method of this article shares some similarities to the shifted boundary method (SBM) proposed in [19] and analysed in [3] in the case of Stokes equations. In particular, SBM also gives an optimal accurate solution (at least with the lowest order finite elements) without introducing the integrals on the cut cells. It is however not evident how can one deal with the computation of the forces on the particles in the SBM framework.

The paper is structured as follows. In the next section, we properly introduce the governing equations, develop an appropriate weak formulation, discretize it (thus introducing our ϕ -FEM scheme), and announce the main theorem about the convergence of the scheme. Section 3 is devoted to the proof of this theorem. As a by-product, we also introduce a ϕ -FEM approach to discretize the Stokes equations alone (on a fixed geometry) on a non-fitted mesh. The details about this (comparatively simple) particular case are given in the Appendix. Finally, in section 4, we illustrate our theoretical results with numerical examples both for the Stokes equations and for the fluid/rigid particle motion problem. ϕ -FEM is also compared there with a standard FEM on fitted meshes, demonstrating the superiority of ϕ -FEM in terms of the accuracy achieved on comparable meshes.

2 Construction of the ϕ -FEM scheme and main results

2.1 Governing equations

We consider the motion of a viscous incompressible fluid around a solid (rigid) particle in the regime of creeping motion, i.e. neglecting all the inertial terms (for simplicity, we restrict ourselves here to the case of one particle, the extension to multiple particles being straight-forward). The particle is mobile and it moves under the action of the forces exerted by the surrounding fluid and the external forces (gravity). Let the fluid occupy (at a given time t) the domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3), the particle occupy the domain $\mathcal{S} \subset \mathbb{R}^d$, and denote $\mathcal{O} = \Omega \cup \mathcal{S}$. Let $\Gamma_w = \partial\mathcal{O}$ be the external boundary of the fluid domain (the immobile wall) where the fluid velocity is assumed to vanish, $\Gamma = \partial\mathcal{S}$ be the fluid/solid interface, and assume that Γ does not touch Γ_w , so that $\partial\Omega$ contains two disjoint components Γ_w and Γ . For simplicity, we assume that the only external body force is gravitation with the constant acceleration g . Hence, the body force density in the fluid is $\rho_f g$ where ρ_f is the constant fluid density. Let ρ_s be the constant density of the solid. Then, the resultant external force on the particle is mg where $m = \rho_s |\mathcal{S}|$ is the mass of the particle, and the resultant moment of the external force with respect to the barycenter of the

particle is 0. Denoting the constant fluid viscosity by ν , the equations governing the motion of the fluid/particle system can be now given as:

$$-2\nu \operatorname{div} D(u) + \nabla p = \rho_f g, \quad \text{in } \Omega \quad (1a)$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega \quad (1b)$$

$$u = U + \psi \times r, \quad \text{on } \Gamma \quad (1c)$$

$$u = 0, \quad \text{on } \Gamma_w \quad (1d)$$

$$-\int_{\Gamma} (2D(u) - pI)n + mg = 0 \quad (1e)$$

$$\int_{\Gamma} (2D(u) - pI)n \times r = 0 \quad (1f)$$

$$\int_{\Omega} p = 0 \quad (1g)$$

Here, the unknowns are the fluid velocity $u : \Omega \rightarrow \mathbb{R}^d$ and the pressure $p : \Omega \rightarrow \mathbb{R}$, the velocity of the particle barycenter $U \in \mathbb{R}^d$, and the angular velocity of the particle $\psi \in \mathbb{R}^{d'}$ ($d' = 1$ if $d = 2$ and $d' = 3$ if $d = 3$). In these equations, $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the strain tensor, r denotes the vector from the barycenter of the solid \mathcal{S} , and n denotes the unit normal on Γ looking into the solid. Equations (1e)–(1f) come from the balance of forces exerted on the particle (the force exerted by the fluid and the gravitational force).

From a numerical simulation perspective, it is natural to introduce an immobile computational mesh on the immobile box \mathcal{O} containing both the fluid and the particle. On the other hand, the solid \mathcal{S} will be moving with velocities $U = U(t)$, $\psi = \psi(t)$ at all-time t , thus permanently changing the shape of the fluid domain Ω . It is therefore interesting to design numerical methods for the system (1a)–(1g) that discretize u and p on a mesh non fitted to Ω .

2.2 A formal derivation of the appropriate weak formulation

Let $\mathcal{T}_h^{\mathcal{O}}$ be a regular simplicial mesh on \mathcal{O} (the background mesh). Introduce the active computational mesh \mathcal{T}_h as a submesh of $\mathcal{T}_h^{\mathcal{O}}$ covering Ω . Let $\Omega_h \supset \Omega$ be the domain of \mathcal{T}_h and G_h the part of the boundary of Ω_h intersecting the solid, cf. Fig. 1 (the actual definition of Ω_h is given in (8), based on an approximation to the levelset ϕ , and may occasionally result in situations where some tiny portions of Ω lie outside Ω_h , but this does not influence the forthcoming derivation of the FE scheme). Assume (on a formal level, just to derive the scheme) that u and p can be extended from Ω to Ω_h as solution to the Stokes equations so that

$$-2\nu \operatorname{div} D(u) + \nabla p = \rho_f g \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega_h.$$

Taking any sufficiently smooth test functions v and q on Ω_h such that $v = 0$ on Γ_w , an integration by parts gives

$$2\nu \int_{\Omega_h} D(u) : D(v) - \int_{\Omega_h} p \operatorname{div} v - \int_{\Omega_h} q \operatorname{div} u - \int_{G_h} (2\nu D(u) - pI)n \cdot v = \int_{\Omega_h} \rho_f g \cdot v. \quad (2)$$

Assuming $u = 0$ on Γ_w , this imposes already the boundary condition (1d) on Γ_w , which we suppose to fit to the mesh \mathcal{T}_h . On the contrary, this formulation does not take into account any boundary conditions on Γ . In order to incorporate boundary conditions (1c) we make the ansatz

$$u = \phi w + \chi(U + \psi \times r) \quad (3)$$

where ϕ is a level set function for the fluid domain $\Omega = \{\phi < 0\}$, and χ is a sufficiently smooth function on \mathcal{O} such that $\chi = 1$ on the solid \mathcal{S} and $\chi = 0$ on Γ_w . This introduces the new vector valued unknown w on Ω_h that should vanish on Γ_w (indeed $w = 0$ on Γ_w implies $u = 0$ on Γ_w thanks to the choice of χ ; in fact, the reason for χ is to decouple the boundary conditions on Γ and Γ_w from one another).

The test functions v in (2) can be represented in the same way as the solution (3):

$$v = \phi s + \chi(V + \omega \times r) \quad (4)$$

for all vector-valued functions s on Ω_h vanishing on Γ_w , and $V \in \mathbb{R}^d$, $\omega \in \mathbb{R}^{d'}$. In particular, the test functions of the form $\chi(V + \omega \times r)$ can be used to take into account the force balance (1e–1f). To this end, we introduce $B_h = \Omega_h \setminus \Omega$, i.e. the strip between Γ and G_h , and use the divergence theorem on B_h to transfer the boundary term in (2) from G_h to Γ where it can be evaluated by (1e–1f):

$$\begin{aligned} \int_{G_h} (2\nu D(u) - pI)n \cdot \chi(V + \omega \times r) &= \int_{G_h} (2\nu D(u) - pI)n \cdot (V + \omega \times r) \\ &= \int_{\Gamma} (2\nu D(u) - pI)n \cdot (V + \omega \times r) + \int_{B_h} \operatorname{div}(2\nu D(u) - pI)n \cdot (V + \omega \times r) \\ &= mg \cdot V - \int_{B_h} \rho_f g \cdot \chi(V + \omega \times r) = \dots \end{aligned} \quad (5)$$

We now remark $B_h = \Omega_h \setminus (\mathcal{O} \setminus \mathcal{S})$ to rewrite the above as

$$\begin{aligned} \dots &= - \int_{\Omega_h} \rho_f g \cdot \chi(V + \omega \times r) + \int_{\mathcal{O}} \rho_f g \cdot \chi(V + \omega \times r) - \int_{\mathcal{S}} \rho_f g \cdot (V + \omega \times r) + mg \cdot V \\ &= - \int_{\Omega_h} \rho_f g \cdot \chi(V + \omega \times r) + \int_{\mathcal{O}} \rho_f g \cdot \chi(V + \omega \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) mg \cdot V. \end{aligned} \quad (6)$$

The last line is justified by observing $\int_{\mathcal{S}} \rho_s g \cdot (V + \omega \times r) = mg \cdot V$ with ρ_s being the constant density of the solid.

Substituting the ansatzes (3)-(4) for u and v into (2) and rewriting the boundary term using (5)-(6) we arrive at the following formal variational formulation of our problem in terms of the new unknowns w, V, ψ : find $w : \Omega_h \rightarrow \mathbb{R}^d$ vanishing on Γ_w , $U \in \mathbb{R}^d$, $\psi \in \mathbb{R}^{d'}$, and $p : \Omega_h \rightarrow \mathbb{R}$ such that

$$\begin{aligned} 2\nu \int_{\Omega_h} D(\phi w + \chi(U + \psi \times r)) : D(\phi s + \chi(V + \omega \times r)) &- \int_{G_h} (2\nu D(\phi w + \chi(U + \psi \times r)) - pI)n \cdot \phi s \\ &- \int_{\Omega_h} p \operatorname{div}(\phi s + \chi(V + \omega \times r)) - \int_{\Omega_h} q \operatorname{div}(\phi w + \chi(U + \psi \times r)) \\ &= \int_{\Omega_h} \rho_f g \cdot \phi s + \int_{\mathcal{O}} \rho_f g \cdot \chi(V + \omega \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) mg \cdot V \end{aligned} \quad (7)$$

for all $s : \Omega_h \rightarrow \mathbb{R}^d$ vanishing on Γ_w , $V \in \mathbb{R}^d$, $\omega \in \mathbb{R}^{d'}$, and $q : \Omega_h \rightarrow \mathbb{R}$. In addition, the pressure p should satisfy the constraint (1g).

Note that the formulation above contains only the integrals on Ω_h , G_h , \mathcal{O} which can be easily approximated by quadrature rules on meshes \mathcal{T}_h and $\mathcal{T}_h^{\mathcal{O}}$. We can thus discretize using the usual finite elements for the trial and test functions, and approximating ϕ , χ by piecewise polynomials.

2.3 The ϕ -FEM scheme: discretization with Taylor-Hood finite elements

We fix an integer $k \geq 2$ and introduce the approximations ϕ_h and χ_h to the levelset ϕ and to the cut-off χ , given by the standard nodal interpolation to the continuous FE spaces of degree k on the mesh $\mathcal{T}_h^{\mathcal{O}}$. The active mesh \mathcal{T}_h and its domain Ω_h are actually defined as follows:

$$\mathcal{T}_h = \{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \{\phi_h < 0\} \neq \emptyset\}, \quad \Omega_h := (\cup_{T \in \mathcal{T}_h} T)^{\circ} \quad (8)$$

as illustrated in Fig. 1. We can now introduce the FE spaces for velocity and pressure on the mesh \mathcal{T}_h :

$$\mathcal{V}_h = \{v_h \in C(\bar{\Omega})^d : v_h|_T \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h, \quad v_h = 0 \text{ on } \Gamma_w\}$$

and

$$\mathcal{M}_h = \left\{ q_h \in C(\bar{\Omega}) : q_h|_T \in \mathbb{P}^{k-1}(T) \quad \forall T \in \mathcal{T}_h, \quad \int_{\Omega} q_h = 0 \right\}.$$

Remark 1. *Note that the definition of the pressure space involves an integral on Ω , which is incompatible with our ϕ -FEM framework since its whole point is to avoid the integrals on Ω and Γ . In practice, we shall rather impose $\int_{\Omega_h} q_h = 0$, introducing a mismatch in the additive pressure constant (which, any way, has no physical meaning) with respect to the exact solution satisfying (1g). We prefer however to keep the unimplementable constraint in the definition above to avoid some technical difficulties in theory. In practice, a special care will have to be taken in the interpretation of the error in pressure. We shall return to this technical point in the numerical results section.*

Moreover, we shall need the collections of the mesh cells \mathcal{T}_h^{Γ} and facets \mathcal{F}_h^{Γ} near the boundary Γ , as illustrated in Fig. 2, to include the appropriate stabilization into the FE scheme. More specifically, we introduce the submesh $\mathcal{T}_h^{\Gamma} \subset \mathcal{T}_h$ and the corresponding subdomain $\Omega_h^{\Gamma} \subset \Omega$ containing the mesh elements intersected by the approximate boundary

$$\Gamma_h = \{\phi_h = 0\},$$

i.e.

$$\mathcal{T}_h^{\Gamma} = \{T \in \mathcal{T}_h : T \cap \Gamma_h \neq \emptyset\}, \quad \Omega_h^{\Gamma} := \left(\cup_{T \in \mathcal{T}_h^{\Gamma}} T \right)^{\circ}. \quad (9)$$

Finally, we set \mathcal{F}_h^{Γ} as the collection of the interior facets of the mesh \mathcal{T}_h either cut by Γ_h or belonging to a cut mesh element

$$\mathcal{F}_h^{\Gamma} = \{E \text{ (an internal facet of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h^{\Gamma} \text{ and } E \in \partial T\}.$$

The stabilized scheme inspired by (7) can be now written as: find $w_h \in \mathcal{V}_h$, $U_h \in \mathbb{R}^d$, $\psi_h \in \mathbb{R}^{d'}$,

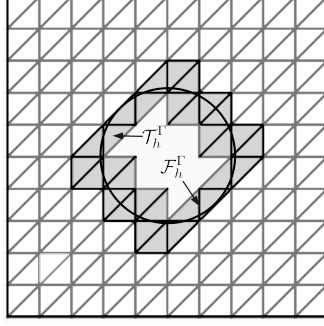


Figure 2: Example of \mathcal{T}_h^Γ and \mathcal{F}_h^Γ with the geometry given in Fig. 1 (left).

