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Lack of local controllability for a water-tank system when the time is not large enough

Jean-Michel Coron ^{*} Armand Koenig [†] Hoai-Minh Nguyen [‡]

February 25, 2022

Abstract

We consider the small-time local controllability property of a water tank modeled by 1D Saint-Venant equations, where the control is the acceleration of the tank. It is known from the work of Dubois et al. that the linearized system is not controllable. Moreover, concerning the linearized system, they showed that a traveling time T^* is necessary to bring the tank from one position to another for which the water is still at the beginning and at the end. Concerning the nonlinear system, Coron showed that local controllability around equilibrium states holds for a time large enough. In this paper, we show that for the local controllability of the nonlinear system around the equilibrium states, the necessary time is at least $2T^*$ even for the tank being still at the beginning and at the end. The key point of the proof is a coercivity property for the quadratic approximation of the water-tank system.

1 Introduction

1.1 Statement of the main result

We consider a water tank problem with a length $L > 0$ in the time interval $(0, T)$ modeled by the following 1D Saint-Venant system (see fig. 1):

$$\begin{cases} \partial_t H + \partial_x(vH) = 0 & \text{for } (t, x) \in (0, T) \times (0, L), \\ \partial_t v + \partial_x\left(gH + \frac{v^2}{2}\right) = -u(t) & \text{for } (t, x) \in (0, T) \times (0, L), \\ v(t, 0) = v(t, L) = 0 & \text{for } t \in (0, T), \end{cases} \quad (1)$$

and

$$\ddot{D}(t) = u(t) \text{ for } t \in (0, T). \quad (2)$$

Here H denotes the height of the water, v is the horizontal velocity field of the water, u is the acceleration that is imposed on the tank, D is the position of the tank, g is the gravity. Given $H_{\text{eq}} > 0$, one can easily check that $(H_{\text{eq}}, 0)$ is a solution of (1) and thus is an equilibrium of (1).

In this article, we are interested in the *local controllability* of this system (see, e.g., [12, Section 1] for a definition). Our main result is the following theorem, concerning the local controllability property around an equilibrium $(H_{\text{eq}}, 0)$ of (1).

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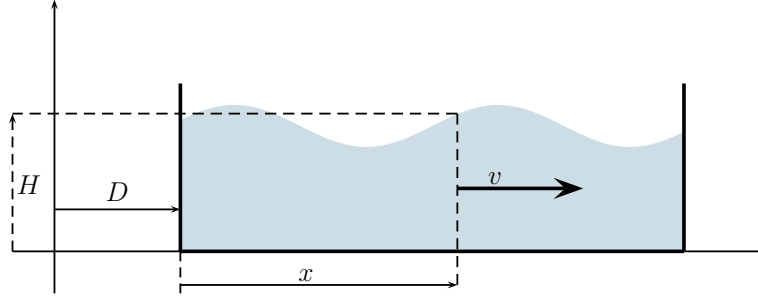


Figure 1: Water tank problem

Theorem 1. Let $L > 0$, $T > 0$, $g > 0$, and $H_{\text{eq}} > 0$. Set

$$T_* := \frac{L}{\sqrt{H_{\text{eq}}g}}. \quad (3)$$

Assume that $T \in (T_*, 2T_*)$. There exists $\eta > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0([0, T])} < \eta$, if the solution $(H, v) \in [C^1([0, T] \times [0, L])]^2$ of the water-tank system (1) with the initial data

$$H(0, x) = H_{\text{eq}} \quad \text{and} \quad v(0, x) = 0 \quad \text{in } (0, L), \quad (4)$$

satisfies

$$H(T, x) = H_{\text{eq}} \quad \text{and} \quad v(T, x) = 0 \quad \text{in } (0, L), \quad (5)$$

and the solution D of (2) satisfies

$$\dot{D}(T) = \dot{D}(0) = 0, \quad (6)$$

then

$$u = 0 \quad \text{in } (0, T). \quad (7)$$

Conditions (4) and (5) reads “ u steers the water-tank system from $(H_{\text{eq}}, 0)$ to $(H_{\text{eq}}, 0)$ at time T ”, while condition (6)¹ reads “the water-tank ends with the same speed as the one it started with”. As a consequence of Theorem 1, the water-tank system is not locally controllable around $(H_{\text{eq}}, 0)$ for time smaller than $2T_*$ (with controls small in $C^0([0, T])$).

Remark 2. 1. The regularity required for the control u is C^0 might be somehow unexpected. Standard well-posedness theorems would assume the source term u (small) in C^1 . The specific form of the source term ($u(t)$ instead of $u(t, x)$) is used for this point.

2. The time T_* is the time needed for waves of the linearized equation to travel from one end of the tank to the other end, as observed in [19].
3. The water-tank system (1) is a hyperbolic system. As such, there is a finite speed of propagation, and it is no surprise local-controllability fails in small time (see remark 22). The interest of this theorem is that the local controllability fails even for times larger than what the finite speed of propagation would suggest.

Open question 3. Is the system (1) and (2) locally controllable for any time T greater than $2T_*$?

¹Which is equivalent to $\int_0^T u(s) ds = 0$.

The controllability of the water-tank system was initially considered by Dubois, Petit and Rouchon [19] where the linearized system was considered. In particular, they proved for the linearized system that, given $T > T_*$, there exists a control that steers an equilibrium $(H_{\text{eq}}, 0)$ back to itself while moving the water-tank.

Concerning the nonlinear system, the local-controllability was investigated by Coron [12] using the *return method*. More precisely, Coron proved that local controllability around equilibrium states $(H_{\text{eq}}, 0)$ for (H, v) starting with $(\dot{D}(0), D(0))$ near (s_0, D_0) and ending with $(\dot{D}(T), D(T))$ near $(s_0, D_0 + Ts_0)$ for a time T large enough. In particular, the local controllability around $(H_{\text{eq}}, 0)$ (for (H, v)) and $(0, 0)$ (for (\dot{D}, D)) holds for a large enough time.

Theorem 1 reveals new properties for the local controllability of the nonlinear water tank problem. First, Theorem 1 reveals that for $T_* < T < 2T_*$, contrary to the linearized system, one cannot steer an equilibrium $H(0, x) = H_{\text{eq}}, v(0, x) = 0$ back to itself if the water-tank ends with the same speed as the one it started with (except for the trivial trajectory where $u = 0$). Theorem 1 also points out that the local controllability around $(H_{\text{eq}}, 0)$ (for (H, v)) and $(0, 0)$ (for (\dot{D}, D)) holds but with at time larger than or equal to $2T_*$.

The optimal time for the boundary controllability of hyperbolic systems have been studied extensively, see [18, 16, 17], where the controls are on one side. This is different from the water tank problem which can be seen as a boundary control problem where the controls are given on two sides, see (34) and (35). Moreover, for the water tank problem the controls are required to be the same on both side, see (35). This rigidity condition yields new phenomena and obstructions that require new ingredients to describe.

1.2 The main ideas of the proof and the organization of the paper

Using standard scaling arguments (see for instance [12, Section 2]), namely setting

$$\begin{aligned} H^*(t, x) &:= \frac{1}{H_{\text{eq}}} H\left(\frac{L}{\sqrt{H_{\text{eq}}g}}t, Lx\right); \\ v^*(t, x) &:= \frac{1}{\sqrt{H_{\text{eq}}g}} H\left(\frac{L}{\sqrt{H_{\text{eq}}g}}t, Lx\right), \end{aligned}$$

we may assume that $L = 1, g = 1$, and $H_{\text{eq}} = 1$ and this will be assumed from now on. Note that in this case, T_* defined in theorem 1 is $T_* = 1$.

The proof has its root in the *power series expansion method*, see, e.g., [14] and [11, Chapter 8]: since the linearized system does not give enough information to conclude about the local-controllability of (1), the second-order approximation is required. Indeed, the linearized system of (1) around the equilibrium $(1, 0)$ is

$$\begin{cases} \partial_t h_1 + \partial_x v_1 = 0 & \text{for } (t, x) \in (0, T) \times (0, 1), \\ \partial_t v_1 + \partial_x h_1 = -u(t) & \text{for } (t, x) \in (0, T) \times (0, 1), \\ v_1(t, 0) = v_1(t, L) = 0 & \text{for } t \in (0, T). \end{cases} \quad (8)$$

Simple computations prove that if $h_1(0, x) = 0$ and $v_1(0, x) = 0$, then $h_1(t, 1 - x) = -h_1(t, x)$ and $v_1(t, 1 - x) = v_1(t, x)$ whatever u is. Thus, the linearized system is not controllable. As usual, the second order approximation system is given as follows

$$\begin{cases} \partial_t h_2 + \partial_x v_2 = -\partial_x(h_1 v_1) & \text{for } (t, x) \in (0, T) \times (0, 1), \\ \partial_t v_2 + \partial_x h_2 = -\partial_x\left(\frac{v_1^2}{2}\right) & \text{for } (t, x) \in (0, T) \times (0, 1), \\ v_2(t, 0) = v_2(t, L) = 0 & \text{for } t \in (0, T). \end{cases} \quad (9)$$

The main idea is to prove that if a control steers the linearized system from 0 to 0, this second order always lies in some half-space, at least when $T < 2T^*$. More precisely, for $T^* < T < 2T^*$, we prove that for well-chosen functions ϕ, ψ , there exists $c > 0$ such that for every control u that steers the linearized system from 0 to 0 and such that $\int_0^T u(s) ds = 0$, with

$$U(t) := \int_0^t u(s) ds, \quad (10)$$

we have

$$(h_2(T, \cdot), \phi) + (v_2(T, \cdot), \psi) \geq c \|U\|_{L^2}^2, \quad (11)$$

This means that the quadratic approximation of the water-tank system cannot be steered into the half-space $\{(h, v) \in L^2(0, 1)^2, (h, \phi) + (v, \psi) < 0\}$. The rest of the proof consists in estimating the difference between the quadratic approximation and the nonlinear system in an appropriate way so that one can reach the optimal control space: smallness of the control being required in $C^0([0, T])$.

The paper is organized as follows:

1. in section 2, we characterize the controls that steers the linearized system from 0 to 0;
2. in section 3, we analyse the second-order term, and prove that it enjoys a “conditional H^{-1} -coercivity” property;
3. in section 4, we study the nonlinear system, and in particular we prove that the error between the nonlinear solution and the second-order approximation cannot counter the positivity of the second-order term.

1.3 Bibliographical comments

Our proof relies on the positivity of a scalar product of the quadratic approximation of the water-tank system (1). This kind of phenomenon was the heart of several lack of small time local controllability results for systems modeled by partial differential equations. Concerning examples in finite dimensional system, we refer to Beauchard and Marbach’s paper [5], and the references therein.

The quadratic obstructions for small-time local controllability was previously observed for the Schrödinger equation with bilinear control [13, 7, 10], the viscous Burgers equation [22], nonlinear heat equations [6] and a KdV system [15] where the speed of the propagation is infinite. All these results share the same core idea: the scalar product of the second-order approximation with appropriate test functions enjoys a coercivity property. Let us detail a little each of these cases.

For the Schrödinger equation with bilinear control, the existing results relies heavily on explicit computation using the eigenfunctions and eigenvalues of the operator $-\partial_x^2$. Note that in Coron’s result [13] as well as Beauchard and Morancey’s result [7], the equivalent of our coercivity estimate (11) also has $\|U\|_{L^2}^2$ in the right-hand side, leading to a lack of small-time local controllability with controls small in L^∞ -norm. Bournissou [10] also has a similar coercivity estimate, with the n th iterated integral of the control instead of U , where n depends on the structure of the potential. This leads to a lack of small-time local controllability with controls small either in $W^{-1, \infty}$ (when $n = 1$) or H^{2n-3} (when $n \geq 2$).

Marbach [22] considered a viscous Burgers equation with control $u(t)$ as a source term. The main difficulty is the fact that the kernel of the quadratic approximation does not seem to be explicitly computable in a usable form. To tackle the problem, he rescaled the equation in time to transform the “small-time” aspect of the problem into a small-viscosity problem. This allows him to compute an asymptotic expansion of the kernel of the quadratic approximation of the viscous Burgers equation in low viscosity limit. Using this, Marbach succeeds in disproving the small-time local controllability

with controls small in L^2 -norm. A striking feature of this result is the equivalent of our coercivity estimate (11) has the $H^{-5/4}$ -norm of the control in the right-hand side, a *noninteger* Sobolev norm.

Beauchard and Marbach [6] considered a class of nonlinear heat equation. They exhibit a range of phenomena. For instance, for some nonlinearities, they prove a coercivity estimate with the H^s -norm of the control for some $s > 0$ that depends on the nonlinearity and that can be fractional. Also, for other nonlinearities, the quadratic term can actually *help* recover the small-time local controllability. This is the first example in which the quadratic term gives the local controllability result.

Concerning the KdV equations [15], we proved that the KdV equation with Dirichlet boundary conditions and Neumann boundary control on the right is not small-time locally controllable with controls small in H^1 for some critical lengths, introduced previously by Rosier [25]. This fact is surprising when compared with known results on internal controls for the corresponding KdV system for which the small time result holds (see e.g. [23]). One of the main difficulties was to characterize the controls that steers the linearized equation from 0 to 0. The analysis is based on a complete characterization of controls which bring 0 to 0 for the linearized system that involves the Paley-Wiener Theorem. The equivalent of the coercivity estimate (11) has the $H^{-2/3}$ -norm of the control in the right-hand side.

