A Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

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1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

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Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\big(\frac{\pi s}{2}\big)\Gamma(1-s)\zeta(1-s) \tag{4}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0$, $\forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1 - s) = \zeta(\bar{s}) = \zeta(1 - \bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{5}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{6}$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \tag{7}$$

[1] This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} \cos\big(\frac{\pi s}{2}\big)\Gamma(s)\zeta(s) \tag{3}$$

And

$$\frac{\sigma-1-i\tau}{(\sigma-1)^2+\tau^2}=\int_1^{+\infty}\frac{\Big(\cos(\tau\ln(x))-i\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx \tag{8}$$

The integral is absolutely convergent. We take the imaginary part of both sides of the above equation and define the function G as following:

$$\begin{aligned} G(\sigma,\tau) &= \frac{\tau}{(\sigma-1)^2+\tau^2} &(9)\\ &= \int_1^{+\infty}\frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx &(10)\end{aligned}$$

We also have $1-\bar{s}=1-\sigma+i\tau=\sigma_1+i\tau_1$ a zero for ζ with a real part σ_1 such that $0<\sigma_1=1-\sigma<1$ and an imaginary part τ_1 such that $\tau_1=\tau$. Therefore

$$\begin{aligned} G(1-\sigma,\tau) &= \frac{\tau_1}{(\sigma_1-1)^2+\tau_1^2} &(11)\\ &= \frac{\tau}{\sigma^2+\tau^2} &(12)\\ &= \int_1^{+\infty}\frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx &(13)\end{aligned}$$

Before we move forward, we need to calculate the following functions $I(\sigma,\tau)$ and $I(1-\sigma,\tau)$ for $\sigma>0$ and $\tau>0$:

$$\begin{aligned} I(\sigma,\tau) &= \int_1^{+\infty}\frac{\sin(\tau\ln(x))}{x^{1+\sigma}}dx &(14)\\ &= \frac{\tau}{(\sigma)^2+\tau^2} &(15)\end{aligned}$$

And

$$\begin{aligned} I(1-\sigma,\tau) &= \int_1^{+\infty}\frac{\sin(\tau\ln(x))}{x^{2-\sigma}}dx &(16)\\ &= \frac{\tau}{(1-\sigma)^2+\tau^2} &(17)\end{aligned}$$

To continue, we will need to prove the following lemmas.

Lemma 1.1. *Let's consider two variables σ and τ such that $\sigma>0$ and $\tau>0$. Let's define two integrals $I(a,\sigma,\tau)$ and $J(a,\sigma,\tau)$ as follows:*

$$\begin{aligned} I(a,\sigma,\tau) &= \int_1^{a}\frac{\sin(\tau\ln(x))}{x^{\sigma}}dx &(18)\\ J(a,\sigma,\tau) &= \int_1^{a}\frac{\cos(\tau\ln(x))}{x^{\sigma}}dx &(19)\end{aligned}$$

Therefore

$$\begin{aligned} I(a,\sigma,\tau) &= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) &(20)\\ J(a,\sigma,\tau) &= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}+\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) &(21)\end{aligned}$$

Where

$$K(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{22}$$

In the context of this lemma, the variables σ and τ are arbitrary and not necessarily related to the zeros of the zeta function.

Proof. Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's take $a > 1$.

$$I(a,\sigma,\tau) = \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx \tag{23}$$

$$= \int_0^{\ln(a)} \sin(\tau x)e^{(1-\sigma)x}dx \tag{24}$$

$$= K(\sigma,\tau)\Big(1 - \frac{\cos(\tau\ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{25}$$

Where

$$K(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{26}$$

And the same for $J(a,\sigma,\tau)$ for $a > 0$:

$$J(a,\sigma,\tau) = \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx \tag{27}$$

$$= \int_0^{\ln(a)} \cos(\tau x)e^{(1-\sigma)x}dx \tag{28}$$

$$= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{29}$$

Where $K(\sigma,\tau)$ is defined above in the equation (25).