$p_h \in \mathcal{M}_h$ such that

$$\begin{aligned}
& 2\nu \int_{\Omega_h} D(\phi_h w_h + \chi_h(U_h + \psi_h \times r)) : D(\phi_h s_h + \chi_h(V_h + \omega_h \times r)) \\
& - \int_{G_h} (2\nu D(\chi_h(U_h + \psi_h \times r) + \phi_h w_h) - p_h I) n \cdot \phi_h s_h - \int_{\Omega_h} p_h \operatorname{div}(\phi_h s_h + \chi_h(V_h + \omega_h \times r)) \\
& \quad - \int_{\Omega_h} q_h \operatorname{div}(\phi_h w_h + \chi_h(U_h + \psi_h \times r)) \\
& \quad + \sigma_u J_u(\chi_h(U_h + \psi_h \times r) + \phi_h w_h, \chi_h(V_h + \omega_h \times r) + \phi_h s_h) \\
& + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (-\nu \Delta(\phi_h w_h + \chi_h(U_h + \psi_h \times r)) + \nabla p_h) \cdot (-\nu \Delta(\phi_h s_h + \chi_h(V_h + \omega_h \times r)) - \nabla q_h) \\
& \quad + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \operatorname{div}(\phi_h w_h + \chi_h(U_h + \psi_h \times r)) \operatorname{div}(\phi_h s_h + \chi_h(V_h + \omega_h \times r)) \\
& = \int_{\Omega_h} \rho_f g \cdot \phi_h s_h + \int_{\mathcal{O}} \rho_f g \cdot \chi_h(V_h + \omega_h \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) m g \cdot V_h \\
& \quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \rho_f g \cdot (-\nu \Delta(\phi_h s_h + \chi_h(V_h + \omega_h \times r)) - \nabla q_h) \quad (10)
\end{aligned}$$

for all $s_h \in \mathcal{V}_h$, $V_h \in \mathbb{R}^d$, $\omega_h \in \mathbb{R}^d$, $q_h \in \mathcal{M}_h$.

Here J_u is the ghost penalties for the velocity, cf. [7]:

$$J_u(u, v) = h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n u] \cdot [\partial_n v] + h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n^2 u] \cdot [\partial_n^2 v].$$

Note that, unlike [7, 8], we do not penalize the jumps of all the derivatives of the velocity; only the derivatives of order up to 2 are included in J_u . There is no penalization on the pressure either. We recall however that the version of the ghost penalty in ϕ -FEM for Poisson problem in [12] is even more reduced: only the jumps of the first order derivatives are penalized there. The inclusion of the second order derivatives in J_u in the present case of Stokes equations allows us to control both velocity and pressure in the forthcoming proofs, cf. Lemma 1.

2.4 Assumptions on the mesh and mains results

We begin with some assumptions on regularity of the mesh and we then state our main results on the numerical convergence of our method.

Assumption 1. *The boundary Γ can be covered by open sets \mathcal{O}_i , $i = 1, \dots, I$ and one can introduce on every \mathcal{O}_i local coordinates ξ_1, \dots, ξ_d with $\xi_d = \phi$ such that all the partial derivatives $\partial^\alpha \xi / \partial x^\alpha$ and $\partial^\alpha x / \partial \xi^\alpha$ up to order $k+1$ are bounded by some $C_0 > 0$. Thus, ϕ is of class C^{k+1} on \mathcal{O} .*

To formulate our next assumption, we introduce an extended band of mesh elements near the boundary Γ , namely the submesh $\mathcal{T}_h^{\Gamma, ext}$ with $\mathcal{T}_h^\Gamma \subset \mathcal{T}_h^{\Gamma, ext} \subset \mathcal{T}_h$ by adding to \mathcal{T}_h^Γ the cells which are neighbors and neighbors of neighbors of cells in \mathcal{T}_h^Γ .

Assumption 2. *$|\nabla \phi| \geq m$, $|\nabla \phi_h| \geq \frac{m}{2}$ on all the mesh cells in $\mathcal{T}_h^{\Gamma, ext}$, $|\phi| \geq mh$ on $\mathcal{T}_h \setminus \mathcal{T}_h^{\Gamma, ext}$, and $|\nabla \phi_h| \leq M$ on Ω_h with some $m, M > 0$.*

We will assume the following regularity on χ

Assumption 3. $\chi \in H^{k+1}(\mathcal{O})$, $\chi = 1$ on \mathcal{S} , $\chi = 0$ on Γ_w .

Assumption 4. *The approximate interface G_h can be covered by element patches $\{\Pi_k\}_{k=1, \dots, N_\Pi}$ having the following properties:*

- *Each Π_k is composed of a mesh element T_k lying inside Ω and some elements cut by Γ , more precisely $\Pi_k = T_k \cup \Pi_k^\Gamma$ where $T_k \in \mathcal{T}_h$, $T_k \subset \bar{\Omega}$, $\Pi_k^\Gamma \subset \mathcal{T}_h^\Gamma$, and Π_k^Γ contains at most N mesh elements;*
- *Each mesh element in a patch Π_k shares at least a facet with another mesh element in the same patch. In particular, T_k shares a facet F_k with an element in Π_k^Γ ;*
- $\mathcal{T}_h^\Gamma = \cup_{k=1}^{N_\Pi} \Pi_k^\Gamma$;
- Π_k and Π_l are disjoint if $k \neq l$.

Assumption 5. *Any mesh cell $T \in \mathcal{T}_h$ has at least d facets not lying on Γ_w .*

Remark 2. *The last assumption is usually required in the theoretical analysis of Taylor-Hood elements for the Stokes equation in the geometrically conforming setting [13], although it can be significantly relaxed, at least in the 2D setting [5]. Note that this assumption only affects the mesh near the outer wall Γ_w , more particularly in the corners of \mathcal{O} , which we treat in the standard geometrically conforming manner any way. It does not impose any further restriction on the active mesh \mathcal{T}_h near the interface Γ , where ϕ -FEM is effectively employed.*

Let us now state our main results:

Theorem 1. *Suppose that Assumptions 1–5 hold true and the mesh \mathcal{T}_h is quasi-uniform. Let $(u, U, \psi, p) \in H^{k+1}(\Omega)^d \times \mathbb{R}^d \times \mathbb{R}^{d'} \times H^k(\Omega)$ be the solution to (1a)-(1g) and $(w_h, U_h, \psi_h, p_h) \in \mathcal{V}_h \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{M}_h$ be the solution to (10). Denoting*

$$u_h := \chi_h(U_h + \psi_h \times r) + \phi_h w_h$$

it holds for $h \leq h_0$

$$|u - u_h|_{1, \Omega \cap \Omega_h} + |p - p_h|_{0, \Omega \cap \Omega_h} \leq Ch^k (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega}) \quad (11)$$

and

$$|U - U_h| + |\psi - \psi_h| \leq Ch^k (\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}) \quad (12)$$

with some $C > 0$ and $h_0 > 0$ depending on the parameters C_0, m, M, N in Assumptions 1–5, on the maximum of the derivatives of ϕ and χ of order up to $k+1$, on the mesh regularity, and on the polynomial degree k , but independent of h, f , and u .

Moreover, supposing $\Omega \subset \Omega_h$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2} (\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}) \quad (13)$$

with a constant $C > 0$ of the same type as above.

Remark 3. The numerical results in Section 4 suggest that convergence order for the particle velocity error and that for the L^2 -error of the fluid velocity is $k+1$. This suggests that both estimates (12) and (13) are not sharp. This is similar to our previous studies [12] and [11].

3 Proofs

From now on, we put the viscosity of the fluid to $\nu = 1$ to simplify the formulas. We shall also tacitly suppose that Assumptions 1–5 holds true.

This section is organized as follows: we start with some technical lemmas in Sections 3.1 and 3.2, essentially adapting the corresponding results from [12]. Note however that the interpolation error bound in Section 3.2 is sharper than its counterpart in [12]; it is now optimal with respect to the Sobolev norm expected from the interpolated function. This is followed by the proofs of Taylor-Hood inf-sup stability, the generalized coercivity of the bilinear form and finally the *a priori* error estimates, respectively, in Sections 3.3, 3.4 and 3.5, thus establishing Theorem 1.

3.1 Some technical lemmas.

Lemma 1. Let T be a triangle/tetrahedron, E one of its sides, v a vector-valued polynomial function on T , q a scalar-valued polynomial function on T such that

$$v = \frac{\partial v}{\partial n} = \frac{\partial^2 v}{\partial n^2} = 0 \text{ on } E \quad (14)$$

and

$$-\Delta v + \nabla q = 0 \text{ and } \operatorname{div} v = 0 \text{ on } T. \quad (15)$$

Then $v = 0$ and $q = \text{const}$ on T .

The proof of this lemma is inspired from the corresponding lemma in Dirichlet ϕ -FEM paper.

Proof. We shall give the proof only in the two dimensional setting, the generalization to the case $d = 3$ being straightforward. Without loss of generality, we can choose the Cartesian coordinates (x, y) such that the edge E lies on the x -axis. We shall denote the components of the vector-valued function v by (v_1, v_2) .

Let us write, for $k \in \{1, 2\}$,

$$v_k = \sum_{i,j \geq 0} v_{ij}^k x^i y^j \text{ and } q = \sum_{i,j \geq 0} q_{ij} x^i y^j.$$

We will prove by strong induction on m that

$$v_{ij}^k = 0 \quad \forall k \in \{1, 2\}, i \geq 0, j \in \{0, \dots, m\}. \quad (S_m)$$

Using (14), it holds

$$\sum_{i \geq 0} v_{i0}^k x^i = \sum_{i \geq 0} v_{i1}^k x^i = \sum_{i \geq 0} v_{i2}^k x^i = 0$$

for all x and $k \in \{1, 2\}$. Hence $v_{i0}^k = v_{i1}^k = v_{i2}^k = 0$ for all $i \geq 0$, $k \in \{1, 2\}$ and we obtain (S_2) . Assume that for a given $m \geq 2$, (S_m) holds true. Thanks to (15), one has

$$-(i+2)(i+1)v_{(i+2)j}^1 - (j+2)(j+1)v_{i(j+2)}^1 + (i+1)q_{(i+1)j} = 0, \quad (16)$$

$$-(i+2)(i+1)v_{(i+2)j}^2 - (j+2)(j+1)v_{i(j+2)}^2 + (j+1)q_{i(j+1)} = 0 \quad (17)$$

and

$$(i+1)v_{(i+1)j}^1 + (j+1)v_{i(j+1)}^2 = 0. \quad (18)$$

From (16) and (17), for each i, j

$$(i+2)v_{(i+2)(j+1)}^1 + \frac{(j+3)(j+2)}{i+1}v_{i(j+3)}^1 = \frac{(i+3)(i+2)}{j+1}v_{(i+3)j}^2 + (j+2)v_{(i+1)(j+2)}^2.$$

The last equality for $j = m - 2$ combined with (S_m) implies that

$$v_{i(m+1)}^1 = 0 \quad \forall i \geq 0.$$

Relation (18) for $j = m$ and (S_m) gives

$$v_{i(m+1)}^2 = 0 \quad \forall i \geq 0,$$

which leads to (S_{m+1}) . Thus $v = 0$. This also implies $\nabla q = 0$ on T thanks to (15). \square

Lemma 2. For any $\beta > 0$ and any integers $s, r \geq 1$ there exists $0 < \alpha < 1$ depending only on the mesh regularity and s, r such that for any continuous vector-valued \mathbb{P}^s FE function v_h on \mathcal{T}_h and any continuous scalar \mathbb{P}^r FE function q_h it holds

$$\begin{aligned} & \|D(v_h)\|_{0, \Omega_h^\Gamma}^2 + (1 - \alpha)h^2|q_h|_{1, \Omega_h^\Gamma}^2 \leq \alpha \|D(v_h)\|_{0, \Omega_h}^2 \\ & + \beta \left(h^2 \| -\Delta v_h + \nabla q_h \|_{0, \Omega_h^\Gamma}^2 + \|\operatorname{div} v_h\|_{0, \Omega_h^\Gamma}^2 + \sum_{E \in \mathcal{F}_h^\Gamma} (h \| [\partial_n v_h] \|_{0, E}^2 + h^3 \| [\partial_n^2 v_h] \|_{0, E}^2) \right). \end{aligned} \quad (19)$$

Proof. Thanks to Assumption 4, the boundary Γ can be covered by elements patches $\{\Pi_k\}_{k=1, \dots, N_\Pi}$. Take $\beta > 0$ and set

$$\alpha := \max_{\Pi_k, v_h, q_h} F(\Pi_k, v_h, q_h), \quad (20)$$

where

$$F(\Pi_k, v_h, q_h) = \frac{\|D(v_h)\|_{0, \Pi_k^\Gamma}^2 + h^2|q_h|_{1, \Pi_k^\Gamma}^2 - \beta G(\Pi_k, v_h, q_h)}{\|D(v_h)\|_{0, \Pi_k}^2 + h^2|q_h|_{1, \Pi_k^\Gamma}^2}$$

with

$$G(\Pi_k, v_h, q_h) = h^2 \| -\Delta v_h + \nabla q_h \|_{0, \Pi_k^\Gamma}^2 + \|\operatorname{div} v_h\|_{0, \Pi_k^\Gamma}^2 + h \| [\partial_n v_h] \|_{\mathcal{F}_k}^2 + h^3 \| [\partial_n^2 v_h] \|_{\mathcal{F}_k}^2.$$

The maximum in (20) is taken over all the continuous vector-valued \mathbb{P}^s FE functions v_h on Π_k , all the continuous scalar \mathbb{P}^r FE functions q_h on Π_k^Γ , such that the denominator in the expression for F does not vanish, and over all the possible configurations of patches Π_k satisfying Assumption

4. The notation \mathcal{F}_k stands for the set of mesh facets inside the patch Π_k which includes thus F_k separating T_k from Π_k^Γ and the other facets inside Π_k^Γ . The norm $\|\cdot\|_{\mathcal{F}_k}$ should be understood as $(\sum_{F \in \mathcal{F}_k} \|\cdot\|_F^2)^{1/2}$.

