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Charaf Ech-Chatbi

► **To cite this version:**

| Charaf Ech-Chatbi. A Simple Proof of the Riemann's Hypothesis. 2022. hal-03587353v8

HAL Id: hal-03587353

<https://hal.science/hal-03587353v8>

Preprint submitted on 29 Jul 2022 (v8), last revised 2 Nov 2023 (v15)

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A Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

Thursday 12 May 2022

Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

*One Raffles Quay, North Tower. 048583 Singapore. Email: charaf.chatbi@gmail.com. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

Or in other terms:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (8)$$

Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (8)$$

$$\phi(x) = \{x\}(1 - \{x\}) \quad (9)$$

$$\Psi(x) = \int_1^x dx \epsilon(x) \quad (10)$$

To continue, we will need to prove some few lemmas.

Lemma 1.1. *The function $\epsilon(x)$ is piecewise continuous on $[0, +\infty)$ and its primitive function $\Psi(x)$ is defined as follows:*

$$\Psi(x) = \frac{1}{2}(x - 1 - \phi(x)) \quad (11)$$

Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:

$$\int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} = \frac{1}{(s-1)(1+s)} \quad (12)$$

And

$$\int_1^{+\infty} dx \frac{\phi(x)}{x^{2+s}} = \frac{1}{s(1-s)} \quad (13)$$

Proof. We will need the function ϕ as you will see later that we need a continuous function instead of a piecewise one like the function ϵ . Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer of part of x . We have $n_x = x - \{x\}$. Therefore, we can write the following:

$$\Psi(x) = \int_1^x \epsilon(x) dx \quad (14)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} \{t\} dt + \int_{n_x}^x \{t\} dt \quad (15)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} (t-n) dt + \int_{n_x}^x (t-n_x) dt \quad (16)$$

$$= \sum_{n=1}^{n_x-1} \frac{1}{2} + \frac{1}{2}(x-n_x)^2 \quad (17)$$

$$= \frac{1}{2}(n_x - 1 + \{x\}^2) \quad (18)$$

$$= \frac{1}{2}(x - 1 - \{x\} + \{x\}^2) \quad (19)$$

This prove the equation (11). ■

Let's prove the second point of the lemma. Let's define the integral $I_\epsilon(s)$ as follows:

$$I_\epsilon(s) = \int_1^{+\infty} dx \frac{\epsilon(x)}{x^{1+s}} \quad (20)$$

The function $x \rightarrow \frac{\epsilon(x)}{x^{1+s}}$ is integrable on $[1, +\infty)$ and thanks to the integration by parts, we can write the following:

$$I_\epsilon(s) = \int_1^{+\infty} dx \frac{\epsilon(x)}{x^{1+s}} \quad (21)$$

$$= \left[\frac{\Psi(x)}{x^{1+s}} \right]_{x=1}^{+\infty} + (1+s) \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (22)$$

$$= (1+s) \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (23)$$

Since, s is a zeta zero, from the equation (6), we have:

$$I_\epsilon(s) = \frac{1}{s-1} \quad (24)$$

Therefore

$$\int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} = \frac{1}{(s-1)(s+1)} \quad (25)$$

Thanks to equation (11), we can write:

$$\int_1^{+\infty} dx \frac{\phi(x)}{x^{2+s}} = \int_1^{+\infty} dx \frac{(x-1)}{x^{2+s}} - 2 \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (26)$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{2}{(s-1)(s+1)} \quad (27)$$

$$= \frac{1}{s(1-s)} \quad (28)$$

■

□

Lemma 1.2. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions ψ_n such that $\psi_0(x) = \phi(x) = \{x\}(1 - \{x\})$ and for each $n \geq 1$:*

$$\psi_{n+1}(x) = \frac{1}{x} \int_0^x \psi_n(t) \quad (29)$$

Therefore:

1. For each $n \geq 1$:

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{s 2^n} + \frac{1}{(1-s) 3^n} \quad (30)$$