□

Lemma 1.2. *Let's consider the function $\epsilon(x)$ that plays a big role in the zeta function as follows:*

$$\epsilon(x) = \{x\} \tag{30}$$

The function $\epsilon(x)$ is piecewise continuous on the interval $[1,+\infty)$ and its primitive function $\Psi(x) = \int_1^x \epsilon(x)dx$ is defined as follows:

$$\Psi(x) = \frac{1}{2}\Big(x - 1 - \{x\}(1-\{x\})\Big) \tag{31}$$

Proof. We will need this function as we want to work with continuous functions instead of piecewise continuous ones.
Let's take $x > 1$ a real number. Let's denote $n_x = \lfloor x \rfloor$ be the integer

of part of x. We have $n_x = x - \{x\}$. Therefore we deduce the primitive function as follows:

$$\begin{aligned}
\Psi(x) &= \int_1^x \epsilon(x)dx && (32)\\
&= \sum_{n=1}^{n_x-1} \int_n^{n+1} \{t\}dt + \int_{n_x}^x \{t\}dt && (33)\\
&= \sum_{n=1}^{n_x-1} \int_n^{n+1} (t-n)dt + \int_{n_x}^x (t-n_x)dt && (34)\\
&= \sum_{n=1}^{n_x-1} \frac{1}{2} + \frac{1}{2}(x-n_x)^2 && (35)\\
&= \frac{1}{2}\Big(n_x - 1 + \{x\}^2\Big) && (36)\\
&= \frac{1}{2}\Big(x - 1 - \{x\} + \{x\}^2\Big) && (37)
\end{aligned}$$

■ □

Lemma 1.3. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define two integrals I_ϵ and J_ϵ as follows:*

$$\begin{aligned}
I_\epsilon(\sigma,\tau) &= \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}}\epsilon(x) && (38)\\
J_\epsilon(\sigma,\tau) &= \int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{1+\sigma}}\epsilon(x) && (39)
\end{aligned}$$

Where the $\epsilon(x)$ is the same function defined in the previous lemma. Therefore:

$$\begin{aligned}
I_\epsilon(\sigma,\tau) &= -\int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}}\Psi(x) && (40)\\
J_\epsilon(\sigma,\tau) &= -\int_1^{+\infty} dx \frac{g_\sigma(x)}{x^{2+\sigma}}\Psi(x) && (41)
\end{aligned}$$

Where

$$\begin{aligned}
f_\sigma(x) &= \tau\cos(\tau\ln(x)) - (1+\sigma)\sin(\tau\ln(x)) && (42)\\
g_\sigma(x) &= -\tau\sin(\tau\ln(x)) - (1+\sigma)\cos(\tau\ln(x)) && (43)
\end{aligned}$$

In the context of this lemma, the variables σ and τ are arbitrary and not necessarily related to the zeros of the zeta function.

Proof. Without loss of generality, let's consider $\sigma > 0$, $\tau > 0$. The function $x \to \frac{\sin(\tau \ln(x))}{x^{1+\sigma}}\epsilon(x)$ is integrable and we can write the following thanks to the integration by parts:

$$\begin{aligned}
I_\epsilon(\sigma,\tau) &= \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}}\epsilon(x) && (44)\\
&= \Big[\frac{\sin(\tau \ln(x))}{x^{1+\sigma}}\Psi(x)\Big]_{x=1}^{+\infty} - \int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}}\Psi(x) && (45)
\end{aligned}$$

Since $\sigma > 0$, we conclude that:

$$I_\epsilon(\sigma,\tau) = -\int_1^{+\infty} dx \frac{f_\sigma(x)}{x^{2+\sigma}}\Psi(x) \quad (46)$$

And the same goes for J_ϵ:

$$J_\epsilon(\sigma,\tau) = -\int_1^{+\infty} dx \frac{g_\sigma(x)}{x^{2+\sigma}}\Psi(x) \quad (47)$$