Since the maximized function F is invariant with respect to the transformation $x \mapsto \frac{1}{h}x$, $v_h \mapsto \frac{1}{h}v_h$, $q_h \mapsto q_h$, we can assume that $h = 1$ in (20). Furthermore $F(\Pi_k, v_h, q_h) = F(\Pi_k, \lambda v_h, \lambda q_h)$ for any $\lambda \neq 0$. Hence the maximum (20) is attained since it can be taken over all admissible patches with $h = 1$ and all v_h, q_h such that $\|D(v_h)\|_{0, \Pi_k}^2 + |q_h|_{1, \Pi_k^\Gamma}^2 = 1$, forming the unit sphere in the finite dimensional space of all (v_h, q_h) factored by rigid body motions on Π_k and constants on Π_k^Γ .

Clearly $\alpha \leq 1$. Let us prove by contradiction that $\alpha < 1$. Assume that $\alpha = 1$. Consider the patch Π_k (with $h = 1$) and v_h, q_h with $\|D(v_h)\|_{0, \Pi_k}^2 + |q_h|_{1, \Pi_k^\Gamma}^2 = 1$ on which the maximum (20) is attained. Then,

$$\|D(v_h)\|_{0, T_k}^2 + \beta G(\Pi_k, v_h, q_h) = 0,$$

since $\Pi_k = T_k \cup \Pi_k^\Gamma$. We deduce that $v_h = \frac{\partial v_h}{\partial n} = \frac{\partial^2 v_h}{\partial n^2} = 0$ on all the facets in \mathcal{F}_k and $-\Delta v_h + \nabla q_h = 0$, $\operatorname{div}(v_h) = 0$ on all $T \in \Pi_k^\Gamma$. Moreover, v_h is a rigid body motion on T_k . Let v_h^{rbm} be the rigid body motion velocity on \mathbb{R}^d coinciding with v_h on T_k . Thanks to Lemma 1 applied to $v_h - v_h^{rbm}$ and to q_h on the cells in Π_k^Γ starting from the cell adjacent in Π_k^Γ to T_k , we have $v_h = v_h^{rbm}$ on Π_k and $q_h = \text{const}$ on Π_k^Γ (recall that q_h is continuous). We have thus reached a contradiction with the assumptions $\|D(v_h)\|_{0, \Pi_k}^2 + |q_h|_{1, \Pi_k^\Gamma}^2 = 1$ and $\alpha = 1$.

This proves that there exists $\alpha < 1$ such that

$$\begin{aligned} \|D(v_h)\|_{0, \Pi_k^\Gamma}^2 + (1 - \alpha)h^2|q_h|_{1, \Pi_k^\Gamma}^2 &\leq \alpha\|D(v_h)\|_{0, \Pi_k}^2 \\ &+ \beta \left(h^2\|-\Delta v_h + \nabla q_h\|_{0, \Pi_k^\Gamma}^2 + \|\operatorname{div} v_h\|_{0, \Pi_k^\Gamma}^2 + h\|[\partial_n v_h]\|_{\mathcal{F}_k}^2 + h^3\|[\partial_n^2 v_h]\|_{\mathcal{F}_k}^2 \right) \end{aligned}$$

on all the patches Π_k and for all v_h, q_h . Summing this over all Π_k gives (19). \square

Lemma 3. For any $v \in H^1(\Omega)^d$ vanishing on Γ_w

$$|v|_{1, \Omega_h} \leq C\|D(v)\|_{0, \Omega_h}. \quad (21)$$

Proof. Since $v = 0$ on Γ_w , we have the following Korn inequality

$$|v|_{1, \Omega} \leq C\|D(v)\|_{0, \Omega} \quad (22)$$

with a constant $C > 0$ depending only on the shape of Ω , cf. [9, Theorem 6.3-4]. This implies

$$|v|_{1, \Omega_h^i} \leq C\|D(v)\|_{0, \Omega_h},$$

where Ω_h^i regroups the mesh cells inside Ω . Now, for any pair of mesh cells T, T' sharing a facet E , we can prove

$$|v|_{1, T} \leq C\|D(v)\|_{0, T} + C|v|_{1, T'} \quad (23)$$

with a constant C independent of h . Indeed, combining the Korn inequalities (22) and the trace theorem on the reference element leads to $|v|_{1, T} \leq C\|D(v)\|_{0, T} + C|v|_{1/2, E}$. Employing again the trace inequality on T' leads to (23).

Let $\Omega_h^{i,1}$ be Ω_h^i plus the cells which are not in Ω_h^i but have a neighbor in Ω_h^i . For any such cell T , we take T' as its neighbor in Ω_h^i , apply the estimate above, sum and compare with estimates yet above. This gives

$$|v|_{1, \Omega_h^{i,1}} \leq C\|D(v)\|_{0, \Omega_h}.$$

Let $\Omega_h^{i,2}$ be $\Omega_h^{i,1}$ plus the cells which are not in $\Omega_h^{i,1}$ but have a neighbor in $\Omega_h^{i,1}$. We have similar to above

$$|v|_{1,\Omega_h^{i,2}} \leq C\|D(v)\|_{0,\Omega_h}$$

and so on. After a finite number of steps, say k , we arrive at $\Omega_h^{i,k} = \Omega_h$. And

$$|v|_{1,\Omega_h} = |v|_{1,\Omega_h^{i,k}} \leq C\|D(v)\|_{0,\Omega_h}.$$

□

Lemma 4. For any $s_h \in \mathcal{V}_h$ and any $V_h \in \mathbb{R}^d, \omega_h \in \mathbb{R}^{d'}$,

$$\|\phi_h s_h\|_{0,\Omega_h^\Gamma} + \left(\sum_{F \in \mathcal{F}_h^\Gamma} h \|\phi_h s_h\|_{0,F}^2 \right)^{1/2} + \sqrt{h} \|\phi_h s_h\|_{0,G_h} \leq Ch \|D(\phi_h s_h + \chi_h(V_h + \omega_h \times r))\|_{0,\Omega_h}.$$

Proof. Take any $s_h \in \mathcal{V}_h, V_h \in \mathbb{R}^d, \omega_h \in \mathbb{R}^{d'}$ and denote $v_h = \phi_h s_h + \chi_h(V_h + \omega_h \times r)$. By [12, Lemma 3.4]

$$\begin{aligned} \|\phi_h s_h\|_{0,\Omega_h^\Gamma} &\leq Ch \|\phi_h s_h\|_{1,\Omega_h^\Gamma} \leq Ch (|v_h|_{1,\Omega_h^\Gamma} + |\chi_h(V_h + \omega_h \times r)|_{1,\Omega_h^\Gamma}) \\ &\leq Ch (|v_h|_{1,\Omega_h^\Gamma} + \|\chi_h\|_{1,\Omega_h^\Gamma} (|V_h| + |\omega_h|)). \end{aligned} \quad (24)$$

By equivalence of norms

$$|V_h| + |\omega_h| \leq C \|V_h + \omega_h \times r\|_{0,\Gamma}. \quad (25)$$

Denote by B_h^Γ the band between Γ and Γ_h . Applying the divergence theorem to the vector field $|V_h + \omega_h \times r|^2 \nabla \phi$ and noting that the normal on Γ (resp. Γ_h) is given by $\pm \frac{\nabla \phi}{|\nabla \phi|}$ (resp. $\pm \frac{\nabla \phi_h}{|\nabla \phi_h|}$) gives

$$\int_\Gamma |V_h + \omega_h \times r|^2 |\nabla \phi| \leq \int_{\Gamma_h} |V_h + \omega_h \times r|^2 \frac{|\nabla \phi \cdot \nabla \phi_h|}{|\nabla \phi_h|} + \left| \int_{B_h^\Gamma} \operatorname{div} (|V_h + \omega_h \times r|^2 \nabla \phi) \right|$$

We now note that $|\nabla \phi|$ (resp. $|\nabla \phi_h|$ and χ_h) are both positive and bounded away from 0 on Γ (resp. on Γ_h) uniformly in h for h small enough, and the measure of B_h^Γ is of order h^{k+1} . The inequality above implies thus

$$\|V_h + \omega_h \times r\|_{0,\Gamma}^2 \leq C (\|\chi_h(V_h + \omega_h \times r)\|_{0,\Gamma_h}^2 + h^{k+1} (|V_h| + |\omega_h|)^2).$$

Combining this with (25) gives, for h small enough

$$|V_h| + |\omega_h| \leq C \|\chi_h(V_h + \omega_h \times r)\|_{0,\Gamma_h} = C \|v_h\|_{0,\Gamma_h}$$

and, by the trace inequality,

$$|V_h| + |\omega_h| \leq C \|v_h\|_{1,\Omega_h^\Gamma}.$$

Substituting this into (24) and combining with the Korn inequality (21) yields the announced estimate for $\|\phi_h s_h\|_{0,\Omega_h^\Gamma}$ since $\|\chi_h\|_{1,\Omega_h^\Gamma}$ is bounded uniformly in h . The remaining part of the estimate follows by trace inverse inequalities as in [12, Lemma 3.5]. □

3.2 Interpolation by finite elements multiplied with the level set

We recall first a Hardy-type inequality, cf. [12].

Lemma 5. *For any integer $s \in [0, k]$ and any $u \in H^{s+1}(\Omega_h)$ vanishing on Γ , it holds $\left\| \frac{u}{\phi} \right\|_{s, \Omega_h} \leq C \|u\|_{s+1, \Omega_h}$ with $C > 0$ depending only on the constants in Assumption 1 and on s .*

This allows us to prove the following bound for interpolation by the products of finite elements with ϕ_h .

Lemma 6. *Let t be an integer $1 \leq t \leq k+1$. For any $v \in H^t(\Omega_h)^d \cap H_0^1(\Omega)$ there exists $w_h \in \mathcal{V}_h$ s.t.*

$$\|v - \phi_h w_h\|_{s, \Omega_h} \leq Ch^{t-s} \|v\|_{t, \Omega_h}, \quad s = 0, 1 \quad (26)$$

with $C > 0$ depending only on t, s , the constants in Assumptions 1–2, and the mesh regularity.

Proof. Let $v \in H^t(\Omega_h)^d$, $v = 0$ on Γ , and set $w = v/\phi$. Thanks to Lemma 5, $w \in H^{t-1}(\Omega_h)^d$ and $\|w\|_{t-1, \Omega_h} \leq C \|v\|_{t, \Omega_h}$. Consider $w_h = I_h^c w$, where I_h^c is a Scott-Zhang interpolation operator. For any $T \in \mathcal{T}_h$, let ω_T denote the patch of mesh cells adjacent to T (not necessarily all the adjacent cells) regrouping the cells affected by the construction of $I_h^c w$ on T , so that $I_h^c w$ on T depends on w only through its restriction to ω_T . The Scott-Zhang interpolation operator can be constructed so that $\omega_T \subset \mathcal{T}_h^{\Gamma, ext}$ for all $T \in \mathcal{T}_h^{\Gamma, ext}$, and $\omega_T \subset \mathcal{T}_h \setminus \mathcal{T}_h^{\Gamma, ext}$ for all $T \in \mathcal{T}_h \setminus \mathcal{T}_h^{\Gamma, ext}$. In what follows, we assume that the operator I_h^c enjoys this property together with the usual interpolation error estimates see for instance [6].