And

2. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-s}} = \frac{1}{(1-s) 2^n} + \frac{1}{s 3^n} \quad (31)$$

3. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} = \frac{1}{\sigma} \frac{1}{2^n} + \frac{1}{1-\sigma} \frac{1}{3^n} + \frac{1}{(2+\sigma)^n} \left[U_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (32)$$

Where $U_0(\sigma)$ is defined as follows:

$$U_0(\sigma) = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma}} \quad (33)$$

And

4. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} = \frac{1}{1-\sigma} \frac{1}{2^n} + \frac{1}{\sigma} \frac{1}{3^n} + \frac{1}{(3-\sigma)^n} \left[U_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (34)$$

Where $U_0(1-\sigma)$ is defined as follows:

$$U_0(1-\sigma) = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{3-\sigma}} \quad (35)$$

Proof. Let's denote the integral $A(s)$ as following:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (36)$$

We use the integration by parts to write the following:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (37)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \psi_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (38)$$

$$= - \int_0^1 dx \psi_0(x) + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (39)$$

$$= - \int_0^1 dx \psi_0(x) - (2+s) \int_0^1 dx \psi_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\psi_2(x)}{x^{2+s}} \quad (40)$$

$$\dots \quad (41)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \psi_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} \quad (42)$$

We could do the above integration by parts. Because the functions ψ_k are continuous functions. We also can prove that the functions ψ_k are non-negative and bounded as we can prove that for each k for each $x > 0$ that:

$$0 < \psi_k(x) \leq \frac{1}{4} \quad (43)$$

This can be proven by recurrence using the fact that for each $x \geq 0$:

$$0 \leq \psi_0(x) = \{x\}(1 - \{x\}) \leq \frac{1}{4} \quad (44)$$

Now, we need to calculate the integrals I_k for $k \geq 0$:

$$I_k = \int_0^1 dx \psi_k(x) \quad (45)$$

For $x \in (0, 1)$, we have:

$$\psi_0(x) = x - x^2 \quad (46)$$

And

$$\psi_1(x) = \frac{x}{2} - \frac{x^2}{3} \quad (47)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\psi_k(x) = \frac{x}{2^k} - \frac{x^2}{3^k} \quad (48)$$

Therefore, for each $k \geq 1$:

$$I_k = \frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \quad (49)$$

Therefore, we can conclude:

$$A(s) = \frac{1}{s(1-s)} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{s 2^{n+1}} + \frac{1}{(1-s) 3^{n+1}} \right) \right] \quad (50)$$

Since $2+s \neq 0$ and $3-s \neq 0$, we conclude the points (1) and (2) of our lemma. \blacksquare

Let's now prove the points (3) and (4) of the lemma. Let's now calculate the integral $\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}}$. Let's $n \geq 1$.

$$U_{n+1} = \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\int_0^x dt \psi_n(t)}{x^{3+\sigma}} \quad (51)$$

$$= \frac{1}{2+\sigma} \int_0^1 dt \psi_n(t) + \frac{1}{2+\sigma} \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} \quad (52)$$

$$= \frac{1}{2+\sigma} \left[\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right] + \frac{U_n}{2+\sigma} \quad (53)$$

Therefore

$$U_{n+1} = \frac{U_0}{(2+\sigma)^{n+1}} + \sum_{k=1}^{n+1} \left[\frac{1}{(2+\sigma)^k 2^{n+2-k}} - \frac{1}{(2+\sigma)^k 3^{n+2-k}} \right] \quad (54)$$

Therefore for each $n \geq 1$:

$$U_n = \frac{1}{\sigma} \frac{1}{2^n} + \frac{1}{1-\sigma} \frac{1}{3^n} + \frac{1}{(2+\sigma)^n} \left[U_0 - \frac{1}{\sigma(1-\sigma)} \right] \quad (55)$$

The point (4) can be proved by replacing σ by $1-\sigma$. \blacksquare

\square

Lemma 1.3. Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions $\tilde{\psi}_n$ such that $\tilde{\psi}_0(x) = \frac{\psi_0(x)}{x^{1-2\sigma}} = \frac{\{x\}(1-\{x\})}{x^{1-2\sigma}}$ and for each $n \geq 1$:

$$\tilde{\psi}_{n+1}(x) = \frac{1}{x} \int_0^x \tilde{\psi}_n(t) dt \quad (56)$$

Where (ψ_k) is the sequence of functions defined in the lemma 1.2. Therefore:

1. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} \quad (57)$$

Where \bar{s} is the complex conjugate of s .