The result of this paper compares to the previous one in the following aspects:

- The control is internal, as was the case for the bilinear Schrödinger equation and the viscous Burgers equation, and unlike the KdV equation (where the control was at the boundary).
- Even if the computations are lengthy, we are able to compute the kernel of the second-order approximation in a very simple closed-form expression, which was more or less the case of the bilinear Schrödinger equation, but was not the case for the viscous Burgers equation and the KdV equation, where only an asymptotic expansion of the kernel was computed in closed form.
- We are able to disprove the small-time local controllability with controls small in C^0 , which is the natural space for the known well-posedness results. This is different from some bilinear Schrödinger equations, some nonlinear heat equations, and the KdV equation, where the existing results require the control quite regular. It is worth noting that less regular controls can change the situation. This is done by Bournissou [9] where the cubic terms surprisingly help recover the local controllability even in the case where the quadratic term gives the obstruction if regular controls are used.

Finally, we note that even with infinite speed of propagation in the linear setting, there might not be small-time controllability when there is a concentration of eigenfunction [4, 3, 20] or when there is condensation of eigenvalues or eigenfunctions [1, 8] and references therein.

2 Preliminary properties of the linearized system

As explained in section 1.2, without loss of generality, we may assume that $g = 1$ and $L = H_{\text{eq}} = 1$. Then the linearization of the system (1) around the equilibrium $(H_{\text{eq}}, 0) = (1, 0)$ is given by the system (8).

This system can be rewritten as $\partial_t F + \mathcal{A}F = U(t)$ with $F = (h_1, v_1) \in (L^2)^2$, $U(t) = (0, -u(t))$ and \mathcal{A} is the unbounded operator on $H = (L^2)^2$ with domain $D(\mathcal{A}) := H^1 \times H_0^1$ and defined by $\mathcal{A}(h, v) = (\partial_x v, \partial_x h)$. One can prove this system is well-posed thanks, e.g., to Lummer-Philips' theorem [24, Theorem 4.3].

2.1 Periodic change of variables

From now on, we denote

$$\mathbb{T} := \mathbb{R}/2\mathbb{Z}. \quad (12)$$

It is convenient to introduce the following periodic change of variables.

Definition 4. Given $F = (h, v) \in [L^2(0, 1)]^2$, define $\mathcal{C}F \in L^2(\mathbb{T})$ by

$$\mathcal{C}F(x) = \begin{cases} h(x) + v(x) & \text{for } 0 < x < 1, \\ h(-x) - v(-x) & \text{for } -1 < x < 0. \end{cases} \quad (13)$$

This change of variables transforms the linearized water-tank system into a transport equation with periodic boundary conditions:

Proposition 5. Let $(H, v) \in C^1([0, T] \times [0, 1])^0$ such that $v(t, 0) = v(t, 1) = 0$ and denote

$$\zeta(t, \cdot) = \mathcal{C}(H(t, \cdot), v(t, \cdot)).$$

Then

- ζ is continuous in $[0, T] \times \mathbb{T}$ and is C^1 in $[0, T] \times (\mathbb{T} \setminus \{0, 1\})$;
- If in addition $U \in L^\infty([0, T] \times [0, 1])^2$ and

$$\partial_t(H, v)(t, x) + \mathcal{A}(H, v)(t, x) = U(t, x) \text{ for } (t, x) \in [0, T] \times [0, 1], \quad (14)$$

then

$$\partial_t \zeta(t, x) + \partial_x \zeta(t, x) = \mathcal{C}U(t, x) \text{ for every } t \geq 0 \text{ and } x \in \mathbb{T} \setminus \{0, 1\}. \quad (15)$$

Proof. The fact that $\zeta = \mathcal{C}(H, v)$ is C^1 in $[0, T] \times (\mathbb{T} \setminus \{0, 1\})$ is a direct consequence of the definition of \mathcal{C} . The continuity at $x = 0$ and $x = 1$ results from the boundary conditions $v(t, 0) = v(t, 1) = 0$.

The second point results from elementary computations. \square

Remark 6. We can check that \mathcal{C} is an isometry (up to a factor 2) from $L^2(0, 1)^2$ to $L^2(\mathbb{T})$, and that if $F = (H, v) \in C^1([0, 1])^2$ with $v(0) = v(1) = 0$, then $\|\mathcal{C}F\|_{W^{1,\infty}} \leq 2\|F\|_{C^1}$.

Using the characteristic method, one can obtain the following formula for the solution of (15).

Lemma 7. Let $w \in L^2((0, T) \times \mathbb{T})$. The solution ζ of $\partial_t \zeta(t, x) + \partial_x \zeta(t, x) = w(t, x)$, $\zeta(0, x) = 0$ is

$$\zeta(t, x) = \int_0^t w(s, x + s - t) ds.$$

The linearized system (8) with zero initial conditions can be rewritten in $\zeta_1(u, t, x) = \mathcal{C}(h_1, v_1)(t, x)$ variable as

$$(\partial_t + \partial_x)\zeta_1(u, t, x) = u(t)\theta(x), \quad \zeta_1(u, 0, \cdot) = 0, \quad (16)$$

where θ is a “square wave” function that is 2-periodic defined by

$$\theta(x) = \begin{cases} 1 & \text{on } (-1, 0), \\ -1 & \text{on } (0, 1). \end{cases} \quad (17)$$

By Lemma 7, we have

$$\zeta_1(u, t, x) = \int_0^t u(s)\theta(x + s - t) ds. \quad (18)$$

We will sometimes denote this $\zeta_1(u, t, x)$ by $\zeta_1(t, x)$, leaving the fact that it depends on u implicit. We will use similar notations for every quantities that depends on the control.

Remark 8. We remark that $\theta(x + 1) = -\theta(x)$, thus, $\zeta_1(t, x + 1) = -\zeta_1(t, x)$.

2.2 Control of the linearized system

We next discuss control properties for the linearized systems. We give a controllability result when the target is 1-antiperiodic and we characterize the controls that steers 0 to 0. We begin with

Lemma 9. *Let $T > 1$. For any $\zeta_T \in H^1(\mathbb{T})$ that is 1-anti-periodic (i.e., $\zeta_T(x+1) = -\zeta_T(x)$), there exists a control $u \in L^2(0, T)$ such that the solution ζ of the linear equation (16) with initial condition 0 satisfies $\zeta(T, \cdot) = \zeta_T$. Moreover, this control u can be chosen such that $\int_0^T u(t) dt = 0$ and, if we set $U(t) := \int_0^t u(s) ds$, such that*

$$\|U\|_{L^2(0, T)} \leq C \|\zeta_T\|_{L^2(\mathbb{T})}$$

for some C independent of ζ_T .

Proof. We construct the control using the so-called *flatness method*. The main point, inspired by Dubois, Petit, and Rouchon [19, Section 3.4], is that if $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is in $H^1(\mathbb{R})$, then the function $\zeta : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$ defined by

$$\zeta(t, x) = \begin{cases} 2\varphi\left(t - x + \frac{1}{2}\right) - \varphi\left(t + \frac{1}{2}\right) - \varphi\left(t - \frac{1}{2}\right) & \text{if } 0 < x < 1 \\ -2\varphi\left(t - x - \frac{1}{2}\right) + \varphi\left(t + \frac{1}{2}\right) + \varphi\left(t - \frac{1}{2}\right) & \text{if } -1 < x < 0 \end{cases}$$

satisfies $(\partial_t + \partial_x)\zeta(t, x) = u(t)\theta(x)$ with $u(t) := -\varphi'(t + 1/2) - \varphi'(t - 1/2)$. We aim to construct a function φ such that the trajectory associated by this formula goes from 0 at time 0 to ζ_T at time T .

To construct φ , for $T - 1/2 < x < T + 1/2$, we set $\varphi(x) := \zeta_T(T - x + 1/2)$, and we extend this as a function in $H^1(\mathbb{R})$, which is still denoted by φ such that $\varphi = 0$ in $(-\infty, 1/2]$.

The first condition ensures that for $0 < x < 1$, the corresponding trajectory ζ satisfies $\zeta(T, x) = \zeta_T(x)$. Since ζ_T is 1-antiperiodic, we also have $\zeta(T, x) = \zeta_T(x)$ for $-1 < x < 0$. The fact that φ is zero on $[-1/2, 1/2]$ ensures that $\zeta(0, \cdot) = 0$.

The corresponding control is $u(t) = -\varphi'(t + 1/2) - \varphi'(t - 1/2)$. Thus,

$$\begin{aligned} \int_0^T u(t) dt &= -\varphi\left(\frac{1}{2}\right) - \varphi\left(-\frac{1}{2}\right) + \varphi\left(T + \frac{1}{2}\right) + \varphi\left(T - \frac{1}{2}\right) \\ &= -0 - 0 + \zeta_T(1) + \zeta_T(0). \end{aligned}$$

Since ζ_T is assumed to be 1-antiperiodic, we do have $\int_0^T u(t) dt = 0$.

The last thing we have to prove is the estimate. We have $U(t) = \int_0^t u(s) ds = -\varphi(t + 1/2) - \varphi(t - 1/2)$, thus, $\|U\|_{L^2(0, T)} \leq 2\|\varphi\|_{L^2(-1/2, T+1/2)} \leq C\|\zeta_T\|_{L^2}$. \square

We now study the controls that steer 0 to 0. We only prove the sufficiency of the following condition, which is all we need, but we could also prove that it is also necessary.

Proposition 10. *Let $T \in (1, 2)$ and let $u \in L^2(0, T)$ such that the solution $\zeta_1(u, \cdot, \cdot)$ of $(\partial_t - \partial_x)\zeta_1(u, t, x) = u(t)\theta(x)$, $\zeta_1(u, 0, \cdot) = 0$ satisfies $\zeta_1(u, T, \cdot) = 0$. Then*

$$u(t) = 0 \text{ for } t \in (T - 1, 1) \quad \text{and} \quad u(t + 1) = u(t) \text{ for } t \in (0, T - 1). \quad (19)$$

If in addition $\int_0^T u(s) ds = 0$, then

$$\int_0^{T-1} u(s) ds = 0.$$

Remark 11. One control that moves the water tank (with the tank ending with the same speed it started with) in time $T = 1$ is $u(t) = \delta'_0(t) + \delta'_1(t)$. In some sense, all controls that steer 0 to 0 are a regularization of this “optimal time” control.

Proof. For notational ease, we will denote in this proof $\zeta_1(u, t, x)$ by $\zeta_1(t, x)$. We also extend u in \mathbb{R} by zero outside $(0, T)$ and still denote this extension by u . By Lemma 7, we have

$$\zeta_1(t, x) = \int_0^t u(s)\theta(x + t - s) ds.$$

Differentiating in x , we have formally (we let the careful reader recast this argument rigorously)

$$\begin{aligned} \partial_x \zeta_1(T, x) &= \int_0^T \partial_x \theta(x + T - s) u(s) ds \\ &= \int_0^T (-2\delta_{x+T-s \bmod 2} + 2\delta_{x+T-s+1 \bmod 2}) u(s) ds \\ &= 2 \sum_{k \in \mathbb{Z}} (-u(x + T + 2k) + u(x + T + 2k + 1)). \end{aligned}$$

It follows that, for $x = T - t$ with $t \in (0, 1)$,

$$\partial_x \zeta_1(T, T - t) = \sum_{k \in \mathbb{Z}} (-2u(t + 2k) + 2u(t + 2k + 1)).$$

Since $0 < t < 1$ and $\text{supp}(u) \subset [0, T] \subset [0, 2]$, the only terms that might be non zero are $u(t)$ and $u(t + 1)$. Thus

$$\partial_x \zeta_1(T, T - t) = -2u(t) + 2u(t + 1). \quad (20)$$

If $T - 1 < t < 1$, $u(t + 1) = 0$. Thus, $\partial_x \zeta_1(T, T - t) = -2u(t)$. Since we assumed that $\zeta_1(T, x) = 0$, this proves the first point.

