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Charaf Ech-Chatbi

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A Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

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Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

*One Raffles Quay, North Tower. 048583 Singapore. Email: charaf.chatbi@gmail.com. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

Or in other terms:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (8)$$

Let's denote the following functions:

$$\epsilon(x) = \{x\} \quad (8)$$

$$\phi(x) = \{x\}(1 - \{x\}) \quad (9)$$

$$\Psi(x) = \int_1^x dx \epsilon(x) \quad (10)$$

To continue, we will need to prove some few lemmas.

Lemma 1.1. *The function $\epsilon(x)$ is piecewise continuous on $[0, +\infty)$ and its primitive function $\Psi(x)$ is defined as follows:*

$$\Psi(x) = \frac{1}{2}(x - 1 - \phi(x)) \quad (11)$$

Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:

$$\int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} = \frac{1}{(s-1)(1+s)} \quad (12)$$

And

$$\int_1^{+\infty} dx \frac{\phi(x)}{x^{2+s}} = \frac{1}{s(1-s)} \quad (13)$$

Proof. We will need the function ϕ as you will see later that we need a continuous function instead of a piecewise one like the function ϵ .

Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer of part of x . We have $n_x = x - \{x\}$. Therefore, we can write the following:

$$\Psi(x) = \int_1^x \epsilon(x) dx \quad (14)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} \{t\} dt + \int_{n_x}^x \{t\} dt \quad (15)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} (t-n) dt + \int_{n_x}^x (t-n_x) dt \quad (16)$$

$$= \sum_{n=1}^{n_x-1} \frac{1}{2} + \frac{1}{2}(x-n_x)^2 \quad (17)$$

$$= \frac{1}{2}(n_x - 1 + \{x\}^2) \quad (18)$$

$$= \frac{1}{2}(x - 1 - \{x\} + \{x\}^2) \quad (19)$$

This prove the equation (11). ■

Let's prove the second point of the lemma. Let's define the integral $I_\epsilon(s)$ as follows:

$$I_\epsilon(s) = \int_1^{+\infty} dx \frac{\epsilon(x)}{x^{1+s}} \quad (20)$$

The function $x \rightarrow \frac{\epsilon(x)}{x^{1+s}}$ is integrable on $[1, +\infty)$ and thanks to the integration by parts, we can write the following:

$$I_\epsilon(s) = \int_1^{+\infty} dx \frac{\epsilon(x)}{x^{1+s}} \quad (21)$$

$$= \left[\frac{\Psi(x)}{x^{1+s}} \right]_{x=1}^{+\infty} + (1+s) \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (22)$$

$$= (1+s) \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (23)$$

Since, s is a zeta zero, from the equation (6), we have:

$$I_\epsilon(s) = \frac{1}{s-1} \quad (24)$$

Therefore

$$\int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} = \frac{1}{(s-1)(s+1)} \quad (25)$$

Thanks to equation (11), we can write:

$$\int_1^{+\infty} dx \frac{\phi(x)}{x^{2+s}} = \int_1^{+\infty} dx \frac{(x-1)}{x^{2+s}} - 2 \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (26)$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{2}{(s-1)(s+1)} \quad (27)$$

$$= \frac{1}{s(1-s)} \quad (28)$$

■

□

Lemma 1.2. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions ψ_n such that $\psi_0(x) = \phi(x) = \{x\}(1 - \{x\})$ and for each $n \geq 1$:*

$$\psi_{n+1}(x) = \frac{1}{x} \int_0^x \psi_n(t) \quad (29)$$

Therefore:

1. For each $n \geq 1$:

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{s 2^n} + \frac{1}{(1-s) 3^n} \quad (30)$$

2. For each n :

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+s}} = \frac{1}{s(1-s)(2+s)^n} \quad (31)$$

Proof. Let's denote the integral $A(s)$ as following:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (32)$$

We use the integration by parts to write the following:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (33)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \psi_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (34)$$

$$= - \int_0^1 dx \psi_0(x) + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (35)$$

$$= - \int_0^1 dx \psi_0(x) - (2+s) \int_0^1 dx \psi_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\psi_2(x)}{x^{2+s}} \quad (36)$$

$$\dots \quad (37)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \psi_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} \quad (38)$$

