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A Simple Proof of the Riemann's Hypothesis

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Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

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Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . There is another way [2] to analytically continue $\zeta(s)$ to the region $\Re(s) > 0$. The idea is to observe that for $\Re(s) > 1$:

$$\left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} - \sum_{n=1}^{+\infty} \frac{2}{(2n)^s} \quad (3)$$

Thus,

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \quad (4)$$

$$= \left(1 - \frac{2}{2^s}\right)^{-1} \eta(s) \quad (5)$$

The Dirichlet eta function $\eta(s)$ converges conditionally when $\Re(s) > 0$ and $s \neq 1 + i\frac{2k\pi}{\ln(2)}$. $\eta(s)$ is used as analytical continuation of the Zeta function on the domain where $\Re(s) > 0$. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (7)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (6)$$

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (8)$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (9)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (10)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)) - i \sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (11)$$

The integral is absolutely convergent. We take the real part and the imaginary part of both sides of the above equation and define the functions F and G as following:

$$F(\sigma, \tau) = \frac{\sigma - 1}{(\sigma - 1)^2 + \tau^2} \quad (12)$$

$$= \int_1^{+\infty} \frac{(\cos(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (13)$$

And

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (14)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (15)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$F(1 - \sigma, \tau) = \frac{\sigma_1 - 1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (16)$$

$$= \frac{-\sigma}{\sigma^2 + \tau^2} \quad (17)$$

$$= \int_1^{+\infty} \frac{(\cos(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (18)$$

And

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (19)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (20)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (21)$$

From now on, σ and τ are not necessarily connected to a ζ zero. Before we move forward, we need to define the following function $I(a, \sigma, \tau)$ for $a > 0$:

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (22)$$

$$= \int_0^{\ln(a)} \sin(\tau x) e^{(1-\sigma)x} dx \quad (23)$$

$$= K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1)}{\tau} \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (24)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma-1)^2 + \tau^2} \quad (25)$$

Let's define the function $J(a, \sigma, \tau)$ for $a > 0$:

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (26)$$

$$= \int_0^{\ln(a)} \cos(\tau x) e^{(1-\sigma)x} dx \quad (27)$$

$$= K(\sigma, \tau) \left(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau} \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (28)$$

Where $K(\sigma, \tau)$ is defined above in the equation (25).

Now, let's write $U(\sigma, \tau)$ as the limit of a sequence as follows:

$$U(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_1^N \frac{\{x\} \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (29)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx \sin(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (30)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (31)$$

$$= \lim_{N \rightarrow +\infty} U(N, \sigma) \quad (32)$$

Where

$$U(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (33)$$

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau} x} \quad (34)$$

$$\epsilon(x) = \{x\} \quad (35)$$

And the same for $V(\sigma, \tau)$:

$$V(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_1^N \frac{\{x\} \cos(\tau \ln(x))}{x^{1+\sigma}} dx \quad (36)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx \cos(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (37)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (38)$$

$$= \lim_{N \rightarrow +\infty} V(N, \sigma) \quad (39)$$

Where

$$V(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (40)$$

Lemma 1.1. *The sequence $\left(\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:*

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) = 1 - \tau U(\sigma, \tau) - \sigma V(\sigma, \tau) \quad (41)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} = J(N, \sigma, \tau) + 1 - \tau U(\sigma, \tau) - \sigma V(\sigma, \tau) + \alpha_N(\sigma, \tau) \quad (42)$$

Where (α_N) is a bounded sequence with limit zero:

$$\alpha_N = \mathcal{O}\left(\frac{1}{N^\sigma}\right) \quad (43)$$

The sequence $\left(\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) = \tau V(\sigma, \tau) - \sigma U(\sigma, \tau) \quad (44)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} = I(N, \sigma, \tau) + \tau V(\sigma, \tau) - \sigma U(\sigma, \tau) + \beta_N(\sigma, \tau) \quad (45)$$

Where (β_N) is a bounded sequence with limit zero:

$$\beta_N = \mathcal{O}\left(\frac{1}{N^\sigma}\right) \quad (46)$$

In case where $s = \sigma + i\tau$ is a zeta zero, we have:

$$\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} = K(\sigma, \tau) \left(\frac{(1-\sigma) \cos(\tau \ln(N))}{\tau N^{\sigma-1}} + \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} \right) + \alpha_N(\sigma, \tau) \quad (47)$$

$$\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} = K(\sigma, \tau) \left(-\frac{\cos(\tau \ln(N))}{N^{\sigma-1}} + \frac{(1-\sigma) \sin(\tau \ln(N))}{\tau N^{\sigma-1}} \right) + \beta_N(\sigma, \tau) \quad (48)$$

■

Proof. Let's study the function g over \mathbb{R} . We have the function g piecewise continuous and its derivatives are also piecewise continuous and:

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau}x} \quad (49)$$

$$g'(x) = \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) - \frac{\sigma}{\tau} g(x) \quad (50)$$

The interval $(0, \tau \ln(N))$ is the union of the intervals $(\tau \ln(n), \tau \ln(n+1))$ where $1 \leq n < N$. We have $\epsilon'(x) = 1$ for each $x \in (\tau \ln(n), \tau \ln(n+1))$ for each $1 \leq n < N$. Therefore, thanks to the integration by parts we can write:

$$U(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (51)$$

$$= \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{N-1} \int_{\tau \ln(n)+\epsilon}^{\tau \ln(n+1)-\epsilon} dx g(x) \sin(x) \quad (52)$$

$$= g(0^+) \cos(0) + \lim_{\epsilon \rightarrow 0} \sum_{n=2}^{N-1} [g(\tau \ln(n) + \epsilon) \cos(\tau \ln(n) + \epsilon) - g(\tau \ln(n) - \epsilon) \cos(\tau \ln(n) - \epsilon)] \quad (53)$$

$$- g(\tau \ln(N)^-) \cos(\tau \ln(N)) + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (54)$$

$$= \lim_{\epsilon \rightarrow 0} \sum_{n=2}^{N-1} [g(\tau \ln(n) + \epsilon) - g(\tau \ln(n) - \epsilon)] \cos(\tau \ln(n)) \quad (55)$$

$$- \frac{\cos(\tau \ln(N))}{\tau N^\sigma} + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (56)$$

$$= -\frac{1}{\tau} \sum_{n=2}^{N-1} \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\cos(\tau \ln(N))}{\tau N^\sigma} + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (57)$$

$$= -\frac{1}{\tau} \sum_{n=2}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\sigma}{\tau} \int_0^{\tau \ln(N)} dx g(x) \cos(x) + \int_0^{\tau \ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) \right) \cos(x) \quad (58)$$

$$= -\frac{1}{\tau} \sum_{n=2}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\sigma}{\tau} V(N, \sigma) + \int_0^{\tau \ln(N)} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \cos(x) \quad (59)$$

Therefore

$$U(N, \sigma) + \frac{\sigma}{\tau} V(N, \sigma) = -\frac{1}{\tau} \sum_{n=2}^N \frac{\cos(\tau \ln(n))}{n^\sigma} + \frac{1}{\tau} J(N, \sigma, \tau) \quad (60)$$

Therefore

$$\sum_{n=2}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) = -\tau U(N, \sigma) - \sigma V(N, \sigma) \quad (61)$$

Therefore, the sequence $\left(\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) = 1 - \tau U(\sigma, \tau) - \sigma V(\sigma, \tau) \quad (62)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} = J(N, \sigma, \tau) + 1 - \tau U(\sigma, \tau) - \sigma V(\sigma, \tau) + \alpha_N \quad (63)$$

Where (α_N) is a bounded sequence with limit zero.

