A Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

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Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

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1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_{1}^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

*One Raffles Quay, North Tower. 048583 Singapore. Email: charaf@melix.net. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x. There is another way [2] to analytically continue $\zeta(s)$ to the region $\Re(s) > 0$. The idea is to observe that for $\Re(s) > 1$:

$$(1 - \frac{2}{2^s})\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} - \sum_{n=1}^{+\infty} \frac{2}{(2n)^s} \tag{3}$$

Thus,

$$\zeta(s) = (1 - \frac{2}{2^s})^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \tag{4}$$

$$= (1 - \frac{2}{2^s})^{-1} \eta(s) \tag{5}$$

The Dirichlet eta function $\eta(s)$ converges conditionally when $\Re(s) > 0$ and $s \neq 1 + i\frac{2k\pi}{\ln(2)}$. $\eta(s)$ is used as analytical continuation of the Zeta function on the domain where $\Re(s) > 0$. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \tag{7}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0$, $\forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

[1] This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \tag{6}$$

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{8}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{9}$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}}dx \tag{10}$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma-1)^2 + \tau^2} = \int_1^{+\infty} \frac{\Big(\cos(\tau\ln(x)) - i\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx \tag{11}$$

The integral is absolutely convergent. We take the real part and the imaginary part of both sides of the above equation and define the functions F and G as following:

$$F(\sigma,\tau) = \frac{\sigma-1}{(\sigma-1)^2+\tau^2} \tag{12}$$

$$= \int_1^{+\infty} \frac{\Big(\cos(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx \tag{13}$$

And

$$G(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{14}$$

$$= \int_1^{+\infty} \frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx \tag{15}$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$F(1-\sigma,\tau) = \frac{\sigma_1 - 1}{(\sigma_1-1)^2+\tau_1^2} \tag{16}$$

$$= \frac{-\sigma}{\sigma^2+\tau^2} \tag{17}$$

$$= \int_1^{+\infty} \frac{\Big(\cos(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx \tag{18}$$

And

$$G(1-\sigma,\tau) = \frac{\tau_1}{(\sigma_1-1)^2+\tau_1^2} \tag{19}$$

$$= \frac{\tau}{\sigma^2+\tau^2} \tag{20}$$

$$= \int_1^{+\infty} \frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx \tag{21}$$

From now on, σ and τ are not necessarily connected to a ζ zero. Before we move forward, we need to define the following function $I(a,\sigma,\tau)$ for $a>0$:

$$I(a,\sigma,\tau) = \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx \tag{22}$$

$$= \int_0^{\ln(a)} \sin(\tau x)e^{(1-\sigma)x}dx \tag{23}$$

$$= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{24}$$

Where

$$K(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{25}$$

Let's define the function $J(a,\sigma,\tau)$ for $a>0$:

$$J(a,\sigma,\tau) = \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx \tag{26}$$

$$= \int_0^{\ln(a)} \cos(\tau x)e^{(1-\sigma)x}dx \tag{27}$$

$$= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}+\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{28}$$

Where $K(\sigma,\tau)$ is defined above in the equation (25).

Now, let's write $U(\sigma,\tau)$ as the limit of a sequence as follows:

$$U(\sigma,\tau) = \lim_{N\to+\infty}\int_1^N \frac{\{x\}\sin(\tau\ln(x))}{x^{1+\sigma}}dx \tag{29}$$

$$= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\,\sin(x)\frac{\epsilon(e^{\frac{x}{\tau}})e^{-\frac{\sigma}{\tau}x}}{\tau} \tag{30}$$

$$= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\,g(x)\sin(x) \tag{31}$$

$$= \lim_{N\to+\infty} U(N,\sigma) \tag{32}$$

Where

$$U(N,\sigma) = \int_0^{\tau\ln(N)} dx\,g(x)\sin(x) \tag{33}$$

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau}e^{-\frac{\sigma}{\tau}x} \tag{34}$$

$$\epsilon(x) = \{x\} \tag{35}$$

And the same for $V(\sigma,\tau)$:

$$\begin{aligned}
V(\sigma,\tau) &= \lim_{N\to+\infty}\int_1^N \frac{\{x\}\cos(\tau\ln(x))}{x^{1+\sigma}}dx && (36)\\
&= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\ \cos(x)\frac{\epsilon(e^{\frac{x}{\tau}})e^{-\frac{\sigma}{\tau}x}}{\tau} && (37)\\
&= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\, g(x)\cos(x) && (38)\\
&= \lim_{N\to+\infty} V(N,\sigma) && (39)
\end{aligned}$$

Where

$$V(N,\sigma) = \int_0^{\tau\ln(N)} dx\, g(x)\cos(x) \qquad (40)$$

Lemma 1.1. *The sequence* $\left(\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} - J(N,\sigma,\tau)\right)_{N\geq 1}$ *converges and its limit is as follows:*

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} - J(N,\sigma,\tau) = 1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau) \qquad (41)$$

Therefore we can write the following:

$$\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} = J(N,\sigma,\tau)+1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N(\sigma,\tau) \qquad (42)$$

Where (α_N) *is a bounded sequence with limit zero:*

$$\alpha_N = \mathcal{O}(\frac{1}{N^\sigma}) \qquad (43)$$

The sequence $\left(\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau)\right)_{N\geq 1}$ *converges and its limit is as follows:*

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau) = \tau V(\sigma,\tau)-\sigma U(\sigma,\tau) \qquad (44)$$

Therefore we can write the following:

$$\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} = I(N,\sigma,\tau)+\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N(\sigma,\tau) \qquad (45)$$

Where (β_N) *is a bounded sequence with limit zero:*

$$\beta_N = \mathcal{O}(\frac{1}{N^\sigma}) \qquad (46)$$

In case where $s=\sigma+i\tau$ *is a zeta zero, we have:*

$$\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^\sigma} = K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\right) + \alpha_N(\sigma,\tau) \quad (47)$$

$$\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))}{n^\sigma} = K(\sigma,\tau)\left(-\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\right) + \beta_N(\sigma,\tau) \quad (48)$$

■

Proof. Let's study the function g over $\mathbb{R}$. We have the function g piecewise continuous and and its derivatives are also peicewise continuous and:

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau}e^{-\frac{\sigma}{\tau}x} \quad (49)$$

$$g\prime(x) = \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}}) - \frac{\sigma}{\tau}g(x) \quad (50)$$

The interval $(0, \tau\ln(N))$ is the union of the intervals $(\tau\ln(n), \tau\ln(n+1))$ where $1 \leq n < N$. We have $\epsilon\prime(x) = 1$ for each $x \in (\tau\log(n), \tau\log(n+1))$ for each $1 \leq n < N$. Therefore, thanks to the integration by parts we can write:

$$U(N,\sigma) = \int_0^{\tau\ln(N)} dx\, g(x)\sin(x) \quad (51)$$

$$= \lim_{\epsilon\to 0}\sum_{n=1}^{N-1}\int_{\tau\ln(n)+\epsilon}^{\tau\ln(n+1)-\epsilon} dx\, g(x)\sin(x) \quad (52)$$

$$= g(0^+)\cos(0) + \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[g(\tau\ln(n)+\epsilon)\cos(\tau\ln(n)+\epsilon) - g(\tau\ln(n)-\epsilon)\cos(\tau\ln(n)-\epsilon)\Big] \quad (53)$$

$$-g(\tau\ln(N)^-)\cos(\tau\ln(N)) + \int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) \quad (54)$$

$$= \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[g(\tau\ln(n)+\epsilon) - g(\tau\ln(n)-\epsilon)\Big]\cos(\tau\ln(n)) \quad (55)$$

$$-\frac{\cos(\tau\ln(N))}{\tau N^\sigma} + \int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) \quad (56)$$

$$= -\frac{1}{\tau}\sum_{n=2}^{N-1}\frac{\cos(\tau\ln(n))}{n^\sigma} - \frac{\cos(\tau\ln(N))}{\tau N^\sigma} + \int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) \quad (57)$$

$$= -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} - \frac{\sigma}{\tau}\int_0^{\tau\ln(N)} dx\, g(x)\cos(x) + \int_0^{\tau\ln(N)} dx\left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}})\right)\cos(x) \quad (58)$$

$$= -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} - \frac{\sigma}{\tau}V(N,\sigma) + \int_0^{\tau\ln(N)} dx\, \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\cos(x) \quad (59)$$

Therefore

$$U(N,\sigma) + \frac{\sigma}{\tau} V(N,\sigma) \quad = \quad -\frac{1}{\tau} \sum_{n=2}^{N} \frac{\cos(\tau \ln(n))}{n^{\sigma}} + \frac{1}{\tau} J(N,\sigma,\tau) \tag{60}$$