We could do the above integration by parts. Because the functions ψ_k are continuous functions. We also can prove that the functions ψ_k are non-negative and bounded as we can prove that for each k for each $x > 0$ that:

$$0 < \psi_k(x) \leq \frac{1}{4} \quad (39)$$

This can be proven by recurrence using the fact that for each $x \geq 0$:

$$0 \leq \psi_0(x) = \{x\}(1 - \{x\}) \leq \frac{1}{4} \quad (40)$$

Now, we need to calculate the integrals I_k for $k \geq 0$:

$$I_k = \int_0^1 dx \psi_k(x) \quad (41)$$

For $x \in (0, 1)$, we have:

$$\psi_0(x) = x - x^2 \quad (42)$$

And

$$\psi_1(x) = \frac{x}{2} - \frac{x^2}{3} \quad (43)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\psi_k(x) = \frac{x}{2^k} - \frac{x^2}{3^k} \quad (44)$$

Therefore, for each $k \geq 1$:

$$I_k = \frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \quad (45)$$

Therefore, we can conclude:

$$A(s) = \frac{1}{s(1-s)} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{s2^{n+1}} + \frac{1}{(1-s)3^{n+1}} \right) \right] \quad (46)$$

Since $2+s \neq 0$ and $3-s \neq 0$, we conclude the result of our lemma. \blacksquare

Let's now prove the second point of the lemma. For $n \geq 1$, we write thanks to the change of the order of integration:

$$\psi_n(x) = \frac{1}{x(n-1)!} \int_0^x dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1} \quad (47)$$

$$= \frac{1}{x(n-1)!} \int_0^1 dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1} + \frac{1}{x(n-1)!} \int_1^x dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1} \quad (48)$$

And then

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_0^1 dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1}}{x^{3+s}} + \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1}}{x^{3+s}} \quad (49)$$

We calculate the first term and deduce the lemma result. We can also change the order of integration and directly calculate the integral:

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_0^1 dt \psi_0(t) \left(\ln \left(\frac{x}{t} \right) \right)^{n-1}}{x^{3+s}} = \frac{1}{(n-1)!} \int_1^{+\infty} dt \psi_0(t) \int_t^{+\infty} dx \frac{\left(\ln \left(\frac{x}{t} \right) \right)^{n-1}}{x^{3+s}} \quad (50)$$

$$= \frac{1}{(2+s)^n} \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+s}} \quad (51)$$

And we conclude using the lemma 1.1

\square

Lemma 1.3. *Let's consider f a continuous function over $[1, +\infty)$. Let's ϕ be a non-null positive function such that $f\phi$ and ϕ are integrable functions over $[1, +\infty)$ with:*

$$0 < \int_1^{+\infty} dx \phi(x) < +\infty \quad (52)$$

And

$$\int_1^{+\infty} dx \phi(x) f(x) < +\infty \quad (53)$$

Therefore, there exists a $c \in (1, +\infty)$ such that:

$$\int_1^{+\infty} dx \phi(x) f(x) = f(c) \int_1^{+\infty} dx \phi(x) \quad (54)$$

Proof. This is just the Mean value theorem for integrals on $[1, +\infty)$. Let's define the real λ as following:

$$\lambda = \frac{\int_1^{+\infty} dx \phi(x) f(x)}{\int_1^{+\infty} dx \phi(x)} \quad (55)$$

We have by construction that:

$$\int_1^{+\infty} dx \phi(x) (f(x) - \lambda) = 0 \quad (56)$$

Therefore, if for each $x > 1$, we have $f(x) > \lambda$, then, we will have:

$$\int_1^{+\infty} dx \phi(x) (f(x) - \lambda) > 0 \quad (57)$$

Which is a contradiction. We will reach a similar contradiction if we assume $f(x) < \lambda$ for each $x > 1$. Therefore, there exists $c \in (1, +\infty)$ such that $f(c) = \lambda$. \blacksquare \square

Lemma 1.4. *Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions $\tilde{\psi}_n$ such that $\tilde{\psi}_0(x) = \frac{\psi_0(x)}{x^{1-2\sigma}} = \frac{\{x\}(1-\{x\})}{x^{1-2\sigma}}$ and for each $n \geq 1$:*

$$\tilde{\psi}_{n+1}(x) = \frac{1}{x} \int_0^x \tilde{\psi}_n(t) \quad (58)$$

Where (ψ_k) is the sequence of functions defined in the lemma 1.2. Therefore:

1. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} \quad (59)$$

Where \bar{s} is the complex conjugate of s .