And

2. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-s}} = \frac{1}{\bar{s}(1+2\sigma)^n} + \frac{1}{(1-\bar{s})(2+2\sigma)^n} \quad (58)$$

Where \bar{s} is the complex conjugate of s .

3. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} = \frac{1}{(1-\sigma)(1+2\sigma)^n} + \frac{1}{\sigma(2+2\sigma)^n} + \frac{1}{(2+\sigma)^n} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (59)$$

Where U_0 is defined as follows:

$$\tilde{U}_0(\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{2+\sigma}} \quad (60)$$

And

4. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{\sigma(1+2\sigma)^n} + \frac{1}{(1-\sigma)(2+2\sigma)^n} + \frac{1}{(3-\sigma)^n} \left[\tilde{U}_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (61)$$

Where $\tilde{U}_0(1-\sigma)$ is defined as follows:

$$\tilde{U}_0(1-\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{3-\sigma}} \quad (62)$$

Proof. The first point can be proved simply by using integration by parts exactly like the proof in the lemma 1.2. Let's denote the integral $\tilde{A}(s)$ as following:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (63)$$

Therefore we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (64)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \tilde{\psi}_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (65)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (66)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) - (2+s) \int_0^1 dx \tilde{\psi}_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\tilde{\psi}_2(x)}{x^{2+s}} \quad (67)$$

$$\dots \quad (68)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \tilde{\psi}_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} \quad (69)$$

Since $0 < \sigma \leq \frac{1}{2}$, the functions $\tilde{\psi}_k$ are continuous functions, bounded and also non-negative as for each k for each $x \geq 0$ that:

$$0 \leq \tilde{\psi}_k(x) \leq 1 \quad (70)$$

To prove the point of the lemma, we need to calculate the integrals \tilde{I}_k for $k \geq 0$:

$$\tilde{I}_k = \int_0^1 dx \tilde{\psi}_k(x) \quad (71)$$

For $x \in (0, 1)$, we have:

$$\tilde{\psi}_0(x) = x^{2\sigma} - x^{1+2\sigma} \quad (72)$$

And

$$\tilde{\psi}_1(x) = \frac{x^{2\sigma}}{1+2\sigma} - \frac{x^{1+2\sigma}}{2+2\sigma} \quad (73)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\tilde{\psi}_k(x) = \frac{x^{2\sigma}}{(1+2\sigma)^k} - \frac{x^{1+2\sigma}}{(2+2\sigma)^k} \quad (74)$$

Therefore, for each $k \geq 1$:

$$\tilde{I}_k = \frac{1}{(1+2\sigma)^{k+1}} - \frac{1}{(2+2\sigma)^{k+1}} \quad (75)$$

Therefore after simplification:

$$\tilde{A}(s) = \frac{1}{\bar{s}(1-\bar{s})} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{(1-\bar{s})(1+2\sigma)^{n+1}} + \frac{1}{\bar{s}(2+2\sigma)^{n+1}} \right) \right] \quad (76)$$

Since \bar{s} is also a zeta zero. So from the lemma 1.1, we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s+1-2\sigma}} = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{3-\bar{s}}} = \frac{1}{\bar{s}(1-\bar{s})} \quad (77)$$

Since $2 + s \neq 0$, we conclude the first point of the lemma. The second point can be proved by replacing s by $1 - s$. ■