Therefore

$$\sum_{n=2}^{N} \frac{\cos(\tau \ln(n))}{n^{\sigma}} - J(N,\sigma,\tau) \quad = \quad -\tau U(N,\sigma) - \sigma V(N,\sigma) \tag{61}$$

Therefore, the sequence $\left(\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^{\sigma}} - J(N,\sigma,\tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^{\sigma}} - J(N,\sigma,\tau) \quad = \quad 1 - \tau U(\sigma,\tau) - \sigma V(\sigma,\tau) \tag{62}$$

Therefore we can write the following:

$$\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^{\sigma}} \quad = \quad J(N,\sigma,\tau) + 1 - \tau U(\sigma,\tau) - \sigma V(\sigma,\tau) + \alpha_N \tag{63}$$

Where (α_N) is a bounded sequence with limit zero.
And the same for $V(N,\sigma)$ as follows:

$$\begin{aligned}
V(N,\sigma) &= \int_0^{\tau \ln(N)} dx\, g(x) \cos(x) && (64) \\
&= -g(0^+) \sin(0) + \lim_{\epsilon \to 0} \sum_{n=2}^{N-1} \big[-g(\tau \ln(n) + \epsilon) \sin(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \sin(\tau \ln(n) - \epsilon) \big] && (65) \\
&\quad + g(\tau \ln(N)^-) \sin(\tau \ln(N)) - \int_0^{\tau \ln(N)} dx\, g\prime(x) \sin(x) && (66) \\
&= \lim_{\epsilon \to 0} \sum_{n=2}^{N-1} \big[-g(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \big] \sin(\tau \ln(n)) && (67) \\
&\quad + \frac{\sin(\tau \ln(N))}{\tau N^{\sigma}} - \int_0^{\tau \ln(N)} dx\, g\prime(x) \sin(x) && (68) \\
&= \frac{1}{\tau} \sum_{n=2}^{N-1} \frac{\sin(\tau \ln(n))}{n^{\sigma}} + \frac{\sin(\tau \ln(N))}{\tau N^{\sigma}} - \int_0^{\tau \ln(N)} dx\, g\prime(x) \sin(x) && (69) \\
&= \frac{1}{\tau} \sum_{n=2}^{N} \frac{\sin(\tau \ln(n))}{n^{\sigma}} + \frac{\sigma}{\tau} \int_0^{\tau \ln(N)} dx\, g(x) \sin(x) - \int_0^{\tau \ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon\prime(e^{\frac{x}{\tau}}) \right) \sin(x) && (70) \\
&= \frac{1}{\tau} \sum_{n=2}^{N} \frac{\sin(\tau \ln(n))}{n^{\sigma}} + \frac{\sigma}{\tau} U(N,\sigma) - \int_0^{\tau \ln(N)} dx\, \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \sin(x) && (71)
\end{aligned}$$

Therefore

$$V(N,\sigma)-\frac{\sigma}{\tau}U(N,\sigma) \quad = \quad \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}}-\frac{1}{\tau}I(N,\sigma,\tau) \tag{72}$$

Therefore

$$\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}}-I(N,\sigma,\tau) \quad = \quad \tau V(N,\sigma)-\sigma U(N,\sigma) \tag{73}$$

Therefore, the sequence $\left(\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}}-I(N,\sigma,\tau)\right)_{N\geq 1}$ converges and its limit is as follows:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}}-I(N,\sigma,\tau) \quad = \quad \tau V(\sigma,\tau)-\sigma U(\sigma,\tau) \tag{74}$$

Therefore we can write the following:

$$\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} \quad = \quad I(N,\sigma,\tau)+\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N \tag{75}$$

Where (β_N) is a bounded sequence with limit zero.
The proof of the result in the equations (43) and (46) can be found in [10]. Another proof of this lemma can be found in [11]. ■ □

2 Dirichlet eta function

From equation (5), the $\zeta(s)$ function is also defined through the Dirichlet eta function $\eta(s)$ as follows:

$$\zeta(s) \quad = \quad (1-\frac{2}{2^s})^{-1}\eta(s) \tag{76}$$

$$= \quad (1-\frac{2}{2^s})^{-1}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^s} \tag{77}$$

Let's define the sequence X_N as follows:

$$X_N(s)=\sum_{n=1}^{N}\frac{1}{n^s} \tag{78}$$

For $N\geq 1$:

$$X_N \quad = \quad \sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^{\sigma}}-i\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} \tag{79}$$

Let's define the sequences U_n, V_n as follows: For $n\geq 1$

$$U_n=\frac{\cos(\tau\ln(n))}{n^{\sigma}},\quad V_n=\frac{\sin(\tau\ln(n))}{n^{\sigma}} \tag{80}$$

Let's define the series A_n, B_n and X_n as follows:

$$A_n = \sum_{k=1}^{n} \frac{\cos(\tau \ln(k))}{k^\sigma}, \quad B_n = \sum_{k=1}^{n} \frac{\sin(\tau \ln(k))}{k^\sigma}, \quad X_n = A_n - iB_n \tag{81}$$

When we are dealing with complex numbers, it is always insightful to work with the norm. So let's develop further the squared norm of the series Z_N as follows:

$$\|X_N\|^2 = A_N^2 + B_N^2 = \left(\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^\sigma}\right)^2 + \left(\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))}{n^\sigma}\right)^2 \tag{82}$$

So

$$\begin{aligned}
A_N^2 &= \left(\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^\sigma}\right)^2 && (83)\\
&= \sum_{n=1}^{N} \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}} + 2\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))}{n^\sigma}\left(A_n - \frac{\cos(\tau \ln(n))}{n^\sigma}\right) && (84)\\
&= -\sum_{n=1}^{N} \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}} + 2\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))A_n}{n^\sigma} && (85)
\end{aligned}$$

And the same calculation for B_N

$$B_N^2 = -\sum_{n=1}^{N} \frac{\sin^2(\tau \ln(n))}{n^{2\sigma}} + 2\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))B_n}{n^\sigma} \tag{86}$$

Hence we have the new expression of square norm of Z_N:

$$\|X_N\|^2 = 2\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))A_n}{n^\sigma} + 2\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))B_n}{n^\sigma} - \sum_{n=1}^{N} \frac{\sin^2(\tau \ln(n)) + \cos^2(\tau \ln(n))}{n^{2\sigma}} \tag{87}$$

which simplies even further to:

$$\|X_N\|^2 = 2\sum_{n=1}^{N} \frac{\cos(\tau \ln(n))A_n}{n^\sigma} + 2\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))B_n}{n^\sigma} - \sum_{n=1}^{N} \frac{1}{n^{2\sigma}} \tag{88}$$

Let's now define F_n and G_n as follows:

$$F_n = \frac{\cos(\tau \ln(n))A_n}{n^\sigma}, \quad G_n = \frac{\sin(\tau \ln(n))B_n}{n^\sigma} \tag{89}$$

Therefore

$$A_N^2 = 2\sum_{n=1}^{N} F_n - \sum_{n=1}^{N} \frac{\cos^2(\tau \ln(n))}{n^{2\sigma}}, \quad B_N^2 = 2\sum_{n=1}^{N} G_n - \sum_{n=1}^{N} \frac{\sin^2(\tau \ln(n))}{n^{2\sigma}} \tag{90}$$

Let's define the sequence Y_N as follows:

$$
\begin{aligned}
Y_N(s) &= X_N(s) - \frac{1}{2^{s-1}} X_{\frac{N}{2}}(s) && (91)\\
&= A_N - \frac{1}{2^{s-1}} A_{\frac{N}{2}} - i\left(B_N - \frac{1}{2^{s-1}} B_{\frac{N}{2}}\right) && (92)\\
&= A_{N,N} - i\, B_{N,N} && (93)
\end{aligned}
$$

Where

$$
A_{N,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\cos(\tau \ln(k)) - 2^{1-\sigma}\cos(\tau\big(\ln(2k)\big))}{k^{\sigma}} + \sum_{k=\frac{N}{2}+1}^{N} \frac{\cos(\tau \ln(k))}{k^{\sigma}} \quad (94)
$$

And

$$
B_{N,N} = \sum_{k=1}^{\frac{N}{2}} \frac{\sin(\tau \ln(k)) - 2^{1-\sigma}\sin(\tau\big(\ln(2k)\big))}{k^{\sigma}} + \sum_{k=\frac{N}{2}+1}^{N} \frac{\sin(\tau \ln(k))}{k^{\sigma}} \quad (95)
$$