2. For each n :

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+s}} = \frac{1}{\bar{s}(1-\bar{s})(2+s)^n} \quad (60)$$

Where \bar{s} is the complex conjugate of s .

3. For each n :

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (61)$$

$$- \frac{1}{c^{1-2\sigma}} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (62)$$

Where $c_n > 1$ and $c > 1$ are real numbers.

4. Let's define the sequence z_n as follows:

$$z_n = \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}}} \quad (63)$$

Let's assume $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in (0, \frac{\pi}{2})$. Let's assume further that the sequence (c_n) converges to c but its terms are different from c such that for each $n \geq 1$, there exist an integer $k \geq n$ such that $c_k \neq c$. Therefore, we can always find an integer n such that $z_n < 0$.

Proof. The first point can be proved simply by using integration by parts exactly like the proof in the lemma 1.2. Let's denote the integral $\tilde{A}(s)$ as following:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (64)$$

Therefore we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (65)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \tilde{\psi}_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (66)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (67)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) - (2+s) \int_0^1 dx \tilde{\psi}_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\tilde{\psi}_2(x)}{x^{2+s}} \quad (68)$$

$$\dots \quad (69)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \tilde{\psi}_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} \quad (70)$$

Since $0 < \sigma \leq \frac{1}{2}$, the functions $\tilde{\psi}_k$ are continuous functions, bounded and also non-negative as for each k for each $x \geq 0$ that:

$$0 \leq \tilde{\psi}_k(x) \leq 1 \quad (71)$$

To prove the point of the lemma, we need to calculate the integrals \tilde{I}_k for $k \geq 0$:

$$\tilde{I}_k = \int_0^1 dx \tilde{\psi}_k(x) \quad (72)$$

For $x \in (0, 1)$, we have:

$$\tilde{\psi}_0(x) = x^{2\sigma} - x^{1+2\sigma} \quad (73)$$

And

$$\tilde{\psi}_1(x) = \frac{x^{2\sigma}}{1+2\sigma} - \frac{x^{1+2\sigma}}{2+2\sigma} \quad (74)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\tilde{\psi}_k(x) = \frac{x^{2\sigma}}{(1+2\sigma)^k} - \frac{x^{1+2\sigma}}{(2+2\sigma)^k} \quad (75)$$

Therefore, for each $k \geq 1$:

$$\tilde{I}_k = \frac{1}{(1+2\sigma)^{k+1}} - \frac{1}{(2+2\sigma)^{k+1}} \quad (76)$$

Therefore after simplification:

$$\tilde{A}(s) = \frac{1}{\bar{s}(1-\bar{s})} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{(1-\bar{s})(1+2\sigma)^{n+1}} + \frac{1}{\bar{s}(2+2\sigma)^{n+1}} \right) \right] \quad (77)$$

Since \bar{s} is also a zeta zero. So from the lemma 1.1, we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s+1-2\sigma}} = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{3-\bar{s}}} = \frac{1}{\bar{s}(1-\bar{s})} \quad (78)$$

Since $2+s \neq 0$, we conclude the first point of the lemma. \blacksquare

The proof of the second point is also very similar to the proof of the second point in the lemma 1.2. \blacksquare

Let's now prove the third point of the lemma. Let's $n \geq 1$. Write the following:

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} = \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (79)$$

$$- \int_1^{+\infty} dx \frac{\int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (80)$$

$$= \int_1^{+\infty} dt \tilde{\psi}_0(t) \int_t^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (81)$$

$$- \int_1^{+\infty} dt \tilde{\psi}_0(t) \int_t^{+\infty} dx \frac{\left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (82)$$

$$= \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma+1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(tx)\right) \left(\ln(x)\right)^{n-1}}{x^{3+\sigma}} \quad (83)$$

$$- \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma+1-2\sigma}} \int_1^{+\infty} dx \frac{\left(\ln(x)\right)^{n-1}}{x^{3+\sigma}} \quad (84)$$

$$= \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma+1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(tx)\right) \left(\ln(x)\right)^{n-1}}{x^{3+\sigma}} \quad (85)$$

$$- \frac{(n-1)!}{(2+\sigma)^n} \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma+1-2\sigma}} \quad (86)$$

Since the functions $t \rightarrow \frac{\psi_0(t)}{t^{2+\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(tx)\right) \left(\ln(x)\right)^{n-1}}{x^{3+\sigma}}$ and $t \rightarrow \frac{\psi_0(t)}{t^{2+\sigma}}$ are non-negative and integrable on $[1, +\infty)$, we apply the lemma

1.3, to write the following:

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(tx)\right) (\ln(x))^{n-1}}{x^{3+\sigma}} \quad (87)$$

$$- \frac{1}{c^{1-2\sigma}} \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma}} \int_1^{+\infty} dx \frac{(\ln(x))^{n-1}}{x^{3+\sigma}} \quad (88)$$

Where $c_n > 1$ and $c > 1$ are real numbers.