If $0 < t < T - 1$, since $\zeta_1(T, x) = 0$, Eq. (20) implies that $u(t) = u(t + 1)$. Thus,

$$0 = \int_0^T u(s) ds = \int_0^{T-1} u(s) ds + \int_1^T u(s) ds = 2 \int_0^{T-1} u(s) ds. \quad \square$$

3 Second-order approximation for the linearized system

3.1 Periodic change of variables

In this section, we deal with the second order approximation system given by (9). Set

$$\zeta_2 := \mathcal{C}(h_2, v_2) \quad (21)$$

Then

$$(\partial_t + \partial_x)\zeta_2(u, t, x) = w_1(u, t, x) := -\mathcal{C}(\partial_x(h_1 v_1), \partial_x(v_1^2/2)). \quad (22)$$

Again, we will leave the fact that ζ_2, w_1 , etc., depend on u implicit. We want to write w_1 as a function of ζ_1 . First, using the definition of \mathcal{C} ,

$$w_1(t, x) = \begin{cases} -\partial_x(h_1 v_1 + v_1^2/2)(t, x) & \text{for } 0 < x < 1 \\ -\partial_x(h_1 v_1 - v_1^2/2)(t, -x) & \text{for } -1 < x < 0. \end{cases}$$

We compute w_1 in term of ζ_1 . We have

$$\begin{aligned} h_1(t, x) &= \frac{1}{2}(\zeta_1(t, x) + \zeta_1(t, -x)) \\ v_1(t, x) &= \frac{1}{2}(\zeta_1(t, x) - \zeta_1(t, -x)). \end{aligned}$$

So,

$$\begin{aligned} h_1 v_1(t, x) &= \frac{1}{4}(\zeta_1^2(t, x) - \zeta_1^2(t, -x)) \\ \frac{1}{2}v_1^2(t, x) &= \frac{1}{8}(\zeta_1(t, x) - \zeta_1(t, -x))^2, \end{aligned}$$

thus,

$$\begin{aligned} h_1 v_1(t, x) + \frac{1}{2}v_1^2(t, x) &= \frac{1}{8}(3\zeta_1^2(t, x) - 2\zeta_1(t, x)\zeta_1(t, -x) - \zeta_1^2(t, -x)) \\ h_1 v_1(t, x) - \frac{1}{2}v_1^2(t, x) &= \frac{1}{8}(\zeta_1^2(t, x) + 2\zeta_1(t, x)\zeta_1(t, -x) - 3\zeta_1^2(t, -x)). \end{aligned}$$

Finally, denoting $r_1(t, x) = (3\zeta_1^2(t, x) - 2\zeta_1(t, x)\zeta_1(t, -x) - \zeta_1^2(t, -x))/8$, we write this as

$$\begin{aligned} h_1 v_1(t, x) + \frac{1}{2}v_1^2(t, x) &= r_1(t, x) \\ h_1 v_1(t, x) - \frac{1}{2}v_1^2(t, x) &= -r_1(t, -x). \end{aligned}$$

Finally, the right-hand side of (22) is

$$w_1(t, x) = \begin{cases} -\partial_x r_1(t, x) & \text{for } 0 < x < 1 \\ -\partial_x r_1(t, x) & \text{for } -1 < x < 0. \end{cases}$$

These computations are not specific to the case of the right-hand side of the second-order equation (22), but are valid whenever $w = -\mathcal{C}(\partial_x(hv), \partial_x(v^2/2))$ and $\zeta = \mathcal{C}(h, v)$.

We summarize these computations in the next lemma.

Lemma 12. *Let Q the quadratic form on \mathbb{R}^2 defined by $Q(a, b) := (3a^2 - 2ab - b^2)/8$. Let $\zeta = \mathcal{C}(h, v)$ for some $(h, v) \in H^1(0, 1) \times H_0^1(0, 1)$. Set*

$$w(x) := \mathcal{C}(-\partial_x(hv), -\partial_x(v^2/2)) \quad \text{and} \quad r(x) := Q(\zeta(x), \zeta(-x)).$$

Then

$$w(x) = -\partial_x r(x).$$

In case $\zeta(x) = \zeta_1(u, t, x)$, we will denote accordingly $w(x)$ by $w_1(u, t, x)$ and $r(x)$ by $r_1(u, t, x)$.

Remark 13. We recall that $\zeta_1(t, x+1) = -\zeta_1(t, x)$, so that $r_1(t, x+1) = r_1(t, x)$. So, w_1 , as well as ζ_2 is 1-periodic in x .

3.2 Kernel for ζ_2

In this section, we express ζ_2 (or more precisely scalar products of ζ_2) via a kernel that we compute explicitly. For $a, b \in \mathbb{R}$, we denote

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\}. \tag{23}$$

We begin with

Lemma 14. Let ϕ be a 1-periodic C^1 function. Let q be the bilinear symmetric form on \mathbb{R}^2 associated to the quadratic form Q defined in lemma 12, i.e., $q(a, b, a', b') = (3aa' - ab' - a'b - bb')/8$. Define $K_t = K_t(\phi)$ by

$$K_t(s_1, s_2) := \int_{\Omega} \phi'(t_1 + t - s_1 \vee s_2) q(\theta(t_1 - |s_2 - s_1|), \theta(t_2 - |s_2 - s_1|), \theta(t_1), \theta(t_2)) dt_1 dt_2, \quad (24)$$

where $\Omega = \{2(s_1 \vee s_2 - t) < t_1 + t_2 < 0, 0 < t_1 - t_2 < 2\}$. Let $u \in L^2(0, T)$ and let $\zeta_2(u, \cdot, \cdot)$ be the second-order correction for the water-tank system, i.e., the solution of $(\partial_t + \partial_x)\zeta_2(u, t, x) = w_1(u, t, x)$, $\zeta_2(u, 0, \cdot) = 0$ (where w_1 was defined in lemma 12). Then,

$$(\zeta_2(u, t, \cdot), \phi)_{L^2(\mathbb{T})} = \int_{[0, t]^2} K_t(s_1, s_2) u(s_1) u(s_2) ds_1 ds_2.$$

Proof. As usual, we will denote $\zeta_2(u, t, x)$ by $\zeta_2(t, x)$, leaving the fact that it depends on u implicit.

This is a mostly straightforward computations using the characteristics formula. Since ζ_2 satisfies the equation $(\partial_t + \partial_x)\zeta_2(t, x) = w_1(t, x)$ with $\zeta_2(0, \cdot) = 0$, then we have according to the characteristics formula:

$$\zeta_2(t, x) = \int_0^t w_1(s, s + x - t) ds.$$

Since $w_1(s, x) = -\partial_x r_1(s, x)$, integrating by parts, we have

$$(\zeta_2(t, \cdot), \phi)_{L^2(\mathbb{T})} = - \int_{\mathbb{T} \times [0, t]} \phi(x) \partial_x r_1(s, x + s - t) dx ds = \int_{\mathbb{T} \times [0, t]} \phi'(x) r_1(s, x + s - t) dx ds.$$

Since the integrand is 1-periodic (r_1 is according to remark 13, and we assumed that ϕ is 1-periodic), we rewrite this as

$$(\zeta_2(t, \cdot), \phi)_{L^2(\mathbb{T})} = 2 \int_{[0, 1] \times [0, t]} \phi'(x) r_1(s, x + s - t) dx ds. \quad (25)$$

Recall that if Q is a quadratic form on \mathbb{C}^d and q is its associated bilinear form, Fubini's theorem implies that for any compact subset X of \mathbb{R}^n and $f : X \rightarrow \mathbb{C}^d$ measurable bounded, we have $Q(\int_X f(s) ds) = \int_{X^2} q(f(s_1), f(s_2)) ds_1 ds_2$. Then, using the fact that $r_1(s, x) = Q(\zeta_1(s, x), \zeta_1(s, -x))$ and $\zeta_1(s, x) = \int_0^s u(s') \theta(x + s' - s) ds'$, we get

$$\begin{aligned} r_1(s, x) &= \int_{[0, s]^2} q(u(s_1) \theta(x + s_1 - s), u(s_1) \theta(-x + s_1 - s), u(s_2) \theta(x + s_2 - s), u(s_2) \theta(-x + s_2 - s)) ds_1 ds_2 \\ &= \int_{[0, s]^2} u(s_1) u(s_2) q(\theta(x + s_1 - s), \theta(-x + s_1 - s), \theta(x + s_2 - s), \theta(-x + s_2 - s)) ds_1 ds_2. \end{aligned}$$

Plugging this into equation (25), we get that the formula $(\zeta_2(t, \cdot), \phi) = \int_{[0, t]^2} K_t(s_1, s_2) u(s_1) u(s_2) ds_1 ds_2$ holds with

$$\begin{aligned} K_t(s_1, s_2) &= 2 \int_{[0, 1] \times [0, t]} \mathbb{1}_{s_1, s_2 \leq s} \phi'(x) q(\theta(x + s_1 - t), \theta(-x + s_1 - 2s + t), \theta(x + s_2 - t), \theta(-x + s_2 - 2s + t)) dx ds \\ &= 2 \int_{[0, 1] \times [s_1 \vee s_2, t]} \phi'(x) q(\theta(x + s_1 - t), \theta(-x + s_1 - 2s + t), \theta(x + s_2 - t), \theta(-x + s_2 - 2s + t)) dx ds. \end{aligned}$$

Since the integrand is 1-periodic in x , the change of variables $x' = x + s - t$ gives

$$K_t(s_1, s_2) = 2 \int_{[0, 1] \times [s_1 \vee s_2, t]} \phi'(x - s + t) q(\theta(x + s_1 - s), \theta(-x + s_1 - s), \theta(x + s_2 - s), \theta(-x + s_2 - s)) dx ds.$$

We see from this expression and the symmetry of q that $K_t(s_1, s_2) = K_t(s_2, s_1)$. So, to simplify the notation, we assume that $s_2 = s_1 \vee s_2$ and $s_1 = s_1 \wedge s_2$. Then, the change of variables $t_1 = x + s_2 - s$, $t_2 = -x + s_2 - s$, that satisfies $dx ds = \frac{1}{2} dt_1 dt_2$ and $x - s + t = t_1 - s_2 + t$ proves

$$K_t(s_1, s_2) = \int_{\Omega} \phi'(t_1 - s_2 + t) q(\theta(t_1 + s_1 - s_2), \theta(t_2 + s_1 - s_2), \theta(t_1), \theta(t_2)) dt_1 dt_2,$$

where Ω is the image of $[0, 1] \times [s_2, t]$. Since $x = (t_1 - t_2)/2$ and $s_2 - s = (t_1 + t_2)/2$, $\Omega = \{0 < t_1 - t_2 < 2, 2(s_2 - t) < t_1 + t_2 < 0\}$. Since we switched s_1 and s_2 so that $s_2 = s_1 \vee s_2$ and $s_1 = s_1 \wedge s_2$, this proves the lemma. \square

Proposition 15. *Let ϕ be a C^1 1-periodic function and $t \in (0, 2)$. The kernel K_t defined in lemma 14 is symmetric and for every $0 < s_1, s_2 < t$ such that $1 < s_2 - s_1$, we have $K_t(s_1, s_2) = -K_t(s_1 + 1, s_2)$. Moreover, for $0 < s_1 < s_2 < t$ and $s_2 - s_1 < 1$, we have*

$$2K_t(s_1, s_2) = \begin{cases} \int_{-2t+2s_2}^0 \phi(s+t-s_2) ds + 2(t-s_2)\phi(t-s_2) - 4(t-s_2)\phi(t-s_1) & \text{if } 2t-1 < s_1+s_2 < 2t \\ \int_{s_2-s_1}^{2-2t+s_2+s_1} \phi(s+t-s_2) ds + (-1+4t-3s_2-s_1)\phi(t-s_2) - (1+2t-3s_2+s_1)\phi(t-s_1) & \text{if } 2t-2 < s_1+s_2 < 2t-1 \\ \int_{2-2t+2s_2}^0 \phi(s+t-s_2) ds + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) & \text{if } 2t-3 < s_1+s_2 < 2t-2 \\ \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s+t-s_2) ds + (-2+4t-3s_2-s_1)\phi(t-s_2) - (2+2t-3s_2+s_1)\phi(t-s_1) & \text{if } 2t-4 < s_1+s_2 < 2t-3 \end{cases} \quad (26)$$

Proof. First, we see from the expression of K_t given in lemma 14 (or from its proof) that $K_t(s_1, s_2) = K_t(s_2, s_1)$. Moreover, if $0 < s_1 < s_2 - 1 < s_2 < t$, we see that setting $s'_1 = s_1 + 1$, $s'_2 = s_2$, we have $s_1 \vee s_2 = s'_1 \vee s'_2$ and that the integration set in formula (24) is the same for $K_t(s_1, s_2)$ and $K_t(s'_1, s'_2)$. Then, using the fact that $\theta(x+1) = -\theta(x)$ and the bilinearity of q , we get that $K_t(s_1 + 1, s_2) = -K_t(s_1, s_2)$. Thus, we only need to compute $K_t(s_1, s_2)$ when $0 < s_2 - s_1 < 1$.

Since $\theta(x)$ only take the value 1 and -1 the term $q(\theta(t_1 - s_2 + s_1), \theta(t_2 - s_2 + s_1), \theta(t_1), \theta(t_2))$ only take a finite number of values. To simplify notations, we set $\sigma = |s_2 - s_1|$, $\tau = t - s_2$ and

$$\alpha_{\sigma}(t_1, t_2) = q(\theta(t_1 - \sigma), \theta(t_2 - \sigma), \theta(t_1), \theta(t_2)).$$

The proof then consists in identifying which values α_{σ} takes and on which subsets of Ω . Then, we integrate $\int \phi'(t_1 + \tau)$ on these sets and sum everything with the right coefficient.

We remark that if a, b, a', b' are equal to ± 1 , then $q(a, b, a', b')$ is equal to 0 or $\pm 1/2$. Indeed, $q(1, 1, 1, 1) = 0$, $q(1, -1, 1, 1) = 1/2$, $q(1, -1, 1, -1) = 1/2$ and we get the other values using the bilinearity and the symmetry of q .

Remark that α_{σ} can only change value when t_1 or t_2 crosses the values $1k$ or $\sigma + 1k$ for some $k \in \mathbb{Z}$. We represent this in fig. 2.