We also have:

$$
\lim_{N\to\infty} Y_N = \lim_{N\to\infty} A_{N,N} - i\, B_{N,N} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} = \eta(s) \quad (96)
$$

Remark. *s is a zeta **zero**, $\zeta(s) = 0$, if and only if*

$$
\lim_{N\to\infty} A_{N,N} = 0 \text{ and } \lim_{N\to\infty} B_{N,N} = 0 \quad (97)
$$

Let's define the following sequences: For $N \geq 1$:

$$
U_{n,N} = \begin{cases} \frac{\cos(\tau \ln(n)) - 2^{1-\sigma}\cos(\tau(\ln(2n)))}{n^{\sigma}}, & \text{if } n \leq \frac{N}{2} \\ \frac{\cos(\tau \ln(n))}{n^{\sigma}} & \text{if } n > \frac{N}{2} \end{cases} \quad (98)
$$

And

$$
V_{n,N} = \begin{cases} \frac{\sin(\tau \ln(n)) - 2^{1-\sigma}\sin(\tau(\ln(2n)))}{n^{\sigma}}, & \text{if } n \leq \frac{N}{2} \\ \frac{\sin(\tau \ln(n))}{n^{\sigma}} & \text{if } n > \frac{N}{2} \end{cases} \quad (99)
$$

And

$$
A_{n,N} = \begin{cases} \sum_{k=1}^{n} \frac{\cos(\tau \ln(k)) - 2^{1-\sigma}\cos(\tau \ln(2k))}{k^{\sigma}}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\cos(\tau \ln(k)) - 2^{1-\sigma}\cos(\tau \ln(2k))}{k^{\sigma}} + \sum_{k=\frac{N}{2}+1}^{n} \frac{\cos(\tau \ln(k))}{k^{\sigma}} & \text{if } n > \frac{N}{2} \\ A_{N,N} & \text{if } n \geq N \end{cases} \quad (100)
$$

And

$$
B_{n,N} = \begin{cases} \sum_{k=1}^{n} \frac{\sin(\tau \ln(k)) - 2^{1-\sigma}\sin(\tau \ln(2k))}{k^{\sigma}}, & \text{if } n \leq \frac{N}{2} \\ \sum_{k=1}^{\frac{N}{2}} \frac{\sin(\tau \ln(k)) - 2^{1-\sigma}\sin(\tau \ln(2k))}{k^{\sigma}} + \sum_{k=\frac{N}{2}+1}^{n} \frac{\sin(\tau \ln(k))}{k^{\sigma}} & \text{if } n > \frac{N}{2} \\ B_{N,N} & \text{if } n \geq N \end{cases} \quad (101)
$$

Let's develop further the squared norm of the series Y_N as follows:

$$\|Y_N\|^2 = A_{N,N}^2 + B_{N,N}^2 = \left(\sum_{n=1}^{N} U_{n,N}\right)^2 + \left(\sum_{n=1}^{N} V_{n,N}\right)^2 \tag{102}$$

So

$$\begin{aligned} A_{N,N}^2 &= \sum_{n=1}^{N} U_{n,N} + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} U_n U_{k,N} &(103)\\ &= -\sum_{n=1}^{N} U_{n,N}^2 + 2\sum_{n=1}^{N} U_{n,N}\, A_{n,N} &(104)\end{aligned}$$

And the same calculation for $B_{N,N}$

$$B_{N,N}^2 = -\sum_{n=1}^{N} V_{n,N}^2 + 2\sum_{n=1}^{N} V_{n,N}\, B_{n,N} \tag{105}$$

Let's now define the sequences $(F_{n,N})$ and $(G_{n,N})$ as follows:

$$F_{n,N} = U_{n,N} A_{n,N}, \quad G_{n,N} = V_{n,N} B_{n,N} \tag{106}$$

Hence we have the new expression of square norm of Z_N^2:

$$\|Y_N\|^2 = 2\sum_{n=1}^{N} F_{n,N} + 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) \tag{107}$$

which simplies even further to:

$$\begin{aligned} \sum_{n=1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) &= \sum_{n=1}^{\frac{N}{2}}\left(U_{n,N}^2 + V_{n,N}^2\right) + \sum_{n=\frac{N}{2}+1}^{N}\left(U_{n,N}^2 + V_{n,N}^2\right) &(108)\\ &= \sum_{n=1}^{N}\frac{1}{n^{2\sigma}} + \sum_{n=1}^{\frac{N}{2}}\frac{2^{2-2\sigma} - 2^{2-\sigma}\Big(\cos(\tau\ln(2))\Big)}{n^{2\sigma}} &(109)\end{aligned}$$

We have $2\sigma > 1$, hence the partial sums $\sum_{n\geq 1}^{N} U_{n,N}^2$ and $\sum_{n\geq 1}^{N} V_{n,N}^2$ converge absolutely. We have

$$A_{N,N}^2 = 2\sum_{n=1}^{N} F_{n,N} - \sum_{n=1}^{N} U_{n,N}^2, \quad B_{N,N}^2 = 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N} V_{n,N}^2 \tag{110}$$

Remark. *The idea is to prove that in the case of a complex s that is in the right hand side of the critical strip $\frac{1}{2} < \sigma \leq 1$ and that is a ζ zero, that the limit $\lim_{n\to\infty} A_{n,n}^2 = +/-\infty$ or the limit $\lim_{n\to\infty} B_{n,n}^2 = +/-\infty$. This will create a contradiction. Because if s is a ζ zero then the $\lim_{n\to\infty} A_{n,n}^2$ should be 0 and the $\lim_{n\to\infty} B_{n,n}^2$ should be 0. And therefore the sequences $(\sum_{n=1}^{N} F_{n,N})_{N\geq 1}$ and $(\sum_{n=1}^{N} G_{n,N})_{N\geq 1}$ should converge and their limits should be: $\lim_{n\to\infty}\sum_{n=1}^{N} F_{n,N} = \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N} U_{n,N}^2 < +\infty$ and $\lim_{n\to\infty}\sum_{n=1}^{N} G_{n,N} = \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N} V_{n,N}^2 < +\infty$.*

Lemma 2.1. *If $\frac{1}{2} < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(\sum_{n=1}^{N} F_{n,N})_{N\geq 1}$:*

$$\sum_{n=1}^{2N} F_{n,2N} = U_0 - \frac{2^{1-\sigma}\sin(\tau\ln(2))\Big(1 - 2^{1-\sigma}\cos(\tau\ln(2))\Big)}{2\big((1-\sigma)^2+\tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (111)$$

Where U_0 is a real number.

Proof. So, let's define the sum $Z_{1,2N,2N}$ for $N \geq 1$:

$$\begin{aligned} Z_{1,2N,2N} &= \sum_{n=1}^{2N} F_{n,2N} & (112)\\ &= \underbrace{\sum_{n=1}^{N} F_{n,2N}}_{Z_{1,N,2N}} + \underbrace{\sum_{n=N+1}^{2N} F_{n,2N}}_{Z_{N+1,2N,2N}} & (113) \end{aligned}$$

And $Z_{N+1,2N,2N}$:

$$\begin{aligned} Z_{N+1,2N,2N} &= \sum_{n=N+1}^{2N} F_{n,2N} & (114)\\ &= \sum_{n=N+1}^{2N} U_{n,2N}\, A_{n,2N} & (115)\\ &= \sum_{n=N+1}^{2N} U_{n,2N}\Big(A_{N,2N} + \sum_{k=N+1}^{n} U_{k,2N}\Big) & (116)\\ &= A_{N,2N}\Big(A_{2N,2N} - A_{N,2N}\Big) + \sum_{n=N+1}^{2N} U_{n,2N} \sum_{k=N+1}^{n} U_{k,2N} & (117) \end{aligned}$$

And

$$\begin{aligned} \sum_{n=N+1}^{2N} U_{n,2N} \sum_{k=N+1}^{n} U_{k,2N} &= \sum_{k=N+1}^{2N} U_{k,2N} \sum_{n=k}^{2N} U_{n,2N} & (118)\\ &= \sum_{k=N+1}^{2N} U_{k,2N}\Big(A_{2N,2N} - A_{k-1,2N}\Big) & (119)\\ &= \sum_{k=N+1}^{2N} U_{k,2N}\Big(A_{2N,2N} - A_{k,2N}\Big) + \sum_{n=N+1}^{2N} U_{n,2N}^2 & (120)\\ &= A_{2N,2N}\Big(A_{2N,2N} - A_{N,2N}\Big) - \sum_{k=N+1}^{2N} U_{k,2N}A_{k,2N} + \sum_{n=N+1}^{2N} U_{n,2N}^2 & (121)\\ &= A_{2N,2N}\Big(A_{2N,2N} - A_{N,2N}\Big) - Z_{N+1,2N,2N} + \sum_{n=N+1}^{2N} U_{n,2N}^2 & (122) \end{aligned}$$

