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# A Simple Proof of the Riemann's Hypothesis

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## Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

**Keywords:** Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

**MSC2020 Classification:** 11Mxx, 11-XX, 26-XX, 30-xx.

## 1 The Riemann Hypothesis

### 1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part  $1/2$ .

### 1.2 Riemann Zeta Function

For a complex number  $s$  where  $\Re(s) > 1$ , the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function  $\zeta(s)$ , by analytical continuation, to an absolutely convergent function in the half plane  $\Re(s) > 0$ , minus a simple pole at  $s = 1$ :

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

Where  $\{x\} = x - [x]$  is the fractional part and  $[x]$  is the integer part of  $x$ . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

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Riemann[1] has shown that Zeta has a functional equation<sup>1</sup>

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where  $\Gamma(s)$  is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of  $\zeta$  are located symmetrically with respect to the line  $\Re(s) = 1/2$ , inside the critical strip  $0 < \Re(s) < 1$ . Riemann has conjectured that all the non trivial-zeros are located on the critical line  $\Re(s) = 1/2$ . In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that  $\zeta(s)$  has no zeros of the form  $s = 1 + it$  for  $t \in \mathbb{R}$ . Some of the known results[2,3] of  $\zeta(s)$  are as follows:

- $\zeta(s)$  has no zero for  $\Re(s) > 1$ .
- $\zeta(s)$  has no zero of the form  $s = 1 + i\tau$ . i.e.  $\zeta(1 + i\tau) \neq 0, \forall \tau$ .
- $\zeta(s)$  has a simple pole at  $s = 1$  with residue 1.
- $\zeta(s)$  has all the trivial zeros at the negative even integers  $s = -2k, k \in \mathbb{N}^*$ .
- The non-trivial zeros are inside the critical strip: i.e.  $0 < \Re(s) < 1$ .
- If  $\zeta(s) = 0$ , then  $1 - s, \bar{s}$  and  $1 - \bar{s}$  are also zeros of  $\zeta$ : i.e.  $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ .

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that  $\zeta$  has no zero on the right hand side  $1/2 < \Re(s) < 1$  of the critical strip.

### 1.3 Proof of the Riemann Hypothesis

Let’s take a complex number  $s$  such that  $s = \sigma + i\tau$ . Unless we explicitly mention otherwise, let’s suppose that  $0 < \sigma < 1, \tau > 0$  and  $\zeta(s) = 0$ .

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have  $s \neq 1, s \neq 0$  and  $\zeta(s) = 0$ , therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

Or in other terms:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

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<sup>1</sup>This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (8)$$

Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (8)$$

To continue, we will prove the following lemmas.

**Lemma 1.1.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $\sigma > 0$  and  $\tau > 0$ . Let's define two integrals  $I(a, \sigma, \tau)$  and  $J(a, \sigma, \tau)$  as follows:*

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (9)$$

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (10)$$

Therefore

$$I(a, \sigma, \tau) = K(\sigma, \tau) \left( 1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1) \sin(\tau \ln(a))}{\tau a^{\sigma-1}} \right) \quad (11)$$

$$J(a, \sigma, \tau) = K(\sigma, \tau) \left( \frac{(\sigma-1)}{\tau} - \frac{(\sigma-1) \cos(\tau \ln(a))}{\tau a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (12)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma-1)^2 + \tau^2} \quad (13)$$

*In the context of this lemma, the variables  $\sigma$  and  $\tau$  are arbitrary and not necessarily related to the zeros of the zeta function.*

*Proof.* Let's consider two variables  $\sigma$  and  $\tau$  such that  $\sigma > 0$  and  $\tau > 0$ . Let's take  $a > 1$ .