Let's now prove the points (3) and (4) of the lemma. Let's now calculate the integral $\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}}$. Let's $n \geq 1$.

$$\tilde{U}_{n+1} = \int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\int_0^x dt \tilde{\psi}_n(t)}{x^{3+\sigma}} \quad (78)$$

$$= \frac{1}{2 + \sigma} \int_0^1 dt \tilde{\psi}_n(t) + \frac{1}{2 + \sigma} \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (79)$$

$$= \frac{1}{2 + \sigma} \left[\frac{1}{(1 + 2\sigma)^{n+1}} - \frac{1}{(2 + 2\sigma)^{n+1}} \right] + \frac{\tilde{U}_n}{2 + \sigma} \quad (80)$$

Therefore

$$\tilde{U}_{n+1} = \frac{\tilde{U}_0}{(2 + \sigma)^{n+1}} + \sum_{k=1}^{n+1} \left[\frac{1}{(2 + \sigma)^k (1 + 2\sigma)^{n+2-k}} - \frac{1}{(2 + \sigma)^k (2 + 2\sigma)^{n+2-k}} \right] \quad (81)$$

Therefore:

$$\tilde{U}_{n+1} = \frac{1}{(1 - \sigma)(1 + 2\sigma)^{n+1}} + \frac{1}{\sigma(2 + 2\sigma)^{n+1}} + \frac{1}{(2 + \sigma)^{n+1}} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1 - \sigma)} \right] \quad (82)$$

The point (4) can be proved by replacing σ by $1 - \sigma$. This ends the proof of our lemma. ■

□

Lemma 1.4. *Let's consider f a continuous function over $[1, +\infty)$. Let's ϕ be a non-null positive function such that $f\phi$ and ϕ are integrable functions over $[1, +\infty)$ with:*

$$0 < \int_1^{+\infty} dx \phi(x) < +\infty \quad (83)$$

And

$$\int_1^{+\infty} dx \phi(x) f(x) < +\infty \quad (84)$$

Therefore, there exists a $c \in (1, +\infty)$ such that:

$$\int_1^{+\infty} dx \phi(x) f(x) = f(c) \int_1^{+\infty} dx \phi(x) \quad (85)$$

Proof. This is just the Mean value theorem for integrals on $[1, +\infty)$. Let's define the real λ as following:

$$\lambda = \frac{\int_1^{+\infty} dx \phi(x) f(x)}{\int_1^{+\infty} dx \phi(x)} \quad (86)$$

We have by construction that:

$$\int_1^{+\infty} dx \phi(x) (f(x) - \lambda) = 0 \quad (87)$$

Therefore, if for each $x > 1$, we have $f(x) > \lambda$, then, we will have:

$$\int_1^{+\infty} dx \phi(x) (f(x) - \lambda) > 0 \quad (88)$$

Which is a contradiction. We will reach a similar contradiction if we assume $f(x) < \lambda$ for each $x > 1$. Therefore, there exists $c \in (1, +\infty)$ such that $f(c) = \lambda$. \blacksquare \square

Lemma 1.5. *Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's consider the sequence of functions (ψ_k) and $(\tilde{\psi}_n)$ defined in the lemmas 1.2 and 1.3. Therefore:*

1. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} - \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} \quad (89)$$

And

2. For each n :

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} \quad (90)$$

Where $c_n > 1$ is a real number.

3. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} - \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} \quad (91)$$

And

4. For each n :

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{d_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (92)$$

Where $d_n > 1$ is a real number.

Proof. The first point of the lemma can be proved using the trigonometric relation $\cos(2x) = 2 \cos^2(x) - 1$. Let's now prove the second point of the lemma. Let's $n \geq 1$. Write the following:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\tau \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma+1-2\sigma}} \quad (93)$$

Since the functions $x \rightarrow \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}}$ is a non-negative and integrable on $[1, +\infty)$, we apply the lemma 1.4, to write the following:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\tau \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (94)$$

Where $c_n > 1$ is a real number. \blacksquare

The points 3 and 4 also can be proved using the same method as in the points 1 and 2. And hence the lemma. \blacksquare \square

Lemma 1.6. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\sigma = \frac{1}{2} \quad (95)$$

Proof. Let's suppose that $\sigma < \frac{1}{2}$. We use the previous lemmas notations. From the lemmas 1.2 and 1.5, we can write the following for $n \geq 1$:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} \quad (96)$$

Where $c_n > 1$ is a real number.