Therefore

$$2\, Z_{N+1,2N,2N} = \Big(A_{2N,2N} + A_{N,2N}\Big)\Big(A_{2N,2N} - A_{N,2N}\Big) + \sum_{n=N+1}^{2N} U_{n,2N}^2 \quad (123)$$

Therefore

$$Z_{1,2N,2N} = \frac{1}{2}\Big(A_{2N,2N}+A_{N,2N}\Big)\Big(A_{2N,2N}-A_{N,2N}\Big)+Z_{1,N,2N}+\sum_{n=N+1}^{2N}U_{n,2N}^2 \quad (124)$$

We also have:

$$A_{2N,2N} = A_{N,2N}+A_{2N}-A_N \quad (125)$$

Therefore

$$Z_{1,2N,2N} = A_{N,2N}\Big(A_{2N}-A_N\Big)+\frac{1}{2}\Big(A_{2N}-A_N\Big)^2+Z_{1,N,2N}+\sum_{n=N+1}^{2N}U_{n,2N}^2 \quad (126)$$

We also have:

$$Z_{1,N,2N} = \sum_{n=1}^{N}F_{n,2N}=\sum_{n=1}^{N}U_{n,2N}\,A_{n,2N} \quad (127)$$

For $n \leq N$, we have:

$$\begin{aligned}
A_{n,2N} &= \sum_{k=1}^{n}U_{k,2N} && (128)\\
&= \sum_{k=1}^{n}\frac{\cos(\tau\ln(k))-2^{1-\sigma}\cos(\tau\ln(2k))}{k^\sigma} && (129)\\
&= \big(1-2^{1-\sigma}\cos(\tau\ln(2))\big)A_n+2^{1-\sigma}\sin(\tau\ln(2))B_n && (130)\\
&= c\,A_n+d\,B_n && (131)
\end{aligned}$$

Where

$$c=1-2^{1-\sigma}\cos(\tau\ln(2)), d=2^{1-\sigma}\sin(\tau\ln(2)) \quad (132)$$

From lemma 1.1 we have:

$$\begin{aligned}
A_n &= K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\epsilon_n}{n^\sigma} && (133)\\
B_n &= K(\sigma,\tau)\left(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\gamma_n}{n^\sigma} && (134)
\end{aligned}$$

$$\begin{aligned}
A_{n,2N} &= c\,K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\epsilon_n}{n^\sigma} && (135)\\
&\quad +d\,K(\sigma,\tau)\left(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\gamma_n}{n^\sigma} && (136)\\
&= a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}+b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\gamma_n}{n^\sigma} && (137)
\end{aligned}$$

Where

$$a=K(\sigma,\tau)\Big(c+\frac{(1-\sigma)}{\tau}d\Big), b=K(\sigma,\tau)\Big(\frac{(1-\sigma)}{\tau}c-d\Big) \quad (138)$$

Therefore

$$Z_{1,N,2N} = \sum_{n=1}^{N} A_{n,2N}\Big(\big(1-2^{1-\sigma}\cos(\tau\ln(2))\big)\frac{\cos(\tau\ln(n))}{n^{\sigma}} + 2^{1-\sigma}\sin(\tau\ln(2))\frac{\sin(\tau\ln(n))}{n^{\sigma}}\Big) \quad (139)$$

$$= \sum_{n=1}^{N}\left(a\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) \quad (140)$$

$$= \sum_{n=1}^{N} X_{n,2N} \quad (141)$$

Where

$$X_{n,2N} = \left(a\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) \quad (142)$$

$$= \frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}} \quad (143)$$

Therefore:

$$Z_{1,N,2N} = \sum_{n=1}^{N}\left[\frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (144)$$

$$= \sum_{n=1}^{N}\left[\alpha\frac{1}{n^{2\sigma-1}} + \beta\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \gamma\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (145)$$

Also, from lemma 3.1 in [10], we can write:

$$Z_{1,N,2N} = \left(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau+\gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma)-\gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2+2(1-\sigma)^2}\right)N^{2-2\sigma} \quad (146)$$

$$+U_0 + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (147)$$

Where U_0 is a real number and:

$$\alpha = \frac{ad+bc}{2},\ \beta = \frac{bc-ad}{2},\ \gamma = \frac{ac+bd}{2} \quad (148)$$

We also have:

$$\sum_{n=N+1}^{2N} U_{n,2N}^2 = \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (149)$$

And

$$A_{N,2N} = a\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\gamma_N}{N^{\sigma}} \quad (150)$$

And

$$A_{2N} - A_N = K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(2N))}{(2N)^{\sigma-1}} + \frac{\sin(\tau\ln(2N))}{(2N)^{\sigma-1}}\right) + \mathcal{O}(\frac{1}{N^\sigma}) \quad (151)$$

$$-K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\right) + \mathcal{O}(\frac{1}{N^\sigma}) \quad (152)$$

$$= K(\sigma,\tau)\left(2^{1-\sigma}\sin(\tau\ln(2)) + \frac{1-\sigma}{\tau}\Big(2^{1-\sigma}\cos(\tau\ln(2)) - 1\Big)\right)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} \quad (153)$$

$$+K(\sigma,\tau)\left(\Big(2^{1-\sigma}\cos(\tau\ln(2)) - 1\Big) - \frac{1-\sigma}{\tau}2^{1-\sigma}\sin(\tau\ln(2))\right)\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma}) \quad (154)$$

$$= e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma}) \quad (155)$$

Where

$$e = -K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big) = -a, f = K(\sigma,\tau)\Big(-\frac{(1-\sigma)}{\tau}c + d\Big) = -b \quad (156)$$

Therefore

$$\big(A_{2N} - A_N\big)^2 = \left[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\right]^2 \quad (157)$$

$$= \frac{e^2+f^2}{2}\frac{1}{N^{2\sigma-2}} + \frac{f^2-e^2}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + ef\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (158)$$

And

$$A_{N,2N}\big(A_{2N} - A_N\big) = \left(a\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\right)\left[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\right] \quad (159)$$

$$= \frac{ae+bf}{2}\frac{1}{N^{2\sigma-2}} + \frac{bf-ae}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + \frac{af+be}{2}\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (160)$$

Therefore

$$A_{N,2N}\big(A_{2N} - A_N\big) + \frac{1}{2}\big(A_{2N} - A_N\big)^2 \quad (161)$$

$$= \Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big)\frac{1}{N^{2\sigma-2}} + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} \quad (162)$$

$$+\Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (163)$$

Therefore

$$Z_{1,2N,2N} = U_0 + \left(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau + \gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma) - \gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2 + 2(1-\sigma)^2}\right)N^{2-2\sigma} \quad (164)$$

$$+ \left[\Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big) + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\cos(2\tau\ln(N))\right. \quad (165)$$

$$\left. + \Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\sin(2\tau\ln(N))\right]N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (166)$$

$$= U_0 + \Big(U + V\sin(2\tau\ln(N)) + W\cos(2\tau\ln(N))\Big)N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (167)$$

Where

$$U = \frac{\alpha}{2(1-\sigma)} + \frac{e^2+f^2}{4} + \frac{ae+bf}{2} \tag{168}$$

$$V = \frac{\beta\tau+\gamma(1-\sigma)}{2\tau^2+2(1-\sigma)^2} + \frac{af+be}{2} + \frac{ef}{2} \tag{169}$$

$$W = \frac{\beta(1-\sigma)-\gamma\tau}{2\tau^2+2(1-\sigma)^2} + \frac{bf-ae}{2} + \frac{f^2-e^2}{4} \tag{170}$$

After simplification, we have:

$$U = 0, W = 0, V = -K(\sigma,\tau)\frac{dc}{2\tau} \tag{171}$$

And

$$Z_{1,2N,2N} = -K(\sigma,\tau)\frac{dc}{2\tau}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{172}$$

■ □

Lemma 2.2. *If $\frac{1}{2} < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(\sum_{n=1}^{N} G_{n,N})_{N\geq 1}$:*

$$\sum_{n=1}^{2N} G_{n,2N} = V_0 + \frac{2^{1-\sigma}\sin(\tau\ln(2))\Big(1-2^{1-\sigma}\cos(\tau\ln(2))\Big)}{2\Big((1-\sigma)^2+\tau^2\Big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{173}$$

Where V_0 is a real number.