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (14)$$

$$= \int_0^{\ln(a)} \sin(\tau x) e^{(1-\sigma)x} dx \quad (15)$$

$$= K(\sigma, \tau) \left( 1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1) \sin(\tau \ln(a))}{\tau a^{\sigma-1}} \right) \quad (16)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma-1)^2 + \tau^2} \quad (17)$$

And the same for  $J(a, \sigma, \tau)$  for  $a > 0$ :

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (18)$$

$$= \int_0^{\ln(a)} \cos(\tau x) e^{(1-\sigma)x} dx \quad (19)$$

$$= K(\sigma, \tau) \left( \frac{(\sigma-1)}{\tau} - \frac{(\sigma-1) \cos(\tau \ln(a))}{\tau a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (20)$$

Where  $K(\sigma, \tau)$  is defined above in the equation (17). □

**Lemma 1.2.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma < 1$ ,  $\tau > 0$  and  $s = \sigma + i\tau$  is a zeta zero. Let's define the sequence of functions  $\epsilon_n$  and  $\bar{\epsilon}_n$  such that  $\epsilon_0(x) = \bar{\epsilon}_0(x) = \{x\}$  and for each  $n \geq 1$ :*

$$\epsilon_{n+1}(x) = \frac{1}{x} \int_0^x \epsilon_n(t) \quad (21)$$

$$\bar{\epsilon}_{n+1}(x) = \frac{1}{x} \int_1^x \bar{\epsilon}_n(t) \quad (22)$$

Therefore:

1. For each  $n \geq 1$ :

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{1+s}} = -\frac{1}{(1-s)2^n} \quad (23)$$

2. For each  $n$ :

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{2-s}} = -\frac{1}{s2^n} \quad (24)$$

3. For each  $x \geq 1$ :

$$\epsilon_n(x) = \bar{\epsilon}_n(x) + \frac{1}{2^n x} \sum_{k=0}^{n-1} \frac{2^k \ln^k(x)}{k!} \quad (25)$$

4. For each  $n$  and  $x \geq 1$ :

$$\bar{\epsilon}_n(x) = \frac{1}{(n-1)!} \frac{1}{x} \int_1^x dt \bar{\epsilon}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1} \quad (26)$$

5. For each  $x \geq 1$ :

$$\lim_{n \rightarrow +\infty} 2^n \epsilon_n(x) = x \quad (27)$$

*Proof.* Let's denote the integral  $A(s)$  as following:

$$A(s) = \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} = -\frac{1}{(1-s)} \quad (28)$$

We use the integration by parts to write the following:

$$A(s) = \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} \quad (29)$$

$$= \left[ \frac{1}{x^{1+s}} \int_0^x dx \epsilon_0(x) \right]_1^{+\infty} + (1+s) \int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} \quad (30)$$

$$= -\int_0^1 dx \epsilon_0(x) + (1+s) \int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} \quad (31)$$

$$= -\int_0^1 dx \epsilon_0(x) - (1+s) \int_0^1 dx \epsilon_1(x) + (1+s)^2 \int_1^{+\infty} dx \frac{\epsilon_2(x)}{x^{1+s}} \quad (32)$$

$$\dots \quad (33)$$

$$= -\sum_{k=0}^n (1+s)^k \int_0^1 dx \epsilon_k(x) + (1+s)^{n+1} \int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{2+s}} \quad (34)$$

We could do the above integration by parts. Because the functions  $\epsilon_k$  are continuous functions. We also can prove that the functions  $\psi_k$  are non-negative and bounded as we can prove that for each  $k$  for each  $x > 0$  that:

$$0 < \epsilon_k(x) \leq 1 \quad (35)$$

This can be proven by recurrence using the fact that for each  $x \geq 0$ :

$$0 \leq \epsilon_0(x) = \{x\} \leq 1 \quad (36)$$

Now, we need to calculate the integrals  $I_k$  for  $k \geq 0$ :

$$I_k = \int_0^1 dx \epsilon_k(x) \quad (37)$$

For  $x \in (0, 1)$ , we have:

$$\epsilon_0(x) = x \quad (38)$$

And

$$\psi_1(x) = \frac{x}{2} \quad (39)$$

Therefore, we can write for each  $k$  for  $x \in (0, 1)$ :

$$\epsilon_k(x) = \frac{x}{2^k} \quad (40)$$

Therefore, for each  $k \geq 1$ :

$$I_k = \frac{1}{2^{k+1}} \quad (41)$$

Therefore, we can conclude:

$$A(s) = -\frac{1}{(1-s)} + (1+s)^{n+1} \left[ \int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{1+s}} + \frac{1}{(1-s)2^{n+1}} \right] \quad (42)$$