We change the order of integration back to the original form and conclude:

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2\left(\frac{\tau}{2} \ln(x)\right) \int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (89)$$

$$- \frac{1}{c^{1-2\sigma}} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) \left(\ln\left(\frac{x}{t}\right)\right)^{n-1}}{x^{3+\sigma}} \quad (90)$$

■

Let's prove now the last point of the lemma. We proceed by contradiction. Therefore for each $n \geq 1$:

$$z_n = \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}}} > 0 \quad (91)$$

Therefore

$$\left(\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}} \right) \left(\cos(\alpha - \beta n) + \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}} \right) > 0 \quad (92)$$

Therefore

$$A_n \cos^2(\beta n) - C_n \sin(\beta n) \cos(\beta n) + B_n > 0 \quad (93)$$

Where

$$A_n = \left[1 - 2 \sin^2(\alpha)\right] \left[1 - \frac{1}{cc_n}\right] + \left[\frac{1}{c_n} - \frac{1}{c}\right] \quad (94)$$

$$B_n = \sin^2(\alpha) \left[\left(1 - \frac{1}{cc_n}\right) - \left(\frac{1}{c_n} - \frac{1}{c}\right) \right] \quad (95)$$

$$C_n = \sin(2\alpha) \left[1 + \frac{1}{cc_n}\right] \quad (96)$$

- Case $\sin(\beta n) \cos(\beta n) > 0$. Therefore for each $n \geq 1$:

$$(A_n^2 + C_n^2) \cos^4(\beta n) + (2A_n B_n - C_n^2) \cos^2(\beta n) + B_n^2 > 0 \quad (97)$$

So we calculate the delta of the second degree equation above:

$$\Delta_n = C_n^2 \left(C_n^2 - 4B_n(A_n + B_n) \right) = \sin^2(2\alpha) \left(\frac{1}{c_n} - \frac{1}{c} \right)^2 > 0 \quad (98)$$

Since the assumption on the sequence (c_n) and c , we have the delta $\Delta_n > 0$. Therefore, the two different solutions of the equation(97) are x_1^n and x_2^n :

$$x_1^n = \frac{C_n^2 - 2A_n B_n - \sqrt{\Delta_n}}{2(A_n^2 + C_n^2)} \quad (99)$$

$$x_2^n = \frac{C_n^2 - 2A_n B_n + \sqrt{\Delta_n}}{2(A_n^2 + C_n^2)} > x_1^n \quad (100)$$

We also have both x_1^n and x_2^n are in the interval $(0, 1)$. Therefore for the equation (97) to be verified for each n , we need $\cos^2(\beta n)$ to be bigger than x_2^n or smaller than x_1^n . Since $\alpha \in (0, \frac{\pi}{2})$, $\beta \in (0, \frac{\pi}{2})$, the sequence (c_n) terms and c are different, we have $x_1^n < x_2^n$ and therefore we can always find n_0 such $\cos^2(\beta n_0) \in (x_1^{n_0}, x_2^{n_0})$. And therefore $(A_{n_0}^2 + C_{n_0}^2) \cos^4(\beta n_0) + (2A_{n_0} B_{n_0} - C_{n_0}^2) \cos^2(\beta n_0) + B_{n_0}^2 < 0$. Which is a contradiction.

- Case $\sin(\beta n) \cos(\beta n) < 0$. The same analysis can be done here.

