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A Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

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Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

To continue, we will need to prove some few lemmas.

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (3)$$

Lemma 1.1. *Let's consider the function $\epsilon(x)$ that plays a big role in the zeta function as follows:*

$$\epsilon(x) = \{x\} \quad (8)$$

The function $\epsilon(x)$ is piecewise continuous on the interval $[1, +\infty)$ and its primitive function $\Psi(x) = \int_1^x \epsilon(x)dx$ is defined as follows:

$$\Psi(x) = \frac{1}{2}(x - 1 - \{x\}(1 - \{x\})) \quad (9)$$

Proof. We will need this function as you will see later. Because we will want to make use of concave functions and we will need a continuous function instead of a piecewise continuous one.

Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer of part of x . We have $n_x = x - \{x\}$. Therefore we deduce the primitive function as follows:

$$\Psi(x) = \int_1^x \epsilon(x)dx \quad (10)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} \{t\}dt + \int_{n_x}^x \{t\}dt \quad (11)$$

$$= \sum_{n=1}^{n_x-1} \int_n^{n+1} (t - n)dt + \int_{n_x}^x (t - n_x)dt \quad (12)$$

$$= \sum_{n=1}^{n_x-1} \frac{1}{2} + \frac{1}{2}(x - n_x)^2 \quad (13)$$

$$= \frac{1}{2}(n_x - 1 + \{x\}^2) \quad (14)$$

$$= \frac{1}{2}(x - 1 - \{x\} + \{x\}^2) \quad (15)$$

■

□

Lemma 1.2. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define two integrals I_ϵ and J_ϵ as follows:*

$$I_\epsilon(s) = \int_1^{+\infty} dx \frac{\epsilon(x)}{x^{1+s}} \quad (16)$$

$$I_\epsilon(\sigma, \tau) = \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x) \quad (17)$$

$$J_\epsilon(\sigma, \tau) = \int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x) \quad (18)$$

Where the $\epsilon(x)$ is the same function defined in the previous lemma. Therefore:

$$I_\epsilon(s) = (1 + s) \int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} \quad (19)$$

$$I_\epsilon(\sigma, \tau) = - \int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}} \Psi(x) \quad (20)$$

$$J_\epsilon(\sigma, \tau) = - \int_1^{+\infty} dx \frac{g_\sigma(x)}{x^{2+\sigma}} \Psi(x) \quad (21)$$

Where

$$f_\sigma(x) = \tau \cos(\tau \ln(x)) - (1 + \sigma) \sin(\tau \ln(x)) \quad (22)$$

$$g_\sigma(x) = -\tau \sin(\tau \ln(x)) - (1 + \sigma) \cos(\tau \ln(x)) \quad (23)$$

In the context of this lemma, the variables σ and τ are arbitrary and not necessarily related to the zeros of the zeta function.

Proof. Without loss of generality, let's consider $\sigma > 0$, $\tau > 0$. The function $x \rightarrow \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x)$ is integrable and we can write the following thanks to the integration by parts:

$$I_\epsilon(\sigma, \tau) = \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x) \quad (24)$$

$$= \left[\frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \Psi(x) \right]_{x=1}^{+\infty} - \int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}} \Psi(x) \quad (25)$$

Since $\sigma > 0$, we conclude that:

$$I_\epsilon(\sigma, \tau) = - \int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}} \Psi(x) \quad (26)$$

And the same goes for J_ϵ :

$$J_\epsilon(\sigma, \tau) = - \int_1^{+\infty} dx \frac{g_\sigma(x)}{x^{2+\sigma}} \Psi(x) \quad (27)$$

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Lemma 1.3. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\int_1^{+\infty} dx \frac{\Psi(x)}{x^{2+s}} = \frac{1}{(s-1)(1+s)} \quad (28)$$

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) = \frac{2\tau\sigma}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (29)$$

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) = - \frac{[(1-\sigma^2) + \tau^2]}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (30)$$