Since  $1+s \neq 0$ , we conclude the point 1. The point 2 can be proved the same way by replacing  $s$  by  $1-s$ .  $\blacksquare$

The point 3) can be proved by recurrence. For  $n = 0$ . We have  $\epsilon_0(x) = \overline{\epsilon_0}(x)$ . Let's assume that it is true till  $n$  and let's prove for  $n+1$ . We have:

$$\epsilon_{n+1}(x) = \frac{1}{x} \int_0^1 dx \epsilon_n(x) + \frac{1}{x} \int_1^x dx \epsilon_n(x) \quad (43)$$

$$= \frac{1}{x 2^{n+1}} + \frac{1}{x 2^n} \int_1^x dt \sum_{k=0}^{n-1} \frac{2^k \ln^k(t)}{k!} + \frac{1}{x} \int_1^x dx \overline{\epsilon_n}(x) \quad (44)$$

$$= \frac{1}{x 2^{n+1}} + \frac{1}{x 2^{n+1}} \sum_{k=0}^{n-1} \frac{2^{k+1} \ln^{k+1}(x)}{(k+1)!} + \overline{\epsilon_{n+1}}(x) \quad (45)$$

$$= \frac{1}{x 2^{n+1}} \sum_{k=0}^n \frac{2^k \ln^k(x)}{(k)!} + \overline{\epsilon_{n+1}}(x) \quad (46)$$

Let's prove the 4<sup>th</sup> point. We proceed by recurrence. For  $n = 1$ , we retrieve the definition of  $\bar{\epsilon}_1(x)$ . Let's assume that it is true up to  $n$  and let's prove it for  $n + 1$ . We have thanks to the integral order change:

$$\bar{\epsilon}_{n+1}(x) = \frac{1}{x} \int_1^x dt \bar{\epsilon}_n(t) \quad (47)$$

$$= \frac{1}{(n-1)!} \frac{1}{x} \int_1^x \frac{dt}{t} \int_1^t ds \bar{\epsilon}_1(s) \left(\ln\left(\frac{t}{s}\right)\right)^{n-1} \quad (48)$$

$$= \frac{1}{(n-1)!} \frac{1}{x} \int_1^x ds \bar{\epsilon}_1(s) \int_s^x \frac{dt}{t} \left(\ln\left(\frac{t}{s}\right)\right)^{n-1} \quad (49)$$

$$= \frac{1}{n!} \frac{1}{x} \int_1^x ds \bar{\epsilon}_1(s) \left(\ln\left(\frac{x}{s}\right)\right)^n \quad (50)$$

And this proves one point of the lemma.  $\blacksquare$

Let's now prove the last point. Let's take  $x > 1$ . From the point (4) above, we can apply the dominated convergence theorem to prove that:

$$\lim_{n \rightarrow +\infty} 2^n \bar{\epsilon}_n(x) = 0 \quad (51)$$

From point (3), we can conclude that:

$$\lim_{n \rightarrow +\infty} 2^n \epsilon_n(x) = \lim_{n \rightarrow +\infty} \left( 2^n \bar{\epsilon}_n(x) + \frac{1}{x} \sum_{k=0}^{n-1} \frac{2^k \ln^k(x)}{k!} \right) \quad (52)$$

$$= \lim_{n \rightarrow +\infty} \frac{1}{x} \sum_{k=0}^{n-1} \frac{\ln^k(x^2)}{k!} \quad (53)$$