This ends the proof of our lemma. ■

□

Lemma 1.5. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\sigma = \frac{1}{2} \quad (101)$$

Proof. Let's suppose that $\sigma < \frac{1}{2}$. From the lemmas 1.2 and 1.4, we can write the following for $n \geq 1$:

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+s}} = \frac{1}{s(1-s)(2+s)^n} \quad (102)$$

And

$$\frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \tilde{\psi}_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+s}} = \frac{1}{\bar{s}(1-\bar{s})(2+s)^n} \quad (103)$$

Where \bar{s} is the complex conjugate of s . And

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \tilde{\psi}_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} = \frac{1}{c_n^{1-2\sigma}} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (104)$$

$$- \frac{1}{c^{1-2\sigma}} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (105)$$

Where $c_n > 1$ and $c > 1$ are real numbers. We also can write:

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x)) \int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} = \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (106)$$

$$- \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (107)$$

Let's write the following:

$$\frac{1}{s(1-s)} = \frac{e^{-i\alpha}}{\|s(1-s)\|} \quad (108)$$

$$\frac{1}{2+s} = \frac{e^{-i\beta}}{\|2+s\|} \quad (109)$$

Where $\alpha \in (0, \pi)$ and $\beta \in (0, \pi)$. Without loss of generality, we can assume α and β are in $(0, \frac{\pi}{2})$ since $0 < \sigma < \frac{1}{2}$ and $\tau > 0$.

Therefore, we can write:

$$\frac{1}{(2+s)^n} = \frac{e^{-i\beta n}}{\|2+s\|^n} \quad (110)$$

Therefore, when we combine the equations (102-107) and get the following:

$$X_n = \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{2 \cos^2(\frac{\tau}{2} \ln(x)) \int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (111)$$

$$= \frac{1}{\|s(1-s)\| \|2+s\|^n} \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\frac{1}{c_n^{1-2\sigma}} - \frac{1}{c^{1-2\sigma}}} \quad (112)$$

And

$$Y_n = \frac{1}{(n-1)!} \int_1^{+\infty} dx \frac{\int_1^x dt \psi_0(t) (\ln(\frac{x}{t}))^{n-1}}{x^{3+\sigma}} \quad (113)$$

$$= \frac{1}{\|s(1-s)\| \|2+s\|^n} \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}}}{\frac{1}{c_n^{1-2\sigma}} - \frac{1}{c^{1-2\sigma}}} \quad (114)$$

We can also calculate X_n and Y_n in another way as follows:

$$X_n = \frac{U_0}{(2+\sigma)^n} + \frac{\cos(\alpha + \beta n)}{\|s(1-s)\| \|2+s\|^n} \quad (115)$$

And

$$Y_n = \frac{U_0}{(2+\sigma)^n} \quad (116)$$

Where

$$U_0 = \int_1^{+\infty} dt \frac{\psi_0(t)}{t^{2+\sigma}} \quad (117)$$

The sequence $(\frac{1}{c_n})_{n \geq 1}$ is bounded. From the equations (113) and (116), we deduce that the limit of $(\frac{1}{c_n})_{n \geq 1}$ is $\frac{1}{c} < 1$. We will distinguish between two cases.

1. Case 1: There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$: $c_n = c$. We have from the equations (111-116) that for $n \geq n_0$:

$$\frac{X_n}{Y_n} = 1 + \frac{(2 + \sigma)^n \cos(\alpha + \beta n)}{U_0 \|s(1-s)\| \|2+s\|^n} = \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}}} \quad (118)$$

Therefore since $c_n = c$, we can write:

$$1 + \frac{(2 + \sigma)^n \cos(\alpha + \beta n)}{U_0 \|s(1-s)\| \|2+s\|^n} = \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}} = 1 \quad (119)$$

Since $\sigma > 0$, therefore for each $n \geq n_0$:

$$\cos(\alpha + \beta n) = \cos(\alpha) \cos(\beta n) - \sin(\alpha) \sin(\beta n) = 0 \quad (120)$$

Since $\beta \in (0, \frac{\pi}{2})$, therefore

$$\sin(\alpha) = \cos(\alpha) = 0 \quad (121)$$

Which is a contradiction.