■ □

Lemma 1.4. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) = \frac{2\tau\sigma}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (48)$$

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) = -\frac{\Big[(1-\sigma^2)+\tau^2\Big]}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (49)$$

And

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}}\phi(x) = \frac{\tau}{\sigma^2+\tau^2} - \frac{\tau}{(1-\sigma)^2+\tau^2} \quad (50)$$

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}}\phi(x) = \frac{\sigma}{\sigma^2+\tau^2} + \frac{(1-\sigma)}{(1-\sigma)^2+\tau^2} \quad (51)$$

Where the function ϕ is defined as $\phi(x) = \{x\}\big(1-\{x\}\big)$ and the function $\Psi(x)$ is defined in lemma 1.2.

Proof. We will use the same notation in the previous lemmas. We combine the result of the equation (8-10) and the lemma 1.3 to write the following equations:

$$\tau\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) + (1+\sigma)\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) = -\frac{(1-\sigma)}{\big((1-\sigma)^2+\tau^2\big)} \quad (52)$$

$$-(1+\sigma)\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) + \tau\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) = -\frac{\tau}{\big((1-\sigma)^2+\tau^2\big)} \quad (53)$$

After simplification, we get the result in the equation (48).

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}}\Psi(x) = \frac{1}{2}\left[\frac{\tau}{(1-\sigma)^2+\tau^2} - \frac{\tau}{(1+\sigma)^2+\tau^2}\right] \quad (54)$$

To get the result of the equation (49), we will use the lemma 1.1 and the definition of $\Psi(x)$. Therefore:

$$\int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \phi(x) = \int_1^{+\infty} dx \frac{(x-1)\sin(\tau \ln(x))}{x^{2+\sigma}} - \frac{4\tau\sigma}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (55)$$

$$= \frac{\tau(1+2\sigma)}{\big((1+\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} - \frac{4\tau\sigma}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (56)$$

$$= \frac{\tau}{\sigma^2+\tau^2} - \frac{\tau}{(1-\sigma)^2+\tau^2} \quad (57)$$

$$= \frac{\tau(1-2\sigma)}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (58)$$

And

$$\int_1^{+\infty} dx \frac{\cos(\tau \ln(x))}{x^{2+\sigma}} \phi(x) = \int_1^{+\infty} dx \frac{(x-1)\cos(\tau \ln(x))}{x^{2+\sigma}} + \frac{2\Big[(1-\sigma^2)+\tau^2\Big]}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (59)$$

$$= \frac{\sigma}{\sigma^2+\tau^2} - \frac{1+\sigma}{(1+\sigma)^2+\tau^2} + \frac{2\Big[(1-\sigma^2)+\tau^2\Big]}{\big((1-\sigma)^2+\tau^2\big)\big((1+\sigma)^2+\tau^2\big)} \quad (60)$$

$$= \frac{\sigma}{\sigma^2+\tau^2} + \frac{\big(1-\sigma\big)}{(1-\sigma)^2+\tau^2} \quad (61)$$

■ □

Lemma 1.5. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence $(x_k)_{k\geq 0}$ such that $x_k = \exp(-\frac{2k\pi}{\tau})$ for each integer k. Therefore:*

$$\int_{x_k}^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \phi(\frac{x}{x_k}) = \frac{\tau(1-2\sigma)\exp(\frac{2k\pi(1+\sigma)}{\tau})}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (62)$$

$$\int_{x_k}^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{3-\sigma}} \phi(\frac{x}{x_k}) = -\frac{\tau(1-2\sigma)\exp(\frac{2k\pi(2-\sigma)}{\tau})}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (63)$$

Where the function ϕ is defined as $\phi(x) = \{x\}\big(1-\{x\}\big)$.
Let's define the function F_σ as follows:

$$F_\sigma(x) = \int_1^x dx \frac{\sin(\tau \ln(x))}{x^{2+\sigma}} \phi(x) \quad (64)$$