$$= \frac{1}{x} \exp(\ln(x^2)) = x \quad (54)$$

Hence the proof of the last point of the lemma.  $\blacksquare$

$\square$

**Lemma 1.3.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma \leq \frac{1}{2}$  and  $\tau > 0$  such that  $s = \sigma + i\tau$  is a zeta zero. Let's define the sequence of functions  $E_{n,\sigma}(x)$ ,  $F_{n,\sigma}(x)$ ,  $G_{n,\sigma}(x)$  and  $H_{n,\sigma}(x)$  over  $[1, +\infty)$  for each  $n \geq 1$  as follows:*

$$E_{n,\sigma}(x) = \int_1^x dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \quad (55)$$

$$F_{n,\sigma}(x) = \int_1^x dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \quad (56)$$

$$G_{n,\sigma}(x) = (E_{n,\sigma}(x))^2 + (F_{n,\sigma}(x))^2 \quad (57)$$

$$H_{n,\sigma}(x) = G_{n,1-\sigma}(x) - G_{n,\sigma}(x) \quad (58)$$

Let's define the sequence of functions  $I_{n,\sigma}(x)$  over  $[0, +\infty)$  for each  $n \geq 1$  as follows:

$$I_{n,\sigma}(x) = H_{n,\sigma}(x + e^{\frac{3}{2\tau}}) \quad (59)$$

We denote  $a_0 = e^{\frac{3}{2\tau}}$  and  $b_0 = e^{\frac{\pi}{2\tau}}$ .

1. For each  $n \geq 1$  there exists  $x_n > b_0$  such as:

$$H_{n,\sigma}(x_n) = 0 \quad (60)$$

2. For each  $n \geq 1$  there exists  $x_n > b_0 - a_0 > 0$  such as:

$$I_{n,\sigma}(x_n) = 0 \quad (61)$$

In other terms:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \quad (62)$$

*Proof.* Let's prove first point. Let's take  $n \geq 1$  an integer. Since  $\sigma \leq \frac{1}{2}$ . We have for each  $x > 1$ :  $\frac{1}{x^{1+\sigma}} > \frac{1}{x^{2-\sigma}}$ . Therefore

$$E_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) > 0 \quad (63)$$

$$F_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) > 0 \quad (64)$$

$$E_{n,\sigma}(b_0) > E_{n,1-\sigma}(b_0) > 0 \quad (65)$$

$$F_{n,\sigma}(b_0) > F_{n,1-\sigma}(b_0) > 0 \quad (66)$$

Therefore

$$(E_{n,\sigma}(b_0))^2 > (E_{n,1-\sigma}(b_0))^2 \quad (67)$$

$$(F_{n,\sigma}(b_0))^2 > (F_{n,1-\sigma}(b_0))^2 \quad (68)$$

And therefore

$$G_{n,\sigma}(b_0) > G_{n,1-\sigma}(b_0) \quad (69)$$

$$H_{n,\sigma}(b_0) < 0 \quad (70)$$

We also have from the lemma 1.2:

$$\lim_{x \rightarrow +\infty} G_{n,\sigma}(x) = \left( \int_1^{+\infty} dt \frac{\sin(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \right)^2 + \left( \int_1^{+\infty} dt \frac{\cos(\tau \ln(t))}{x^{1+\sigma}} \epsilon_n(x) \right)^2 \quad (71)$$

$$= \frac{1}{2^{2n} \|1-s\|^2} \quad (72)$$

$$\lim_{x \rightarrow +\infty} G_{n,1-\sigma}(x) = \left( \int_1^{+\infty} dt \frac{\sin(\tau \ln(t))}{x^{2-\sigma}} \epsilon_n(x) \right)^2 + \left( \int_1^{+\infty} dt \frac{\cos(\tau \ln(t))}{x^{2-\sigma}} \epsilon_n(x) \right)^2 \quad (73)$$

$$= \frac{1}{2^{2n} \|s\|^2} \quad (74)$$

Therefore

$$\lim_{x \rightarrow +\infty} H_{n,\sigma}(x) = \frac{1}{2^{2n}} \left( \frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (75)$$