And the same for $V(N, \sigma)$ as follows:

$$V(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (64)$$

$$= -g(0^+) \sin(0) + \lim_{\epsilon \rightarrow 0} \sum_{n=2}^{N-1} \left[-g(\tau \ln(n) + \epsilon) \sin(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \sin(\tau \ln(n) - \epsilon) \right] \quad (65)$$

$$+ g(\tau \ln(N)^-) \sin(\tau \ln(N)) - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (66)$$

$$= \lim_{\epsilon \rightarrow 0} \sum_{n=2}^{N-1} \left[-g(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \right] \sin(\tau \ln(n)) \quad (67)$$

$$+ \frac{\sin(\tau \ln(N))}{\tau N^\sigma} - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (68)$$

$$= \frac{1}{\tau} \sum_{n=2}^{N-1} \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sin(\tau \ln(N))}{\tau N^\sigma} - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (69)$$

$$= \frac{1}{\tau} \sum_{n=2}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sigma}{\tau} \int_0^{\tau \ln(N)} dx g(x) \sin(x) - \int_0^{\tau \ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon I(e^{\frac{x}{\tau}}) \right) \sin(x) \quad (70)$$

$$= \frac{1}{\tau} \sum_{n=2}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sigma}{\tau} U(N, \sigma) - \int_0^{\tau \ln(N)} dx \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \sin(x) \quad (71)$$

Therefore

$$V(N, \sigma) - \frac{\sigma}{\tau} U(N, \sigma) = \frac{1}{\tau} \sum_{n=2}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - \frac{1}{\tau} I(N, \sigma, \tau) \quad (72)$$

Therefore

$$\sum_{n=2}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) = \tau V(N, \sigma) - \sigma U(N, \sigma) \quad (73)$$

Therefore, the sequence $\left(\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) = \tau V(\sigma, \tau) - \sigma U(\sigma, \tau) \quad (74)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} = I(N, \sigma, \tau) + \tau V(\sigma, \tau) - \sigma U(\sigma, \tau) + \beta_N \quad (75)$$

Where (β_N) is a bounded sequence with limit zero.

The proof of the result in the equations (43) and (46) can be found in [10]. Another proof of this lemma can be found in [11]. ■ □

Lemma 1.2. *Let's $\alpha > -1$, β , a , b and c real numbers. Let's define the function as follows:*

$$f(t) = \frac{a + b \sin(\beta \ln(t)) + c \cos(\beta \ln(t))}{t^\alpha} \quad (76)$$

$$\sum_{n=1}^N f(n) = c_0 + \int_1^N f(t) dt - \frac{1}{2} f(N) + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (77)$$

Where $c_0 \in \mathbb{R}$.

Proof. The function f is infinitely differentiable. The starting point of the proof is that the first order Taylor theorem. Namely, for a twice continuously differentiable function $f(x)$, for x, x_0 such that $x > x_0 > 0$, we have:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x f''(u)(x - u) du \quad (78)$$

For each $n \geq 1$. Let's apply Taylor theorem on the interval $(n, n + 1)$. We have for each $x \in (n, n + 1)$:

$$f(x) = f(n) + f'(n)(x - n) + \int_n^x f''(u)(x - u) du \quad (79)$$

We integrate the equation above to get:

$$\int_n^{n+1} f(x)dx = \int_n^{n+1} f(n)dx + \int_n^{n+1} f'(n)(x-n)dx + \int_n^{n+1} dx \int_n^x f''(u)(x-u)du \quad (80)$$

$$= f(n) + \frac{1}{2}f'(n) + \int_n^{n+1} dx \int_n^x f''(u)(x-u)du \quad (81)$$

We have the functions f' and f'' defined as follows:

$$f'(t) = \frac{-\alpha a - (\alpha b + c\beta) \sin(\beta \ln(t)) + (\beta b - \alpha c) \cos(\beta \ln(t))}{t^{1+\alpha}} \quad (82)$$

$$= \frac{a_1 + b_1 \sin(\beta \ln(t)) + c_1 \cos(\beta \ln(t))}{t^{1+\alpha}} \quad (83)$$

$$f''(t) = \frac{-(1+\alpha)a_1 - ((1+\alpha)b_1 + c_1\beta) \sin(\beta \ln(t)) + (\beta b_1 - (1+\alpha)c_1) \cos(\beta \ln(t))}{t^{2+\alpha}} \quad (84)$$

Therefore there exists a constant $C > 0$ such that for each $n \geq 1$:

$$|f''(n)| \leq \frac{C}{n^{2+\alpha}} \quad (85)$$

Therefore

$$\left| \int_n^{n+1} dx \int_n^x f''(u)(x-u)du \right| \leq \frac{C}{n^{2+\alpha}} \int_n^{n+1} dx \int_n^x (x-u)du \quad (86)$$

$$\leq \frac{C}{6n^{2+\alpha}} \quad (87)$$

Therefore, combining with the equation (81), we have:

$$\left| \int_n^{n+1} f(x)dx - f(n) + \frac{1}{2}f'(n) \right| \leq \frac{C}{6n^{2+\alpha}} \quad (88)$$

Therefore

$$\int_n^{n+1} f(x)dx = f(n) + \frac{1}{2}f'(n) + \mathcal{O}\left(\frac{1}{n^{2+\alpha}}\right) \quad (89)$$

We have $\alpha > -1$. Therefore:

$$\int_1^N f(t)dt = c_0 + \sum_{n=1}^N f(n) + \frac{1}{2} \sum_{n=1}^N f'(n) + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (90)$$

We apply this result to the function f' , to get:

$$\int_1^N f'(t)dt = c_1 + \sum_{n=1}^N f'(n) + \frac{1}{2} \sum_{n=1}^N f''(n) + \mathcal{O}\left(\frac{1}{N^{2+\alpha}}\right) \quad (91)$$

Thanks to (85), we can write:

$$\sum_{n=1}^N f''(n) = a_0 + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (92)$$

Where $a_0 \in \mathbb{R}$. Therefore

$$\int_1^N f'(t)dt = c_2 + \sum_{n=1}^N f'(n) + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (93)$$

Therefore

$$\sum_{n=1}^N f'(n) = f(N) - f(1) - c_2 + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (94)$$

We combining with the equation (90), we get:

$$\int_1^N f(t)dt = \sum_{n=1}^N f(n) + \frac{1}{2}f(N) - c_3 + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (95)$$

Therefore

$$\sum_{n=1}^N f(n) = c_3 + \int_1^N f(t)dt - \frac{1}{2}f(N) + \mathcal{O}\left(\frac{1}{N^{1+\alpha}}\right) \quad (96)$$

■
□

2 Dirichlet eta function

From equation (5), the $\zeta(s)$ function is also defined through the Dirichlet eta function $\eta(s)$ as follows:

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \eta(s) \quad (97)$$

$$= \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \quad (98)$$

Let's define the sequence X_N as follows:

$$X_N(s) = \sum_{n=1}^N \frac{1}{n^s} \quad (99)$$

For $N \geq 1$:

$$X_N = \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - i \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} \quad (100)$$

Let's define the sequences U_n, V_n as follows: For $n \geq 1$

$$U_n = \frac{\cos(\tau \ln(n))}{n^\sigma}, \quad V_n = \frac{\sin(\tau \ln(n))}{n^\sigma} \quad (101)$$

Let's define the series A_n, B_n and X_n as follows:

$$A_n = \sum_{k=1}^n \frac{\cos(\tau \ln(k))}{k^\sigma}, \quad B_n = \sum_{k=1}^n \frac{\sin(\tau \ln(k))}{k^\sigma}, \quad X_n = A_n - iB_n \quad (102)$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the series Z_N as follows:

$$\|X_N\|^2 = A_N^2 + B_N^2 = \left(\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} \right)^2 + \left(\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} \right)^2 \quad (103)$$

So

$$A_N^2 = \left(\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} \right)^2 \quad (104)$$

$$= \sum_{n=1}^N \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}} + 2 \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} \left(A_n - \frac{\cos(\tau \ln(n))}{n^\sigma} \right) \quad (105)$$

$$= - \sum_{n=1}^N \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}} + 2 \sum_{n=1}^N \frac{\cos(\tau \ln(n))A_n}{n^\sigma} \quad (106)$$

And the same calculation for B_N

$$B_N^2 = - \sum_{n=1}^N \frac{\sin^2(\tau \ln(n))}{n^{2\sigma}} + 2 \sum_{n=1}^N \frac{\sin(\tau \ln(n))B_n}{n^\sigma} \quad (107)$$

Hence we have the new expression of square norm of Z_N :

$$\|X_N\|^2 = 2 \sum_{n=1}^N \frac{\cos(\tau \ln(n))A_n}{n^\sigma} + 2 \sum_{n=1}^N \frac{\sin(\tau \ln(n))B_n}{n^\sigma} - \sum_{n=1}^N \frac{\sin^2(\tau \ln(n)) + \cos^2(\tau \ln(n))}{n^{2\sigma}} \quad (108)$$

which simplifies even further to:

$$\|X_N\|^2 = 2 \sum_{n=1}^N \frac{\cos(\tau \ln(n))A_n}{n^\sigma} + 2 \sum_{n=1}^N \frac{\sin(\tau \ln(n))B_n}{n^\sigma} - \sum_{n=1}^N \frac{1}{n^{2\sigma}} \quad (109)$$

Let's now define F_n and G_n as follows:

$$F_n = \frac{\cos(\tau \ln(n))A_n}{n^\sigma}, \quad G_n = \frac{\sin(\tau \ln(n))B_n}{n^\sigma} \quad (110)$$

Therefore

$$A_N^2 = 2 \sum_{n=1}^N F_n - \sum_{n=1}^N \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}}, \quad B_N^2 = 2 \sum_{n=1}^N G_n - \sum_{n=1}^N \frac{\sin^2(\tau \ln(n))}{n^{2\sigma}} \quad (111)$$

Let's define the sequence Y_N as follows:

$$Y_N(s) = X_N(s) - \frac{1}{2^{s-1}} X_{\frac{N}{2}}(s) \quad (112)$$

$$= A_N - \frac{1}{2^{s-1}} A_{\frac{N}{2}} - i \left(B_N - \frac{1}{2^{s-1}} B_{\frac{N}{2}} \right) \quad (113)$$

$$= A_{N,N} - i B_{N,N} \quad (114)$$

Where

$$A_{N,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\cos(\tau \ln(k)) - 2^{1-\sigma} \cos(\tau(\ln(2k)))}{k^\sigma} + \sum_{k=\frac{N}{2}+1}^N \frac{\cos(\tau \ln(k))}{k^\sigma} \quad (115)$$

And

$$B_{N,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\sin(\tau \ln(k)) - 2^{1-\sigma} \sin(\tau(\ln(2k)))}{k^\sigma} + \sum_{k=\frac{N}{2}+1}^N \frac{\sin(\tau \ln(k))}{k^\sigma} \quad (116)$$

We also have:

$$\lim_{N \rightarrow \infty} Y_N = \lim_{N \rightarrow \infty} A_{N,N} - i B_{N,N} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{(-1)^{n+1}}{n^s} = \eta(s) \quad (117)$$

Remark. s is a zeta zero, $\zeta(s) = 0$, if and only if

$$\lim_{N \rightarrow \infty} A_{N,N} = 0 \text{ and } \lim_{N \rightarrow \infty} B_{N,N} = 0 \quad (118)$$

Let's define the following sequences: For $N \geq 1$:

$$U_{n,N} = \begin{cases} \frac{\cos(\tau \ln(n)) - 2^{1-\sigma} \cos(\tau(\ln(2n)))}{n^\sigma}, & \text{if } n \leq \frac{N}{2} \\ \frac{\cos(\tau \ln(n))}{n^\sigma} & \text{if } n > \frac{N}{2} \end{cases} \quad (119)$$

And

$$V_{n,N} = \begin{cases} \frac{\sin(\tau \ln(n)) - 2^{1-\sigma} \sin(\tau(\ln(2n)))}{n^\sigma}, & \text{if } n \leq \frac{N}{2} \\ \frac{\sin(\tau \ln(n))}{n^\sigma} & \text{if } n > \frac{N}{2} \end{cases} \quad (120)$$

And

$$A_{n,N} = \begin{cases} \sum_{k=1}^n \frac{\cos(\tau \ln(k)) - 2^{1-\sigma} \cos(\tau \ln(2k))}{k^\sigma}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\cos(\tau \ln(k)) - 2^{1-\sigma} \cos(\tau \ln(2k))}{k^\sigma} + \sum_{k=\frac{N}{2}+1}^n \frac{\cos(\tau \ln(k))}{k^\sigma} & \text{if } n > \frac{N}{2} \\ A_{N,N} & \text{if } n \geq N \end{cases} \quad (121)$$

And

$$B_{n,N} = \begin{cases} \sum_{k=1}^n \frac{\sin(\tau \ln(k)) - 2^{1-\sigma} \sin(\tau \ln(2k))}{k^\sigma}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\sin(\tau \ln(k)) - 2^{1-\sigma} \sin(\tau \ln(2k))}{k^\sigma} + \sum_{k=\frac{N}{2}+1}^n \frac{\sin(\tau \ln(k))}{k^\sigma} & \text{if } n > \frac{N}{2} \\ B_{N,N} & \text{if } n \geq N \end{cases} \quad (122)$$

Let's develop further the squared norm of the series Y_N as follows:

$$\|Y_N\|^2 = A_{N,N}^2 + B_{N,N}^2 = \left(\sum_{n=1}^N U_{n,N} \right)^2 + \left(\sum_{n=1}^N V_{n,N} \right)^2 \quad (123)$$

So

$$A_{N,N}^2 = \sum_{n=1}^N U_{n,N} + 2 \sum_{n=1}^N \sum_{k=1}^{n-1} U_{n,N} U_{k,N} \quad (124)$$

$$= - \sum_{n=1}^N U_{n,N}^2 + 2 \sum_{n=1}^N U_{n,N} A_{n,N} \quad (125)$$

And the same calculation for $B_{N,N}$

$$B_{N,N}^2 = - \sum_{n=1}^N V_{n,N}^2 + 2 \sum_{n=1}^N V_{n,N} B_{n,N} \quad (126)$$