And

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{\psi_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} \quad (97)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-s}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\sigma}} \quad (98)$$

Since \bar{s} is also a zeta zero, therefore we can use the lemma 1.2 to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{3-\sigma}} = \frac{1}{2} \left(\frac{1}{(1-s)2^n} + \frac{1}{s3^n} + \frac{1}{(1-\bar{s})2^n} + \frac{1}{\bar{s}3^n} \right) \quad (99)$$

$$+ \frac{1}{(1-\sigma)2^n} + \frac{1}{\sigma 3^n} + \frac{1}{(3-\sigma)^n} \left[U_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (100)$$

$$= \frac{(1-\sigma)}{\|1-s\|^2 2^n} + \frac{1}{(1-\sigma)2^n} + \frac{\sigma}{\|s\|^2 3^n} + \frac{1}{\sigma 3^n} + \frac{1}{(3-\sigma)^n} \left[U_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (101)$$

Where $U_0(1-\sigma)$ is defined as follows:

$$U_0(1-\sigma) = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{3-\sigma}} \quad (102)$$

And the same

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\psi_n(x) \cos(\tau \ln(x))}{x^{2+\sigma}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} \quad (103)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+\sigma}} \quad (104)$$

Since \bar{s} is also a zeta zero, therefore we can use the lemma 1.2 to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \psi_n(x)}{x^{2+\sigma}} = \frac{1}{2} \left(\frac{1}{s2^n} + \frac{1}{(1-s)3^n} + \frac{1}{\bar{s}2^n} + \frac{1}{(1-\bar{s})3^n} \right) \quad (105)$$

$$+ \frac{1}{\sigma 2^n} + \frac{1}{(1-\sigma)3^n} + \frac{1}{(2+\sigma)^n} \left[U_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (106)$$

$$= \frac{\sigma}{\|s\|^2 2^n} + \frac{1}{\sigma 2^n} + \frac{(1-\sigma)}{\|1-s\|^2 3^n} + \frac{1}{(1-\sigma)3^n} + \frac{1}{(2+\sigma)^n} \left[U_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (107)$$

We have for each n : $2^n < (2 + \sigma)^n < 3^n$ and $2^n < (3 - \sigma)^n < 3^n$. Therefore, from the equations (96), (101) and (107), we conclude:

$$\frac{1}{c_n^{1-2\sigma}} \sim_{n \rightarrow +\infty} \frac{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}}{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}} \quad (108)$$

Since $c_n > 1$ for each n , we conclude that:

$$\frac{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}}{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}} \leq 1 \quad (109)$$

And we do the same for $\tilde{\psi}_n$. From the lemmas 1.3 and 1.5, we can write the following for $n \geq 1$

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{d_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (110)$$

Where $d_n > 1$ is a real number.

And

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x) \cos(\tau \ln(x))}{x^{3-\sigma}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} \quad (111)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-s}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{3-\sigma}} \quad (112)$$

Since \bar{s} is also a zeta zero, therefore we can use the lemma 1.3 to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{3-\sigma}} = \frac{1}{2} \left(\frac{1}{\bar{s}(1+2\sigma)^n} + \frac{1}{(1-\bar{s})(2+2\sigma)^n} + \frac{1}{s(1+2\sigma)^n} + \frac{1}{(1-s)(2+2\sigma)^n} \right) \quad (113)$$

$$+ \frac{1}{\sigma(1+2\sigma)^n} + \frac{1}{(1-\sigma)(2+2\sigma)^n} + \frac{1}{(3-\sigma)^n} \left[\tilde{U}_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (114)$$

$$= \frac{\sigma}{\|s\|^2(1+2\sigma)^n} + \frac{1}{\sigma(1+2\sigma)^n} + \frac{1-\sigma}{\|1-s\|^2(2+2\sigma)^n} + \frac{1}{(1-\sigma)(2+2\sigma)^n} \quad (115)$$

$$+ \frac{1}{(3-\sigma)^n} \left[\tilde{U}_0(1-\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (116)$$