Our first goal is to prove (26) for $s = 0$. Taking any $T \in \mathcal{T}_h$. Recall that ϕ is supposed to be of class (at least) C^t so that $\|\phi - \phi_h\|_{\infty, T} \leq Ch^t$. Hence,

$$\begin{aligned} \|v - \phi_h w_h\|_{0, T} &= \|\phi w - \phi_h w_h\|_{0, T} \\ &\leq \|\phi\|_{\infty, T} \|w - w_h\|_{0, T} + \|\phi - \phi_h\|_{\infty, T} \|w_h\|_{0, T} \\ &\leq \|\phi\|_{\infty, T} \|w - w_h\|_{0, T} + Ch^t \|w\|_{0, \omega_T}. \end{aligned} \quad (27)$$

To continue this proof, we distinguish two cases: the cells $T \in \mathcal{T}_h^{\Gamma, ext}$ close to Γ_h and the remaining cells, which are at the distance of at least order h from Γ_h .

- (i) Consider $T \in \mathcal{T}_h^{\Gamma, ext}$. We have $\|\phi\|_{\infty, T} \leq Ch$ on these cells since they are at the distance $\sim h$ from Γ . Noting that $\|w - w_h\|_{0, T} \leq Ch^{t-1} |w|_{t-1, \omega_T}$ by the usual interpolation estimate, we derive from (27)

$$\|v - \phi_h w_h\|_{0, T} \leq Ch^t (|w|_{t-1, \omega_T} + \|w\|_{0, \omega_T}). \quad (28)$$

- (ii) Now consider $T \in \mathcal{T}_h \setminus \mathcal{T}_h^{\Gamma, ext}$. We note that ϕ does not vanish on ω_T for such T (recall that $\omega_T \subset \mathcal{T}_h \setminus \mathcal{T}_h^{\Gamma, ext}$), so that $w \in H^t(\omega_T)$ and, by (27) and the usual approximation estimates,

$$\|v - \phi_h w_h\|_{0, T} \leq Ch^t (\|\phi\|_{\infty, T} |w|_{t, \omega_T} + \|w\|_{0, \omega_T}). \quad (29)$$

In order to bound $|w|_{t, \omega_T}$ here, we note the Leibniz formula valid for any multi-index $\alpha \in \mathbb{N}^d$, $\alpha \neq 0$

$$\partial^\alpha w = \frac{1}{\phi} \partial^\alpha v - \sum_{\substack{\beta \in \mathbb{N}^d \\ \beta \leq \alpha, \beta \neq 0}} C_\alpha^\beta \frac{\partial^\beta \phi}{\phi} \partial^{\alpha-\beta} w \quad (30)$$

where C_α^β are the binomial coefficients depending only on the multi-indices α and β . Applying (30) to w on ω_T gives

$$|w|_{t,\omega_T} \leq \frac{1}{\min_{\omega_T} |\phi|} (|v|_{t,\omega_T} + C\|w\|_{t-1,\omega_T}).$$

Hence, by (29),

$$\|v - \phi_h w_h\|_{0,T} \leq Ch^t \left(\frac{\|\phi\|_{\infty,T}}{\min_{\omega_T} |\phi|} (|v|_{t,\omega_T} + \|w\|_{t-1,\omega_T}) + \|w\|_{0,\omega_T} \right). \quad (31)$$

Recall that $\min_{\omega_T} |\phi| \geq mh$ by Assumption 2. The distance between any point on T and any point on ω_T is at most $2h$ so that

$$\frac{\|\phi\|_{\infty,T}}{\min_{\omega_T} |\phi|} = 1 + \frac{\max_T |\phi| - \min_{\omega_T} |\phi|}{\min_{\omega_T} |\phi|} \leq 1 + \frac{2Mh}{mh} \leq 1 + \frac{2M}{m}$$

with M denoting an upper bound on $|\nabla\phi|$. Substituting this into (31) gives

$$\|v - \phi_h w_h\|_{0,T} \leq Ch^t (|v|_{t,\omega_T} + \|w\|_{t-1,\omega_T}). \quad (32)$$

Summing (28) over all the cells $T \in \mathcal{T}_h^{\Gamma,ext}$ and (32) over all the remaining cells of mesh \mathcal{T}_h gives

$$\|v - \phi_h w_h\|_{0,\Omega_h} \leq Ch^t (|v|_{t,\Omega_h} + \|w\|_{t-1,\Omega_h}).$$

This yields (26) with $s = 0$ thanks to the estimate $\|w\|_{t-1,\Omega_h} \leq \|v\|_{t,\Omega_h}$ given by Lemma 5.

Let us now prove (26) for $s = 1$. Introduce $v_h = I_h^c v$ so that

$$|v - v_h|_{1,\Omega_h} \leq Ch^{t-1} |v|_{t,\Omega_h} \quad \text{and} \quad \|v - v_h\|_{0,\Omega_h} \leq Ch^t |v|_{t,\Omega_h}.$$

Then, combining the already proven estimate (26) for $s = 0$, the inverse inequality, and the interpolation estimates above, we obtain

$$\begin{aligned} |v - \phi_h w_h|_{1,\Omega_h} &\leq |v_h - \phi_h w_h|_{1,\Omega_h} + |v - v_h|_{1,\Omega_h} \\ &\leq \frac{C}{h} \|v_h - \phi_h w_h\|_{0,\Omega_h} + |v - v_h|_{1,\Omega_h} \\ &\leq \frac{C}{h} \|v - \phi_h w_h\|_{0,\Omega_h} + \frac{C}{h} \|v - v_h\|_{0,\Omega_h} + |v - v_h|_{1,\Omega_h} \\ &\leq Ch^{t-1} \|v\|_{t,\Omega_h}. \end{aligned}$$

□

3.3 An adaptation of the Taylor-Hood inf-sup stability to ϕ -FEM

In this section, we prove some Taylor-Hood inf-sup conditions which will be used to establish the coercivity of our bilinear form in the next section. The proofs are inspired by [17].

Lemma 7. *There exists an h -independent constant $C > 0$ such that $\forall p_h \in \mathcal{M}_h, \exists w_h \in \mathcal{V}_h$ satisfying*

$$h^2 |p_h|_{1,\Omega_h}^2 - Ch^2 |p_h|_{1,\Omega_h^\Gamma}^2 \leq \int_{\Omega_h} \nabla p_h \cdot (\phi_h w_h) \quad \text{and} \quad |\phi_h w_h|_{1,\Omega_h} \leq Ch |p_h|_{1,\Omega_h}. \quad (33)$$

Proof. Let us introduce the submesh $\mathcal{T}_h^i = \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma$ and denote by \mathcal{E}_h^i the set of the edges of mesh \mathcal{T}_h^i including those shared with \mathcal{T}_h^Γ , but excluding those lying on Γ_w . For any edge $E \in \mathcal{E}_h^i$, let t_E be the unit tangent vector to E (any of two, but fixed), x_E be the midpoint of E , $\omega(E)$ be the set of the mesh cells sharing E , and ψ_E be the piecewise quadratic function such that $\psi_E(x_E) = 1$ and ψ_E vanishes at all the other edge midpoints and at all the nodes of \mathcal{T}_h . Moreover, defined for all $E \in \mathcal{E}_h^i$,

$$\bar{\phi}_E = \begin{cases} -h, & \text{if } E \text{ belongs to a cell from } \mathcal{T}_h^{\Gamma, ext}, \\ \frac{1}{|E|} \int_E \phi_h, & \text{otherwise.} \end{cases}$$

Taking any $p_h \in \mathcal{M}_h$ and set $w_h \in \mathcal{V}_h$ as

$$w_h = \sum_{E \in \mathcal{E}_h^i} \frac{h^2}{\bar{\phi}_E} \psi_E (t_E \cdot \nabla p_h) t_E. \quad (34)$$

We have indeed $w_h \in \mathcal{V}_h$, since the pressure tangential derivative $t_E \cdot \nabla p_h$ is a continuous piecewise polynomial of degree $\leq k-1$ on $\omega(E)$ and ψ_E is a continuous piecewise polynomial of degree 2, vanishing outside $\omega(E)$. Note also that $w_h = 0$ on Γ_w since \mathcal{E}_h^i does not contain the edges lying on Γ_w . It holds for any cell $T \in \omega(E) \cap \mathcal{T}_h^i$

$$\int_T \frac{\phi_h}{\bar{\phi}_E} \psi_E |t_E \cdot \nabla p_h|^2 \geq c \int_T |t_E \cdot \nabla p_h|^2. \quad (35)$$

Here and elsewhere, the constants $c > 0$ depend only on the polynomial degree k , the shape regularity, and the parameters of Assumption 2. This inequality can be proven by considering

$$\min_{T, \phi_h, q_h} \frac{\int_T \frac{\phi_h}{\bar{\phi}_E} \psi_E q_h^2}{\int_T q_h^2}$$

taken over all the simplexes T permitted by the mesh regularity, polynomials ϕ_h satisfying the restrictions of Assumption 2, and all the polynomials $q_h := t_E \cdot \nabla p_h$ of degree $k-2$. By homogeneity and rescaling, one can safely assume that $\text{diam } T = 1$ and $\left\| \frac{\phi_h}{\bar{\phi}_E} \right\|_{0,T} = \|q_h\|_{0,T} = 1$.

Additional constraints on ϕ_h are imposed differently in the two cases: if $T \in \mathcal{T}_h^{\Gamma, ext}$, then $-\phi_h \geq 0$ on T , and (after rescaling) $\left| \nabla \frac{\phi_h}{\bar{\phi}_E} \right| \geq \frac{m}{2}$; if $T \notin \mathcal{T}_h^{\Gamma, ext}$, then again $-\phi_h \geq 0$ on T , and $\frac{\max_T |\phi_h|}{\min_T |\phi_h|} = 1 + \frac{\max_T |\phi_h| - \min_T |\phi_h|}{\min_T |\phi_h|} \leq 1 + \frac{M}{m}$ so that $\frac{\phi_h}{\bar{\phi}_E}$ is uniformly bounded away from 0. In both cases, the set of possible ϕ_h is bounded and closed, so that the minimum above is indeed attained and is positive.

Thanks to (35), we have, setting $\Omega_h^i = \Omega_h \setminus \Omega_h^\Gamma$ and denoting by $\mathcal{E}(T)$ the set of edges of a cell T excluding the edges on Γ_w ,

$$\int_{\Omega_h^i} \phi_h w_h \cdot \nabla p_h = \sum_{E \in \mathcal{E}_h^i} h^2 \int_{\omega(E) \cap \Omega_h^i} \frac{\phi_h}{\bar{\phi}_E} \psi_E |t_E \cdot \nabla p_h|^2 \geq c \sum_{T \in \mathcal{T}_h^i} \sum_{E \in \mathcal{E}(T)} h^2 \int_T |t_E \cdot \nabla p_h|^2.$$

Taking into account Assumption 5, we have by scaling and the equivalence of norms on all $T \in \mathcal{T}_h^i$

$$\sum_{E \in \mathcal{E}(T)} \int_T |t_E \cdot \nabla p_h|^2 \geq c |p_h|_{1,T}^2.$$

Hence,

$$\int_{\Omega_h^i} \phi_h w_h \cdot \nabla p_h \geq c h^2 |p_h|_{1, \Omega_h^i}^2. \quad (36)$$

Now, on a mesh cell $T \in \mathcal{T}_h^\Gamma$ having common edges with cells of \mathcal{T}_h^i , definition (34) clearly gives

$$\left| \int_T \phi_h w_h \cdot \nabla p_h \right| = \left| \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h^i} h \int_T \phi_h \psi_E |t_E \cdot \nabla p_h|^2 \right| \leq Ch^2 |p_h|_{1,T}^2$$

since ϕ_h is of order h on such cells. Combining this with (36) gives

$$\begin{aligned} ch^2 |p_h|_{1,\Omega_h}^2 &\leq \int_{\Omega_h} \phi_h w_h \cdot \nabla p_h - \int_{\Omega_h^\Gamma} \phi_h w_h \cdot \nabla p_h + ch^2 |p_h|_{1,\Omega_h^\Gamma}^2 \\ &\leq \int_{\Omega_h} \phi_h w_h \cdot \nabla p_h + Ch^2 |p_h|_{1,\Omega_h^\Gamma}^2. \end{aligned} \quad (37)$$

We also have

$$\begin{aligned} |\phi_h w_h|_{1,\Omega_h}^2 &= \int_{\Omega_h} |(\nabla \phi_h) w_h + \phi_h (\nabla w_h)|^2 \\ &\leq C \sum_{T \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h^i} h^4 \int_T \left(\frac{|\nabla \phi_h|^2}{\bar{\phi}_E^2} |\nabla p_h|^2 + \frac{\phi_h^2}{\bar{\phi}_E^2} |\nabla \psi_E|^2 |\nabla p_h|^2 + \frac{\phi_h^2}{\bar{\phi}_E^2} |\nabla^2 p_h|^2 \right) \\ &\leq C \sum_{T \in \mathcal{T}_h} h^2 \int_T |\nabla p_h|^2 = Ch^2 |p_h|_{1,\Omega_h}^2. \end{aligned} \quad (38)$$

We have used here the finite element inverse estimates on ϕ_h , ψ_E , p , and a uniform upper bound on $\frac{\max_T |\phi_h|}{|\bar{\phi}_E|}$ that can be established similarly to (35).