Therefore:

$$\int_{x_k}^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{3-\sigma}} \phi(\frac{x}{x_k}) = (1-2\sigma)\exp(\frac{2k\pi(1+\sigma)}{\tau}) \int_{x_k}^{+\infty} dx \frac{F_\sigma(\frac{x}{x_k})}{x^{2-2\sigma}} \quad (65)$$

Proof. The proof is straightforward as we can deduce the results from the lemma 1.4 using the variable change $u = x\, x_k$. ■ □

Lemma 1.6. *Let's σ be such that $0 < \sigma < 1$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\sigma = \frac{1}{2} \quad (66)$$

And the Riemann's Hypothesis is true.

Proof. From the lemma 1.5, we can write the following from the equations (62-63):

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{\sin(\tau\ln(x))}{x^{2+\sigma}}\phi(\frac{x}{x_k}) = \lim_{k\to+\infty}\frac{\tau(1-2\sigma)\exp(\frac{2k\pi(1+\sigma)}{\tau})}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (67)$$

$$= +\infty \quad (68)$$

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{\sin(\tau\ln(x))}{x^{3-\sigma}}\phi(\frac{x}{x_k}) = \lim_{k\to+\infty}-\frac{\tau(1-2\sigma)\exp(\frac{2k\pi(2-\sigma)}{\tau})}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (69)$$

$$= -\infty \quad (70)$$

On the other hand, from the equation (65):

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{\sin(\tau\ln(x))}{x^{3-\sigma}}\phi(\frac{x}{x_k}) = \lim_{k\to+\infty}(1-2\sigma)\exp(\frac{2k\pi(1+\sigma)}{\tau})\int_{x_k}^{+\infty}dx\frac{F_\sigma(\frac{x}{x_k})}{x^{2-2\sigma}} \quad (71)$$

We have for each $x>0$:

$$F_\sigma(\frac{x}{x_k}) \sim_{k\to+\infty} F_\sigma(+\infty) = \frac{\tau(1-2\sigma)}{\big((1-\sigma)^2+\tau^2\big)\big(\sigma^2+\tau^2\big)} \quad (72)$$

Therefore

$$\frac{F_\sigma(\frac{x}{x_k})}{x^{2-2\sigma}} \sim_{x\to0^+} \frac{F_\sigma(+\infty)}{x^{2-2\sigma}} \quad (73)$$

Since we have $2-2\sigma = 1+\underbrace{1-2\sigma}_{>0}$. Therefore, the function $x\to\frac{F_\sigma(+\infty)}{x^{2-2\sigma}}$ is not integrable around 0^+ and we have:

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{F_\sigma(\frac{x}{x_k})}{x^{2-2\sigma}} = \lim_{k\to+\infty}\int_{x_k}^{+\infty}\frac{F_\sigma(+\infty)}{x^{2-2\sigma}} \quad (74)$$

$$= \lim_{k\to+\infty}F_\sigma(+\infty)\exp(\frac{2k\pi(1-2\sigma)}{\tau}) \quad (75)$$

$$= +\infty \quad (76)$$

From the equation (71) we can conclude that:

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{\sin(\tau\ln(x))}{x^{3-\sigma}}\phi(\frac{x}{x_k}) = +\infty \quad (77)$$

But from the equation (70) we have:

$$\lim_{k\to+\infty}\int_{x_k}^{+\infty}dx\frac{\sin(\tau\ln(x))}{x^{3-\sigma}}\phi(\frac{x}{x_k}) = -\infty \quad (78)$$

Therefore $1-2\sigma$ must be negative:

$$1-2\sigma < 0 \quad (79)$$

Which is a contradiction. Therefore

$$\sigma = \frac{1}{2} \quad (80)$$

And the Riemann's Hypothesis is true.

■ □

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$*. In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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