Let's now define the sequences $(F_{n,N})$ and $(G_{n,N})$ as follows:

$$F_{n,N} = U_{n,N} A_{n,N}, \quad G_{n,N} = V_{n,N} B_{n,N} \quad (127)$$

Hence we have the new expression of square norm of Z_N^2 :

$$\|Y_N\|^2 = 2 \sum_{n=1}^N F_{n,N} + 2 \sum_{n=1}^N G_{n,N} - \sum_{n=1}^N (U_{n,N}^2 + V_{n,N}^2) \quad (128)$$

which simplifies even further to:

$$\sum_{n=1}^N (U_{n,N}^2 + V_{n,N}^2) = \sum_{n=1}^{\frac{N}{2}} (U_{n,N}^2 + V_{n,N}^2) + \sum_{n=\frac{N}{2}+1}^N (U_{n,N}^2 + V_{n,N}^2) \quad (129)$$

$$= \sum_{n=1}^N \frac{1}{n^{2\sigma}} + \sum_{n=1}^{\frac{N}{2}} \frac{2^{2-2\sigma} - 2^{2-\sigma} (\cos(\tau \ln(2)))}{n^{2\sigma}} \quad (130)$$

We have $2\sigma > 1$, hence the partial sums $\sum_{n \geq 1}^N U_{n,N}^2$ and $\sum_{n \geq 1}^N V_{n,N}^2$ converge absolutely. We have

$$A_{N,N}^2 = 2 \sum_{n=1}^N F_{n,N} - \sum_{n=1}^N U_{n,N}^2, \quad B_{N,N}^2 = 2 \sum_{n=1}^N G_{n,N} - \sum_{n=1}^N V_{n,N}^2 \quad (131)$$

Remark. The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < \sigma \leq 1$ and that is a ζ zero, that the limit $\lim_{n \rightarrow \infty} A_{n,n}^2 = +/\infty$ or the limit $\lim_{n \rightarrow \infty} B_{n,n}^2 = +/\infty$. This will create a contradiction. Because if s is a ζ zero then the $\lim_{n \rightarrow \infty} A_{n,n}^2$ should be 0 and the $\lim_{n \rightarrow \infty} B_{n,n}^2$ should be 0. And therefore the sequences $(\sum_{n=1}^N F_{n,N})_{N \geq 1}$ and $(\sum_{n=1}^N G_{n,N})_{N \geq 1}$ should converge and their limits should be: $\lim_{n \rightarrow \infty} \sum_{n=1}^N F_{n,N} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N U_{n,N}^2 < +\infty$ and $\lim_{n \rightarrow \infty} \sum_{n=1}^N G_{n,N} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N V_{n,N}^2 < +\infty$.

Lemma 2.1. *If $0 < \sigma < 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(\sum_{n=1}^N F_{n,N})_{N \geq 1}$:*

$$\sum_{n=1}^{2N} F_{n,2N} = U_0 + \left[U_1 + V_1 \sin(2\tau \ln(N)) + W_1 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (132)$$

Where

$$U_1 = -\frac{\left[\left(1 - 2^{-\sigma} \cos(\tau \ln(2))\right)^2 + 2^{-2\sigma} \right]}{2(1-2\sigma)} < 0 \quad (133)$$

and U_0, V_1 and W_1 are real constants.

Proof. So, let's define the sum $Z_{1,2N,2N}$ for $N \geq 1$:

$$Z_{1,2N,2N} = \sum_{n=1}^{2N} F_{n,2N} \quad (134)$$

$$= \underbrace{\sum_{n=1}^N F_{n,2N}}_{Z_{1,N,2N}} + \underbrace{\sum_{n=N+1}^{2N} F_{n,2N}}_{Z_{N+1,2N,2N}} \quad (135)$$

And $Z_{N+1,2N,2N}$:

$$Z_{N+1,2N,2N} = \sum_{n=N+1}^{2N} F_{n,2N} \quad (136)$$

$$= \sum_{n=N+1}^{2N} U_{n,2N} A_{n,2N} \quad (137)$$

$$= \sum_{n=N+1}^{2N} U_{n,2N} \left(A_{N,2N} + \sum_{k=N+1}^n U_{k,2N} \right) \quad (138)$$

$$= A_{N,2N} \left(A_{2N,2N} - A_{N,2N} \right) + \sum_{n=N+1}^{2N} U_{n,2N} \sum_{k=N+1}^n U_{k,2N} \quad (139)$$

And

$$\sum_{n=N+1}^{2N} U_{n,2N} \sum_{k=N+1}^n U_{k,2N} = \sum_{k=N+1}^{2N} U_{k,2N} \sum_{n=k}^{2N} U_{n,2N} \quad (140)$$

$$= \sum_{k=N+1}^{2N} U_{k,2N} \left(A_{2N,2N} - A_{k-1,2N} \right) \quad (141)$$

$$= \sum_{k=N+1}^{2N} U_{k,2N} \left(A_{2N,2N} - A_{k,2N} \right) + \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (142)$$

$$= A_{2N,2N} \left(A_{2N,2N} - A_{N,2N} \right) - \sum_{k=N+1}^{2N} U_{k,2N} A_{k,2N} + \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (143)$$

$$= A_{2N,2N} \left(A_{2N,2N} - A_{N,2N} \right) - Z_{N+1,2N,2N} + \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (144)$$

Therefore

$$2Z_{N+1,2N,2N} = (A_{2N,2N} + A_{N,2N})(A_{2N,2N} - A_{N,2N}) + \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (145)$$

Therefore

$$Z_{1,2N,2N} = \frac{1}{2}(A_{2N,2N} + A_{N,2N})(A_{2N,2N} - A_{N,2N}) + Z_{1,N,2N} + \frac{1}{2} \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (146)$$

We also have:

$$A_{2N,2N} = A_{N,2N} + A_{2N} - A_N \quad (147)$$

Therefore

$$Z_{1,2N,2N} = A_{N,2N}(A_{2N} - A_N) + \frac{1}{2}(A_{2N} - A_N)^2 + Z_{1,N,2N} + \frac{1}{2} \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (148)$$

We also have:

$$Z_{1,N,2N} = \sum_{n=1}^N F_{n,2N} = \sum_{n=1}^N U_{n,2N} A_{n,2N} \quad (149)$$

For $n \leq N$, we have:

$$A_{n,2N} = \sum_{k=1}^n U_{k,2N} \quad (150)$$

$$= \sum_{k=1}^n \frac{\cos(\tau \ln(k)) - 2^{1-\sigma} \cos(\tau \ln(2k))}{k^\sigma} \quad (151)$$

$$= (1 - 2^{1-\sigma} \cos(\tau \ln(2)))A_n + 2^{1-\sigma} \sin(\tau \ln(2))B_n \quad (152)$$

$$= cA_n + dB_n \quad (153)$$

Where

$$c = 1 - 2^{1-\sigma} \cos(\tau \ln(2)), d = 2^{1-\sigma} \sin(\tau \ln(2)) \quad (154)$$