And

$$\int_1^{+\infty} dx \frac{\phi(x)}{x^{2+s}} = \frac{1}{s(1-s)} \quad (31)$$

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \phi(x) = \frac{\tau}{\sigma^2 + \tau^2} - \frac{\tau}{(1-\sigma)^2 + \tau^2} \quad (32)$$

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}} \phi(x) = \frac{\sigma}{\sigma^2 + \tau^2} + \frac{(1-\sigma)}{(1-\sigma)^2 + \tau^2} \quad (33)$$

Where

$$\phi(x) = \{x\}(1 - \{x\}) \quad (34)$$

Proof. We will use the same notation in the previous lemmas. We use the result of the lemma 1.2 to write the following equations:

$$\tau \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) + (1+\sigma) \int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) = -\frac{(1-\sigma)}{((1-\sigma)^2 + \tau^2)} \quad (35)$$

$$-(1+\sigma) \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) + \tau \int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) = -\frac{\tau}{((1-\sigma)^2 + \tau^2)} \quad (36)$$

After simplification, we get the result in the equation (29).

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \Psi(x) = \frac{1}{2} \left[\frac{\tau}{(1-\sigma)^2 + \tau^2} - \frac{\tau}{(1+\sigma)^2 + \tau^2} \right] \quad (37)$$

To get the result of the equations (32-33), we will use the lemma 1.1 and the definition of $\Psi(x)$. Therefore:

$$\int_1^{+\infty} dx \frac{\phi(x) \sin(\tau \ln(x))}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{(x-1) \sin(\tau \ln(x))}{x^{2+\sigma}} - \frac{4\tau\sigma}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (38)$$

$$= \frac{\tau(1+2\sigma)}{((1+\sigma)^2 + \tau^2)(\sigma^2 + \tau^2)} - \frac{4\tau\sigma}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (39)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} - \frac{\tau}{(1-\sigma)^2 + \tau^2} \quad (40)$$

$$= \frac{\tau(1-2\sigma)}{((1-\sigma)^2 + \tau^2)(\sigma^2 + \tau^2)} \quad (41)$$

And

$$\int_1^{+\infty} dx \frac{\phi(x) \cos(\tau \ln(x))}{x^{2+\sigma}} = \int_1^{+\infty} dx \frac{(x-1) \cos(\tau \ln(x))}{x^{2+\sigma}} + \frac{2[(1-\sigma^2) + \tau^2]}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (42)$$

$$= \frac{\sigma}{\sigma^2 + \tau^2} - \frac{1+\sigma}{(1+\sigma)^2 + \tau^2} + \frac{2[(1-\sigma^2) + \tau^2]}{((1-\sigma)^2 + \tau^2)((1+\sigma)^2 + \tau^2)} \quad (43)$$

$$= \frac{\sigma}{\sigma^2 + \tau^2} + \frac{(1-\sigma)}{(1-\sigma)^2 + \tau^2} \quad (44)$$

We can also just apply the integration by parts on the complex integral and directly deduce the lemma results.

■

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Lemma 1.4. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions ψ_n such that $\psi_0(x) = \phi(x) = \{x\}(1 - \{x\})$ and for each $n \geq 1$:*

$$\psi_{n+1}(x) = \frac{1}{x} \int_0^x \psi_n(t) \quad (45)$$

Let's denote the integral $A(s)$ as following:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (46)$$

From the lemma 1.4 we have:

$$A(1-s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{3-s}} = \frac{1}{s(1-s)} = A(s) \quad (47)$$

Therefore:

1. For each $n \geq 1$:

$$A(s) = - \sum_{k=0}^n (2+s)^k \int_0^1 dx \psi_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} \quad (48)$$

$$A(1-s) = - \sum_{k=0}^n (3-s)^k \int_0^1 dx \psi_k(x) + (3-s)^{n+1} \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{3-s}} \quad (49)$$

2. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{2+s}} = \frac{1}{s 2^n} + \frac{1}{(1-s) 3^n} \quad (50)$$

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-s}} = \frac{1}{(1-s) 2^n} + \frac{1}{s 3^n} \quad (51)$$