Since the function  $H_{n,\sigma}$  is continuous over  $[1, +\infty)$ . From (71) and (76) and thanks to the Mean value theorem, we can conclude that there exists an  $x_n > b_0$  such that:

$$H_{n,\sigma}(x_n) = 0 \quad (76)$$



To prove the second point, we proceed by the same approach as in the first point. We remark that since  $b_0 - a_0 > 0$ :

$$I_{n,\sigma}(b_0 - a_0) = H_{n,\sigma}(a_0 + b_0 - a_0) = H_{n,\sigma}(b_0) < 0 \quad (77)$$

And the same like above:

$$\lim_{x \rightarrow +\infty} I_{n,\sigma}(x) = \lim_{x \rightarrow +\infty} H_{n,\sigma}(a_0 + x) \quad (78)$$

$$= \frac{1}{2^{2n}} \left( \frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2} \right) > 0 \quad (79)$$

And the same we conclude thanks to the Mean value theorem.  $\blacksquare$

$\square$

**Lemma 1.4.** *Let's consider two variables  $\sigma$  and  $\tau$  such that  $0 < \sigma < 1$  and  $\tau \geq 1$  such that  $s = \sigma + i\tau$  is a zeta zero. Therefore for each  $n \geq 1$ :*

$$\sigma = \frac{1}{2} \quad (80)$$

*Proof.* Let's suppose that  $\sigma < \frac{1}{2}$ . We use the previous lemma notations. From the lemma 1.3, we can construct a sequence  $(x_n > b_0 - a_0)$  such that:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \quad (81)$$

We can distinguish between two cases:

1. The sequence  $(x_n)$  is unbounded.
2. The sequence  $(x_n)$  is bounded.

**Case 1: The sequence  $(x_n)$  is unbounded** In this case, without loss of generality, we can assume that the limit of  $(x_n)$  is  $+\infty$ . From the lemma 1.2, we have

$$\lim_{n \rightarrow +\infty} 2^n \epsilon_n(x) = x \quad (82)$$

Therefore, thanks to the integration theory of calculus, we can conclude that:

$$G_{n,\sigma}(x_n + a_0) \sim_{n \rightarrow +\infty} \left( \int_1^{x_n + a_0} dt \frac{\sin(\tau \ln(t))}{x^\sigma} \right)^2 + \left( \int_1^{x_n + a_0} dt \frac{\cos(\tau \ln(t))}{x^\sigma} \right)^2 \quad (83)$$

From the lemma 1.1, we can write:

$$G_{n,\sigma}(x_n + a_0) \sim_{n \rightarrow +\infty} \frac{(x_n + a_0)^{2(1-\sigma)} - 2(x_n + a_0)^{(1-\sigma)} \cos(\tau \ln(x_n + a_0)) + 1}{(1-\sigma)^2 + \tau^2} \quad (84)$$

$$\sim_{n \rightarrow +\infty} \frac{(x_n)^{2(1-\sigma)}}{(1-\sigma)^2 + \tau^2} \quad (85)$$

And the same

$$G_{n,1-\sigma}(x_n + a_0) \sim_{n \rightarrow +\infty} \frac{(x_n + a_0)^{2\sigma} - 2(x_n + a_0)^\sigma \cos(\tau \ln(x_n + a_0)) + 1}{\sigma^2 + \tau^2} \quad (86)$$

$$\sim_{n \rightarrow +\infty} \frac{(x_n)^{2\sigma}}{\sigma^2 + \tau^2} \quad (87)$$

From the relation (82), we conclude:

$$x_n^{2(1-2\sigma)} \sim_{n \rightarrow +\infty} \frac{(1-\sigma)^2 + \tau^2}{\sigma^2 + \tau^2} \quad (88)$$

Which is a contradiction since the limit of  $(x_n)$  is  $+\infty$ . Therefore  $1 - 2\sigma$  must be equal zero. Therefore  $\sigma = \frac{1}{2}$ .