From lemma 1.1 and lemma 1.2 we can write:

$$A_n = K(\sigma, \tau) \left(\frac{(1-\sigma) \cos(\tau \ln(n))}{\tau n^{\sigma-1}} + \frac{\sin(\tau \ln(n))}{n^{\sigma-1}} \right) - \frac{1}{2} \frac{\cos(\tau \ln(n))}{n^\sigma} + \mathcal{O}\left(\frac{1}{n^{1+\sigma}}\right) \quad (155)$$

$$B_n = K(\sigma, \tau) \left(-\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma) \sin(\tau \ln(n))}{\tau n^{\sigma-1}} \right) - \frac{1}{2} \frac{\sin(\tau \ln(n))}{n^\sigma} + \mathcal{O}\left(\frac{1}{n^{1+\sigma}}\right) \quad (156)$$

Therefore, by combining equations (152), (154) and (155) we get:

$$A_{n,2N} = cK(\sigma, \tau) \left(\frac{(1-\sigma) \cos(\tau \ln(n))}{\tau n^{\sigma-1}} + \frac{\sin(\tau \ln(n))}{n^{\sigma-1}} \right) - \frac{c}{2} \frac{\cos(\tau \ln(n))}{n^\sigma} \quad (157)$$

$$+ dK(\sigma, \tau) \left(-\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma) \sin(\tau \ln(n))}{\tau n^{\sigma-1}} \right) - \frac{d}{2} \frac{\sin(\tau \ln(n))}{n^\sigma} + \mathcal{O}\left(\frac{1}{n^{1+\sigma}}\right) \quad (158)$$

$$= a \frac{\sin(\tau \ln(n))}{n^{\sigma-1}} + b \frac{\cos(\tau \ln(n))}{n^{\sigma-1}} - \frac{c}{2} \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{d}{2} \frac{\sin(\tau \ln(n))}{n^\sigma} + \mathcal{O}\left(\frac{1}{n^{1+\sigma}}\right) \quad (159)$$

Where

$$a = K(\sigma, \tau) \left(c + \frac{(1-\sigma)}{\tau} d \right), b = K(\sigma, \tau) \left(\frac{(1-\sigma)}{\tau} c - d \right) \quad (160)$$

Therefore by combining equation (149) and the definition (119), we get:

$$Z_{1,N,2N} = \sum_{n=1}^N A_{n,2N} \left((1 - 2^{1-\sigma} \cos(\tau \ln(n))) \frac{\cos(\tau \ln(n))}{n^\sigma} + 2^{1-\sigma} \sin(\tau \ln(n)) \frac{\sin(\tau \ln(n))}{n^\sigma} \right) \quad (161)$$

$$= \sum_{n=1}^N \left(a \frac{\sin(\tau \ln(n))}{n^{\sigma-1}} + b \frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \mathcal{O}\left(\frac{1}{n^{1+\sigma}}\right) \right) \left(c \frac{\cos(\tau \ln(n))}{n^\sigma} + d \frac{\sin(\tau \ln(n))}{n^\sigma} \right) \quad (162)$$

$$- \frac{1}{2} \sum_{n=1}^N \left(c \frac{\cos(\tau \ln(n))}{n^\sigma} + d \frac{\sin(\tau \ln(n))}{n^\sigma} \right)^2 \quad (163)$$

$$= \sum_{n=1}^N X_{n,2N} - Y_{n,2N} \quad (164)$$

Where

$$X_{n,2N} = \left(a \frac{\sin(\tau \ln(n))}{n^{\sigma-1}} + b \frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \mathcal{O}\left(\frac{1}{n^\sigma}\right) \right) \left(c \frac{\cos(\tau \ln(n))}{n^\sigma} + d \frac{\sin(\tau \ln(n))}{n^\sigma} \right) \quad (165)$$

$$= \frac{ad+bc}{2} \frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2} \frac{\cos(2\tau \ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2} \frac{\sin(2\tau \ln(n))}{n^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{n^{1+2\sigma}}\right) \quad (166)$$

And

$$Y_{n,2N} = \frac{1}{2} \left(d \frac{\sin(\tau \ln(n))}{n^\sigma} + c \frac{\cos(\tau \ln(n))}{n^\sigma} \right)^2 \quad (167)$$

$$= \frac{c^2+d^2}{4} \frac{1}{n^{2\sigma}} + \frac{c^2-d^2}{4} \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \frac{cd}{2} \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \quad (168)$$

Therefore:

$$Z_{1,N,2N} = \sum_{n=1}^N \left[\frac{ad+bc}{2} \frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2} \frac{\cos(2\tau \ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2} \frac{\sin(2\tau \ln(n))}{n^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{n^{1+2\sigma}}\right) \right] \quad (169)$$

$$- \sum_{n=1}^N \left[\frac{c^2+d^2}{4} \frac{1}{n^{2\sigma}} + \frac{c^2-d^2}{4} \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \frac{cd}{2} \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \right] \quad (170)$$

$$= \sum_{n=1}^N \left[\alpha \frac{1}{n^{2\sigma-1}} + \beta \frac{\cos(2\tau \ln(n))}{n^{2\sigma-1}} + \gamma \frac{\sin(2\tau \ln(n))}{n^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{n^{1+2\sigma}}\right) \right] \quad (171)$$

$$- \sum_{n=1}^N \left[\alpha_1 \frac{1}{n^{2\sigma}} + \beta_1 \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \gamma_1 \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \right] \quad (172)$$

Also, from lemma 1.2, we can write:

$$Z_{1,N,2N} = \left(\frac{\alpha}{2(1-\sigma)} + \frac{(\beta\tau + \gamma(1-\sigma)) \sin(2\tau \ln(N)) + (\beta(1-\sigma) - \gamma\tau) \cos(2\tau \ln(N))}{2\tau^2 + 2(1-\sigma)^2} \right) N^{2-2\sigma} \quad (173)$$

$$- \frac{1}{2} \left[\alpha + \beta \cos(2\tau \ln(N)) + \gamma \sin(2\tau \ln(N)) \right] N^{1-2\sigma} \quad (174)$$

$$- \left(\frac{\alpha_1}{(1-2\sigma)} + \frac{(2\beta_1\tau + \gamma_1(1-2\sigma)) \sin(2\tau \ln(N)) + (\beta_1(1-2\sigma) - 2\gamma_1\tau) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right) N^{1-2\sigma} \quad (175)$$

$$+ U_0 + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (176)$$

Where U_0 is a real number and:

$$\alpha = \frac{ad+bc}{2}, \beta = \frac{bc-ad}{2}, \gamma = \frac{ac+bd}{2} \quad (177)$$

And

$$\alpha_1 = \frac{c^2+d^2}{4}, \beta_1 = \frac{c^2-d^2}{4}, \gamma_1 = \frac{cd}{2} \quad (178)$$