Proof. The equations (48-49) can be proved simply by using integration by parts. Therefore we can write:

$$A(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s}} \quad (52)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \psi_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (53)$$

$$= - \int_0^1 dx \psi_0(x) + (2+s) \int_1^{+\infty} dx \frac{\psi_1(x)}{x^{2+s}} \quad (54)$$

$$= - \int_0^1 dx \psi_0(x) - (2+s) \int_0^1 dx \psi_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\psi_2(x)}{x^{2+s}} \quad (55)$$

$$\dots \quad (56)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \psi_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} \quad (57)$$

We could do the above integration by parts. Because the functions ψ_k are continuous functions and also we can also prove that they are non-negative and bounded as we can prove that for each k for each $x > 0$ that:

$$0 < \psi_k(x) \leq \frac{1}{4} \quad (58)$$

This can be proven by recurrence using the fact that for each $x \geq 0$:

$$0 \leq \psi_0(x) = \{x\}(1 - \{x\}) \leq \frac{1}{4} \quad (59)$$

The same can be done for $A(1 - s)$. To prove the second point of the lemma, we need to calculate the integrals I_k for $k \geq 0$:

$$I_k = \int_0^1 dx \psi_k(x) \quad (60)$$

For $x \in (0, 1)$, we have:

$$\psi_0(x) = x - x^2 \quad (61)$$

And

$$\psi_1(x) = \frac{x}{2} - \frac{x^2}{3} \quad (62)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\psi_k(x) = \frac{x}{2^k} - \frac{x^2}{3^k} \quad (63)$$

Therefore, for each $k \geq 1$:

$$I_k = \frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \quad (64)$$

Therefore, from the first point of the lemma we can conclude:

$$A(s) = \frac{1}{s(1-s)} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{s 2^{n+1}} + \frac{1}{(1-s) 3^{n+1}} \right) \right] \quad (65)$$

$$A(1-s) = \frac{1}{s(1-s)} + (3-s)^{n+1} \left[\int_1^{+\infty} dx \frac{\psi_{n+1}(x)}{x^{3-s}} - \left(\frac{1}{(1-s) 2^{n+1}} + \frac{1}{s 3^{n+1}} \right) \right] \quad (66)$$

Since $2+s \neq 0$ and $3-s \neq 0$, we conclude the second point of the lemma. \blacksquare

\square

Lemma 1.5. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions $\tilde{\psi}_n$ such that $\tilde{\psi}_0(x) = \frac{\psi_0(x)}{x^{1-2\sigma}} = \frac{\{x\}(1-\{x\})}{x^{1-2\sigma}}$ and for each $n \geq 1$:*

$$\tilde{\psi}_{n+1}(x) = \frac{1}{x} \int_0^x \tilde{\psi}_n(t) \quad (67)$$

Where (ψ_k) is the sequence of functions defined in the previous lemma. Let's denote the integral $\tilde{A}(s)$ as following:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (68)$$

Therefore:

1. For each n :

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} \quad (69)$$

Where \bar{s} is the complex conjugate of s .

2. For each n :

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} = \frac{1}{3-\bar{s}} \left[\frac{1}{2^n} - \frac{1}{3^n} \right] + \frac{1}{3-\bar{s}} \int_1^{+\infty} dx \frac{\psi_0(x)f(x,n)}{x^{3-\bar{s}}} \quad (70)$$

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{2+s} \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\psi_0(x)f(x,n)}{x^{3-\bar{s}}} \quad (71)$$

Where $f(x, n)$ is found in the definition of $\psi_n(x)$ and $\tilde{\psi}_n(x)$ as follows:

$$\psi_n(x) = \frac{1}{x} \int_0^x dt_n \psi_0(t_n) f(t_n, n) \quad (72)$$

$$\tilde{\psi}_n(x) = \frac{1}{x} \int_0^x dt_n \frac{\psi_0(t_n)}{t_n^{1-2\sigma}} f(t_n, n) \quad (73)$$