Proof. So, let's define the sum $Z_{1,2N,2N}$ for $N \geq 1$:

$$Z_{1,2N,2N} = \sum_{n=1}^{2N} G_{n,2N} \tag{174}$$

$$= \underbrace{\sum_{n=1}^{N} G_{n,2N}}_{Z_{1,N,2N}} + \underbrace{\sum_{n=N+1}^{2N} G_{n,2N}}_{Z_{N+1,2N,2N}} \tag{175}$$

And $Z_{N+1,2N,2N}$:

$$Z_{N+1,2N,2N} = B_{N,2N}\Big(B_{2N,2N} - B_{N,2N}\Big) + \sum_{n=N+1}^{2N} V_{n,2N}\sum_{k=N+1}^{n} V_{k,2N} \tag{176}$$

And

$$\sum_{n=N+1}^{2N} V_{n,2N}\sum_{k=N+1}^{n} V_{k,2N} = B_{2N,2N}\Big(B_{2N,2N} - B_{N,2N}\Big) - Z_{N+1,2N,2N} + \sum_{n=N+1}^{2N} V_{n,2N}^2 \tag{177}$$

Therefore

$$2\,Z_{N+1,2N,2N} = \Big(B_{2N,2N} + B_{N,2N}\Big)\Big(B_{2N,2N} - B_{N,2N}\Big) + \sum_{n=N+1}^{2N} V_{n,2N}^2 \tag{178}$$

Therefore

$$Z_{1,2N,2N} = \frac{1}{2}\Big(B_{2N,2N}+B_{N,2N}\Big)\Big(B_{2N,2N}-B_{N,2N}\Big)+ZZ_{1,N,2N}+\sum_{n=N+1}^{2N}V_{n,2N}^2 \quad (179)$$

We also have:

$$B_{2N,2N} = B_{N,2N}+B_{2N}-B_N \quad (180)$$

Therefore

$$Z_{1,2N,2N} = B_{N,2N}\Big(B_{2N}-B_N\Big)+\frac{1}{2}\Big(B_{2N}-B_N\Big)^2+ZZ_{1,N,2N}+\sum_{n=N+1}^{2N}V_{n,2N}^2 \quad (181)$$

We also have:

$$Z_{1,N,2N} = \sum_{n=1}^{N}G_{n,2N}=\sum_{n=1}^{N}V_{n,2N}\,B_{n,2N} \quad (182)$$

For $n \leq N$, we have:

$$\begin{aligned} B_{n,2N} &= \sum_{k=1}^{n}V_{k,2N} && (183)\\ &= \sum_{k=1}^{n}\frac{\sin(\tau\ln(k))-2^{1-\sigma}\sin(\tau\ln(2k))}{k^\sigma} && (184)\\ &= \big(1-2^{1-\sigma}\cos(\tau\ln(2))\big)B_n-2^{1-\sigma}\sin(\tau\ln(2))A_n && (185)\\ &= c\,A_n+d\,B_n && (186)\end{aligned}$$

Where

$$c=-2^{1-\sigma}\sin(\tau\ln(2)),\, d=1-2^{1-\sigma}\cos(\tau\ln(2)) \quad (187)$$

Remark. *We note here that c and d are slightly different from the terms c and d in the proof of the lemma 2.1.*

From lemma 1.1 we have:

$$\begin{aligned} A_n &= K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\epsilon_n}{n^\sigma} && (188)\\ B_n &= K(\sigma,\tau)\left(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\gamma_n}{n^\sigma} && (189)\end{aligned}$$

$$\begin{aligned} B_{n,2N} &= c\,K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\epsilon_n}{n^\sigma} && (190)\\ &\quad +d\,K(\sigma,\tau)\left(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\right)+\frac{\gamma_n}{n^\sigma} && (191)\\ &= a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}+b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}}+\frac{\gamma_n}{n^\sigma} && (192)\end{aligned}$$

Where

$$a = K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big), b = K(\sigma,\tau)\Big(\frac{(1-\sigma)}{\tau}c - d\Big) \tag{193}$$

Remark. *We note here that a and b have the same expressions like in the case of the lemma 2.1 but they are also different because the definition of c and d are different.*

Therefore

$$\begin{aligned}
Z_{1,N,2N} &= \sum_{n=1}^{N} B_{n,2N}\Big(\big(1-2^{1-\sigma}\sin(\tau\ln(2))\big)\frac{\cos(\tau\ln(n))}{n^{\sigma}} - 2^{1-\sigma}\cos(\tau\ln(2))\frac{\sin(\tau\ln(n))}{n^{\sigma}}\Big) &(194)\\
&= \sum_{n=1}^{N}\left(a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) &(195)\\
&= \sum_{n=1}^{N} Y_{n,2N} &(196)
\end{aligned}$$

Where

$$\begin{aligned}
Y_{n,2N} &= \left(a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) &(197)\\
&= \frac{ad+bc}{2}\,\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\,\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\,\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}} &(198)
\end{aligned}$$

Therefore

$$\begin{aligned}
Z_{1,N,2N} &= \sum_{n=1}^{N}\left[\frac{ad+bc}{2}\,\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\,\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\,\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] &(199)\\
&= \sum_{n=1}^{N}\left[\alpha\,\frac{1}{n^{2\sigma-1}} + \beta\,\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \gamma\,\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] &(200)\\
&= \left(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau+\gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma)-\gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2+2(1-\sigma)^2}\right)N^{2-2\sigma} &(201)\\
&\quad +V_0 + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) &(202)
\end{aligned}$$

Where V_0 is a real number and:

$$\alpha = \frac{ad+bc}{2},\ \beta = \frac{bc-ad}{2},\ \gamma = \frac{ac+bd}{2} \tag{203}$$

$$\sum_{n=N+1}^{2N} V_{n,2N}^2 = \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{204}$$

$$B_{N,2N} = a\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\gamma_N}{N^{\sigma}} \tag{205}$$

And

$$\begin{aligned}
B_{2N}-B_N &= K(\sigma,\tau)\Bigg(-\frac{\cos(\tau\ln(2N))}{(2N)^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(2N))}{(2N)^{\sigma-1}}\Bigg)+\mathcal{O}(\frac{1}{N^\sigma}) && (206)\\
&\quad -K(\sigma,\tau)\Bigg(-\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\Bigg)+\mathcal{O}(\frac{1}{N^\sigma}) && (207)\\
&= K(\sigma,\tau)\Bigg(\frac{1-\sigma}{\tau}2^{1-\sigma}\sin(\tau\ln(2))+\Big(1-2^{1-\sigma}\cos(\tau\ln(2))\Big)\Bigg)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} && (208)\\
&\quad +K(\sigma,\tau)\Bigg(\frac{1-\sigma}{\tau}\Big(2^{1-\sigma}\cos(\tau\ln(2))-1\Big)+2^{1-\sigma}\sin(\tau\ln(2))\Bigg)\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}+\mathcal{O}(\frac{1}{N^\sigma}) && (209)\\
&= e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}+f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\mathcal{O}(\frac{1}{N^\sigma}) && (210)
\end{aligned}$$

Where

$$e=-K(\sigma,\tau)\Big(c+\frac{(1-\sigma)}{\tau}d\Big)=-a,\; f=K(\sigma,\tau)\Big(-\frac{(1-\sigma)}{\tau}c+d\Big)=-b \qquad (211)$$

Therefore

$$\begin{aligned}
\big(B_{2N}-B_N\big)^2 &= \Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}+f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\mathcal{O}(\frac{1}{N^\sigma})\Bigg]^2 && (212)\\
&= \frac{e^2+f^2}{2}\frac{1}{N^{2\sigma-2}}+\frac{f^2-e^2}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}}+ef\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}}+\mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (213)
\end{aligned}$$

And

$$\begin{aligned}
B_{N,2N}\big(B_{2N}-B_N\big) &= \Bigg(a\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}+b\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\mathcal{O}(\frac{1}{N^\sigma})\Bigg)\Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}+f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\mathcal{O}(\frac{1}{N^\sigma})\Bigg] && (214)\\
&= \frac{ae+bf}{2}\frac{1}{N^{2\sigma-2}}+\frac{bf-ae}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}}+\frac{af+be}{2}\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}}+\mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (215)
\end{aligned}$$