We remark that the set where $\alpha_{\sigma} = 1/2$ is the intersection of three rectangles and Ω :

$$\underbrace{\Omega \cap [-2 + \sigma, -1] \times [-3 + \sigma, -2]}_{\Omega_{11}} \cup \underbrace{\Omega \cap [-1 + \sigma, 0] \times [-2, -1 + \sigma]}_{\Omega_{12}} \cup \underbrace{\Omega \cap [\sigma, 1] \times [-1, 0]}_{\Omega_{13}}$$

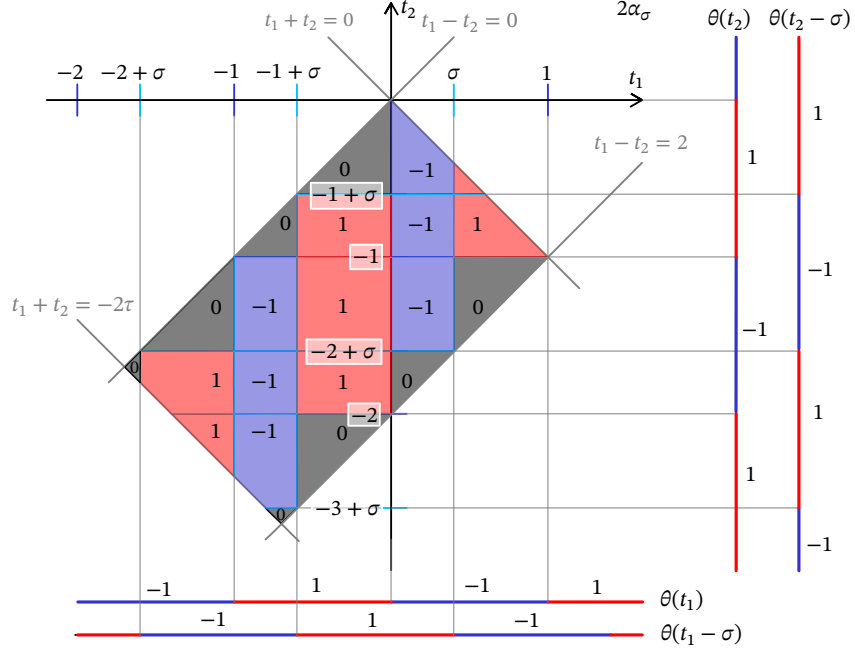


Figure 2: In light blue, the potential threshold for t_1 and t_2 where α_σ might change value. On the right, the values of $\theta(t_2 - \sigma)$ and $\theta(t_2)$. At the bottom, the values of $\theta(t_1 - \sigma)$ and $\theta(t_1)$. The diagonally placed rectangle is Ω . Inside Ω , we write what is the value of $2\alpha_\sigma(t_1, t_2)$.

while the set where $\alpha_\sigma = -1/2$ is the intersection of three rectangles and Ω :

$$\underbrace{\Omega \cap [-2, -2 + \sigma] \times [-3, -2 + \sigma]}_{\Omega_{-11}} \cup \underbrace{\Omega \cap [-1, -1 + \sigma] \times [-3 + \sigma, -1]}_{\Omega_{-12}} \cup \underbrace{\Omega \cap [0, \sigma] \times [-2 + \sigma, 0]}_{\Omega_{-13}}.$$

In other word, with the notations above,

$$K_t(s_1, s_2) = \frac{1}{2} \int_{\Omega_{11} \cup \Omega_{12} \cup \Omega_{13}} \phi'(t_1 + \tau) dt_1 dt_2 - \frac{1}{2} \int_{\Omega_{-11} \cup \Omega_{-12} \cup \Omega_{-13}} \phi'(t_1 + \tau) dt_1 dt_2.$$

Using Green's theorem, we get

$$K_t(s_1, s_2) = \frac{1}{2} \sum_{i=1}^3 \oint_{\partial\Omega_{1i}} \phi(t_1 + \tau) dt_2 - \frac{1}{2} \sum_{i=1}^3 \oint_{\partial\Omega_{-1i}} \phi(t_1 + \tau) dt_2.$$

The only thing left to do is identify the different cases where the $\Omega_{i,j}$ are empty, triangles, some other 4-polygon or 5-polygon and compute each of these integrals.

We detail one case, and give the result for the other with just a figure as explanation.

Step 1: Case $2t - 1 < s_1 + s_2 < 2t$ (fig. 3). In this case, the domains $\Omega_{i,j}$ look like the one of fig. 3. We have:

$$2K_t(s_1, s_2) = \underbrace{\int_0^\sigma \phi(s + \tau) ds - \int_0^\sigma \phi(s + \tau) ds}_{\text{"Diagonal" part of } \int_{\partial\Omega_{-1,3}}} + \underbrace{2\tau\phi(\tau) - 2\tau\phi(\sigma + \tau)}_{\text{"Vertical" part of } \int_{\partial\Omega_{-1,3}}}$$

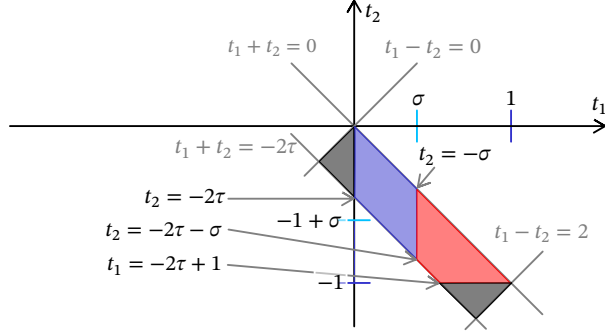


Figure 3: The equivalent of fig. 2 when $2t - 1 < s_1 + s_2$.

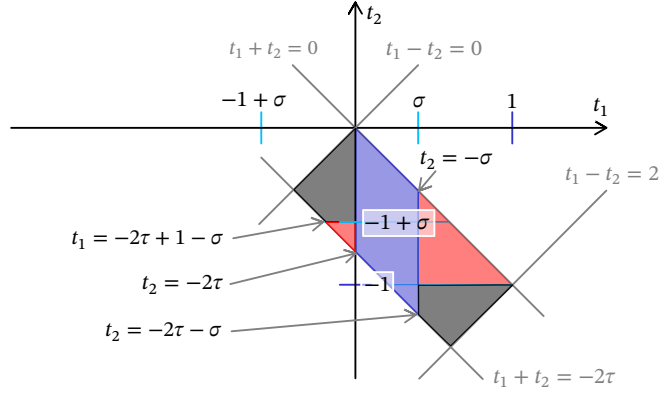


Figure 4: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 1 < 2s_1 + 1$.

$$\begin{aligned}
& \underbrace{- \int_{\sigma}^{-2\tau+1} \phi(s + \tau) ds + \int_{\sigma}^1 \phi(s + \tau) ds}_{\text{"Diagonal" part of } \int_{\partial\Omega_{13}}} - \underbrace{2\tau\phi(\sigma + \tau)}_{\text{"Vertical" part of } \int_{\partial\Omega_{13}}} \\
&= \int_{-2\tau+1}^1 \phi(s + \tau) ds + 2\tau\phi(\tau) - 4\tau\phi(\sigma + \tau) \\
&= \int_{-2t+2s_2}^0 \phi(s + t - s_2) ds + 2(t - s_2)\phi(t - s_2) - 4(t - s_2)\phi(t - s_1)
\end{aligned}$$

Step 2: Case $s_1 + s_2 < 2t - 1 < 2s_1 + 1$ (fig. 4).

$$2K_t(s_1, s_2) = \int_{s_2-s_1}^{2-2t+s_2+s_1} \phi(s - s_2 + t) ds + (4t - 1 - 3s_2 - s_1)\phi(t - s_2) - (1 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 3: Case $2s_1 < 2t - 2 < s_1 + s_2$ (fig. 5).

$$2K_t(s_1, s_2) = - \int_{2-2t+s_2+s_1}^{s_2-s_1} \phi(s - s_2 + t) ds + (4t - 1 - 3s_2 - s_1)\phi(t - s_2) - (1 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 4: Case $s_1 + s_2 < 2t - 2 < 2s_2$ (fig. 6).

$$2K_t(s_1, s_2) = - \int_0^{2-2t+2s_2} \phi(s + t - s_2) ds + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

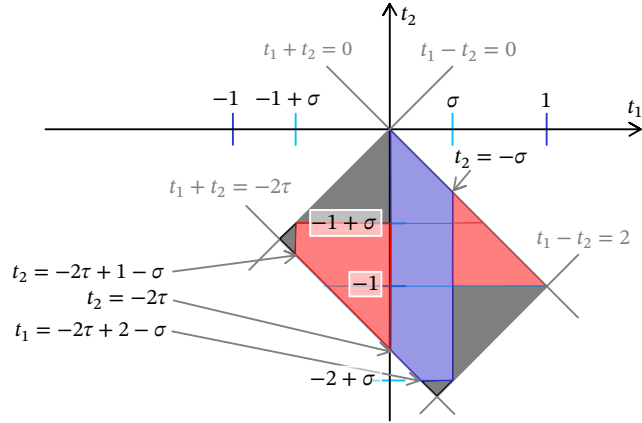


Figure 5: The equivalent of fig. 2 when $2s_1 < 2t - 2 < s_1 + s_2$.

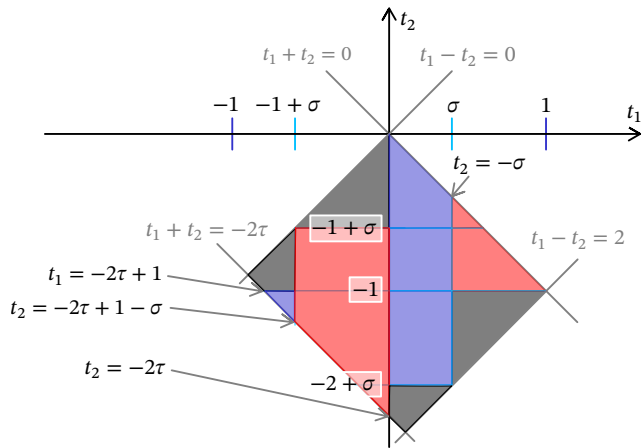


Figure 6: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 2 < 2s_2$.

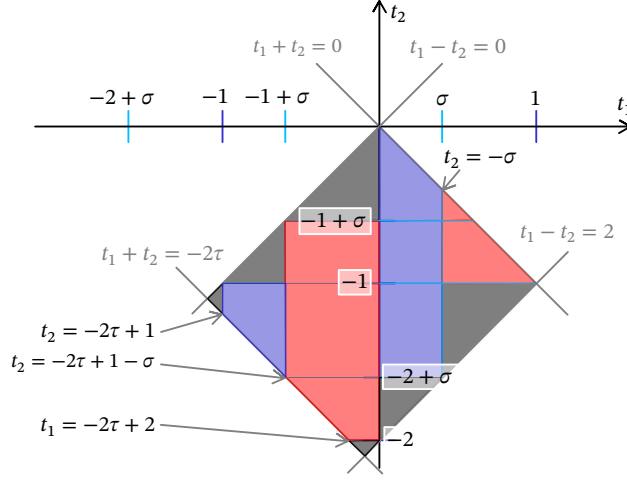


Figure 7: The equivalent of fig. 2 when $2s_2 - 1 < 2t - 3 < s_1 + s_2$.

Step 5: Case $2s_2 - 1 < 2t - 3 < s_1 + s_2$ (fig. 7).

$$2K_t(s_1, s_2) = \int_{2-2t+2s_2}^0 \phi(s - s_2 + t) ds + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

3

Step 6: Case $s_1 + s_2 < 2t - 3 < 2s_1 + 1$ (fig. 8).

$$2K_t(s_1, s_2) = \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s + t - s_2) ds + (-2 + 4t - 3s_2 - s_1)\phi(t - s_2) - (2 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 7: Case $2s_1 < 2t - 4 < s_1 + s_2$ (fig. 9).

$$2K_t(s_1, s_2) = \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s + t - s_2) ds + (-2 + 4t - 3s_2 - s_1)\phi(t - s_2) - (2 + 2t - 3s_2 + s_1)\phi(t - s_1). \quad \square$$

When the control u steers the linearized equation (8) from 0 to 0, we can prove that this kernel acts as another, simpler one.

Proposition 16. Let $T \in (1, 2)$. Let $\phi \in C^1(\mathbb{T})$ that is 1-periodic. We define the reduced kernel $K_T^{\text{red}}: [0, T - 1]^2 \rightarrow \mathbb{R}$ by

$$K_T^{\text{red}}(s_1, s_2) := \frac{3}{2}(1 - |s_2 - s_1|)(\phi(T - s_1 \vee s_2) - \phi(T - s_1 \wedge s_2)).$$

Let $u \in L^2(0, T)$ that steers the linearized equation (16) from 0 to 0 (i.e., $\zeta_1(u, T, \cdot) = 0$). Let $\zeta_2(u, \cdot, \cdot)$ be the second-order correction for the water-tank system, i.e., the solution of (22). Then,

$$(\zeta_2(u, T, \cdot), \phi)_{L^2(\mathbb{T})} = \int_{[0, T-1]^2} K_T^{\text{red}}(s_1, s_2) u(s_1) u(s_2) ds_1 ds_2. \quad (27)$$

The two important points of this formula, is that the expression of the reduced kernel is simpler, and that we integrate on $[0, T - 1]^2$ instead of $[0, T]^2$.

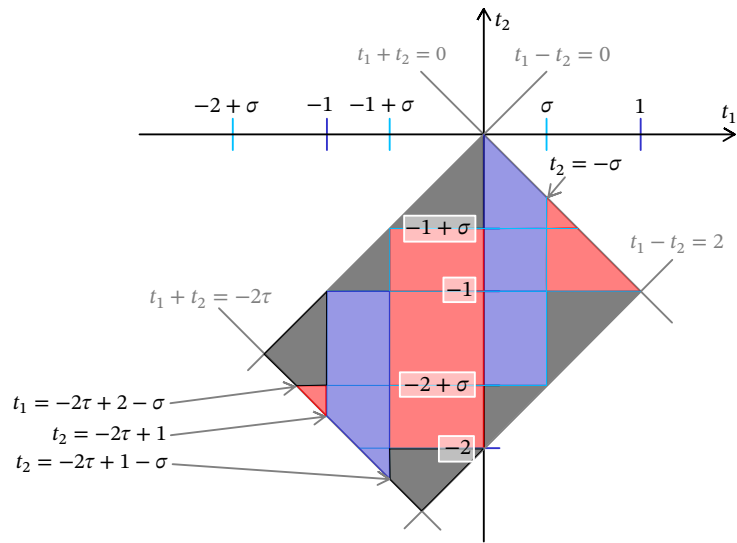


Figure 8: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 3 < 2s_1 + 1$.