Proof. The first point can be proved simply by using integration by parts exactly like the proof in the lemma 1.5. Therefore we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x)}{x^{2+s}} \quad (74)$$

$$= \left[\frac{1}{x^{2+s}} \int_0^x dx \tilde{\psi}_0(x) \right]_1^{+\infty} + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (75)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) + (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_1(x)}{x^{2+s}} \quad (76)$$

$$= - \int_0^1 dx \tilde{\psi}_0(x) - (2+s) \int_0^1 dx \tilde{\psi}_1(x) + (2+s)^2 \int_1^{+\infty} dx \frac{\tilde{\psi}_2(x)}{x^{2+s}} \quad (77)$$

$$\dots \quad (78)$$

$$= - \sum_{k=0}^n (2+s)^k \int_0^1 dx \tilde{\psi}_k(x) + (2+s)^{n+1} \int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} \quad (79)$$

The functions $\tilde{\psi}_k$ are continuous functions and also non-negative as for each k for each $x > 1$ that:

$$\tilde{\psi}_k(x) \geq 0 \quad (80)$$

To prove the point of the lemma, we need to calculate the integrals \tilde{I}_k for $k \geq 0$:

$$\tilde{I}_k = \int_0^1 dx \tilde{\psi}_k(x) \quad (81)$$

For $x \in (0, 1)$, we have:

$$\tilde{\psi}_0(x) = x^{2\sigma} - x^{1+2\sigma} \quad (82)$$

And

$$\tilde{\psi}_1(x) = \frac{x}{1+2\sigma} - \frac{x^2}{2+2\sigma} \quad (83)$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\tilde{\psi}_k(x) = \frac{x}{(1+2\sigma)^k} - \frac{x^2}{(2+2\sigma)^k} \quad (84)$$

Therefore, for each $k \geq 1$:

$$\tilde{I}_k = \frac{1}{(1+2\sigma)^{k+1}} - \frac{1}{(2+2\sigma)^{k+1}} \quad (85)$$

Therefore after simplification:

$$\tilde{A}(s) = \frac{1}{\bar{s}(1-\bar{s})} + (2+s)^{n+1} \left[\int_1^{+\infty} dx \frac{\tilde{\psi}_{n+1}(x)}{x^{2+s}} - \left(\frac{1}{(1-\bar{s})(1+2\sigma)^{n+1}} + \frac{1}{\bar{s}(2+2\sigma)^{n+1}} \right) \right] \quad (86)$$

Since \bar{s} is also a zeta zero. So from the lemmas 1.3 and 1.4 we can write:

$$\tilde{A}(s) = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{2+s+1-2\sigma}} = \int_1^{+\infty} dx \frac{\psi_0(x)}{x^{3-\bar{s}}} = \frac{1}{\bar{s}(1-\bar{s})} \quad (87)$$

Since $2+s \neq 0$, we conclude the first point of the lemma. \blacksquare

Let's now prove the second point of the lemma. Let's $n \geq 1$. We also use integration by parts.

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} = \int_1^{+\infty} \frac{dx}{x^{3-\bar{s}}} \frac{1}{x} \underbrace{\int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \dots \frac{dt_{n-1}}{t_{n-1}} \int_0^{t_{n-1}} dt_n \psi_0(t_n)}_{\psi_{n-1}(t_1)} \quad (88)$$

We change the order of integration in the integral of $\psi_{n-1}(t_1)$ as follows:

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} = \int_1^{+\infty} \frac{dx}{x^{3-\bar{s}}} \frac{1}{x} \int_0^x dt_n \psi_0(t_n) \underbrace{\int_{\dots} \frac{dt_{n-1}}{t_{n-1}} \dots \int_{\dots} \int_{\dots} \frac{dt_1}{t_1}}_{f(t_n, n)} \quad (89)$$

$$= \int_1^{+\infty} \frac{dx}{x^{4-\bar{s}}} \int_0^x dt_n \psi_0(t_n) f(t_n, n) \quad (90)$$

$$= \int_0^{+\infty} dt_n \psi_0(t_n) f(t_n, n) \int_{\max(1, t_n)}^{+\infty} \frac{dx}{x^{4-\bar{s}}} \quad (91)$$