Redefining w_h as w_h/c with the constant c from (37) shows (33) as a combination of (37) and (38). \square

Lemma 8. *There exists an h -independent constant $C > 0$ such that $\forall p_h \in \mathcal{M}_h \exists s_h \in \mathcal{V}_h$*

$$\|p_h\|_{0,\Omega_h}^2 - Ch^2 |p_h|_{1,\Omega_h^\Gamma}^2 \leq \int_{\Omega_h} \nabla p_h \cdot (\phi_h s_h) \quad \text{and} \quad |\phi_h s_h|_{1,\Omega_h} \leq C \|p_h\|_{0,\Omega_h}. \quad (39)$$

Proof. Taking $p_h \in \mathcal{M}_h$. By continuous velocity-pressure inf-sup (recall that $\int_\Omega p_h = 0$), there exists $v \in H_0^1(\Omega)$ s.t.

$$\operatorname{div} v = -p_h \text{ on } \Omega, \text{ and } \|v\|_{1,\Omega} \leq C \|p_h\|_{0,\Omega}.$$

Let $\tilde{v} \in H^1(\Omega_h)$ be the extension of v by 0 outside Ω . Lemma 6 with $t = 1$ implies $\exists v_h \in \mathcal{V}_h$ s.t.

$$\|\tilde{v} - \phi_h v_h\|_{0,\Omega_h} \leq Ch \|\tilde{v}\|_{1,\Omega_h} \leq Ch \|p_h\|_{0,\Omega_h} \quad \text{and} \quad |\phi_h v_h|_{1,\Omega_h} \leq C \|\tilde{v}\|_{1,\Omega_h} \leq C \|p_h\|_{0,\Omega_h}.$$

Thus,

$$\begin{aligned} \|p_h\|_{0,\Omega}^2 &= - \int_\Omega p_h \operatorname{div} v = \int_\Omega \nabla p_h \cdot \tilde{v} = \int_{\Omega_h} \nabla p_h \cdot (\phi_h v_h) + \int_{\Omega_h} \nabla p_h \cdot (\tilde{v} - \phi_h v_h) \\ &\leq \int_{\Omega_h} \nabla p_h \cdot (\phi_h v_h) + Ch |p_h|_{1,\Omega_h} \|p_h\|_{0,\Omega_h}. \end{aligned} \quad (40)$$

By an argument similar to that used in the proof of Lemma 2, we have

$$\|p_h\|_{0,\Omega_h}^2 \leq C_1 (\|p_h\|_{0,\Omega}^2 + h^2 |p_h|_{1,\Omega_h^\Gamma}^2).$$

We can thus pass from the norm on Ω to that on Ω_h in (40):

$$\|p_h\|_{0,\Omega_h}^2 \leq C_1 \int_{\Omega_h} \nabla p_h \cdot (\phi_h v_h) + Ch|p_h|_{1,\Omega_h} \|p_h\|_{0,\Omega_h} + C_1 h^2 |p_h|_{1,\Omega_h^\Gamma}^2.$$

Hence, by Young inequality,

$$\begin{aligned} \frac{1}{2} \|p_h\|_{0,\Omega_h}^2 &\leq C_1 \int_{\Omega_h} \nabla p_h \cdot (\phi_h v_h) + C_2 h^2 |p_h|_{1,\Omega_h}^2 + C_1 h^2 |p_h|_{1,\Omega_h^\Gamma}^2 \\ &\leq C_1 \int_{\Omega_h} \nabla p_h \cdot (\phi_h v_h) + C_2 \int_{\Omega_h} \nabla p_h \cdot (\phi_h w_h) + Ch^2 |p_h|_{1,\Omega_h^\Gamma}^2 \end{aligned}$$

with w_h given by Lemma 7. Thus,

$$\frac{1}{2} \|p_h\|_{0,\Omega_h}^2 - Ch^2 |p_h|_{1,\Omega_h^\Gamma}^2 \leq \int_{\Omega_h} \nabla p_h \cdot \phi_h (C_1 v_h + C_2 w_h) \quad \text{and} \quad |\phi_h (C_1 v_h + C_2 w_h)|_{1,\Omega_h} \leq C \|p_h\|_{0,\Omega_h}$$

since $|\phi_h w_h|_{1,\Omega_h} \leq Ch|p_h|_{1,\Omega_h} \leq C \|p_h\|_{0,\Omega_h}$ by inverse finite element estimates.

Setting $s_h = 2(C_1 v_h + C_2 w_h)$ proves (39). \square

3.4 The generalized coercivity (the inf-sup condition) for the bilinear form.

To ease the forthcoming calculations, let us introduce the finite element space of velocities combining the rigid body motion on the approximate boundary and the contributions involving the level set:

$$\mathcal{V}_h^{rbm} = \{\chi_h(V_h + \omega_h \times r) + \phi_h s_h \text{ with } s_h \in \mathcal{V}_h, V_h \in \mathbb{R}^d, \omega_h \in \mathbb{R}^{d'}\}. \quad (41)$$

In the scheme (10), we shall now combine the test functions s_h, V_h, ω_h into $v_h \in \mathcal{V}_h^{rbm}$ as in the definition above. Similarly, we shall combine the trial functions w_h, U_h, ψ_h into $u_h \in \mathcal{V}_h^{rbm}$ setting $u_h = \chi_h(U_h + \psi_h \times r) + \phi_h w_h$. Scheme (10) can be then rewritten in the compact form: find $u_h \in \mathcal{V}_h^{rbm}$ and $p_h \in \mathcal{M}_h$ such that

$$c_h(u_h, p_h; v_h, q_h) = L_h(v_h, q_h), \quad \forall v_h \in \mathcal{V}_h^{rbm}, q_h \in \mathcal{M}_h, \quad (42)$$

where the bilinear form c_h is given by

$$\begin{aligned} c_h(u_h, p_h; v_h, q_h) &= 2 \int_{\Omega_h} D(u_h) : D(v_h) - \int_{G_h} (2D(u_h) - p_h I) n \cdot \phi_h s_h \\ &\quad - \int_{\Omega_h} q_h \operatorname{div} u_h - \int_{\Omega_h} p_h \operatorname{div} v_h \\ &\quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (-\Delta u_h + \nabla p_h) \cdot (-\Delta v_h - \nabla q_h) + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\operatorname{div} u_h)(\operatorname{div} v_h) \\ &\quad + \sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n u_h] \cdot [\partial_n v_h] + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n^2 u_h] \cdot [\partial_n^2 v_h] \end{aligned}$$

and the linear form L_h is given by

$$\begin{aligned} L_h(v_h, q_h) &= \int_{\Omega_h} \rho_f g \cdot \phi_h s_h + \int_{\mathcal{O}} \rho_f g \cdot \chi_h (V_h + \omega_h \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) mg \cdot V_h \\ &\quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \rho_f g \cdot (-\Delta v_h - \nabla q_h). \quad (43) \end{aligned}$$

In both expressions above, s_h , V_h , and ω_h are related to $v_h \in \mathcal{V}_h^{rbm}$ as in (41).

Lemma 9. *Introduce the norm on $\mathcal{V}_h^{rbm} \times \mathcal{M}_h$*

$$\| \| v_h, q_h \| \|_h := \left(|v_h|_{1,\Omega_h}^2 + \|q_h\|_{0,\Omega_h}^2 + h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \| -\Delta v_h + \nabla q_h \|_{0,T}^2 + J_u(v_h, v_h) \right)^{1/2}.$$

The following inf-sup condition holds

$$\forall (u_h, p_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h \quad \exists (v_h, q_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h$$

such that

$$\frac{c_h(u_h, p_h; v_h, q_h)}{\| \| v_h, q_h \| \|_h} \geq \theta \| \| u_h, p_h \| \|_h \quad (44)$$

with a constant $\theta > 0$ depending only on the mesh regularity.

Proof. Let us take $p_h \in \mathcal{M}_h$ and $u_h = \chi_h(U_h + \psi_h \times r) + \phi_h w_h \in \mathcal{V}_h^{rbm}$ with $w_h \in \mathcal{V}_h$, $U_h \in \mathbb{R}^d$ and $\psi_h \in \mathbb{R}^{d'}$.

Step 1: controlling the velocity. By choosing (u_h, p_h) as the trial function and $(u_h, -p_h)$ as the test function in the bilinear form c_h , we obtained:

$$\begin{aligned} c_h(u_h, p_h; u_h, -p_h) &= 2 \int_{\Omega_h} |D(u_h)|^2 - \int_{G_h} (2D(u_h) - p_h I) n \cdot \phi_h w_h \\ &\quad + \sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E |[\partial_n u_h]|^2 + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E |[\partial_n^2 u_h]|^2 \\ &\quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T |-\Delta u_h + \nabla p_h|^2 + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T |\operatorname{div} u_h|^2. \end{aligned} \quad (45)$$

Let \tilde{B}_h be the strip between $\Gamma_h = \{\phi_h = 0\}$ and G_h , i.e. $\tilde{B}_h = \{\phi_h > 0\} \cap \Omega_h$. Since $\phi_h w_h = 0$ on Γ_h ,

$$\begin{aligned} \int_{G_h} D(u_h) n \cdot \phi_h w_h &= \int_{\partial \tilde{B}_h} D(u_h) n \cdot \phi_h w_h \\ &= \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\partial(\tilde{B}_h \cap T)} D(u_h) n_T \cdot \phi_h w_h - \sum_{T \in \mathcal{T}_h^\Gamma} \sum_{E \in \mathcal{F}_h^{cut}(T)} \int_{\tilde{B}_h \cap E} D(u_h) n_T \cdot \phi_h w_h, \end{aligned}$$

where \mathcal{T}_h^Γ is defined in (9), $\mathcal{F}_h^{cut}(T)$ regroups the facets of a mesh element T cut by Γ_h , and n_T is the unit normal pointing outside of T on the boundary of a mesh cell T . Applying the divergence theorem to the integrals on $\partial(\tilde{B}_h \cap T)$ and regrouping the integrals on the facets gives

$$\begin{aligned} \int_{G_h} D(u_h) n \cdot \phi_h w_h &= \int_{\tilde{B}_h} D(u_h) : D(\phi_h w_h) + \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\tilde{B}_h \cap T} \operatorname{div} D(u_h) \cdot \phi_h w_h \\ &\quad - \sum_{E \in \mathcal{F}_h^\Gamma} \int_{E \cap \tilde{B}_h} [D(u_h) n] \cdot \phi_h w_h \\ &= \int_{\tilde{B}_h} |D(u_h)|^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\tilde{B}_h \cap T} \operatorname{div} D(u_h) \cdot \phi_h w_h - \sum_{E \in \mathcal{F}_h^\Gamma} \int_{E \cap \tilde{B}_h} [D(u_h) n] \cdot \phi_h w_h, \end{aligned}$$

since $u_h - \phi_h w_h$ is the velocity of a rigid motion on \tilde{B}_h .

Similarly (and simpler)

$$\int_{G_h} p_h n \cdot \phi_h w_h = \int_{\partial \tilde{B}_h} p_h n \cdot \phi_h w_h = \int_{\tilde{B}_h} p_h \operatorname{div} u_h + \int_{\tilde{B}_h} \nabla p_h \cdot \phi_h w_h.$$

Substituting this into (45) yields

$$\begin{aligned} c_h(u_h, p_h; u_h, -p_h) &= 2 \int_{\Omega_h} |D(u_h)|^2 - 2 \int_{\tilde{B}_h} |D(u_h)|^2 - \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\tilde{B}_h \cap T} (2 \operatorname{div} D(u_h) - \nabla p_h) \cdot \phi_h w_h \\ &\quad + 2 \sum_{F \in \mathcal{F}_h^\Gamma} \int_{F \cap \tilde{B}_h} [D(u_h)n] \cdot \phi_h w_h + \int_{\tilde{B}_h} p_h \operatorname{div} u_h \\ &\quad + \sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n u_h]^2 + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E |[\partial_n^2 u_h]|^2 \\ &\quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T |\Delta u_h - \nabla p_h|^2 + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T |\operatorname{div} u_h|^2. \end{aligned}$$

With several Young inequalities, the estimates of Lemma 4, and observing that

$$2 \operatorname{div} D(u_h) - \nabla p_h = \Delta u_h - \nabla p_h + \nabla \operatorname{div} u_h$$

on any cell T , we arrive at

$$\begin{aligned} c_h(u_h, p_h; u_h, -p_h) &\geq \left(2 - C \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{2}\right) \|D(u_h)\|_{0, \Omega_h}^2 - 2 \|D(u_h)\|_{0, \Omega_h^\Gamma}^2 - \frac{\varepsilon_3}{2} \|p_h\|_{0, \Omega_h^\Gamma}^2 \\ &\quad + h^2 \left(\sigma - \frac{1}{2\varepsilon_1}\right) \sum_{T \in \mathcal{T}_h^\Gamma} \|\Delta u_h - \nabla p_h\|_{0, T}^2 + \left(\sigma - \frac{1}{2\varepsilon_3}\right) \|\operatorname{div} u_h\|_{0, \Omega_h^\Gamma}^2 \\ &\quad + h \left(\sigma_u - \frac{1}{2\varepsilon_2}\right) \sum_{E \in \mathcal{F}_h^\Gamma} \|\partial_n u_h\|_{0, E}^2 + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n^2 u_h]\|_{0, E}^2 \end{aligned}$$

valid for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. Thanks to Lemma 2, this can be further bounded as

$$\begin{aligned} c_h(u_h, p_h; u_h, -p_h) &\geq \left(2(1 - \alpha) - C \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{2}\right) \|D(u_h)\|_{0, \Omega_h}^2 + 2(1 - \alpha) h^2 |p_h|_{1, \Omega_h^\Gamma}^2 \\ &\quad - \frac{\varepsilon_3}{2} \|p_h\|_{0, \Omega_h^\Gamma}^2 + h^2 \left(\sigma - \frac{1}{2\varepsilon_1} - 2\beta\right) \sum_{T \in \mathcal{T}_h^\Gamma} \|\Delta u_h - \nabla p_h\|_{0, T}^2 \\ &\quad + \left(\sigma - \frac{1}{2\varepsilon_3} - 2\beta\right) \|\operatorname{div} u_h\|_{0, \Omega_h^\Gamma}^2 + h \left(\sigma_u - \frac{1}{2\varepsilon_2} - 2\beta\right) \sum_{E \in \mathcal{F}_h^\Gamma} \|\partial_n u_h\|_{0, E}^2 \\ &\quad + h^3 (\sigma_u - 2\beta) \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n^2 u_h]\|_{0, E}^2 \quad (46) \end{aligned}$$

with some $\beta > 0$ and $\alpha \in (0, 1)$.