**Case 2: The sequence  $(x_n)$  is bounded** In such case, without loss of generality, we can assume that the limit of  $(x_n)$  is  $a \geq b_0 - a_0 > 0$ .

Similarly like in the first case, we can write that:

$$\frac{(a + a_0)^{2\sigma} - 2(a + a_0)^\sigma \cos(\tau \ln(a + a_0)) + 1}{\sigma^2 + \tau^2} = \frac{(a + a_0)^{2(1-\sigma)} - 2(a + a_0)^{(1-\sigma)} \cos(\tau \ln(a + a_0)) + 1}{(1-\sigma)^2 + \tau^2} \quad (89)$$

Let's study the function  $f: x \rightarrow \frac{(a+a_0)^{2x} - 2(a+a_0)^x \cos(\tau \ln(a+a_0)) + 1}{x^2 + \tau^2}$ .  
Therefore:

$$(x^2 + \tau^2)^2 f'(x) = 2 \ln(a + a_0)(a + a_0)^x \left( (a + a_0)^x - \cos(\tau \ln(a + a_0)) \right) (x^2 + \tau^2) \quad (90)$$

$$-2x \left( (a + a_0)^{2x} - 2(a + a_0)^x \cos(\tau \ln(a + a_0)) + 1 \right) \quad (91)$$

Since  $a + a_0 \geq e^{\frac{\pi}{2\tau}} > e^{\frac{3}{2\tau}}$ . For  $x > 0$ , we have:

$$\left( (a + a_0)^x - \cos(\tau \ln(a + a_0)) \right) > 0 \quad (92)$$

And we can write:

$$2(a + a_0)^x \left( (a + a_0)^x - \cos(\tau \ln(a + a_0)) \right) = (a + a_0)^{2x} - 2(a + a_0)^x \cos(\tau \ln(a + a_0)) + 1 + \underbrace{(a + a_0)^{2x} - 1}_{>0} \quad (93)$$

$$> \left( (a + a_0)^{2x} - 2(a + a_0)^x \cos(\tau \ln(a + a_0)) + 1 \right) > 0 \quad (94)$$

And we also have:

$$x^2 + \tau^2 > 2x\tau \quad (95)$$

$$\ln(a + a_0) > \ln(a_0) = \frac{3}{2\tau} \quad (96)$$

Therefore by combining the inequations (94 – 96), we conclude:

$$2 \ln(a + a_0)(a + a_0)^x \left( (a + a_0)^x - \cos(\tau \ln(a + a_0)) \right) (x^2 + \tau^2) > \frac{3}{2\tau} \left( (a + a_0)^{2x} - 2(a + a_0)^x \cos(\tau \ln(a + a_0)) + 1 \right) 2x\tau \quad (97)$$

$$> 2x \left( (a + a_0)^{2x} - 2(a + a_0)^x \cos(\tau \ln(a + a_0)) + 1 \right) \quad (98)$$

Therefore for each  $x > 0$ :

$$(x^2 + \tau^2)^2 f'(x) > 0 \quad (99)$$

Therefore, the function  $f$  is a strictly increasing function over  $(0, +\infty)$ . From the equation (89), we have  $f(\sigma) = f(1-\sigma)$ . Which is a contradiction if  $\sigma \neq 1 - \sigma$ . Therefore  $\sigma$  must be equal  $1 - \sigma$ . Hence

$$\sigma = 1 - \sigma = \frac{1}{2} \quad (100)$$

This ends the proof of the Riemann Hypothesis. ■.

□

## 1.4 Conclusion

We saw that if  $s$  is a zeta zero, then real part  $\Re(s)$  can only be  $\frac{1}{2}$ . Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of  $\zeta(s)$  have real part equal to  $\frac{1}{2}$ .* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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