$$= \int_0^1 dt_n \psi_0(t_n) f(t_n, n) \int_1^{+\infty} \frac{dx}{x^{4-\bar{s}}} + \int_1^{+\infty} dt_n \psi_0(t_n) f(t_n, n) \int_{t_n}^{+\infty} \frac{dx}{x^{4-\bar{s}}} \quad (92)$$

$$= \frac{1}{3-\bar{s}} \int_0^1 \psi_0(x) f(x, n) + \frac{1}{3-\bar{s}} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (93)$$

$$= \frac{1}{3-\bar{s}} \int_0^1 \psi_{n-1}(x) + \frac{1}{3-\bar{s}} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (94)$$

$$= \frac{1}{3-\bar{s}} \left[\frac{1}{2^n} - \frac{1}{3^n} \right] + \frac{1}{3-\bar{s}} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (95)$$

The move from the equation (93) to (94) is a change of the integration order back to the original order of the integral $\tilde{\psi}_{n-1}$. Hence the first formula of the second point. \blacksquare

And we do the same for the integral with $\tilde{\psi}_n(x)$:

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \int_1^{+\infty} \frac{dx}{x^{2+s}} \frac{1}{x} \int_0^x \underbrace{\frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \dots \int_0^{t_{n-1}} \frac{dt_{n-1}}{t_{n-1}} \int_0^{t_{n-1}} dt_n \tilde{\psi}_0(t_n)}_{\tilde{\psi}_{n-1}(t_1)} \quad (96)$$

We change the order of integration in the integral of $\tilde{\psi}_{n-1}(t_1)$ as follows:

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \int_1^{+\infty} \frac{dx}{x^{2+s}} \frac{1}{x} \int_0^x dt_n \tilde{\psi}_0(t_n) \underbrace{\int_{\dots} \frac{dt_{n-1}}{t_{n-1}} \dots \int_{\dots} \int_{\dots} \frac{dt_1}{t_1}}_{f(t_n, n)} \quad (97)$$

$$= \int_1^{+\infty} \frac{dx}{x^{3+s}} \int_0^x dt_n \tilde{\psi}_0(t_n) f(t_n, n) \quad (98)$$

$$= \int_0^{+\infty} dt_n \tilde{\psi}_0(t_n) f(t_n, n) \int_{\max(1, t_n)}^{+\infty} \frac{dx}{x^{3+s}} \quad (99)$$

$$= \int_0^1 dt_n \tilde{\psi}_0(t_n) f(t_n, n) \int_1^{+\infty} \frac{dx}{x^{3+s}} + \int_1^{+\infty} dt_n \tilde{\psi}_0(t_n) f(t_n, n) \int_{t_n}^{+\infty} \frac{dx}{x^{3+s}} \quad (100)$$

$$= \frac{1}{2+s} \int_0^1 \tilde{\psi}_0(x) f(x, n) + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x) f(x, n)}{x^{2+s}} \quad (101)$$

$$= \frac{1}{2+s} \int_0^1 \tilde{\psi}_{n-1}(x) + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\tilde{\psi}_0(x) f(x, n)}{x^{2+s}} \quad (102)$$

$$= \frac{1}{2+s} \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{2+s+1-2\sigma}} \quad (103)$$

$$= \frac{1}{2+s} \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (104)$$

Also, the move from the equation (101) to (102) is a change of the integration order back to the original order of the integral $\tilde{\psi}_{n-1}$. \blacksquare

Lemma 1.6. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\sigma = \frac{1}{2} \quad (105)$$

Proof. Let's suppose that $\sigma < \frac{1}{2}$. From the lemma 1.5 and 1.6 we can write the following for $n \geq 1$:

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} = \frac{1}{3-\bar{s}} \left[\frac{1}{2^n} - \frac{1}{3^n} \right] + \frac{1}{3-\bar{s}} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (106)$$

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{2+s} \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] + \frac{1}{2+s} \int_1^{+\infty} dx \frac{\psi_0(x) f(x, n)}{x^{3-\bar{s}}} \quad (107)$$

Therefore:

$$(3 - \bar{s}) \int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} - (2+s) \int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \left[\frac{1}{2^n} - \frac{1}{3^n} \right] - \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] \quad (108)$$