We also have from lemma 1.2:

$$\sum_{n=N+1}^{2N} U_{n,2N}^2 = \frac{1}{2} \sum_{n=N+1}^{2N} \frac{1 + \cos(2\tau \ln(n))}{n^{2\sigma}} \quad (179)$$

$$= \frac{1}{2} \sum_{n=N+1}^{2N} \left[\alpha_2 \frac{1}{n^{2\sigma}} + \beta_2 \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \gamma_2 \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \right] \quad (180)$$

$$= \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) + \frac{1}{2} \left(\frac{\alpha_2 2^{1-2\sigma}}{(1-2\sigma)} - \frac{\alpha_2}{(1-2\sigma)} \right) N^{1-2\sigma} \quad (181)$$

$$+ \frac{1}{2} \left(\frac{(2\beta_2\tau + \gamma_2(1-2\sigma)) \sin(2\tau \ln(2N)) + (\beta_2(1-2\sigma) - 2\gamma_2\tau) \cos(2\tau \ln(2N))}{4\tau^2 + (1-2\sigma)^2} \right) (2N)^{1-2\sigma} \quad (182)$$

$$- \frac{1}{2} \left(\frac{(2\beta_2\tau + \gamma_2(1-2\sigma)) \sin(2\tau \ln(N)) + (\beta_2(1-2\sigma) - 2\gamma_2\tau) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right) N^{1-2\sigma} \quad (183)$$

$$= \frac{1}{2} \left(\frac{1}{(1-2\sigma)} + \frac{2\tau \sin(2\tau \ln(2N)) + (1-2\sigma) \cos(2\tau \ln(2N))}{4\tau^2 + (1-2\sigma)^2} \right) (2N)^{1-2\sigma} \quad (184)$$

$$- \frac{1}{2} \left(\frac{1}{(1-2\sigma)} + \frac{2\tau \sin(2\tau \ln(N)) + (1-2\sigma) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right) N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (185)$$

$$= \frac{1}{2} \left(\frac{2^{1-2\sigma} - 1}{1-2\sigma} + c_1 \cos(2\tau \ln(N)) + d_1 \sin(2\tau \ln(N)) \right) N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (186)$$

Where

$$\alpha_2 = 1, \beta_2 = 1, \gamma_2 = 0 \quad (187)$$

And

$$c_1 = 2\tau 2^{1-2\sigma} \sin(2\tau \ln(2)) + (1-2\sigma) \left(2^{1-2\sigma} \cos(2\tau \ln(2)) - 1 \right) \quad (188)$$

$$d_1 = 2\tau \left(2^{1-2\sigma} \cos(2\tau \ln(2)) - 1 \right) - (1-2\sigma) 2^{1-2\sigma} \sin(2\tau \ln(2)) \quad (189)$$

And from equation (159) we can write:

$$A_{N,2N} = a \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} + b \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} - \frac{c \cos(\tau \ln(N))}{2} \frac{1}{N^\sigma} - \frac{d \sin(\tau \ln(N))}{2} \frac{1}{N^\sigma} + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (190)$$

And from equation (155):

$$A_{2N} - A_N = K(\sigma, \tau) \left(\frac{(1-\sigma) \cos(\tau \ln(2N))}{\tau} \frac{1}{(2N)^{\sigma-1}} + \frac{\sin(\tau \ln(2N))}{(2N)^{\sigma-1}} \right) - \frac{1}{2} \frac{\cos(\tau \ln(2N))}{(2N)^\sigma} + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (191)$$

$$-K(\sigma, \tau) \left(\frac{(1-\sigma) \cos(\tau \ln(N))}{\tau} \frac{1}{N^{\sigma-1}} + \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} \right) + \frac{1}{2} \frac{\cos(\tau \ln(N))}{N^\sigma} + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (192)$$

$$= K(\sigma, \tau) \left(2^{1-\sigma} \sin(\tau \ln(2)) + \frac{1-\sigma}{\tau} (2^{1-\sigma} \cos(\tau \ln(2)) - 1) \right) \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} \quad (193)$$

$$+ K(\sigma, \tau) \left((2^{1-\sigma} \cos(\tau \ln(2)) - 1) - \frac{1-\sigma}{\tau} 2^{1-\sigma} \sin(\tau \ln(2)) \right) \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} \quad (194)$$

$$+ \frac{1}{2} \left(d \frac{\sin(\tau \ln(N))}{N^\sigma} + c \frac{\cos(\tau \ln(N))}{N^\sigma} \right) + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (195)$$

$$= e \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} + f \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} + \frac{1}{2} \left(d' \frac{\sin(\tau \ln(N))}{N^\sigma} + c' \frac{\cos(\tau \ln(N))}{N^\sigma} \right) + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (196)$$

Where

$$e = -K(\sigma, \tau) \left(c + \frac{(1-\sigma)d}{\tau} \right) = -a, f = K(\sigma, \tau) \left(-\frac{(1-\sigma)c}{\tau} + d \right) = -b \quad (197)$$

And

$$c' = 1 - 2^{-\sigma} \cos(\tau \ln(2)) = \frac{1+c}{2}, d' = 2^{-\sigma} \sin(\tau \ln(2)) = \frac{d}{2} \quad (198)$$

Therefore

$$A_{2N} - A_N = -a \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} - b \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} + \frac{1}{2} \left(d' \frac{\sin(\tau \ln(N))}{N^\sigma} + c' \frac{\cos(\tau \ln(N))}{N^\sigma} \right) + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \quad (199)$$

Therefore

$$(A_{2N} - A_N)^2 = \left[e \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} + f \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} + \frac{1}{2} \left(d' \frac{\sin(\tau \ln(N))}{N^\sigma} + c' \frac{\cos(\tau \ln(N))}{N^\sigma} \right) + \mathcal{O}\left(\frac{1}{N^{1+\sigma}}\right) \right]^2 \quad (200)$$

$$= \frac{e^2 + f^2}{2} \frac{1}{N^{2\sigma-2}} + \frac{f^2 - e^2}{2} \frac{\cos(2\tau \ln(N))}{N^{2\sigma-2}} + ef \frac{\sin(2\tau \ln(N))}{N^{2\sigma-2}} \quad (201)$$

$$+ \left[e \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} + f \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} \right] \left(d' \frac{\sin(\tau \ln(N))}{N^\sigma} + c' \frac{\cos(\tau \ln(N))}{N^\sigma} \right) + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (202)$$

$$= \frac{e^2 + f^2}{2} \frac{1}{N^{2\sigma-2}} + \frac{f^2 - e^2}{2} \frac{\cos(2\tau \ln(N))}{N^{2\sigma-2}} + ef \frac{\sin(2\tau \ln(N))}{N^{2\sigma-2}} \quad (203)$$

$$+ \frac{fc' + ed'}{2} \frac{1}{N^{2\sigma-1}} + \frac{fd' - edc'}{2} \frac{\cos(2\tau \ln(N))}{N^{2\sigma-1}} + \frac{ec' + fd'}{2} \frac{\sin(2\tau \ln(N))}{N^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (204)$$