Where $\tilde{U}_0(1-\sigma)$ is defined as follows:

$$\tilde{U}_0(1-\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{3-\sigma}} \quad (117)$$

And the same

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x) \cos(\tau \ln(x))}{x^{2+\sigma}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (118)$$

$$= \frac{1}{2} \left(\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\bar{s}}} \right) + \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+\sigma}} \quad (119)$$

Since \bar{s} is also a zeta zero, therefore we can use the lemma 1.3 to write:

$$\int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \tilde{\psi}_n(x)}{x^{2+\sigma}} = \frac{1}{2} \left(\frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} + \frac{1}{(1-s)(1+2\sigma)^n} + \frac{1}{s(2+2\sigma)^n} \right) \quad (120)$$

$$+ \frac{1}{(1-\sigma)(1+2\sigma)^n} + \frac{1}{\sigma(2+2\sigma)^n} + \frac{1}{(2+\sigma)^n} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (121)$$

$$= \frac{1-\sigma}{\|1-s\|^2(1+2\sigma)^n} + \frac{1}{(1-\sigma)(1+2\sigma)^n} + \frac{\sigma}{\|s\|^2(2+2\sigma)^n} + \frac{1}{\sigma(2+2\sigma)^n} \quad (122)$$

$$+ \frac{1}{(2+\sigma)^n} \left[\tilde{U}_0(\sigma) - \frac{1}{\sigma(1-\sigma)} \right] \quad (123)$$

Where $\tilde{U}_0(\sigma)$ is defined as follows:

$$\tilde{U}_0(\sigma) = \int_1^{+\infty} dt \frac{\tilde{\psi}_0(t)}{t^{2+\sigma}} \quad (124)$$

We have for each $n \geq 1$: $(1+2\sigma)^n < (2+\sigma)^n < (3-\sigma)^n$ and $(1+2\sigma)^n < (2+\sigma)^n < (2+2\sigma)^n$. Therefore, from the equations (110), (115) and (122), we conclude:

$$\frac{1}{d_n^{1-2\sigma}} \sim_{n \rightarrow +\infty} \frac{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}}{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}} \quad (125)$$

Since $d_n > 1$ for each n , we conclude that:

$$\frac{\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma}}{\frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)}} \leq 1 \quad (126)$$

From the equations (109) and (126), we conclude that:

$$\frac{\sigma}{\|s\|^2} + \frac{1}{\sigma} = \frac{(1-\sigma)}{\|1-s\|^2} + \frac{1}{(1-\sigma)} \quad (127)$$

Since the function $g(x) \rightarrow \frac{x}{x^2+\tau^2} + \frac{1}{x}$ is strictly decreasing since its first derivative is strictly negative: for each $x > 0$:

$$g'(x) = \frac{-2x^4 - \tau^4 - x^2\tau^2}{x^2(x^2+\tau^2)^2} < 0 \quad (128)$$

$$\sigma = 1 - \sigma = \frac{1}{2} \quad (129)$$

This ends the proof of the Riemann Hypothesis. \blacksquare

\square

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

Acknowledgments

I would like to thank Farhat Latrach, Giampiero Esposito, Jacques G elinas, Michael Milgram, L eo Ag elas, Ronald F. Fox, Kim Y.G, Masumi Nakajima, Maksym Radziwill and Shekhar Suman for thoughtful comments and discussions on my paper versions on the RH. All errors are mine.

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