From the lemma 1.5 and lemma 1.6 we can write for each $n \geq 1$:

$$\int_1^{+\infty} dx \frac{\tilde{\psi}_n(x)}{x^{2+s}} = \frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} \quad (109)$$

$$\int_1^{+\infty} dx \frac{\psi_n(x)}{x^{3-\bar{s}}} = \frac{1}{(1-\bar{s})2^n} + \frac{1}{\bar{s}3^n} \quad (110)$$

Therefore we can write the equation (108) as follows:

$$(2+s) \left[\frac{1}{(1-\bar{s})(1+2\sigma)^n} + \frac{1}{\bar{s}(2+2\sigma)^n} \right] - \left[\frac{1}{(1+2\sigma)^n} - \frac{1}{(2+2\sigma)^n} \right] = (3-\bar{s}) \left[\frac{1}{(1-\bar{s})2^n} + \frac{1}{\bar{s}3^n} \right] - \left[\frac{1}{2^n} - \frac{1}{3^n} \right] \quad (111)$$

As $\sigma < \frac{1}{2}$, we have $1+2\sigma < 2$ and $2+2\sigma < 3$. Therefore, we multiply the terms of the equation by $(1+2\sigma)^n$:

$$\lim_{n \rightarrow +\infty} (2+s) \left[\frac{1}{(1-\bar{s})} + \frac{(1+2\sigma)^n}{\bar{s}(2+2\sigma)^n} \right] - \left[1 - \frac{(1+2\sigma)^n}{(2+2\sigma)^n} \right] = \lim_{n \rightarrow +\infty} (3-\bar{s}) \left[\frac{(1+2\sigma)^n}{(1-\bar{s})2^n} + \frac{(1+2\sigma)^n}{\bar{s}3^n} \right] - \left[\frac{(1+2\sigma)^n}{2^n} - \frac{(1+2\sigma)^n}{3^n} \right] \quad (112)$$

And therefore

$$\frac{2+s}{1-\bar{s}} = 1 \quad (113)$$

Therefore

$$\sigma = -\frac{1}{2} \quad (114)$$

Which is a contradiction.

Let's suppose now that $\sigma > \frac{1}{2}$. In this case, we have $1+2\sigma > 2$ and $2+2\sigma > 3$. Therefore, we multiply the terms of the equation by 2^n :

$$\lim_{n \rightarrow +\infty} (2+s) \left[\frac{2^n}{(1-\bar{s})(1+2\sigma)^n} + \frac{2^n}{\bar{s}(2+2\sigma)^n} \right] - \left[\frac{2^n}{(1+2\sigma)^n} - \frac{2^n}{(2+2\sigma)^n} \right] = \lim_{n \rightarrow +\infty} (3-\bar{s}) \left[\frac{1}{(1-\bar{s})} + \frac{2^n}{\bar{s}3^n} \right] - \left[1 - \frac{2^n}{3^n} \right] \quad (115)$$

And therefore

$$\frac{3-\bar{s}}{1-\bar{s}} = 1 \quad (116)$$

Therefore

$$3 = 1 \quad (117)$$

Which is a contradiction.

Therefore, σ can only be $\frac{1}{2}$. And in such case we have $\bar{s} = 1-s$ and the equation (108) well satisfied too:

$$(2+s) \left[\frac{1}{(1-\bar{s})} \underbrace{\frac{1}{(1+2\sigma)^n}}_2 + \frac{1}{\bar{s}} \underbrace{\frac{1}{(2+2\sigma)^n}}_3 \right] - \left[\underbrace{\frac{1}{(1+2\sigma)^n}}_2 - \underbrace{\frac{1}{(2+2\sigma)^n}}_3 \right] = \underbrace{(3-\bar{s})}_{2+s} \left[\frac{1}{(1-\bar{s})2^n} + \frac{1}{\bar{s}3^n} \right] - \left[\frac{1}{2^n} - \frac{1}{3^n} \right] \quad (118)$$

This ends the proof of the Riemann Hypothesis. \blacksquare

\square

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$* . In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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