Step 2: controlling the pressure. Let now $s_h \in \mathcal{V}_h$ be the function given by Lemma 8 and set $v_h^p = \phi_h s_h$. Noting that

$$-\int_{\Omega_h} p_h \operatorname{div} v_h^p + \int_{G_h} p_h n \cdot v_h^p = \int_{\Omega_h} \nabla p_h \cdot v_h^p \geq \|p_h\|_{0, \Omega_h}^2 - Ch^2 |p_h|_{1, \Omega_h^\Gamma}^2$$

we get

$$\begin{aligned}
c_h(u_h, p_h; v_h^p, 0) &\geq \|p_h\|_{0, \Omega_h}^2 - Ch^2 |p_h|_{1, \Omega_h^\Gamma}^2 + 2 \int_{\Omega_h} D(u_h) : D(v_h^p) - 2 \int_{G_h} D(u_h) n \cdot v_h^p \\
&\quad + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\Delta u_h - \nabla p_h) \cdot \Delta v_h^p + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\operatorname{div} u_h) (\operatorname{div} v_h^p) \\
&\quad + \sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n u_h] \cdot [\partial_n v_h^p] + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n^2 u_h] \cdot [\partial_n^2 v_h^p].
\end{aligned}$$

Recalling that $|v_h^p|_{1, \Omega_h} \leq C \|p_h\|_{0, \Omega_h}$, remarking that $\|v_h^p\|_{0, G_h} \leq \frac{C}{\sqrt{h}} |v_h^p|_{1, \Omega_h^\Gamma}$, and applying Young and inverse inequalities allows us to conclude

$$\begin{aligned}
c_h(u_h, p_h; v_h^p, 0) &\geq \frac{1}{2} \|p_h\|_{0, \Omega_h}^2 - Ch^2 |p_h|_{1, \Omega_h^\Gamma}^2 - C \|D(u_h)\|_{0, \Omega_h}^2 \\
&\quad - C \left(\sigma^2 h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \|-\Delta u_h + \nabla p_h\|_{0, T}^2 + \sigma^2 \sum_{T \in \mathcal{T}_h^\Gamma} \|\operatorname{div} u_h\|_{0, T}^2 \right. \\
&\quad \left. + \sigma_u^2 h \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n u_h]\|_{0, E}^2 + \sigma_u^2 h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n^2 u_h]\|_{0, E}^2 \right). \quad (47)
\end{aligned}$$

Step 3: combining the estimates. Multiply (47) by $\lambda > 0$ and add it to (46). This gives

$$\begin{aligned}
c_h(u_h, p_h; u_h + \lambda v_h^p, -p_h) &\geq \left(2(1 - \alpha) - C \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{2} - C\lambda \right) \|D(u_h)\|_{0, \Omega_h}^2 \\
&\quad + \frac{\lambda - \varepsilon_3}{2} \|p_h\|_{0, \Omega_h}^2 + (2(1 - \alpha) - C\lambda) h^2 |p_h|_{1, \Omega_h^\Gamma}^2 \\
&\quad + h^2 \left(\sigma - \frac{1}{2\varepsilon_1} - 2\beta - C\lambda\sigma^2 \right) \sum_{T \in \mathcal{T}_h^\Gamma} \|\Delta u_h - \nabla p_h\|_{0, T}^2 + \left(\sigma - \frac{1}{2\varepsilon_3} - 2\beta - C\lambda\sigma^2 \right) \|\operatorname{div} u_h\|_{0, \Omega_h^\Gamma}^2 \\
&\quad + h \left(\sigma_u - \frac{1}{2\varepsilon_2} - 2\beta - C\lambda\sigma_u^2 \right) \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n u_h]\|_{0, E}^2 + (\sigma_u - 2\beta - C\lambda\sigma_u^2) h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \|[\partial_n^2 u_h]\|_{0, E}^2.
\end{aligned}$$

Taking $\lambda, \varepsilon_1, \varepsilon_2$ small enough, $\varepsilon_3 = \lambda/2$ and σ, σ_u big enough, and recalling Korn inequality (21), this amounts to

$$c_h(u_h, p_h; w_h + \lambda v_h^p, -p_h) \geq c \| \| u_h, p_h \| \|_h^2$$

with some $c > 0$. We also have easily

$$\| \| w_h + \lambda v_h^p, -p_h \| \|_h \leq C \| \| w_h, p_h \| \|_h,$$

hence the inf-sup estimate (44) with $v_h = u_h + \lambda v_h^p$, $q_h = -p_h$. \square

3.5 *A priori* error estimates.

In this section, we will prove Theorem 1 following the argumentation of [12], which is ameliorated since we require only the optimal regularity $H^{k+1}(\Omega)^d \times H^k(\Omega)$ for the velocity-pressure pair (u, p) given by (1a)-(1g).

Proof of the H^1 a priori error estimate (11): Let $(u, p) \in H^{k+1}(\Omega)^d \times H^k(\Omega)$ with $u = U + \psi \times r$ on Γ be the solution to the continuous problem (1a)-(1g) and $(u_h, p_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h$ with $u_h = \phi_h w_h + \chi_h(U_h + \psi_h \times r)$ be the solution to the discrete problem (10). Choose sufficiently smooth extension \tilde{u} and \tilde{p} of u and p on Ω_h such that $\tilde{u} = u$, $\tilde{p} = p$ on Ω , and

$$\|\tilde{u}\|_{k+1, \Omega_h} \leq C\|u\|_{k+1, \Omega}, \quad \|\tilde{p}\|_{k, \Omega_h} \leq C\|p\|_{k, \Omega}.$$

Applying Lemma 6 to $\tilde{u} - \chi(U + \psi \times r)$, which vanishes on Γ and on Γ_w , we see that there exists $\tilde{w}_h \in \mathcal{V}_h$ such that

$$\|\tilde{u} - \chi(U + \psi \times r) - \phi_h \tilde{w}_h\|_{1, \Omega_h} \leq Ch^k \|\tilde{u} - \chi(U + \psi \times r)\|_{k+1, \Omega_h}.$$

This allows us to introduce $\tilde{u}_h = \phi_h \tilde{w}_h + \chi_h(U + \psi \times r) \in \mathcal{V}_h^{rbm}$ satisfying

$$\begin{aligned} \|\tilde{u} - \tilde{u}_h\|_{1, \Omega_h} &\leq \|\tilde{u} - \chi(U + \psi \times r) - \phi_h \tilde{w}_h\|_{1, \Omega_h} + \|(\chi - \chi_h)(U + \psi \times r)\|_{1, \Omega_h} \\ &\leq Ch^k (\|\tilde{u}\|_{k+1, \Omega_h} + \|\chi\|_{k+1, \Omega_h} (|U| + |\psi|)) \\ &\leq Ch^k \|\tilde{u}\|_{k+1, \Omega_h} \leq Ch^k \|u\|_{k+1, \Omega}, \end{aligned} \quad (48)$$

thanks to the standard interpolation of $\chi \in H^{k+1}(\Omega_h)$ and to the bounds $|U|, |\psi| \leq C\|u\|_{1, \Omega}$ valid by the trace inequality (recall that $u = U + \psi \times r$ on Γ).

Similarly, $\|\tilde{u} - \tilde{u}_h\|_{0, \Omega_h} \leq Ch^{k+1} \|u\|_{k+1, \Omega}$. We define moreover $\tilde{p}_h \in \mathcal{M}_h$ by the standard FE nodal interpolation $\tilde{p}_h = I_h \tilde{p}$ such that

$$\|\tilde{p} - \tilde{p}_h\|_{0, \Omega_h} \leq Ch^k \|p\|_{k, \Omega}. \quad (49)$$

Thanks to Lemma 9, $\exists (v_h, q_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h$ such that

$$\|\tilde{u}_h - u_h, \tilde{p}_h - p_h\|_h \leq C \frac{c_h(\tilde{u}_h - u_h, \tilde{p}_h - p_h; v_h, q_h)}{\|(v_h, q_h)\|_h}. \quad (50)$$

We should now substitute (\tilde{u}, \tilde{p}) into the form c_h . To this end, we introduce the fictitious right-hand sides \tilde{F} and \tilde{Q} on Ω_h so that

$$-2\nu \operatorname{div} D(\tilde{u}) + \nabla \tilde{p} = \tilde{F} \text{ and } \operatorname{div} \tilde{u} = \tilde{Q} \text{ in } \Omega_h.$$

We observe then, taking any $v_h = \phi_h s_h + \chi_h(V_h + \omega_h \times r) \in \mathcal{V}_h^{rbm}$, $q_h \in \mathcal{M}_h$,

$$2 \int_{\Omega_h} D(\tilde{u}) : D(v_h) - \int_{\Omega_h} \tilde{p} \operatorname{div} v_h - \int_{G_h} (2D(\tilde{u}) - \tilde{p}I)n \cdot v_h = \int_{\Omega_h} \tilde{F} \cdot v_h$$

and, recalling $B_h = \Omega_h \setminus \Omega$,

$$\begin{aligned} \int_{G_h} (2D(\tilde{u}) - \tilde{p}I)n \cdot (V_h + \omega_h \times r) &= \int_{\Gamma} (2D(\tilde{u}) - pI)n \cdot (V_h + \omega_h \times r) \\ &+ \int_{B_h} \operatorname{div}(2D(\tilde{u}) - \tilde{p}I)n \cdot (V_h + \omega_h \times r) = mg \cdot V_h - \int_{B_h} \tilde{F} \cdot (V_h + \omega_h \times r). \end{aligned}$$

Hence,

$$\begin{aligned} c_h(\tilde{u}, \tilde{p}; v_h, q_h) &= \int_{\Omega_h} \tilde{F} \cdot v_h - \int_{B_h} \tilde{F} \cdot (V_h + \omega_h \times r) + mg \cdot V_h - \int_{\Omega_h} q_h \operatorname{div} \tilde{u} \\ &+ \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \tilde{F} \cdot (-\Delta v_h - \nabla q_h) + \sigma \int_{\Omega_h^\Gamma} (\operatorname{div} \tilde{u})(\operatorname{div} v_h). \end{aligned}$$

Also note that the RHS (43) of the scheme (42) can be rewritten as

$$L_h(v_h, q_h) = \int_{\Omega_h} \rho_f g \cdot v_h - \int_{B_h} \rho_f g \cdot (V_h + \omega_h \times r) + mg \cdot V_h \\ + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \rho_f g \cdot (-\Delta v_h - \nabla q_h)$$

This allows us to establish the following Galerkin orthogonality relation, valid for all $v_h \in \mathcal{V}_h^{rbm}$, $q_h \in \mathcal{M}_h$,

$$c_h(\tilde{u} - u_h, \tilde{p} - p_h; v_h, q_h) = R_h(v_h, q_h), \quad (51)$$

where

$$R_h(v_h, q_h) = \int_{B_h} (\tilde{F} - \rho_f g) \cdot \phi_h s_h - \int_{B_h} q_h \operatorname{div} \tilde{u} \\ + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\tilde{F} - \rho_f g) \cdot (-\Delta v_h - \nabla q_h) + \sigma \int_{\Omega_h^\Gamma} (\operatorname{div} \tilde{u})(\operatorname{div} v_h).$$

The integrals of $\tilde{F} - \rho_f g$ and $\operatorname{div} \tilde{u}$ on Ω_h have been rewritten as integrals on B_h since both $\tilde{F} - \rho_f g$ and $\operatorname{div} \tilde{u}$ vanish on Ω .