Therefore

$$A_{N,2N}(A_{2N} - A_N) + \frac{1}{2}(A_{2N} - A_N)^2 \quad (205)$$

$$= -\frac{e^2 + f^2}{4} \frac{1}{N^{2\sigma-2}} - \frac{(f^2 - e^2) \cos(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-2}} - \frac{ef \sin(2\tau \ln(N))}{2} \frac{1}{N^{2\sigma-2}} \quad (206)$$

$$- \frac{(f(c+ct) + e(d+dt))}{4} \frac{1}{N^{2\sigma-1}} - \frac{(f(c+ct) - e(d+dt)) \cos(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} \quad (207)$$

$$- \frac{(e(c+ct) + f(d+dt)) \sin(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (208)$$

$$+ \frac{fct + edt}{4} \frac{1}{N^{2\sigma-1}} + \frac{fct - edt \cos(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} + \frac{ect + fdt \sin(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (209)$$

$$= -\frac{e^2 + f^2}{4} \frac{1}{N^{2\sigma-2}} - \frac{(f^2 - e^2) \cos(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-2}} - \frac{ef \sin(2\tau \ln(N))}{2} \frac{1}{N^{2\sigma-2}} \quad (210)$$

$$- \frac{(fc + ed)}{4} \frac{1}{N^{2\sigma-1}} - \frac{(fc - ed) \cos(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} - \frac{(ec + fd) \sin(2\tau \ln(N))}{4} \frac{1}{N^{2\sigma-1}} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (211)$$

Therefore

$$Z_{1,2N,2N} = U_0 + \left[\frac{\alpha}{2(1-\sigma)} + \frac{(\beta\tau + \gamma(1-\sigma)) \sin(2\tau \ln(N)) + (\beta(1-\sigma) - \gamma\tau) \cos(2\tau \ln(N))}{2\tau^2 + 2(1-\sigma)^2} \right] N^{2-2\sigma} \quad (212)$$

$$- \left[\frac{(e^2 + f^2)}{4} + \frac{(f^2 - e^2)}{4} \cos(2\tau \ln(N)) + \frac{ef}{2} \sin(2\tau \ln(N)) \right] N^{2-2\sigma} \quad (213)$$

$$- \frac{1}{2} \left[\alpha + \beta \cos(2\tau \ln(N)) + \gamma \sin(2\tau \ln(N)) \right] N^{1-2\sigma} \quad (214)$$

$$- \left[\frac{\alpha_1}{(1-2\sigma)} + \frac{(2\beta_1\tau + \gamma_1(1-2\sigma)) \sin(2\tau \ln(N)) + (\beta_1(1-2\sigma) - 2\gamma_1\tau) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right] N^{1-2\sigma} \quad (215)$$

$$- \left[\frac{(fc + ed)}{4} + \frac{(fc - ed)}{4} \cos(2\tau \ln(N)) + \frac{(ec + fd) \sin(2\tau \ln(N))}{4} \right] N^{1-2\sigma} \quad (216)$$

$$+ \frac{1}{4} \left[\frac{2^{1-2\sigma} - 1}{1-2\sigma} + c_1 \cos(2\tau \ln(N)) + d_1 \sin(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (217)$$

$$= U_0 + \left[U + V \sin(2\tau \ln(N)) + W \cos(2\tau \ln(N)) \right] N^{2-2\sigma} \quad (218)$$

$$+ \left[U_1 + V_1 \sin(2\tau \ln(N)) + W_1 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (219)$$

Where

$$U = \frac{\alpha}{2(1-\sigma)} + \frac{e^2 + f^2}{4} + \frac{ae + bf}{2} \quad (220)$$

$$V = \frac{\beta\tau + \gamma(1-\sigma)}{2\tau^2 + 2(1-\sigma)^2} + \frac{af + be}{2} + \frac{ef}{2} \quad (221)$$

$$W = \frac{\beta(1-\sigma) - \gamma\tau}{2\tau^2 + 2(1-\sigma)^2} + \frac{bf - ae}{2} + \frac{f^2 - e^2}{4} \quad (222)$$

After simplification, we have:

$$U = 0, W = 0, V = 0 \quad (223)$$

And Where

$$U_1 = \frac{1}{4} \left(\frac{2^{1-2\sigma} - 1}{1-2\sigma} \right) - \frac{\alpha}{2} - \frac{\alpha_1}{1-2\sigma} - \frac{(fc+ed)}{4} \quad (224)$$

$$V_1 = \frac{d_1}{4} - \frac{\gamma}{2} - \frac{(2\beta_1\tau + \gamma_1(1-2\sigma))}{4\tau^2 + (1-2\sigma)^2} - \frac{(ec+fd)}{4} \quad (225)$$

$$W_1 = \frac{c_1}{4} - \frac{\beta}{2} - \frac{(\beta_1(1-2\sigma) - 2\gamma_1\tau)}{4\tau^2 + (1-2\sigma)^2} - \frac{(fc-ed)}{4} \quad (226)$$

After simplification, we have:

$$U_1 = - \frac{\left[(1 - 2^{-\sigma} \cos(\tau \ln(2)))^2 + 2^{-2\sigma} \right]}{2(1-2\sigma)} < 0 \quad (227)$$

$$V_1 = \frac{d_1}{4} - \frac{(2\beta_1\tau + \gamma_1(1-2\sigma))}{4\tau^2 + (1-2\sigma)^2} \quad (228)$$

$$W_1 = \frac{c_1}{4} - \frac{(\beta_1(1-2\sigma) - 2\gamma_1\tau)}{4\tau^2 + (1-2\sigma)^2} \quad (229)$$

Therefore

$$Z_{1,2N,2N} = U_0 + \left[U_1 + V_1 \sin(2\tau \ln(N)) + W_1 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (230)$$

■

□

Lemma 2.2. *If $0 < \sigma < 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(A_{N,N})_{N \geq 1}$:*

$$\frac{1}{2} A_{2N,2N}^2 = \tilde{U}_0 + \left[U_2 + V_2 \sin(2\tau \ln(N)) + W_2 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (231)$$

Where

$$U_2 = - \frac{2\alpha_1}{(1-2\sigma)} \quad (232)$$

$$V_2 = - \frac{2(2\beta_1\tau + \gamma_1(1-2\sigma))}{4\tau^2 + (1-2\sigma)^2} \quad (233)$$

$$W_2 = - \frac{2(\beta_1(1-2\sigma) - 2\gamma_1\tau)}{4\tau^2 + (1-2\sigma)^2} \quad (234)$$

and \tilde{U}_0 is a real constant. And

$$\alpha_1 = \frac{c^2 + d^2}{4}, \beta_1 = \frac{c^2 - d^2}{4}, \gamma_1 = \frac{cd}{2} \quad (235)$$

$$c = 1 - 2^{1-\sigma} \cos(\tau \ln(2)), d = 2^{1-\sigma} \sin(\tau \ln(2)) \quad (236)$$

Proof. From the equation (131), we have:

$$\frac{1}{2} A_{2N,2N}^2 = \underbrace{\sum_{n=1}^{2N} F_{n,2N}}_{Z_{1,2N,2N}} - \frac{1}{2} \underbrace{\sum_{n=1}^{2N} U_{n,2N}^2}_{ZU_{1,2N}} \quad (237)$$

$Z_{1,2N,2N}$ is dealt with in the previous lemma. Let's now concentrate on $ZU_{1,2N}$:

$$ZU_{1,2N} = \frac{1}{2} \sum_{n=1}^N U_{n,2N}^2 + \frac{1}{2} \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (238)$$

From equation (186), we have:

$$\frac{1}{2} \sum_{n=N+1}^{2N} U_{n,2N}^2 = \frac{1}{4} \left(\frac{2^{1-2\sigma} - 1}{1 - 2\sigma} + c_1 \cos(2\tau \ln(N)) + d_1 \sin(2\tau \ln(N)) \right) N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (239)$$