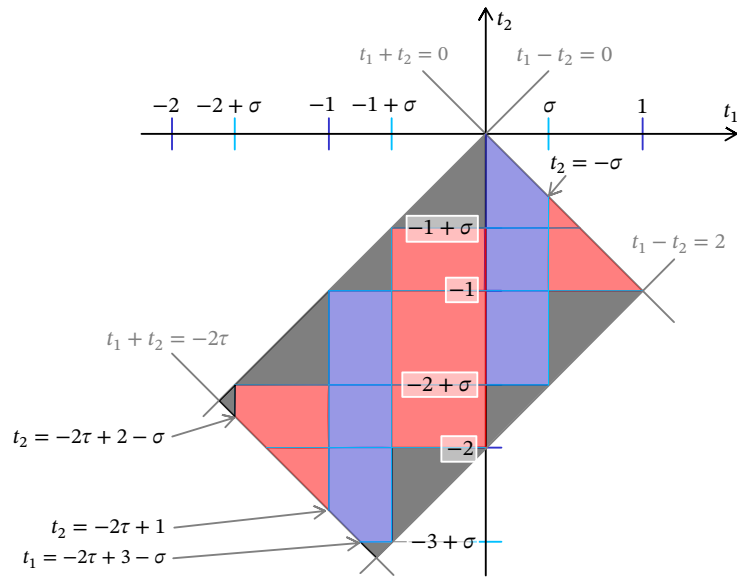


Figure 9: The equivalent of fig. 2 when $2s_1 < 2t - 4 < s_1 + s_2$.

Proof. Step 1: Expression of K_T^{red} as a function of K_T . According to proposition 10, we have for every $T-1 < s < 1$, $u(s) = 0$ and $u(s+1) = u(s)$. Thus, according to the proposition 15 we have

$$\begin{aligned} (\zeta_2(T, \cdot), \phi) &= \int_{[0, T]^2} K_T(s_1, s_2) u(s_1) u(s_2) ds_1 ds_2 \\ &= \int_{[0, T-1]^2} (K_T(s_1, s_2) + K_T(1+s_1, s_2) + K_T(s_1, 1+s_2) + K_T(1+s_1, 1+s_2)) u(s_1) u(s_2) ds_1 ds_2. \end{aligned}$$

Thus, equation (27) holds with $K_T^{\text{red}}(s_1, s_2) = K_T(s_1, s_2) + K_T(1+s_1, s_2) + K_T(s_1, 1+s_2) + K_T(1+s_1, 1+s_2)$. Since K_T (and also K_T^{red}) are symmetric in s_1, s_2 , we may assume that $s_1 \leq s_2$. Then, with $s'_2 := 1+s_2$ and $s'_1 := s_1$, we have $s'_1+1 \leq s'_2$, thus, according to proposition 15, we have $K_T(s'_1, s'_2) = -K_T(1+s'_1, s'_2)$. Thus, $K_T(s_1, 1+s_2) + K_T(1+s_1, 1+s_2) = 0$ and $K_T^{\text{red}}(s_1, s_2) = K_T(s_1, s_2) + K_T(1+s_1, s_2)$.

We end the computation by using the formula for K_T of proposition 15. We have $0 < s_1 \leq s_2 < T-1$ and $1 < T < 2$. So $2T-4 < 0 < s_1+s_2 < 2T-2$. We consider two cases: $2T-3 < s_1+s_2 < 2T-2$ and $2T-4 < s_1+s_2 < 2T-3$.

Step 2: Case $2T-3 < s_1+s_2 < 2T-2$. To compute $K_T(s_1, s_2)$, we use the third case of the expression (26) of K_T . To compute $K_T(1+s_1, s_2)$, we remark that with $s'_1 := s_2$ and $s'_2 := 1+s_1$, we have $s'_1 < s'_2$ and $2T-2 < s'_1+s'_2 < 2T-1$. Thus, $K_T(1+s_1, s_2) = K_T(s'_1, s'_2)$ is computed with the second case of the expression (26) of K_T . We get

$$\begin{aligned} 2K_T^{\text{red}}(s_1, s_2) &= 2K_T(s_1, s_2) + 2K_T(s'_1, s'_2) \\ &= \int_{2-2t+2s_2}^0 \phi(s+t-s_2) ds + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) \\ &\quad + \int_{s'_2-s'_1}^{2-2t+s'_2+s'_1} \phi(s+t-s'_2) ds + (-1+4t-3s'_2-s'_1)\phi(t-s'_2) - (1+2t-3s'_2+s'_1)\phi(t-s'_1) \\ &= \int_{2-2t+2s_2}^0 \phi(s+t-s_2) ds + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) \\ &\quad + \int_{1+s_1-s_2}^{3-2t+s_1+s_2} \phi(s+t-s_1) ds + (-4+4t-3s_1-s_2)\phi(t-s_1) - (-2+2t-3s_1+s_2)\phi(t-s_2) \\ &= \int_{2-2t+2s_2}^0 \phi(s+t-s_2) ds + \int_{1+s_1-s_2}^{3-2t+s_1+s_2} \phi(s+t-s_1) ds \\ &\quad + (3-3s_2+3s_1)\phi(t-s_2) - (3-3s_2+3s_1)\phi(t-s_1). \end{aligned}$$

In the second integral, we make the change of variables $s' = s + s_2 - s_1$:

$$\begin{aligned} 2K_T^{\text{red}}(s_1, s_2) &= \int_{2-2t+2s_2}^0 \phi(s+t-s_2) ds + \int_1^{3-2t+2s_2} \phi(s+t-s_2) ds + 3(1-s_2+s_1)(\phi(t-s_2) - \phi(t-s_1)) \\ &= 3(1-s_2+s_1)(\phi(t-s_2) - \phi(t-s_1)), \end{aligned}$$

where we used the 1-periodicity of ϕ to cancel the two integrals. Since we switched s_1 and s_2 to have $s_1 = s_1 \vee s_2$ and $s_2 = s_1 \wedge s_2$, this is indeed the claimed formula.

Step 3: Case $2T-4 < s_1+s_2 < 2T-3$. This case is treated in the same way, the only difference being that $K_T(s_1, s_2)$ is computed using the fourth case of the expression (26) of K_T , and $K_T(1+s_1, s_2)$ is computed using the third case of the same expression. We get the same formula. \square

3.3 Coercivity of the kernel

In this section, we use the expression of $(\zeta_2(u, T, \cdot), \phi)$ given in proposition 16 to prove that when $1 < T < 2$, $|\zeta_2(u, T, \cdot)|$ is lower-bounded by essentially $\|u\|_{H^{-1}}^2$. To do that, we first have to choose the right function ϕ .

Definition 17. Let $1 < T < 2$ and let ϕ be a C^∞ 1-periodic function such that $\phi(s) = s$ in $[0, T]$.

Proposition 18. If $1 < T < 2$ and $u \in L^2(0, T)$ steers a solution of the linearized equation (16) from 0 to 0 (i.e., $\zeta_1(u, T, \cdot) = 0$) and if $\int_0^T u(t) dt = 0$, then denoting $U(t) = \int_0^t u(s) ds$,

$$(\zeta_2(u, T, \cdot), \phi)_{L^2} \geq 3(2 - T)\|U\|_{L^2}^2,$$

where ϕ is a function given in definition 17.

Proof. We first simplify the expression of K_T^{red} given by proposition 16. For $0 < s_1, s_2 < T - 1$, we have $1 < T - s_1 \vee s_2 \leq T - s_1 \wedge s_2 < T$, thus, according to the definition of ϕ , we have for $0 < s_1, s_2 < T - 1$,

$$\begin{aligned} K_T^{\text{red}}(s_1, s_2) &= \frac{3}{2}(1 - |s_2 - s_1|)((T - s_1 \vee s_2) - (T - s_1 \wedge s_2)) \\ &= -\frac{3}{2}(1 - |s_2 - s_1|)|s_2 - s_1| \\ &= \frac{3}{2}(-|s_2 - s_1| + (s_2 - s_1)^2). \end{aligned}$$

Thus, according to proposition 16, if u is as in the statement of proposition 18,

$$(\zeta_2(u, T, \cdot), \phi) = -\frac{3}{2} \int_{[0, T-1]^2} |s_2 - s_1| u(s_1) u(s_2) ds_1 ds_2 + \frac{3}{2} \int_{[0, T-1]^2} (s_2 - s_1)^2 u(s_1) u(s_2) ds_1 ds_2 \quad (28)$$

The main point of the proof at hand is the following computations:

Lemma 19. Let $a > b$, let $u \in L^2(a, b)$ with $\int_a^b u(s) ds = 0$, and let U be a primitive of u with $U(a) = U(b) = 0$. We have

$$\int_{[a, b]^2} |s_1 - s_2| u(s_1) u(s_2) ds_1 ds_2 = -2\|U\|_{L^2(a, b)}^2; \quad (29)$$

$$\int_{[a, b]^2} (s_1 - s_2)^2 u(s_1) u(s_2) ds_1 ds_2 = -2 \left(\int_a^b U(s) ds \right)^2. \quad (30)$$

Proof. The fact that $\int_a^b u(s) ds = 0$ implies that $U(x) = \int_a^x u(s) ds = -\int_x^b u(s) ds$. Then, the proof of

the first formula essentially consists in integrating by parts in s_1 :

$$\begin{aligned}
\int_{[a,b]^2} |s_1 - s_2| u(s_1) u(s_2) &= \int_a^b u(s_2) \left(\int_a^{s_2} (s_2 - s_1) u(s_1) ds_1 + \int_{s_2}^b (s_1 - s_2) u(s_1) ds_1 \right) ds_2 \\
&= \int_a^b u(s_2) \left(\int_a^{s_2} U(s_1) ds_1 - \int_{s_2}^b U(s_1) ds_1 \right) ds_2 \\
&= \int_{[a,b]^2} \text{sgn}(s_2 - s_1) U(s_1) u(s_2) ds_1 ds_2 \\
&= \int_a^b U(s_1) \left(- \int_a^{s_1} u(s_2) ds_2 + \int_{s_1}^b u(s_2) ds_2 \right) ds_1 \\
&= -2 \int_a^b U(s_1)^2 ds_1.
\end{aligned}$$

For the second formula, we develop $(s_2 - s_1)^2$:

$$\begin{aligned}
&\int_{[a,b]^2} (s_2 - s_1)^2 u(s_1) u(s_2) ds_1 ds_2 \\
&= \int_a^b s_1^2 u(s_1) ds_1 \int_a^b u(s_2) ds_2 - 2 \int_a^b s_1 u(s_1) ds_1 \int_a^b s_2 u(s_2) ds_2 + \int_a^b u(s_1) ds_1 \int_a^b s_2^2 u(s_2) ds_2. \\
&= -2 \left(\int_a^b s u(s) ds \right)^2,
\end{aligned}$$

where we used that $\int_a^b u(s) ds = 0$. Finally, we integrate by parts:

$$\int_{[a,b]^2} (s_2 - s_1)^2 u(s_1) u(s_2) ds_1 ds_2 = -2 \left(\int_a^b U(s) ds \right)^2. \quad \diamond$$

Plugging the two formulas of lemma 19 into the expression (28), we get

$$(\zeta_2(u, T, \cdot), \phi) = 3 \|U\|_{L^2}^2 - 3 \left(\int_0^{T-1} U(s) ds \right)^2. \quad (31)$$

According to Cauchy-Schwarz inequality, we have $|\int_0^{T-1} U(s) ds| \leq \sqrt{T-1} \|U\|_{L^2}$. Thus

$$(\zeta_2(u, T, \cdot), \phi) \geq 3(2-T) \|U\|_{L^2}^2 \quad \square$$

Remark 20. How did we choose the ϕ of definition 17? Long story short, it turns out that if ϕ is monotone on $[1, T]$, the “conditional H^{-1} -coercivity” of the kernel of proposition 16 is equivalent to the condition $\int_1^T \phi'(s) ds \int_1^T (\phi'(s))^{-1} ds < (3-T)^2$ (we sketch the proof of this fact in appendix A). Hence, the smaller the left-hand-side of this condition, the larger the time of non-local-controllability. With some calculus of variations, we can see that if ϕ minimizes the left-hand side, then ϕ' is constant on $[1, T]$, hence our choice of ϕ .

4 Nonlinear equation

The proposition 18 shows that if the time T smaller than 2 and if u steers the linearized equation (16) from 0 to 0, then $\|\zeta_2(u, T, \cdot)\|_{L^2} \geq c\|U\|_{L^2(0,T)}^2$ (where $U(t) = \int_0^t u(s) ds$). As in the previous section, we fix $T \in (1, 2)$. Our aim now is to prove that the solution of the *nonlinear* equation also have this property, as long as $\|u\|_{C^0}$ is small enough. As a consequence, one cannot move the water-tank in time T with a control small in C^0 -norm, and that finishes the proof of theorem 1.

To this end, we use the fact that if $\|u\|$ is small enough, the solution of the nonlinear equation is well approximated by $(h_1, v_1) + (h_2, v_2)$, where (h_1, v_1) solves the linearized system (8) and (h_2, v_2) solves the second order system (9).