2. Case 2: For each $n_0 \in \mathbb{N}$ there exists $n \geq n_0$: $c_n \neq c$.

We have from the equation (118):

$$\frac{X_n}{Y_n} = \frac{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c^{1-2\sigma}}}{\cos(\alpha - \beta n) - \frac{\cos(\alpha + \beta n)}{c_n^{1-2\sigma}}} \quad (122)$$

Since $0 < \sigma < \frac{1}{2}$ and $\tau > 0$, we have $\alpha \in (0, \frac{\pi}{2})$, $\beta \in (0, \frac{\pi}{2})$, $\sin(\alpha) \neq 0$ and $\cos(\alpha) \neq 0$. And since the sequence (c_n) terms and c are different, from the lemma 1.4, we can find n_0 such that $\cos(\alpha - \beta n_0) - \frac{\cos(\alpha + \beta n_0)}{c^{1-2\sigma}} > 0$ and $\cos(\alpha - \beta n_0) - \frac{\cos(\alpha + \beta n_0)}{c^{1-2\sigma}} < 0$ or $\cos(\alpha - \beta n_0) - \frac{\cos(\alpha + \beta n_0)}{c^{1-2\sigma}} < 0$ and $\cos(\alpha - \beta n_0) - \frac{\cos(\alpha + \beta n_0)}{c^{1-2\sigma}} > 0$.

Therefore, in such case, we will have $\frac{X_{n_0}}{Y_{n_0}} < 0$. Which means either $X_{n_0} < 0$ or $Y_{n_0} < 0$. Which is a contradiction.

We can also write the sequence $(\frac{X_n}{Y_n})$ as follows:

$$\frac{X_n}{Y_n} = \frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)} \quad (123)$$

Where the sequences (A_n) and (B_n) converge respectively to the limits A and B .

$$A_n = \cos(\alpha) \left(1 - \frac{1}{c_n^{1-2\sigma}}\right), B_n = \sin(\alpha) \left(1 + \frac{1}{c_n^{1-2\sigma}}\right) \quad (124)$$

$$A = \cos(\alpha) \left(1 - \frac{1}{c^{1-2\sigma}}\right), B = \sin(\alpha) \left(1 + \frac{1}{c^{1-2\sigma}}\right) \quad (125)$$

From the equation (118), we deduce that the sequence $(\frac{X_n}{Y_n})$ converges to 1. And therefore the sequence $(\frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)})$ converges to 1. But $\sin(\alpha) \neq 0$ and $\cos(\alpha) \neq 0$, the sequence $(\frac{A \cos(\beta n) + B \sin(\beta n)}{A_n \cos(\beta n) + B_n \sin(\beta n)})$ cannot converge. Hence the contradiction.

In contrast, in case $\alpha = 0$ or $\alpha = \frac{\pi}{2}$, we will have $\frac{X_n}{Y_n} = \frac{1 - \frac{1}{c_n^{1-2\sigma}}}{1 - \frac{1}{c_n^{1-2\sigma}}}$ always non-negative. And since $\sin(\alpha) = \frac{\tau(1-2\sigma)}{(\sigma(1-\sigma)+\tau^2)^2 + (\tau(1-2\sigma))^2} < 1$, α cannot be $\frac{\pi}{2}$. Therefore α should be zero and:

$$\sigma = \frac{1}{2} \tag{126}$$

In case of $\sigma > \frac{1}{2}$. We work with $1 - s$ that is also a zeta zero and here we have $1 - \sigma < \frac{1}{2}$. And we will reach the same contradiction by following the same steps above. Therefore, σ can only be $\frac{1}{2}$. This ends the proof of the Riemann Hypothesis. ■
□

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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References

- [1] Bernhard Riemann. *On the Number of Prime Numbers less than a Given Quantity*
<https://www.claymath.org/sites/default/files/ezeta.pdf>
- [2] Aleksandar Ivic. *The Riemann Zeta-Function: Theory and Applications*
- [3] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*
http://wayback.cecm.sfu.ca/~pborwein/TEMP_PROTECTED/book.pdf
- [4] J orgen Veisdal. *The Riemann Hypothesis, explained*
<https://medium.com/cantors-paradise/the-riemann-hypothesis-explained-fa01c1f75d3f>
- [5] Thai Pham. *Dirichlet's Theorem on Arithmetic Progressions*
https://web.stanford.edu/~thaipham/papers/MIT_18.104_Review_Paper.pdf
- [6] G. H. Hardy. *The general theory of dirichlet series.*
<https://archive.org/details/generaltheoryofd029816mbp/page/n9>
- [7] Garrett, Paul. *Primes in arithmetic progressions, 2011.*
[http : //www.math.umn.edu/~garrett/m/mfms/notes_c/dirichlet.pdf](http://www.math.umn.edu/~garrett/m/mfms/notes_c/dirichlet.pdf)