Therefore

$$\begin{aligned}
& B_{N,2N}\big(B_{2N}-B_N\big)+\frac{1}{2}\big(B_{2N}-B_N\big)^2 && (216)\\
&= \Big(\frac{ae+bf}{2}+\frac{e^2+f^2}{4}\Big)\frac{1}{N^{2\sigma-2}}+\Big(\frac{bf-ae}{2}+\frac{f^2-e^2}{4}\Big)\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} && (217)\\
&\quad +\Big(\frac{af+be}{2}+\frac{ef}{2}\Big)\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}}+\mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (218)
\end{aligned}$$

Therefore, like in the proof of the lemma 2.1, we have:

$$\begin{aligned}
Z_{1,2N,2N} &= V_0+\Bigg(\frac{\alpha}{2(1-\sigma)}+\frac{\Big(\beta\tau+\gamma(1-\sigma)\Big)\sin(2\tau\ln(N))+\Big(\beta(1-\sigma)-\gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2+2(1-\sigma)^2}\Bigg)N^{2-2\sigma} && (219)\\
&+ \Bigg[\Big(\frac{ae+bf}{2}+\frac{e^2+f^2}{4}\Big)+\Big(\frac{bf-ae}{2}+\frac{f^2-e^2}{4}\Big)\cos(2\tau\ln(N)) && (220)\\
&+ \Big(\frac{af+be}{2}+\frac{ef}{2}\Big)\sin(2\tau\ln(N))\Bigg]N^{2-2\sigma}+\mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (221)\\
&= V_0+\Big(U+V\sin(2\tau\ln(N))+W\cos(2\tau\ln(N))\Big)N^{2-2\sigma}+\mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (222)
\end{aligned}$$

Where

$$U = \frac{\alpha}{2(1-\sigma)} + \frac{e^2+f^2}{4} + \frac{ae+bf}{2} \tag{223}$$

$$V = \frac{\beta\tau+\gamma(1-\sigma)}{2\tau^2+2(1-\sigma)^2} + \frac{af+be}{2} + \frac{ef}{2} \tag{224}$$

$$W = \frac{\beta(1-\sigma)-\gamma\tau}{2\tau^2+2(1-\sigma)^2} + \frac{bf-ae}{2} + \frac{f^2-e^2}{4} \tag{225}$$

After simplification, we have:

$$U = 0, W = 0, V = -K(\sigma,\tau)\frac{dc}{2\tau} \tag{226}$$

And

$$Z_{1,2N,2N} = -K(\sigma,\tau)\frac{dc}{2\tau}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{227}$$

■ □

Lemma 2.3. *Let's σ be such that $0 < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

- *if $\sigma \neq \frac{1}{2}$, then $\tau = \frac{k\pi}{\ln(2)}$ with $k \in \mathbb{N}^*$.*
- *if $\tau \neq \frac{k\pi}{\ln(2)}$ with $k \in \mathbb{N}^*$, then $\sigma = \frac{1}{2}$.*

Proof. Let's suppose $\sigma > \frac{1}{2}$. From the lemma 2.1, we have:

$$\sum_{n=1}^{2N} F_{n,2N} = U_0 - \frac{2^{1-\sigma}\sin(\tau\ln(2))\Big(1-2^{1-\sigma}\cos(\tau\ln(2))\Big)}{2\big((1-\sigma)^2+\tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{228}$$

We have $s = \sigma + i\tau$ is a zeta zero. Therefore, from equation (110), we have:

$$\lim_{N\to+\infty} A^2_{N,N} = 2\sum_{n=1}^{N} F_{n,N} - \sum_{n=1}^{N} U^2_{n,N} = 0 \tag{229}$$

Therefore as $\sigma > \frac{1}{2}$

$$\lim_{N\to+\infty}\sum_{n=1}^{N} F_{n,N} = \lim_{N\to+\infty}\frac{1}{2}\sum_{n=1}^{N} U^2_{n,N} < +\infty \tag{230}$$

Therefore, in the asymptotic expansion in the equation(230), we must have:

$$\sin(\tau\ln(2))\Big(1-2^{1-\sigma}\cos(\tau\ln(2))\Big) = 0 \tag{231}$$

Otherwise the sequences $(A^2_{N,N})_{N\geq 1}$ and $(B^2_{N,N})_{N\geq 1}$ will diverge. Therefore, from lemma 2.2, similarly we have:

$$\sum_{n=1}^{2N} G_{n,2N} = V_0 + \frac{2^{1-\sigma}\sin(\tau\ln(2))\Big(1-2^{1-\sigma}\cos(\tau\ln(2))\Big)}{2\big((1-\sigma)^2+\tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{232}$$

Case: $\tau \neq \frac{k\pi}{\ln(2)}$, $k \in \mathbb{N}^*$ In this case we have:

$$\cos(\tau \ln(2)) = \frac{1}{2^{1-\sigma}} \tag{233}$$

We also have $1 - \overline{s} = 1 - \sigma + i\tau = \tilde{\sigma} + i\tau$ also a zeta zero. Therefore, as $\tilde{\sigma} > \frac{1}{2}$, by the same method, we have:

$$\sin(\tau \ln(2))\Big(1 - 2^{1-\tilde{\sigma}} \cos(\tau \ln(2))\Big) = 0 \tag{234}$$

Therefore

$$\cos(\tau \ln(2)) = \frac{1}{2^{\sigma}} \tag{235}$$

Therefore by combining (233) and (235), we get:

$$2^{1-\sigma} = 2^{\sigma} \tag{236}$$

Hence a contradiction with $\sigma > \frac{1}{2}$. Therefore we conclude the following:

- if $\sigma \neq \frac{1}{2}$, then we must have $\tau = \frac{k\pi}{\ln(2)}$ with $k \in \mathbb{N}^*$.
- if $\tau \neq \frac{k\pi}{\ln(2)}$ with $k \in \mathbb{N}^*$, then we must have $\sigma = \frac{1}{2}$.

■

□

3 Widder lambda function

We will prove similar results for Widder lambda function like in the case of Dirichlet eta function above.

The $\zeta(s)$ function is also defined through the Widder lambda function[8, 9] $\lambda(s)$ as follows:

$$\zeta(s) = (1 - \frac{3}{3^s})^{-1}\lambda(s) \tag{237}$$

$$= (1 - \frac{3}{3^s})^{-1}\sum_{n=0}^{+\infty}\left(\frac{1}{(3n+1)^s} + \frac{1}{(3n+2)^s} - \frac{2}{(3n+3)^s}\right) \tag{238}$$

We use the same notation from previous section. Typically, we will use the same notation for the sequences: A_n, B_n and X_n. Let's define the sequence Y_N as follows:

$$Y_N(s) = X_N(s) - \frac{1}{3^{s-1}}X_{\frac{N}{3}}(s) \tag{239}$$

$$= A_N - \frac{1}{3^{s-1}}A_{\frac{N}{3}} - i\left(B_N - \frac{1}{3^{s-1}}B_{\frac{N}{3}}\right) \tag{240}$$

$$= A_{N,N} - i\,B_{N,N} \tag{241}$$

Where

$$A_{N,N} = \sum_{k=1}^{\frac{N}{3}} \frac{\cos(\tau \ln(k)) - 3^{1-\sigma}\cos(\tau\big(\ln(3k)\big))}{k^\sigma} + \sum_{k=\frac{N}{3}+1}^{N} \frac{\cos(\tau \ln(k))}{k^\sigma} \tag{242}$$

And

$$B_{N,N} = \sum_{k=1}^{\frac{N}{3}} \frac{\sin(\tau \ln(k)) - 3^{1-\sigma}\sin(\tau\big(\ln(3k)\big))}{k^\sigma} + \sum_{k=\frac{N}{3}+1}^{N} \frac{\sin(\tau \ln(k))}{k^\sigma} \tag{243}$$

We also have:

$$\lim_{N\to\infty} Y_N = \lim_{N\to\infty} A_{N,N} - i\, B_{N,N} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} = \lambda(s) \tag{244}$$

Remark. *s is a zeta **zero**, $\zeta(s) = 0$, if and only if*

$$\lim_{N\to\infty} A_{N,N} = 0 \text{ and } \lim_{N\to\infty} B_{N,N} = 0 \tag{245}$$

Let's define the following sequences: For $N \geq 1$:

$$U_{n,N} = \begin{cases} \frac{\cos(\tau \ln(n)) - 3^{1-\sigma}\cos(\tau(\ln(3n)))}{n^\sigma}, & \text{if } n \leq \frac{N}{3} \\ \frac{\cos(\tau \ln(n))}{n^\sigma} & \text{if } n > \frac{N}{3} \end{cases} \tag{246}$$