4.1 Well-posedness of the water-tank system and properties of the first and second order approximation system

In this section, we state several basic results on the nonlinear and the linear systems related to the water-tank system (1). We begin with the well-posedness of the water-tank system, where, as in the rest of the article, $g = 1$ and $L = 1$.

Proposition 21. *Let $T > 0$. There exists $\epsilon > 0$ such that for $(H_0, v_0) \in [C^1([0, 1])]^2$ that satisfies*

$$\|u\|_{C^0[0,T]} + \|(H_0, v_0) - (1, 0)\|_{C^1[0,1]} < \epsilon,$$

as well as the compatibility conditions

$$\partial_x H_0(0) = \partial_x H_0(1) = -u(0),$$

there exists a unique solution $(H_{nl}, v_{nl}) \in [C^1([0, T] \times [0, 1])]^2$ of the water-tank system (1) with $H_{nl}(0, x) = H_0(x)$ and $v_{nl}(0, x) = v_0(x)$. Moreover,

$$\|(H_{nl}, v_{nl}) - (1, 0)\|_{C^1([0,T] \times [0,1])} \leq C \left(\|u\|_{C^0[0,T]} + \|(H_0, v_0) - (1, 0)\|_{C^1[0,1]} \right), \quad (32)$$

for some positive constant C depending only on T .

Proof. In this proof, we drop the index nl and write just (H, v) for (H_{nl}, v_{nl}) .

Standard results for the well-posedness of hyperbolic systems assume that all coefficients are at least C^1 , but here we assume that u is only C^0 . In order to achieve that, we note that if (H, v) solves the water tank system (1), then with V defined by $v(t, x) = V(t, x) - U(t)$, where, as usual, $U(t) = \int_0^t u(s) ds$, the water-tank system becomes

$$\begin{cases} \partial_t H + \partial_x((V - U)H) = 0; \\ \partial_t V + \partial_x \left(H + \frac{(V-U)^2}{2} \right) = 0; \\ V(t, 0) = V(t, 1) = U(t), \end{cases} \quad (33)$$

where all the coefficients are now C^1 . This system can be written in the form

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \begin{pmatrix} V - U & H \\ 1 & V - U \end{pmatrix} \partial_x \begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0, T] \times [0, 1] \quad (34)$$

and

$$V(t, 0) = V(t, 1) = U(t) \text{ in } [0, T]. \quad (35)$$

System (34) and (35) is not in the standard form of the quasilinear hyperbolic system since the control U also appears in the nonlinearity. Nevertheless, the proof can be derived from the standard fixed point arguments, see, e.g., [21, Chapter 4] and [16, The proof of Lemma 2.2]. We now outline the proof. Set

$$(H^{(0)}, V^{(0)})(t, x) = (H_0(x), V_0(x)) \quad \text{in } [0, T] \times [0, 1],$$

and define $(H^{(n)}, V^{(n)})$ in $[0, T] \times [0, L]$ for $n \geq 1$ by

$$\partial_t \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} + \begin{pmatrix} V^{(n-1)} - U & H^{(n-1)} \\ 1 & V^{(n-1)} - U \end{pmatrix} \partial_x \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0, T] \times [0, 1], \quad (36)$$

with the corresponding boundary conditions. Using the characteristic method, we have, if

$$\|(H^{(n-1)} - 1, V^{(n-1)}, U)\|_{C^1([0, T] \times [0, 1])} \leq C\epsilon,$$

then

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^0([0, T] \times [0, 1])} \leq C(\|U\|_{C^0[0, T]} + \|(H_0, V_0) - (1, 0)\|_{C^0[0, 1]})$$

and by taking the derivative of the equation with respect to t , we also obtain

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0, T] \times [0, 1])} \leq C(\|u\|_{C^0[0, T]} + \|(H_0, V_0) - (1, 0)\|_{C^1[0, 1]}).$$

We derive that

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0, T] \times [0, 1])} \leq C\epsilon.$$

Set

$$\rho_n(r) := \sup_{\substack{|(t,x)-(s,y)| < r \\ (t,x), (s,y) \in [0, T] \times [0, 1]}} |(H^{(n)}(t, x) - H^{(n)}(s, y), V^{(n)}(t, x) - V^{(n)}(s, y))|$$

Using the characteristic, we can prove that there exists a positive constant γ depending only on T such that for ϵ sufficiently small,

$$\rho_n(r) \leq C \left(\sup_{\substack{|t-s| < \gamma r \\ t, s \in [0, T]}} |u(t) - u(s)| + \sup_{\substack{|x-y| < \gamma r \\ x, y \in [0, L]}} |(H_0, V_0) - (1, 0)| \right).$$

Using Ascoli's theorem, one can conclude that there exists up to a subsequence $(H^{(n)}, V^{(n)})$ converges to (H, V) in $C^1([0, T] \times [0, 1])$.

Considering the system solved by $(H^{(n+1)} - H^{(n)}, V^{(n+1)} - V^{(n)})$, one can check that

$$\begin{aligned} \|(H^{(n+1)} - H^{(n)}, V^{(n+1)} - V^{(n)})\|_{C^0([0, T] \times [0, 1])} \\ \leq C\epsilon \|(H^{(n)} - H^{(n-1)}, V^{(n)} - V^{(n-1)})\|_{C^0([0, T] \times [0, 1])}. \end{aligned} \quad (37)$$

Thus $(H^{(n)}, V^{(n)})$ converges to (H, V) in $C^0([0, T] \times [0, L])$. We thus derive that $(H, V) \in C^1([0, T] \times [0, 1])$ is the corresponding solution.

The uniqueness follows as in (37). \square

Remark 22. We do not need this for the proofs below, but it is worth noting that standard methods using the propagation along characteristics can be used to prove the lack of local-controllability around equilibrium states in time $T < T_*$. Let us sketch it.

Consider the characteristic speeds λ_{\pm} and Riemann invariants R_{\pm} , which are given by²

$$\begin{aligned}\lambda_{\pm} &= v \pm \sqrt{H}; \\ R_{\pm} &= v \pm 2\sqrt{H} + U.\end{aligned}$$

We have

$$\begin{cases} (\partial_t + \lambda_{\pm} \partial_x) R_{\pm} = 0; \\ R_{\pm}(t, 0) = -R_{\mp}(t, 1) + 2U. \end{cases}$$

Consider also the characteristics, i.e., the solutions x_{\pm} of the Cauchy problem

$$\begin{cases} \partial_t x_{\pm}(t, t_0, x_{t_0}) = \lambda_{\pm}(x_{\pm}(t, t_0, x_{t_0})); \\ x_{\pm}(t_0, t_0, x_{t_0}) = x_{t_0}. \end{cases}$$

Then, differentiating in t and using the equation for R_{+} , we get that $R_{+}(t, x_{+}(t, t_0, x_0))$ does not depend on t (as long as $x_{+}(t, t_0, 0)$ is defined, i.e., stays inside $[0, 1]$). Hence

$$R_{+}(t, x(t, t_0, 0)) = R_{+}(t_0, 0) = -R_{-}(t_0, 1) + 2U(t_0).$$

Hence, if $R_{\pm}(T, \cdot) = 0$, $0 < t_0 < T$, and if $x_{+}(T, t_0, 0)$ is defined, $U(t_0) = 0$. The characteristic speed depends on the solution, and thus on the control, but if the control is small, the characteristic speeds are $\lambda_{\pm}(t, x) = \pm 1 + O(\|u\|_{C^0})$, which implies that $x_{+}(t, t_0, 0) = t - t_0 + O(\|u\|_{C^0})$. Hence, the computations outlined above are valid if $T < 1 - O(\|u\|_{C^0})$.

In what follows, for $T > 0$, we use the notations $L_t^2 L_x^2$ and $L_t^{\infty} L_x^2$ as a shorthand for $L^2(0, T; L^2(\mathbb{T}))$ and $L^{\infty}(0, T; L^2(\mathbb{T}))$.

For the linearized system (16), we have the following proposition.

Proposition 23. *Let $T > 0$. The solution ζ of $(\partial_t + \partial_x)\zeta = w$ satisfies:*

$$\|\zeta\|_{L_t^2 L_x^2} \leq C \|w\|_{L_t^2 L_x^2}. \quad (38)$$

Moreover, in case the right-hand side is $w(t, x) = u(t)\vartheta(x)$, if we set $U(t) := \int_0^t u(s) ds$, then the solution $\zeta_1(u, \cdot, \cdot)$ satisfies

$$\|\zeta_1(u)\|_{L_t^2 L_x^2} \leq C \|U\|_{L^2}. \quad (39)$$

Proof. The first inequality is standard, and is proved with the characteristic formula (lemma 7) and Cauchy-Schwarz inequality:

$$\begin{aligned}\|\zeta\|_{L_t^2 L_x^2}^2 &= \int_{[0, T]^3 \times \mathbb{T}} \mathbb{1}_{s_1, s_2 \leq t} w(s_1, x + s_1 - t) w(s_2, x + s_2 - t) ds_1 ds_2 dt dx \\ &\leq \int_{[0, T]^3 \times \mathbb{T}} w(s_1, x + s_1 - t)^2 ds_1 ds_2 dt dx \\ &= \int_{[0, T]^3 \times \mathbb{T}} w(s_1, x')^2 ds_1 ds_2 dt dx',\end{aligned}$$

where we also used the change of variables $x' = x + s_1 - t$. This implies the claimed estimate (38).

For the second inequality, we claim that for $0 < x < 1$ and $t > 0$,

$$\zeta_1(u, t, x) = -U(t) + 2 \sum_{k=0}^{+\infty} (-1)^k U(t - x - k), \quad (40)$$

²The Riemann invariant as defined in [2, Section 1.4] do not have the $+U$ term. But in our case, it is convenient to add it.

where we extended u by 0 in $(-\infty, 0]$.³ Indeed, if we define $\tilde{\zeta}_1$ as the right hand side of this formula, we see that $\tilde{\zeta}_1(u, t, 1) = -\tilde{\zeta}_1(u, t, 0)$, so that the 1-antiperiodic extension of $\tilde{\zeta}_1$ is continuous in $(t, x) \in [0, T] \times \mathbb{T}$. Moreover, we see that for $0 < x < 1$ and $t > 0$

$$(\partial_t + \partial_x)\tilde{\zeta}_1(u, t, x) = -u(t).$$

Thus, if we still denote by $\tilde{\zeta}_1$ the 1-antiperiodic extension of $\tilde{\zeta}_1$, we have $(\partial_t + \partial_x)\tilde{\zeta}_1 = u(t)\theta(x)$. Thus, $\tilde{\zeta}_1 = \zeta_1$. This expression (40) implies the claimed estimate (39). \square

4.2 Proof of theorem 1

In this section, $(H_{\text{nl}}(u), v_{\text{nl}}(u)) = (1 + h_{\text{nl}}(u), v_{\text{nl}}(u))$ is the solution of the water-tank system (1) with control u . We will often conflate this solution and $\zeta_{\text{nl}}(u) := \mathcal{C}(h_{\text{nl}}(u), v_{\text{nl}}(u))$. The same will be done for the solution $(h_1(u), v_1(u))$ of the linearized system (8) and $\zeta_1(u) := \mathcal{C}(h_1(u), v_1(u))$ (solution of (16)), as well as the solution $(h_2(u), v_2(u))$ of (9) and $\zeta_2(u) := \mathcal{C}(h_2(u), v_2(u))$. If anything, this will make the notations more lightweight.

We will also set $w_{\text{nl}}(u) := -\mathcal{C}(\partial_x(h_{\text{nl}}(u)v_{\text{nl}}(u)), \partial_x(v_{\text{nl}}(u)^2/2))$, so that $\zeta_{\text{nl}}(u)$ satisfies $(\partial_t + \partial_x)\zeta_{\text{nl}}(u, t, x) = w_{\text{nl}}(u, t, x) + u(t)\theta(x)$. We also denote the right-hand side of the equation (22) satisfied by $\zeta_2(u)$ by $w_1(u, t, x)$, i.e., $w_1(u) = -\mathcal{C}(\partial_x(h_1(u)v_1(u)), \partial_x(v_1(u)^2/2))$. Finally, we set $\delta_1(u) := \zeta_{\text{nl}}(u) - \zeta_1(u)$ and $\delta_2(u) := \zeta_{\text{nl}}(u) - \zeta_1(u) - \zeta_2(u)$.

The basic idea of the proof of theorem 1 is that $|\zeta_2(u, T, \cdot)| \geq c\|U\|_{L^2}^2$ and that $\zeta_1(u) + \zeta_2(u)$ is (supposed to be) a good approximation of $\zeta_{\text{nl}}(u)$.

We start with an estimate for the nonlinear equation, which is a consequence of the nonlinear well-posedness (proposition 21) and the linear estimates (proposition 23):

Corollary 24. *Let $T > 0$. There exists $\eta > 0$ and $C > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0([0, T])} < \delta$, there exists a unique solution $(H_{\text{nl}}, v_{\text{nl}}) \in [C^1([0, T] \times [0, 1])]^2$ of the water-tank system (1) with $(H_{\text{nl}}, v_{\text{nl}})(0, \cdot) = (1, 0)$. Moreover, with the notation ζ_{nl} defined at the beginning of this section, we have*

$$\|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \leq C\|U\|_{L^2(0, T)}.$$

Proof. The existence and uniqueness is a consequence of the well-posedness (proposition 21). Let us now prove the inequality. We write $\zeta_{\text{nl}} = \zeta_1 + \delta_1$.