Now, let's focus on the remaining term. From the definition in the equation (119), we can write the following:

$$\frac{1}{2} \sum_{n=1}^N U_{n,2N}^2 = \sum_{n=1}^N \frac{1}{2} \left(d \frac{\sin(\tau \ln(n))}{n^\sigma} + c \frac{\cos(\tau \ln(n))}{n^\sigma} \right)^2 \quad (240)$$

$$= \sum_{n=1}^N Y_{n,2N} \quad (241)$$

Where c , d and $Y_{n,2N}$ are defined above in the previous lemma. From the equation (168), we have:

$$Y_{n,2N} = \frac{c^2 + d^2}{4} \frac{1}{n^{2\sigma}} + \frac{c^2 - d^2}{4} \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \frac{cd}{2} \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \quad (242)$$

And from the equation ():

$$\sum_{n=1}^N Y_{n,2N} = \sum_{n=1}^N \left[\alpha_1 \frac{1}{n^{2\sigma}} + \beta_1 \frac{\cos(2\tau \ln(n))}{n^{2\sigma}} + \gamma_1 \frac{\sin(2\tau \ln(n))}{n^{2\sigma}} \right] \quad (243)$$

Also, from lemma 1.2, we can write:

$$\sum_{n=1}^N Y_{n,2N} = \left[\frac{\alpha_1}{(1-2\sigma)} + \frac{(2\beta_1\tau + \gamma_1(1-2\sigma)) \sin(2\tau \ln(N)) + (\beta_1(1-2\sigma) - 2\gamma_1\tau) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right] N^{1-2\sigma} \quad (244)$$

$$+ U_{l_0} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (245)$$

Where U_{l_0} is a real number. Therefore from the equations (237), (242) and (243):

$$ZU_{1,2N} = \frac{1}{4} \left(\frac{2^{1-2\sigma} - 1}{1 - 2\sigma} + c_1 \cos(2\tau \ln(N)) + d_1 \sin(2\tau \ln(N)) \right) N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (246)$$

$$+ \left[\frac{\alpha_1}{(1-2\sigma)} + \frac{(2\beta_1\tau + \gamma_1(1-2\sigma)) \sin(2\tau \ln(N)) + (\beta_1(1-2\sigma) - 2\gamma_1\tau) \cos(2\tau \ln(N))}{4\tau^2 + (1-2\sigma)^2} \right] N^{1-2\sigma} \quad (247)$$

$$+ U_{l_0} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (248)$$

From the equations (212-217) and the equations (244-246), we conclude:

$$\frac{1}{2} A_{2N,2N} = \tilde{U}_0 + \left[U_2 + V_2 \sin(2\tau \ln(N)) + W_2 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (249)$$

Where

$$U_2 = -\frac{2\alpha_1}{(1-2\sigma)} < 0 \quad (250)$$

$$V_2 = -\frac{2(2\beta_1\tau + \gamma_1(1-2\sigma))}{4\tau^2 + (1-2\sigma)^2} \quad (251)$$

$$W_2 = -\frac{2(\beta_1(1-2\sigma) - 2\gamma_1\tau)}{4\tau^2 + (1-2\sigma)^2} \quad (252)$$

and $\tilde{U}_0 = U_0 - U_{I_0}$ is a real constant. And

$$\alpha_1 = \frac{c^2 + d^2}{4}, \beta_1 = \frac{c^2 - d^2}{4}, \gamma_1 = \frac{cd}{2} \quad (253)$$

$$c = 1 - 2^{1-\sigma} \cos(\tau \ln(2)), d = 2^{1-\sigma} \sin(\tau \ln(2)) \quad (254)$$

Remark. If s was not a zeta zero, we will have an asymptotic expansion with the dominant term $N^{(1-\sigma)}$ instead as follows:

$$\frac{1}{2} A_{2N,2N}^2 = \tilde{U}_0 + (\dots) N^{1-\sigma} + \left[U_2 + V_2 \sin(2\tau \ln(N)) + W_2 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (255)$$

I am mentioning this because one may wonder why a positive sequence $A_{2N,2N}^2$ is having a negative dominant term $N^{1-2\sigma}$ that can go to $-\infty$ since $U_2 < 0$. This will not be problematic in the case of $\sigma \geq \frac{1}{2}$ and also in the case where s is not a zeta zero thanks to the presence of the dominant term $N^{(1-\sigma)}$.

□

Lemma 2.3. Let's σ be such that $0 < \sigma < 1$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:

$$\sigma = \frac{1}{2} \quad (256)$$

And the Riemann's Hypothesis is true.

Proof. Let's suppose $0 < \sigma < \frac{1}{2}$. From the lemma 2.2, we have the following symptotic expansion:

$$\frac{1}{2} A_{2N,2N} = \tilde{U}_0 + \left[U_2 + V_2 \sin(2\tau \ln(N)) + W_2 \cos(2\tau \ln(N)) \right] N^{1-2\sigma} + \mathcal{O}\left(\frac{1}{N^{2\sigma}}\right) \quad (257)$$

Where U_2 , V_2 and W_2 are defined lemma 2.2. $s = \sigma + i\tau$ is a zeta zero. Therefore, the limit of the sequence $(A_{2N,2N})_{N \geq 1}$ is zero. Therefore in the equation (257), we must have $U_2 = V_2 = W_2 = 0$. Otherwise, the sequence

$(A_{N,N})_{N \geq 1}$ will diverge as we can always construct a sequence of integers strictly increasing $(N_n)_{n \geq 1}$ such that: $\lim_{n \rightarrow \infty} V_1 \sin(2\tau \ln(N_n)) + W_1 \cos(2\tau \ln(N_n)) = 0$ And therefore: $\lim_{n \rightarrow \infty} A_{2N_n, 2N_n} = -\infty$ since $U_1 = -\frac{2\alpha_1}{(1-2\sigma)} < 0$. And this is in contradiction with the fact that the sequence $(A_{N,N})$ converges with limit zero. Therefore $U_2 = V_2 = W_2 = 0$. Therefore $c = d = 0$. Therefore σ must be 1. And this is a contradiction with our initial assumption of $0 < \sigma < \frac{1}{2}$. Therefore σ cannot be smaller than $\frac{1}{2}$. If $\sigma > \frac{1}{2}$, we work with $\tilde{\sigma} = 1 - \sigma < \frac{1}{2}$ and we will reach to the same conclusion as well. Therefore σ can only be $\frac{1}{2}$:

$$\sigma = \frac{1}{2} \tag{258}$$

Hence, the Riemann's Hypothesis is true. ■

Remark. We can use similar functions λ_p like Dirichlet eta function or Widder lambda function[8, 9]. For a prime p we define λ_p :

$$\lambda_p(s) = \sum_{n=0}^{+\infty} \left(\sum_{i=1}^{p-1} \frac{1}{(pn+i)^s} - \frac{p-1}{(pn+p)^s} \right) \tag{259}$$

We have the following relationship between the zeta ζ function and the λ_p function:

$$\zeta(s) = \left(1 - \frac{p}{p^s}\right)^{-1} \lambda_p(s) \tag{260}$$

□

2.1 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH) making use of the lambda functions λ_p .

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