Combination of (50) and (51) entails

$$\| \tilde{u}_h - u_h, \tilde{p}_h - p_h \|_h \leq C \frac{c_h(\tilde{u}_h - \tilde{u}, \tilde{p}_h - \tilde{p}; v_h, q_h) + R_h(v_h, q_h)}{\| v_h, q_h \|_h}.$$

We can now use interpolation inequalities as in [12, Section 3.4]. In particular, the term with c_h in the nominator of the fraction above is bounded by $Ch^k(\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega})\|v_h, q_h\|_h$ thanks to (48)–(49) and to the estimates of Lemma 4. To bound $R_h(v_h, q_h)$ we recall that $\tilde{F} - \rho_f g$ and $\operatorname{div} \tilde{u}$ vanish on Ω . Thus, thanks to [12, Lemma 3.6]

$$\|\tilde{F} - \rho_f g\|_{0, \Omega_h^\Gamma} \leq Ch^{k-1} \|\tilde{F} - \rho_f g\|_{k-1, \Omega_h^\Gamma} \leq Ch^{k-1} (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega})$$

and

$$\|\operatorname{div} \tilde{u}\|_{0, \Omega_h^\Gamma} \leq Ch^k \|\operatorname{div} \tilde{u}\|_{k, \Omega_h^\Gamma} \leq Ch^k \|u\|_{k+1, \Omega}. \quad (52)$$

This, combined with the estimates of Lemma 4, in particular $\|\phi_h s_h\|_{0, \Omega_h^\Gamma} \leq Ch|v_h|_{1, \Omega_h^\Gamma}$, leads to $|R_h(v_h, q_h)| \leq Ch^k(\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega})\|v_h, q_h\|_h$. Recalling again the interpolation estimates (48)–(49), we obtain

$$\| \tilde{u} - u_h, \tilde{p} - p_h \|_h \leq Ch^k (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega}) \quad (53)$$

giving the error estimates for u and p , announced by (11).

Proof of the a priori error estimate (12) on the velocity of the solid: Let us now estimate the error for the translation and the rotation of the solid. Thanks to the equivalence of norms, and recalling $\chi = 1$ on Γ ,

$$|U - U_h| + |\psi - \psi_h| \leq C \|U - U_h + (\psi - \psi_h) \times r\|_{0, \Gamma} \\ \leq C \|\chi(U + \psi \times r) - \chi_h(U_h + \psi_h \times r)\|_{0, \Gamma} + C \|(\chi - \chi_h)(U_h + \psi_h \times r)\|_{0, \Gamma} \\ \leq C \|u - u_h\|_{0, \Gamma} + C \|(\phi - \phi_h)w_h\|_{0, \Gamma} + C \|\chi - \chi_h\|_{0, \Gamma} (|U_h| + |\psi_h|) \\ \leq Ch^k (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega} + \|\chi\|_{k+1, \Omega} (|U_h| + |\psi_h|))$$

with the last estimate valid by the trace inequality, the already established H^1 -error estimate for u , an interpolation estimate for $\chi - \chi_h$ and $\phi - \phi_h$ and the Hardy inequality. Incorporating $\|\chi\|_{k+1,\Omega}$ into the constant C , we deduce

$$(1 - Ch^k)(|U - U_h| + |\psi - \psi_h|) \leq Ch^k(\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega} + |U| + |\psi|)$$

which proves (12) for h small enough since $|U| + |\psi| \leq \|u\|_{1,\Omega}$ by the trace inequality.

Proof of the L^2 a priori error estimate (13): Let $(v, q, V, \omega) \in H^2(\Omega)^d \times H^1(\Omega) \times \mathbb{R}^d \times \mathbb{R}^d$ the solution to

$$\begin{cases} -2 \operatorname{div} D(v) + \nabla q = u - u_h, & \text{in } \Omega, \\ \operatorname{div} v = 0, & \text{in } \Omega, \\ v = V + \omega \times r, & \text{on } \Gamma, \\ v = 0, & \text{on } \Gamma_w, \\ \int_{\Gamma} (2D(v) - qI)n = 0, \\ \int_{\Gamma} (2D(v) - qI)n \times r = 0, \\ \int_{\Omega} q = 0. \end{cases}$$

An integration by parts gives

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= \int_{\Omega} (u - u_h)(-2 \operatorname{div} D(v) + \nabla q) \\ &= 2 \int_{\Omega} D(u - u_h) : D(v) - \int_{\Omega} q \operatorname{div}(u - u_h) - \int_{\Omega} (p - p_h) \operatorname{div} v. \end{aligned} \quad (54)$$

Note that the boundary term $\int_{\Gamma} (u - u_h) \cdot (2D(v) - qI)n$ vanishes since $u - u_h$ is a rigid body motion on Γ .

Let $(\tilde{v}, \tilde{q}) \in H^2(\Omega_h)^d \times H^1(\Omega_h)$ coincide with (v, q) on Ω . They can be constructed by a bounded extension operator in $H^2 \times H^1$ so that

$$\|\tilde{v}\|_{2,\Omega_h} + \|\tilde{q}\|_{1,\Omega_h} \leq C(\|v\|_{2,\Omega} + \|q\|_{1,\Omega}) \leq C\|u - u_h\|_{0,\Omega}. \quad (55)$$

We now further rewrite (54) using Galerkin orthogonality (51) with the test functions $v_h = \phi_h s_h + \chi_h(V_h + \omega_h \times r) \in \mathcal{V}_h^{rbm}$ and $q_h \in \mathcal{M}_h$ and recalling $B_h = \Omega_h \setminus \Omega$,

$$\begin{aligned} &\|u - u_h\|_{0,\Omega}^2 \\ &= \underbrace{\int_{\Omega_h} 2D(\tilde{u} - u_h) : D(\tilde{v} - v_h) - \int_{\Omega_h} (\tilde{q} - q_h) \operatorname{div}(\tilde{u} - u_h) - \int_{\Omega_h} (\tilde{p} - p_h) \operatorname{div}(\tilde{v} - v_h)}_I \\ &\quad - \underbrace{\left(\int_{B_h} 2D(\tilde{u} - u_h) : D(\tilde{v}) - \int_{B_h} \tilde{q} \operatorname{div}(\tilde{u} - u_h) - \int_{B_h} (\tilde{p} - p_h) \operatorname{div} \tilde{v} \right)}_{II} \\ &\quad + \underbrace{\int_{G_h} (2D(\tilde{u} - u_h) - (\tilde{p} - p_h)I)n \cdot \phi_h s_h}_{III} \quad (56) \end{aligned}$$

$$\begin{aligned}
& \underbrace{-\sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n} (\tilde{u} - u_h) \right] \cdot \left[\frac{\partial v_h}{\partial n} \right] - \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial^2}{\partial n^2} (\tilde{u} - u_h) \right] \cdot \left[\frac{\partial^2 v_h}{\partial n^2} \right]}_{IV} \\
& \quad - \underbrace{\sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T -\Delta(\tilde{u} - u_h) + \nabla(\tilde{p} - p_h)) \cdot (-\Delta v_h - \nabla q_h)}_V \\
& \quad \quad - \underbrace{\sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \operatorname{div}(\tilde{u} - u_h) \operatorname{div} v_h}_{VI} + \underbrace{R_h(v_h, q_h)}_{VII}.
\end{aligned}$$

We now take $V_h = V, \omega_h = \omega$ and set $s_h \in \mathcal{V}_h$ so that $\phi_h s_h$ is an optimal interpolant of $v - \chi_h(V + \omega \times r)$, as guaranteed by Lemma 6. We also set $q_h = \tilde{I}_h \tilde{q}$ using an appropriate Clément interpolation \tilde{I}_h . We can now estimate all the terms of (56) using the already proven estimate (53) and the interpolation estimates for $\tilde{v} - v_h$ and $\tilde{p} - p_h$. This gives

$$\|u - u_h\|_{0,\Omega}^2 \leq Ch^{k+1/2} (\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}) (\|\tilde{v}\|_{2,\Omega_h} + \|\tilde{q}\|_{1,\Omega_h}). \quad (57)$$

Rather than go to the details of the tedious calculations leading to this estimate, we prefer here to refer to the similar arguments used in [12] to estimate the terms in eq. (3.24). Indeed, the terms $I - III$ in (56) can be treated as the terms $I - III$ in eq. (3.24) of [12]. Terms $IV - V$ in (56) can be treated as term IV in eq. (3.24) of [12]. Term VI in (56) is also similar to the latter (note, in particular, $\|\operatorname{div} \tilde{v}\|_{0,\Omega_h^\Gamma} \leq Ch\|v\|_{2,\Omega}$ similarly to (52)). Finally, term VII in (56) can be treated as term V in eq. (3.24) of [12].

Combining (57) with (55) proves (13).

4 Numerical tests

In this section, we present numerical results, first in the particular case of a fixed particle, i.e. the Stokes equations alone (see the Appendix and the ϕ -FEM scheme (63)), and second in the case of the particulate flows (equations (1a)-(1g) and the ϕ -FEM scheme (10)). These schemes will be compared with a standard FE schemes on fitted meshes. In the case of Stokes equations, the error is measured with respect to a manufactured solution, while a reference solution obtained by standard FEM on a fitted fine mesh is used in the case of particulate flows. We have implemented ϕ -FEM in `multiphénics` [4]. The implementation scripts can be consulted on GitHub.¹

The fluid/solid domain in our both test cases is $\mathcal{O} = (0,1)^2 \subset \mathbb{R}^2$ and we consider for the solid S a disk centered in $(0.5, 0.5)$ and of radius $R = 0.21$. Then $\Omega = [0,1]^2 \setminus S$. The geometry is presented in Fig. 1 (left).

4.1 Particular case of a fixed particle: Stokes equations

We start by Stokes equations (62) in the domain Ω , as above, with the right-hand side such that the exact solution is as follows, cf. [14],

$$\begin{aligned}
u_{exact}(x, y) &= (\cos(\pi x) \sin(\pi y), -\sin(\pi x) \cos(\pi y)), \\
p_{exact}(x, y) &= (y - 0.5) \cos(2\pi x) + (x - 0.5) \sin(2\pi y).
\end{aligned}$$

¹<https://github.com/michelduprez/phi-FEM-particulate-flows-Stokes.git>

We shall test the ϕ -FEM scheme given by (63) in the Appendix. The level-set of the solid is defined for each $(x, y) \in \mathbb{R}^2$ by

$$\phi(x, y) = R^2 - (x - 0.5)^2 - (y - 0.5)^2. \quad (58)$$

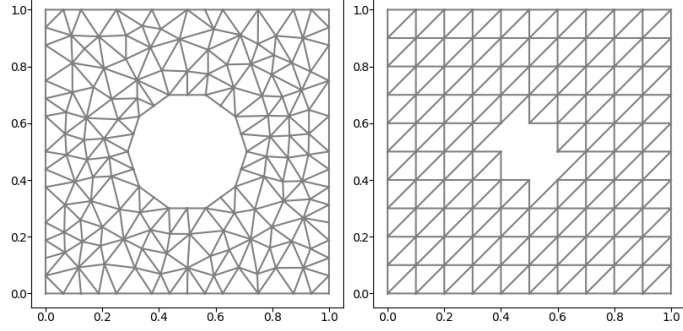


Figure 3: Mesh used for the standard FEM formulation (left) and mesh used in the ϕ -FEM schemes (right).

This will be compared with a standard Taylor-Hood scheme on a fitted mesh constructed as follows. We define first the Taylor-Hood FE space on a triangular mesh \mathcal{T}_h^{fit} fitted to Ω . The domain occupied by this mesh, denoted by Ω_h^{fit} , is an approximation of Ω , since the interface Γ is curvilinear and cannot be represented exactly by the straight edges. We fix an integer $k \geq 2$ and introduce the FE spaces

$$\mathcal{V}_h^{fit, uD} = \left\{ v_h \in C(\bar{\Omega}_h^{fit})^d : v_h|_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h^{fit}, \quad v_h = I_h u_D \text{ on } \Gamma_w \cap \Gamma \right\}$$

and

$$\mathcal{M}_h^{fit} = \left\{ q_h \in C(\bar{\Omega}_h^{fit}) : q_h|_T \in \mathbb{P}^{k-1}(T) \quad \forall T \in \mathcal{T}_h^{fit}, \quad \int_{\Omega_h^{fit}} q_h = 0 \right\}. \quad (59)$$

A standard Taylor-Hood finite element formulation for meshes fitting the boundary of the domain can be written as: find $(u_h, p_h) \in \mathcal{V}_h^{fit, uD} \times \mathcal{M}_h^{fit}$ such that

$$\int_{\Omega_h^{fit}} 2D(u_h) : D(v_h) - \int_{\Omega_h^{fit}} p_h \operatorname{div} v_h - \int_{\Omega_h^{fit}} q_h \operatorname{div} u_h = \int_{\Omega} f v_h, \quad (60)$$

for all $(v_h, q_h) \in \mathcal{V}_h^{fit, 0} \times \mathcal{M}_h^{fit}$.