According to proposition 23, we have $\|\zeta_1\|_{L_t^2 L_x^2} \leq C\|U\|_{L^2}$ and $\|\delta_1\|_{L_t^2 L_x^2} \leq C\|w_{\text{nl}}\|_{L^2}$. Since w_{nl} can be written as $-\partial_x r_{\text{nl}}$ where $r_{\text{nl}}(t, x)$ is a quadratic form of $\zeta_{\text{nl}}(t, x)$ and $\zeta_{\text{nl}}(t, -x)$ (lemma 12), we have $\|w_{\text{nl}}\|_{L_t^2 L_x^2} \leq C\|\partial_x \zeta_{\text{nl}}\|_{L^\infty} \|\zeta_{\text{nl}}\|_{L_t^2 L_x^2}$. Thus,

$$\|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \leq C(\|U\|_{L^2} + \|\partial_x \zeta_{\text{nl}}\|_{L^\infty} \|\zeta_{\text{nl}}\|_{L_t^2 L_x^2}).$$

Finally, since $\|\partial_x \zeta_{\text{nl}}\|_{L^\infty} \leq \|\mathcal{C}(h_{\text{nl}}, v_{\text{nl}})\|_{W^{1, \infty}} \leq 2\|(h_{\text{nl}}, v_{\text{nl}})\|_{C^1}$ (see remark 6), we have according to the well-posedness estimate of proposition 21 $\|\partial_x \zeta_{\text{nl}}\|_{L^\infty} \leq C\|u\|_{C^0} \leq C\eta$. Thus,

$$\|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \leq C\|U\|_{L^2} + C\eta\|\zeta_{\text{nl}}\|_{L_t^2 L_x^2},$$

which implies for η small enough

$$\|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \leq \frac{C}{1 - C\eta}\|U\|_{L^2}. \quad \square$$

³Note that with this extension, we have for $t \leq 0$, $U(t) = 0$, so that there is only a finite number of non-zero terms in the previous sum.

Next, we prove the approximation property:

Lemma 25. *Let $\phi \in C^1(\mathbb{T})$. Let $T > 0$ and $u \in H_0^1(0, T)$ with $\|u\|_{H_0^1} < \eta$, and set $U(t) := \int_0^t u(s) ds$. Then, with the notations above, for some $C > 0$ independent of u ,*

$$\|\delta_1(u, \cdot, \cdot)\|_{L_t^\infty L_x^2} \leq C \|U\|_{L^2} \|u\|_{C^0}; \quad (41)$$

$$|(\delta_2(u, T, \cdot), \phi)| \leq C \|U\|_{L^2}^2 \|u\|_{C^0}. \quad (42)$$

Proof. Step 1: Estimate of δ_1 in L^2 -norm. We have $(\partial_t + \partial_x)\delta_1 = w_{\text{nl}}$, thus, using Duhammel's formula,

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C \|w_{\text{nl}}\|_{L_t^1 L_x^2}.$$

Since $w_{\text{nl}} = -\mathcal{C}(\partial_x(h_{\text{nl}}v_{\text{nl}}), \partial_x(v_{\text{nl}}^2/2))$, we can use lemma 12 to write $w_{\text{nl}} = -\partial_x r_{\text{nl}}$ with $r_{\text{nl}}(t, x) = Q(\zeta_{\text{nl}}(t, x), \zeta_{\text{nl}}(t, -x))$. Thus,

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C \|\partial_x r_{\text{nl}}\|_{L_t^1 L_x^2}.$$

Since Q is a quadratic form (see lemma 12), $\partial_x r_{\text{nl}}$ is a sum of products of ζ_{nl} and $\partial_x \zeta_{\text{nl}}$ evaluated at (t, x) or $(t, -x)$. Thus, we get

$$\begin{aligned} \|\delta_1\|_{L_t^\infty L_x^2} &\leq C \|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \|\partial_x \zeta_{\text{nl}}\|_{L_t^\infty L_x^\infty} \\ &\leq C \|\zeta_{\text{nl}}\|_{L_t^2 L_x^2} \|(h_{\text{nl}}, v_{\text{nl}})\|_{C^1([0, T] \times [0, 1])}, \end{aligned}$$

where we used that the change of variables \mathcal{C} (see definition 4) is such that for $(h, v) \in C^1([0, 1])$ with $v(0) = v(1) = 0$, then $\|\mathcal{C}(h, v)\|_{W^{1, \infty}} \leq 2\|(h, v)\|_{C^1}$. Finally, using the well-posedness estimates of proposition 21, we get

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C \|U\|_{L^2} \|u\|_{C^0}. \quad (43)$$

Step 2: Estimation on (δ_2, ϕ) . The function δ_2 is solution of $(\partial_t + \partial_x)\delta_2 = w_{\text{nl}} - w_1$. Thus, using the characteristics formula (lemma 7),

$$\begin{aligned} (\delta_2(u, T, \cdot), \phi) &= \int_{x \in \mathbb{T}} \delta_2(u, T, x) \phi(x) dx \\ &= \int_{[0, T] \times \mathbb{T}} (w_{\text{nl}} - w_1)(u, s, x + s - T) \phi(x) ds dx. \end{aligned}$$

We can use lemma 12 to write $w_{\text{nl}} = -\partial_x r_{\text{nl}}$ with $r_{\text{nl}}(t, x) = Q(\zeta_{\text{nl}}(t, x), \zeta_{\text{nl}}(t, -x))$ and similarly for w_1 . Thus, integrating by parts,

$$\begin{aligned} (\delta_2(u, T, \cdot), \phi) &= - \int_{[0, T] \times \mathbb{T}} (\partial_x r_{\text{nl}} - \partial_x r_1)(u, s, x + s - T) \phi(x) ds dx \\ &= \int_{[0, T] \times \mathbb{T}} (r_{\text{nl}} - r_1)(u, s, x + s - T) \partial_x \phi(x) ds dx. \end{aligned}$$

Thus,

$$|(\delta_2(u, T, \cdot), \phi)| \leq \|r_{\text{nl}}(u) - r_1(u)\|_{L_t^1 L_x^1} \|\phi(x)\|_{C^1}.$$

We recall that $r_{\text{nl}}(t, x) = Q(\zeta_{\text{nl}}(t, x), \zeta_{\text{nl}}(t, -x))$ where Q is a quadratic form, and similarly for r_1 . Thus, writing $aa' - bb' = ((a - b)(a' + b') + (a' - b')(a + b))/2$, we get

$$\begin{aligned} |(\delta_2(u, T, \cdot), \phi)| &\leq C \left(\|(\zeta_1 - \zeta_{\text{nl}})(t, x)(\zeta_1(t, x) + \zeta_{\text{nl}}(t, x))\|_{L_t^1 L_x^1} \right. \\ &\quad + \|(\zeta_1 - \zeta_{\text{nl}})(t, -x)(\zeta_1(t, x) + \zeta_{\text{nl}}(t, x))\|_{L_t^1 L_x^1} \\ &\quad + \|(\zeta_1 - \zeta_{\text{nl}})(t, x)(\zeta_1(t, -x) + \zeta_{\text{nl}}(t, -x))\|_{L_t^1 L_x^1} \\ &\quad \left. + \|(\zeta_1 - \zeta_{\text{nl}})(t, -x)(\zeta_1(t, -x) + \zeta_{\text{nl}}(t, -x))\|_{L_t^1 L_x^1} \right) \\ &\leq C \|\zeta_1 - \zeta_{\text{nl}}\|_{L_t^2 L_x^2} (\|\zeta_1\|_{L_t^2 L_x^2} + \|\zeta_{\text{nl}}\|_{L_t^2 L_x^2}). \end{aligned}$$

Finally, using the estimate on δ_1 we obtained in the first step and the regularity estimate on ζ_1 of proposition 23 and the estimate on ζ_{nl} of corollary 24,

$$|(\delta_2(u, T, \cdot), \phi)| \leq C \|U\|_{L^2} \|u\|_{C^0} \|U\|_{L^2}. \quad \square$$

We will also need to estimate $\zeta_2(u) - \zeta_2(\tilde{u})$.

Lemma 26. *Let $\phi \in C^1(\mathbb{T})$, $T > 0$ and $u, \tilde{u} \in L^2$. With the notations of lemma 25, and with $U(t) := \int_0^t u(s) ds$ and $\tilde{U}(t) := \int_0^t \tilde{u}(s) ds$, for some $C > 0$ independent of u, \tilde{u} ,*

$$|(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \leq C \|U - \tilde{U}\|_{L^2} (\|U\|_{L^2} + \|\tilde{U}\|_{L^2}).$$

Proof. We use the same notations w_1 and r_1 as lemma 12. The function $\zeta_2(u) - \zeta_2(\tilde{u})$ satisfies

$$(\partial_t + \partial_x)(\zeta_2(u) - \zeta_2(\tilde{u})) = w_1(u) - w_1(\tilde{u}).$$

Thus, according to the characteristics formula

$$(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi) = \int_{[0, T] \times \mathbb{T}} (w_1(u, s, x + s - T) - w_1(\tilde{u}, s, x + s - T)) \phi ds dx.$$

Since $w_1(u) = -\partial_x r_1(u)$, we integrate by parts to get

$$(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi) = \int_{[0, T] \times \mathbb{T}} (r_1(u, s, x + s - T) - r_1(\tilde{u}, s, x + s - T)) \partial_x \phi ds dx.$$

Thus,

$$|(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \leq C |r_1(u) - r_1(\tilde{u})|_{L_t^1 L_x^1}.$$

Recall that $r_1(u, t, x)$ is a linear combination of quadratic terms involving $\zeta_1(u, t, x)$ and $\zeta_1(u, t, -x)$, (see lemma 12). Thus, writing $aa' - bb' = ((a - b)(a' + b') + (a' - b')(a + b))/2$, and using Hölder's inequality, we get

$$|(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \leq C \|\zeta_1(u - \tilde{u})\|_{L_t^2 L_x^2} (\|\zeta_1(u)\|_{L_t^2 L_x^2} + \|\zeta_1(\tilde{u})\|_{L_t^2 L_x^2}).$$

Finally, the regularity estimate for the linear equation (proposition 23) proves the theorem. \square

We finally prove theorem 1.

Proof of theorem 1. Let $T \in (1, 2)$.⁴ According to proposition 18, for some fixed 1-periodic function $\phi \in C^\infty(\mathbb{T})$, if \tilde{u} steers the linearized equation (8) from 0 to 0 and if $\int_0^T \tilde{u}(t) dt = 0$, then

$$|(\zeta_2(\tilde{u}, T, \cdot), \phi)_{L^2}| \geq c \|\tilde{U}\|_{L^2}^2, \quad (44)$$

where $\tilde{U}(s) = \int_0^s \tilde{u}(s') ds'$.

Let $u \in L^2(0, T)$ with $\int_0^T u(s) ds = 0$ such that the solution of the water-tank system (1) with zero initial condition satisfies $H_{\text{nl}}(T, \cdot) = 1$ and $v_{\text{nl}}(T, \cdot) = 0$ (we assumed that the average height of the fluid is 1). We will use the notations defined at the beginning of this subsection.

Step 1: There exists a “modified control” close to the original one that steers the linearized equation from 0 to 0. We look for a control \tilde{u} close to u such that $\zeta_1(\tilde{u}, T, \cdot) = 0$. Writing $\tilde{u} = u + v$, this condition is equivalent to $\zeta_1(u, T, \cdot) = -\zeta_1(v, T, \cdot)$. Since $\zeta_{\text{nl}}(u, T, \cdot) = 0$, we rewrite this as

$$\zeta_1(v, T, \cdot) = \delta_1(u, T, \cdot). \quad (45)$$

According to lemma 9, such a control v exists, and we can also choose it such that $\int_0^T v(t) dt = 0$ and such that $\mathcal{V}(t) := \int_0^t v(s) ds$ satisfies $\|\mathcal{V}\|_{L^2} \leq C \|\delta_1(u, T, \cdot)\|_{L^2}$. According to the estimate on δ_1 of lemma 25, this control is such that

$$\|\mathcal{V}\|_{L^2} \leq C \|U\|_{L^2} \|u\|_{C^0}. \quad (46)$$

Step 2: Estimating the difference $(\zeta_2(u, T, \cdot), \phi) - (\zeta_2(\tilde{u}, T, \cdot), \phi)$. We aim to prove that it is small enough, so that it does not counter the positivity of $(\zeta_2(\tilde{u}, T, 2), \phi)$. Since $\zeta_1(u, T, \cdot)$ is 1-antiperiodic (remark 8), and since ϕ is 1-periodic, $(\zeta_1(u, T, \cdot), \phi) = 0$. Thus using the triangle inequality

$$\begin{aligned} |(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| &= |(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_1(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \\ &\leq |(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_1(u, T, \cdot) - \zeta_2(u, T, \cdot), \phi)| + |(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \end{aligned}$$

The first term of the right-hand side is $|(\delta_2(u, T, \cdot), \phi)|$, and according to lemma 25, we have $|(\delta_2(u, T, \cdot), \phi)| \leq C \|U\|_{L^2}^2 \|u\|_{C^0}$. According to lemma 26, the second term is bounded by $C \|\mathcal{V}\|_{L^2} (\|U\|_{L^2} + \|\mathcal{V}\|_{L^2})$. Thus,

$$|(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \leq C \|U\|_{L^2}^2 \|u\|_{C^0} + C \|\mathcal{V}\|_{L^2} (\|U\|_{L^2} + \|\mathcal{V}\|_{L^2}).$$

Now, plugging the estimate (46) on $|\mathcal{V}|$, we get

$$\begin{aligned} |(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| &\leq C \|U\|_{L^2}^2 \|u\|_{C^0} + C \|U\|_{L^2} \|u\|_{C^0} (\|U\|_{L^2} \|u\|_{C^0} + \|U\|_{L^2}) \\ &= C \|U\|_{L^2}^2 \|u\|_{C^0} + C \|U\|_{L^2}^2 \|u\|_{C^0} (\|u\|_{C^0} + 1). \end{aligned}$$

If we assume $\|u\|_{C^0} < 1$, we have

$$|(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \leq C \|U\|_{L^2}^2 \|u\|_{C^0}. \quad (47)$$

Step 3: Conclusion using the coercivity of the kernel. Now, we assume that $\|u\|_{C^0} < \eta$ for some $0 < \eta < 1$ that will be chosen later.