And

$$V_{n,N} = \begin{cases} \frac{\sin(\tau \ln(n)) - 3^{1-\sigma}\sin(\tau(\ln(3n)))}{n^\sigma}, & \text{if } n \leq \frac{N}{3} \\ \frac{\sin(\tau \ln(n))}{n^\sigma} & \text{if } n > \frac{N}{3} \end{cases} \tag{247}$$

And

$$A_{n,N} = \begin{cases} \sum_{k=1}^{n} \frac{\cos(\tau \ln(k)) - 3^{1-\sigma}\cos(\tau \ln(3k))}{k^\sigma}, & \text{if } n \leq \frac{N}{3} \\ \sum_{k=1}^{\frac{N}{3}} \frac{\cos(\tau \ln(k)) - 3^{1-\sigma}\cos(\tau \ln(3k))}{k^\sigma} + \sum_{k=\frac{N}{3}+1}^{n} \frac{\cos(\tau \ln(k))}{k^\sigma} & \text{if } n > \frac{N}{3} \\ A_{N,N} & \text{if } n \geq N \end{cases} \tag{248}$$

And

$$B_{n,N} = \begin{cases} \sum_{k=1}^{n} \frac{\sin(\tau \ln(k)) - 3^{1-\sigma}\sin(\tau \ln(3k))}{k^\sigma}, & \text{if } n \leq \frac{N}{3} \\ \sum_{k=1}^{\frac{N}{3}} \frac{\sin(\tau \ln(k)) - 3^{1-\sigma}\sin(\tau \ln(3k))}{k^\sigma} + \sum_{k=\frac{N}{3}+1}^{n} \frac{\sin(\tau \ln(k))}{k^\sigma} & \text{if } n > \frac{N}{3} \\ B_{N,N} & \text{if } n \geq N \end{cases} \tag{249}$$

Let's develop further the squared norm of the series Y_N as follows:

$$\|Y_N\|^2 = A_{N,N}^2 + B_{N,N}^2 = \left(\sum_{n=1}^{N} U_{n,N}\right)^2 + \left(\sum_{n=1}^{N} V_{n,N}\right)^2 \tag{250}$$

So

$$\begin{aligned} A_{N,N}^2 &= \sum_{n=1}^{N} U_{n,N} + 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} U_n U_{k,N} && (251) \\ &= -\sum_{n=1}^{N} U_{n,N}^2 + 2\sum_{n=1}^{N} U_{n,N}\, A_{n,N} && (252) \end{aligned}$$

And the same calculation for $B_{N,N}$

$$B_{N,N}^2 = -\sum_{n=1}^{N} V_{n,N}^2 + 2\sum_{n=1}^{N} V_{n,N}\, B_{n,N} \tag{253}$$

Let's now define the sequences $(F_{n,N})$ and $(G_{n,N})$ as follows:

$$F_{n,N} = U_{n,N} A_{n,N}, \quad G_{n,N} = V_{n,N} B_{n,N} \tag{254}$$

Hence we have the new expression of square norm of Z_N^2:

$$\|Y_N\|^2 = 2\sum_{n=1}^{N} F_{n,N} + 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N}\Big(U_{n,N}^2 + V_{n,N}^2\Big) \tag{255}$$

which simplies even further to:

$$\sum_{n=1}^{N}\Big(U_{n,N}^2 + V_{n,N}^2\Big) = \sum_{n=1}^{\frac{N}{3}}\Big(U_{n,N}^2 + V_{n,N}^2\Big) + \sum_{n=\frac{N}{3}+1}^{N}\Big(U_{n,N}^2 + V_{n,N}^2\Big) \tag{256}$$

$$= \sum_{n=1}^{N}\frac{1}{n^{2\sigma}} + \sum_{n=1}^{\frac{N}{3}}\frac{3^{2-2\sigma} - 3^{2-\sigma}\Big(\cos(\tau\ln(3))\Big)}{n^{2\sigma}} \tag{257}$$

We have $2\sigma > 1$, hence the partial sums $\sum_{n\geq 1}^{N} U_{n,N}^2$ and $\sum_{n\geq 1}^{N} V_{n,N}^2$ converge absolutely. We have

$$A_{N,N}^2 = 2\sum_{n=1}^{N} F_{n,N} - \sum_{n=1}^{N} U_{n,N}^2, \quad B_{N,N}^2 = 2\sum_{n=1}^{N} G_{n,N} - \sum_{n=1}^{N} V_{n,N}^2 \tag{258}$$

Lemma 3.1. *If $\frac{1}{2} < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(\sum_{n=1}^{N} F_{n,N})_{N\geq 1}$:*

$$\sum_{n=1}^{3N} F_{n,3N} = U_0 - \frac{3^{1-\sigma}\sin(\tau\ln(3))\Big(1 - 3^{1-\sigma}\cos(\tau\ln(3))\Big)}{2\big((1-\sigma)^2 + \tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \tag{259}$$

Where U_0 is a real number.

Proof. So, let's define the sum $Z_{1,3N,3N}$ for $N \geq 1$:

$$Z_{1,3N,3N} = \sum_{n=1}^{3N} F_{n,3N} \tag{260}$$

$$= \underbrace{\sum_{n=1}^{N} F_{n,3N}}_{Z_{1,N,3N}} + \underbrace{\sum_{n=N+1}^{3N} F_{n,3N}}_{Z_{N+1,3N,3N}} \tag{261}$$

And $Z_{N+1,3N,3N}$:

$$Z_{N+1,3N,3N} = \sum_{n=N+1}^{3N} F_{n,3N} \tag{262}$$

$$= \sum_{n=N+1}^{3N} U_{n,3N}\, A_{n,3N} \tag{263}$$

$$= \sum_{n=N+1}^{3N} U_{n,3N}\Big(A_{N,3N} + \sum_{k=N+1}^{n} U_{k,3N}\Big) \tag{264}$$

$$= A_{N,3N}\Big(A_{3N,3N} - A_{N,3N}\Big) + \sum_{n=N+1}^{3N} U_{n,3N} \sum_{k=N+1}^{n} U_{k,3N} \tag{265}$$

And

$$\sum_{n=N+1}^{3N} U_{n,3N} \sum_{k=N+1}^{n} U_{k,3N} = \sum_{k=N+1}^{3N} U_{k,3N} \sum_{n=k}^{3N} U_{n,3N} \tag{266}$$

$$= \sum_{k=N+1}^{3N} U_{k,3N}\Big(A_{3N,3N} - A_{k-1,3N}\Big) \tag{267}$$

$$= \sum_{k=N+1}^{3N} U_{k,3N}\Big(A_{3N,3N} - A_{k,3N}\Big) + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{268}$$

$$= A_{3N,3N}\Big(A_{3N,3N} - A_{N,3N}\Big) - \sum_{k=N+1}^{3N} U_{k,3N} A_{k,3N} + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{269}$$

$$= A_{3N,3N}\Big(A_{3N,3N} - A_{N,3N}\Big) - Z_{N+1,3N,3N} + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{270}$$

Therefore

$$2\, Z_{N+1,3N,3N} = \Big(A_{3N,3N} + A_{N,3N}\Big)\Big(A_{3N,3N} - A_{N,3N}\Big) + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{271}$$

Therefore

$$Z_{1,3N,3N} = \frac{1}{2}\Big(A_{3N,3N} + A_{N,3N}\Big)\Big(A_{3N,3N} - A_{N,3N}\Big) + Z_{1,N,3N} + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{272}$$

We also have:

$$A_{3N,3N} = A_{N,3N} + A_{3N} - A_N \tag{273}$$

Therefore

$$Z_{1,3N,3N} = A_{N,3N}\Big(A_{3N} - A_N\Big) + \frac{1}{2}\Big(A_{3N} - A_N\Big)^2 + Z_{1,N,3N} + \sum_{n=N+1}^{3N} U_{n,3N}^2 \tag{274}$$

We also have:

$$Z_{1,N,3N} = \sum_{n=1}^{N} F_{n,3N} = \sum_{n=1}^{N} U_{n,3N}\, A_{n,3N} \tag{275}$$

For $n \leq N$, we have:

$$\begin{aligned}
A_{n,3N} &= \sum_{k=1}^{n} U_{k,3N} && (276)\\
&= \sum_{k=1}^{n} \frac{\cos(\tau \ln(k)) - 3^{1-\sigma}\cos(\tau \ln(3k))}{k^{\sigma}} && (277)\\
&= \big(1 - 3^{1-\sigma}\cos(\tau\ln(3))\big)A_n + 3^{1-\sigma}\sin(\tau\ln(3))B_n && (278)\\
&= c\,A_n + d\,B_n && (279)
\end{aligned}$$