Remark 4. As said in Remark 1, it is impossible to impose that the integral of the discrete pressures p_h zero on Ω , so we look for a discrete pressure p_h zero integral on the discrete domain Ω_h , i.e. $\int_{\Omega_h} p_h = 0$ (see Definition (59) of \mathcal{M}_h^{fit}). For this, we will use Lagrange multipliers. To compute the relative error on the L^2 norm of the pressure we will subtract the integral on the discrete domain Ω_h of the real pressure p_{exact} to itself, i.e.

$$\frac{\left(\int_{\Omega_h} (p_h - p_{exact} - \int_{\Omega_h} p_{exact})^2 \right)^{1/2}}{\left(\int_{\Omega_h} p_{exact}^2 \right)^{1/2}}.$$

Examples of meshes for the standard FEM formulation and the ϕ -FEM scheme are given in Fig. 3 (left) and (right), respectively. In Fig. 4, we present some results of convergence for the standard Taylor-Hood FEM scheme (60) and for the ϕ -FEM scheme (63) in the case of Stokes equations. We recover the theoretical rates of convergence for the ϕ -FEM scheme stated in Theorem 2 and they are better than the standard FEM scheme.

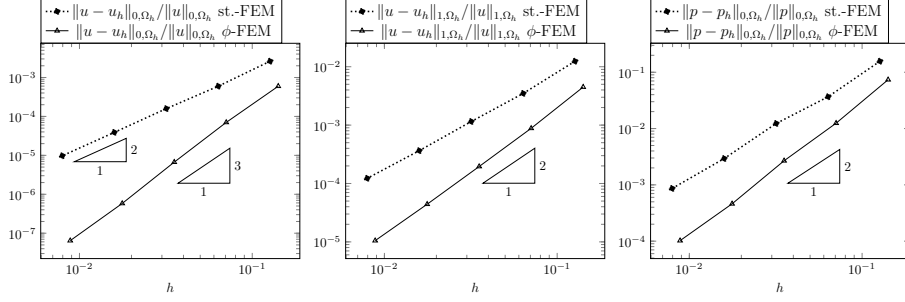


Figure 4: Rates of convergence for the standard Taylor-Hood FEM scheme (60) and the ϕ -FEM scheme (63) in the case of Stokes equations. The L^2 relative error of the velocity (left), the H^1 relative error of the velocity (middle) and the L^2 relative error of the pressure (right).

4.2 Particulate flows

We now turn to the creeping particulate flow equations (1a)-(1g). The vertical gravity is assumed to be equal to 10. The density of the fluid and the solid are equal to $\rho_f = 1$ and $\rho_s = 2$, respectively. We deduce that the mass of the solid is equal to $m = \rho_s \pi^2 R^2$. The level-set of the solid is again defined by (58). For the cut-off χ , we consider the radial polynomial of degree 5 on the interval (r_0, r_1) with $r_0 = R$ and $r_1 = 0.45$ such that $\chi(r_0) = 1$ and $\chi'(r_0) = \chi''(r_0) = \chi(r_1) = \chi'(r_1) = \chi''(r_1) = 0$, that is,

$$\chi(r) = \begin{cases} 1, & \text{for } r < r_0 \\ 1 + \frac{f(r_0, r_1)}{(r_1 - r_0)^5}, & \text{for } r_0 < r < r_1 \\ 0, & \text{for } r > r_1 \end{cases}$$

where

$$f(r_0, r_1) = (-6r^5 + 15(r_0 + r_1)r^4 - 10(r_0^2 + 4r_0r_1 + r_1^2)r^3 + 30r_0r_1(r_0 + r_1)r^2 - 30r_0^2r_1^2r + r_0^3(r_0^2 - 5r_1r_0 + 10r_1^2)).$$

Let us introduce a Taylor-Hood scheme which will be compared with our ϕ -FEM scheme. The boundary conditions on the solid are imposed thanks to Lagrange multipliers. We define first the Taylor-Hood FE space on meshes fitting the domain. For a mesh Ω_h^{fit} of Ω fitting its boundary with \mathcal{T}_h^{fit} the set of its cells, we fix an integer $k \geq 2$ and introduce the FE spaces

$$\mathcal{V}_h^{fit} = \left\{ v_h \in C(\bar{\Omega}_h^{fit})^d : v_h|_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h^{fit}, \quad v_h = 0 \text{ on } \Gamma_w \right\},$$

$$\mathcal{M}_h^{fit} = \left\{ q_h \in C(\bar{\Omega}_h^{fit}) : q_h|_T \in \mathbb{P}^{k-1}(T) \quad \forall T \in \mathcal{T}_h^{fit}, \quad \int_{\Omega_h^{fit}} q_h = 0 \right\}$$

and

$$\Lambda_h^{fit} = \left\{ \mu_h \in C(\tilde{\Omega}_h^{fit}) : \mu_h|_F \in \mathbb{P}^{k-1}(F) \quad \forall F \in \mathcal{F}_h^{fit} \right\},$$

where \mathcal{F}_h^{fit} is the set of the external facet on the solid. A standard Taylor-Hood finite element formulation for meshes fitting the boundary of the domain can be written as: find $(u_h, p_h, \lambda_h, U_h, \psi_h) \in \mathcal{V}_h^{fit} \times \mathcal{M}_h^{fit} \times \Lambda_h^{fit} \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\begin{aligned} \int_{\Omega_h^{fit}} 2D(u_h) : D(v_h) - \int_{\Omega_h^{fit}} p_h \operatorname{div} v_h - \int_{\Gamma_h^{fit}} (2D(u_h) - p_h I) n \cdot v_h - \int_{\Omega_h^{fit}} q_h \operatorname{div} u_h \\ + \int_{\Gamma_h^{fit}} \lambda_h (v_h - V_h - \omega_h \times r) + \int_{\Gamma_h^{fit}} \mu_h (u_h - U_h - \psi_h \times r) = \int_{\Omega} f v_h, \end{aligned} \quad (61)$$

for all $(v_h, q_h, \mu_h, V_h, \omega_h) \in \mathcal{V}_h^{fit} \times \mathcal{M}_h^{fit} \times \Lambda_h^{fit} \times \mathbb{R}^d \times \mathbb{R}^d$.

Remark 5. As for the Stokes equations, we look for a discrete pressure of zero integral on the discrete domain i.e. $\int_{\Omega_h} p_h = 0$ thanks to Lagrange multipliers and, the relative error on the L^2 norm of the pressure is computed as follows

$$\frac{\left(\int_{\tilde{\Omega}_h} (\tilde{p} - p_h - \int_{\tilde{\Omega}_h} p_h)^2 \right)^{1/2}}{\left(\int_{\tilde{\Omega}_h} \tilde{p}^2 \right)^{1/2}},$$

where $\tilde{\Omega}_h$ is the fine discrete space and \tilde{p} the fine solution defined on $\tilde{\Omega}_h$ which are used to compute the error.

We present in Fig. 5 the velocity obtained with the standard Taylor-Hood FEM scheme (61). Again we give some examples of meshes for the standard FEM formulation and the ϕ -FEM scheme are given in Fig. 3 (left) and (right), respectively.

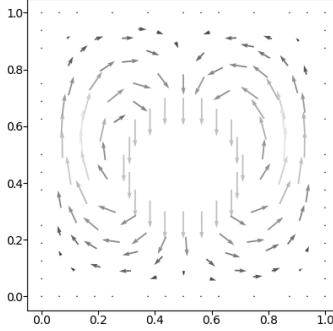


Figure 5: Velocity obtained with the standard Taylor-Hood FEM scheme (61).

In Figures 6 and 7, we give some results of convergence for the standard Taylor-Hood FEM scheme (61) and for the ϕ -FEM scheme (10) (we do not compute the relative error of the rotation since the real rotation is equal to 0). The degree of polynomial is $k = 2$ for w_h, ϕ_h, χ_h in both schemes. Moreover the approximation is exact for ϕ in this case. The stabilization parameters are $\sigma = \sigma_u = 20$ (as in [12]). The error is computed by a comparison with a fine solution of the standard FEM scheme. We observe that we recover the theoretical convergence rates established

for ϕ -FEM for the velocity (in H^1 norm) and the pressure (in L^2 norm), while it seems that our theoretical estimates for the L^2 -error of the fluid velocity and that of the particle velocity are not sharp.

Remark 6. *As observed in [12], better numerical results can be obtained if ϕ is interpolated in a FE space with a higher polynomial degree than the solution. This phenomenon is not studied here since the approximation of ϕ by \mathbb{P}_2 FE is already exact.*

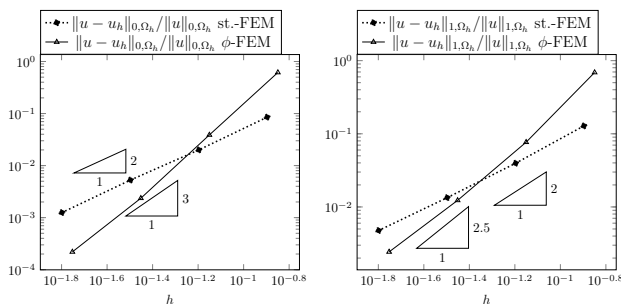


Figure 6: Rates of convergence for the standard Taylor-Hood FEM scheme (61) and the ϕ -FEM scheme (10) in the case of particulate flows. The L^2 relative error of the velocity (left) and the H^1 relative error of the velocity (right).

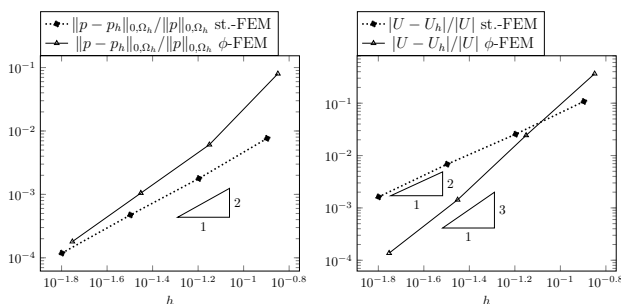


Figure 7: Rates of convergence for the standard Taylor-Hood FEM scheme (60) and the ϕ -FEM scheme (63) in the case of particulate flows. The L^2 relative error of the pressure (left) and relative error of the displacement of the solid (right).

A The particular case of a fixed particle : Stokes equation

In this section, we propose a ϕ -FEM scheme for the simpler case of a fixed solid in the fluid. The governing equations are the non-homogeneous Stokes equations given by

$$\begin{cases} -2 \operatorname{div}(D(u)) + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = u_D, & \text{on } \Gamma \cup \Gamma_w. \end{cases} \quad (62)$$

Assume that u_D and f are defined in the whole discrete domain Ω_h . Inspired by the ϕ -FEM scheme for particulate flow given in (10), we can derive the following ϕ -FEM scheme for the non-homogeneous Stokes equations (62): find $w_h \in \mathcal{V}_h$, $p_h \in \mathcal{M}_h$ satisfying

$$\begin{aligned}
& \int_{\Omega_h} 2D(u_D + \phi_h w_h) : D(\phi_h s_h) - \int_{G_h} (2D(u_D + \phi_h w_h) - p_h I) n \cdot \phi_h s_h \quad (63) \\
& - \int_{\Omega_h} p_h \operatorname{div}(\phi_h s_h) - \int_{\Omega_h} q_h \operatorname{div}(u_D + \phi_h w_h) + \sigma_u J_u(u_D + \phi_h w_h, \phi_h s_h) \\
& + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (-\Delta(u_D + \phi_h w_h) + \nabla p_h) \cdot (-\Delta(\phi_h s_h) - \nabla q_h) \\
& + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \operatorname{div}(u_D + \phi_h w_h) \operatorname{div}(\phi_h s_h) \\
& = \int_{\Omega_h} f \phi_h s_h + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T f (-\Delta(u_D + \phi_h w_h) - \nabla q_h),
\end{aligned}$$

for all $s_h \in \mathcal{V}_h$, $q_h \in \mathcal{M}_h$.

We now state our second main result for the Stokes equations:

Theorem 2. *Suppose that Assumptions 1, 2, 4 and 5 hold true, the mesh \mathcal{T}_h is quasi-uniform. Let $(u, p) \in H^{k+1}(\Omega)^d \times H^k(\Omega)$ be the solution to (62) and $(w_h, p_h) \in \mathcal{V}_h \times \mathcal{M}_h$ be the solution to (63). Denoting $u_h := \phi_h w_h$, it holds*

$$|u - u_h|_{1, \Omega \cap \Omega_h} + |p - p_h|_{0, \Omega \cap \Omega_h} \leq Ch^k (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega})$$

with a constant $C > 0$ depending on the C_0 , m , M in Assumptions 1, 4, on the maximum of the derivatives of ϕ , on the mesh regularity, and on the polynomial degree k , but independent of h , f , and u .

Moreover, supposing $\Omega \subset \Omega_h$

$$\|u - u_h\|_{0, \Omega} \leq Ch^{k+1} (\|u\|_{k+1, \Omega} + \|p\|_{k, \Omega})$$

with a constant $C > 0$ of the same type.

The proof of Theorem 2 can be adapted from the proof of Theorem 1. It is even more simple.

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