Since we constructed \tilde{u} such that $\int_0^T \tilde{u}(t) dt = 0$ and $\zeta_1(\tilde{u}, T, \cdot) = 0$, we can use the lower bound (44) on the quadratic approximation: we have $|(\zeta_2(\tilde{u}, T, \cdot), \phi)| \geq c \|\tilde{U}\|_{L^2}^2$. Combined with the estimate (47) from previous step and the inverse triangle inequality, we get

$$\begin{aligned} \|\zeta_{\text{nl}}(u, T, \cdot)\|_{L^2} &\geq c |(\zeta_{\text{nl}}(u, T, \cdot), \phi)| \\ &\geq c |(\zeta_2(\tilde{u}, T, \cdot), \phi)| - c |(\zeta_{\text{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi)| \\ &\geq c \|\tilde{U}\|_{L^2}^2 - C \eta \|U\|_{L^2}^2. \end{aligned} \quad (48)$$

⁴Recall that we rescaled the problem and that we have $L = H_{\text{eq}} = g = 1$.

To conclude, we bound from below $|\tilde{U}|$ by $|U|$. Since $\tilde{u} = u + \nu$, we have $\|U\|_{L^2}^2 \leq 2|\tilde{U}|_{L^2}^2 + 2\|\nu\|_{L^2}^2$. Thus $\|\tilde{U}\|_{L^2}^2 \geq \frac{1}{2}\|U\|_{L^2}^2 - \|\nu\|_{L^2}^2$. Plugging this into (48), we get

$$\|\zeta_{\text{nl}}(u, T, \cdot)\|_{L^2} \geq c\|U\|_{L^2}^2 - C\|\nu\|_{L^2}^2 - C\eta\|U\|_{L^2}^2.$$

Now, according to the estimate (46) on \mathcal{V} , we get

$$\begin{aligned} \|\zeta_{\text{nl}}(u, T, \cdot)\|_{L^2} &\geq c\|U\|_{L^2}^2 - C\|U\|_{L^2}^2\|u\|_{C^0}^2 - C\eta\|U\|_{L^2}^2 \\ &\geq c\|U\|_{L^2}^2 - C\eta\|U\|_{L^2}^2. \end{aligned}$$

In conclusion, if η is chosen small enough, we get

$$\|\zeta_{\text{nl}}(u, T, \cdot)\|_{L^2} \geq c\|U\|_{L^2}^2.$$

But we assumed that $\zeta_{\text{nl}}(u, T, \cdot) = 0$. Thus, $u = 0$. □

A On the positivity of a class of quadratic forms

In this appendix, we sketch the proof of the following proposition.

Proposition 27. *Let $I = [a, b]$ with $a < b$, let $\phi : I \rightarrow \mathbb{R}$ be C^1 and such that $\phi' \geq c > 0$ and ϕ' non constant, and let $\epsilon \in \mathbb{R}$. Set $K(s_1, s_2) := (1 + \epsilon|s_2 - s_1|)(\phi(s_1 \wedge s_2) - \phi(s_1 \vee s_2))$, and denote by Q_K the associated quadratic form, i.e., $Q_K(u) := \int_{I^2} K(s_1, s_2)u(s_1)u(s_2) ds_1 ds_2$. The following assertions are equivalent:*

1. *There exists $c > 0$ such that for every $u \in L^2(I)$ with $\int_a^b u(t) dt = 0$, $Q_K(u) > c\|U\|_{L^2(I)}^2$, where $U(t) := \int_a^t u(s) ds$;*
2. $\int_I \phi'(s) ds \int_I \frac{ds}{\phi'(s)} < (b - a + 2\epsilon^{-1})^2$.

On the other hand, if $\int_I \phi'(s) ds \int_I (\phi'(s))^{-1} ds > (b - a + 2\epsilon^{-1})^2$, there exists $u_1, u_2 \in L^2(I)$ with $\int_I u_1(s) ds = \int_I u_2(s) ds = 0$ such that $Q_K(u_1) > 0$ and $Q_K(u_2) < 0$.

If $\epsilon = 0$, the term $(b - a + 2\epsilon^{-1})^2$ should be understood as $+\infty$. The hypothesis that ϕ' is not constant is useful to avoid some degeneracy several times in the proof, but the result still holds if ϕ' is constant by perturbing ϕ .

When $K \in L^2(I^2)$, we denote the associated operator also by K , i.e., $Ku(x) = \int_I K(x, y)u(y) dy$. We first start by recasting the quadratic form in a more manageable way for us. This is done thanks to the following lemma.

Lemma 28. *Let $I = (a, b)$ with $a < b$, and let $K \in H^1(I^2)$ such that $\partial_{s_1, s_2} K$ is in $L^2(s_1 < s_2)$ and $L^2(s_1 > s_2)$. Let $R \in L^2(I^2)$ such that for $s_1 \neq s_2$, $R(s_1, s_2) = \partial_{s_1, s_2} K(s_1, s_2)$, and let $g(s) := \partial_{s_2} K(s+0, s) - \partial_{s_2} K(s-0, 0)$. Then, for every $U \in H_0^1(a, b)$, we have*

$$(KU', U') = \int_I g(s)|U(s)|^2 ds + (RU, U).$$

The proof of this lemma formally consists in integrating by parts in s_1 and s_2 , with, formally, $\partial_{s_1 s_2} K = R + g\delta_{s_1=s_2}$. The details are left to the reader. In our case, this allows us to rewrite Q_K in the following way.

Corollary 29. Define Q_K as in proposition 27. Then, for every $u \in L^2(I)$ with $\int_0^T u(t) dt = 0$,

$$Q_K(u) = 2 \int_I \phi'(s)(U(s))^2 ds + 2\epsilon \int_I \phi'(s)U(s) ds \int_I U(s) ds,$$

where $U(t) := \int_a^t u(s) ds$. We will denote the right-hand side of the expression as $\tilde{Q}_K(U)$ which makes sense for each $U \in L^2(I)$. With this notation, $Q_K(u) = \tilde{Q}_K(U)$.

This formula actually holds without the assumption $\phi'(s) \geq c > 0$; ϕ can even be complex-valued, with the same proof. Moreover, we see that if $\epsilon = 0$, and $\phi' \geq c > 0$, $Q_K(u) \geq 2c\|U\|_{L^2}^2$, so proposition 27 is trivial in this case. From now on, we assume that $\epsilon \neq 0$.

Sketch of the proof. With K as in proposition 27 and g, R as in lemma 28, routine computations show that $g(s) = 2\phi'(s)$ and $R(s_1, s_2) = \epsilon(\phi'(s_1) + \phi'(s_2))$. \square

The expression of this corollary suggests that we work in the weighted space $L_{\phi'}^2 := L^2(I, \phi'(s) ds)$. This is where the hypothesis $\phi'(s) > 0$ is useful: to make sense of this space. We will denote $\|\cdot\|_{\phi'}$ the norm in $L_{\phi'}^2$ and $(\cdot, \cdot)_{\phi'}$ the scalar product. The main consequence of working in this space is that on a space of codimension 2, $Q_K(u) = 2\|U\|_{\phi'}^2$.

Lemma 30. Let \tilde{Q}_K as in corollary 29. Let S be the symmetric operator (for the $L_{\phi'}^2$ scalar product) associated with \tilde{Q}_K . Let $E := \{U \in L_{\phi'}^2, \int_I U(s) ds = \int_I \phi'(s)U(s) ds = 0\}$ and $F := \text{Span}(1, (\phi')^{-1})$. Then:

- E is the orthogonal of F (for the $L_{\phi'}^2$ scalar product);
- E and F are stable by S ;
- the restriction of S on E is $S|_E = 2I$.

Sketch of the proof. The orthogonality of E and F results from simple computations. Since E is of codimension 2, $E + F = L_{\phi'}^2$. For the other two points, let us denote $M(U)$ the constant function equal to $\int_I U(s) ds$ and M^* the adjoint of this operator M for the $L_{\phi'}^2$ -scalar product. Routine computations show that

$$S(U) = 2U + \epsilon(M(U) + M^*(U)) = 2U + \epsilon \left(\int_I U(s) ds + \frac{1}{\phi'} \int_I \phi'(s)U(s) ds \right).$$

With this expression of S , the last two points are immediate. \square

With these lemmas, we can prove proposition 27.

Sketch of the proof of proposition 27. The main idea is that according to lemma 30, the only possible counter examples to the coercivity inequality $Q_K(U) \geq c\|U\|_{\phi'}^2$ are in F , thus we are left to study whether a 2×2 matrix is positive.

Let us first compute the matrix of the restriction of \tilde{Q}_K to F in the basis $(1, (\phi')^{-1})$. Here, we use the fact that ϕ' is not constant; otherwise, the family $(1, (\phi')^{-1})$ would not be linearly independent. For simplicity, write $U_1 := 1$, $U_2 := (\phi')^{-1}$, and $M(U)$ the constant function equal to $\int_I U(s) ds$. Then,

$$\begin{aligned} A &:= \text{Matrix}_{(U_1, U_2)}(\tilde{Q}_K)|_F \\ &= \begin{pmatrix} 2|U_1|^2 + 2\epsilon(M(U_1), U_1) & 2(U_1, U_2) + \epsilon(M(U_1), U_2) + \epsilon(U_1, M(U_2))) \\ 2(U_1, U_2) + \epsilon(M(U_1), U_2) + \epsilon(U_1, M(U_2)) & 2|U_2|^2 + 2\epsilon(M(U_2), U_2) \end{pmatrix}, \end{aligned}$$

where all the norms and scalar products are taken in $L^2_{\phi'}$. Finally, if we set $\alpha := \int_I \phi'(s) ds$ and $\beta := \int_I (\phi'(s))^{-1} ds$, some routine (again) computations prove that this matrix is

$$A = \begin{pmatrix} 2\alpha(1 + \epsilon(b-a)) & 2(b-a) + \epsilon(b-a)^2 + \epsilon\alpha\beta \\ 2(b-a) + \epsilon(b-a)^2 + \epsilon\alpha\beta & 2\beta(1 + \epsilon(b-a)) \end{pmatrix}.$$

To study the positivity of \tilde{Q}_K , we compute the trace and determinant of A . Routine computations show that:

$$\text{Tr}(A) = 2(\alpha + \beta)(1 + \epsilon(b-a)) \quad (49)$$

$$\det(A) = -\epsilon^2(\alpha\beta - (b-a)^2)(\alpha\beta - (b-a - 2\epsilon^{-1})^2) \quad (50)$$

Finally, let us note that thanks to Cauchy-Schwarz inequality, $(b-a)^2 < \alpha\beta$, where the inequality is strict because we assumed that ϕ' is not constant.

Step 1: 1. \implies 2. If assertion 1 holds, \tilde{Q}_K is positive definite, thus, the matrix A is positive definite. Hence, $\det(A) > 0$. Since $(b-a)^2 < \alpha\beta$, according to the expression (50) of $\det(A)$, we have $\alpha\beta < (b-a + 2\epsilon^{-1})^2$, which is exactly assertion 2.

Step 2: 2. \implies 1. If assertion 2 holds, according to expression (50) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) > 0$. Moreover, since $(b-a)^2 < \alpha\beta < (b-a + 2\epsilon^{-1})^2$, we have $b-a < |b-a + 2\epsilon^{-1}|$, i.e., $b-a < b-a + 2\epsilon^{-1}$ or $b-a < -(b-a) - 2\epsilon^{-1}$. In both cases, we get $1 + \epsilon(b-a) > 0$. Hence, according to the expression (49) of $\text{Tr}(A)$, we have $\text{Tr}(A) > 0$. Thus, A is positive definite. Finally, according to lemma 30, we deduce that for each $U \in L^2$, $\tilde{Q}_K(U) > c\|U\|_{\phi'}^2$. Since $\phi' \geq c > 0$, the $L^2_{\phi'}$ and L^2 norm are equivalent, hence assertion 1 holds.

Step 3: Last assertion. If $\alpha\beta > (b-a + 2\epsilon^{-1})^2$, according to expression (50) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) < 0$, hence A has a positive and a negative eigenvalue, and so do \tilde{Q}_K . Hence, we can find $\tilde{U}_1, \tilde{U}_2 \in L^2(I)$ such that $\tilde{Q}_K(\tilde{U}_1) > 0$ and $\tilde{Q}_K(\tilde{U}_2) < 0$. By approximating in L^2 -norm \tilde{U}_i by some $U_i \in H_0^1(I)$, we find $U_1, U_2 \in H_0^1(I)$ such that $\tilde{Q}_K(U_1) > 0$ and $\tilde{Q}_K(U_2) < 0$. Since $Q_K(U') = \tilde{Q}_K(U)$, this proves the proposition. \square

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