Where

$$c = 1 - 3^{1-\sigma}\cos(\tau\ln(3)),\, d = 3^{1-\sigma}\sin(\tau\ln(3)) \tag{280}$$

From lemma 1.1 we have:

$$\begin{aligned}
A_n &= K(\sigma,\tau)\bigg(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\bigg) + \frac{\epsilon_n}{n^{\sigma}} && (281)\\
B_n &= K(\sigma,\tau)\bigg(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\bigg) + \frac{\gamma_n}{n^{\sigma}} && (282)
\end{aligned}$$

$$\begin{aligned}
A_{n,3N} &= c\,K(\sigma,\tau)\bigg(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\bigg) + \frac{\epsilon_n}{n^{\sigma}} && (283)\\
&\quad + d\,K(\sigma,\tau)\bigg(-\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^{\sigma-1}}\bigg) + \frac{\gamma_n}{n^{\sigma}} && (284)\\
&= a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}} && (285)
\end{aligned}$$

Where

$$a = K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big),\, b = K(\sigma,\tau)\Big(\frac{(1-\sigma)}{\tau}c - d\Big) \tag{286}$$

Therefore

$$\begin{aligned}
Z_{1,N,3N} &= \sum_{n=1}^{N} A_{n,3N}\Big(\big(1 - 3^{1-\sigma}\cos(\tau\ln(3))\big)\frac{\cos(\tau\ln(n))}{n^{\sigma}} + 3^{1-\sigma}\sin(\tau\ln(3))\frac{\sin(\tau\ln(n))}{n^{\sigma}}\Big) && (287)\\
&= \sum_{n=1}^{N}\bigg(a\,\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\,\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\bigg)\bigg(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\bigg) && (288)\\
&= \sum_{n=1}^{N} X_{n,3N} && (289)
\end{aligned}$$

Where

$$X_{n,3N} = \left(a\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^\sigma}\right)\left(c\frac{\cos(\tau\ln(n))}{n^\sigma} + d\frac{\sin(\tau\ln(n))}{n^\sigma}\right) \quad (290)$$

$$= \frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}} \quad (291)$$

Therefore:

$$Z_{1,N,3N} = \sum_{n=1}^{N}\left[\frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (292)$$

$$= \sum_{n=1}^{N}\left[\alpha\frac{1}{n^{2\sigma-1}} + \beta\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \gamma\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (293)$$

Also, from lemma 3.1 in [10], we can write:

$$Z_{1,N,3N} = \left(\frac{\alpha}{2(1-\sigma)} + \frac{\left(\beta\tau+\gamma(1-\sigma)\right)\sin(2\tau\ln(N)) + \left(\beta(1-\sigma)-\gamma\tau\right)\cos(2\tau\ln(N))}{2\tau^2+2(1-\sigma)^2}\right)N^{2-2\sigma} \quad (294)$$

$$+U_0 + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (295)$$

Where U_0 is a real number and:

$$\alpha = \frac{ad+bc}{2},\ \beta = \frac{bc-ad}{2},\ \gamma = \frac{ac+bd}{2} \quad (296)$$

We also have:

$$\sum_{n=N+1}^{3N} U_{n,3N}^2 = \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (297)$$

And

$$A_{N,3N} = a\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\gamma_N}{N^\sigma} \quad (298)$$

And

$$A_{3N} - A_N = K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(3N))}{(3N)^{\sigma-1}} + \frac{\sin(\tau\ln(3N))}{(3N)^{\sigma-1}}\right) + \mathcal{O}(\frac{1}{N^\sigma}) \quad (299)$$

$$-K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\right) + \mathcal{O}(\frac{1}{N^\sigma}) \quad (300)$$

$$= K(\sigma,\tau)\left(3^{1-\sigma}\sin(\tau\ln(3)) + \frac{1-\sigma}{\tau}\left(3^{1-\sigma}\cos(\tau\ln(3))-1\right)\right)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} \quad (301)$$

$$+K(\sigma,\tau)\left(\left(3^{1-\sigma}\cos(\tau\ln(3))-1\right) - \frac{1-\sigma}{\tau}3^{1-\sigma}\sin(\tau\ln(3))\right)\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma}) \quad (302)$$

$$= e\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma}) \quad (303)$$

Where

$$e = -K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big) = -a, f = K(\sigma,\tau)\Big(-\frac{(1-\sigma)}{\tau}c + d\Big) = -b \tag{304}$$

Therefore

$$\begin{aligned}
\big(A_{3N} - A_N\big)^2 &= \Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\Bigg]^2 && (305)\\
&= \frac{e^2+f^2}{2}\frac{1}{N^{2\sigma-2}} + \frac{f^2-e^2}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + ef\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (306)
\end{aligned}$$

And

$$\begin{aligned}
A_{N,3N}\big(A_{3N} - A_N\big) &= \Bigg(a\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\Bigg)\Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^\sigma})\Bigg] && (307)\\
&= \frac{ae+bf}{2}\frac{1}{N^{2\sigma-2}} + \frac{bf-ae}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + \frac{af+be}{2}\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (308)
\end{aligned}$$

Therefore

$$\begin{aligned}
& A_{N,3N}\big(A_{3N} - A_N\big) + \frac{1}{2}\big(A_{3N} - A_N\big)^2 && (309)\\
= & \Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big)\frac{1}{N^{2\sigma-2}} + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} && (310)\\
& +\Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (311)
\end{aligned}$$

Therefore

$$\begin{aligned}
Z_{1,3N,3N} &= U_0 + \Bigg(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau + \gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma) - \gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2 + 2(1-\sigma)^2}\Bigg)N^{2-2\sigma} && (312)\\
&+ \Bigg[\Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big) + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\cos(2\tau\ln(N)) && (313)\\
&+ \Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\sin(2\tau\ln(N))\Bigg]N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (314)\\
&= U_0 + \Big(U + V\sin(2\tau\ln(N)) + W\cos(2\tau\ln(N))\Big)N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) && (315)
\end{aligned}$$

Where

$$\begin{aligned}
U &= \frac{\alpha}{2(1-\sigma)} + \frac{e^2+f^2}{4} + \frac{ae+bf}{2} && (316)\\
V &= \frac{\beta\tau + \gamma(1-\sigma)}{2\tau^2 + 2(1-\sigma)^2} + \frac{af+be}{2} + \frac{ef}{2} && (317)\\
W &= \frac{\beta(1-\sigma) - \gamma\tau}{2\tau^2 + 2(1-\sigma)^2} + \frac{bf-ae}{2} + \frac{f^2-e^2}{4} && (318)
\end{aligned}$$

After simplification, we have:

$$U = 0, W = 0, V = -K(\sigma,\tau)\frac{dc}{2\tau} \tag{319}$$

And

$$Z_{1,3N,3N} = -K(\sigma,\tau)\frac{dc}{2\tau}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (320)$$

■ □

Lemma 3.2. *If $\frac{1}{2} < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero, we have the following asymptotic expansion for the sequence $(\sum_{n=1}^{N} G_{n,N})_{N\geq 1}$:*

$$\sum_{n=1}^{3N} G_{n,3N} = V_0 + \frac{3^{1-\sigma}\sin(\tau\ln(3))\Big(1-3^{1-\sigma}\cos(\tau\ln(3))\Big)}{2\Big((1-\sigma)^2+\tau^2\Big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (321)$$

Where V_0 is a real number.

Proof. So, let's define the sum $Z_{1,3N,3N}$ for $N \geq 1$:

$$Z_{1,3N,3N} = \sum_{n=1}^{3N} G_{n,3N} \quad (322)$$

$$= \underbrace{\sum_{n=1}^{N} G_{n,3N}}_{Z_{1,N,3N}} + \underbrace{\sum_{n=N+1}^{3N} G_{n,3N}}_{Z_{N+1,3N,3N}} \quad (323)$$

And $Z_{N+1,3N,3N}$:

$$Z_{N+1,3N,3N} = B_{N,3N}\Big(B_{3N,3N} - B_{N,3N}\Big) + \sum_{n=N+1}^{3N} V_{n,3N}\sum_{k=N+1}^{n} V_{k,3N} \quad (324)$$

And

$$\sum_{n=N+1}^{3N} V_{N,3N}\sum_{k=N+1}^{n} V_{k,3N} = B_{3N,3N}\Big(B_{3N,3N} - B_{N,3N}\Big) - Z_{N+1,3N,3N} + \sum_{n=N+1}^{3N} V_{N,3N}^2 \quad (325)$$

Therefore

$$2\,Z_{N+1,3N,3N} = \Big(B_{3N,3N} + B_{N,3N}\Big)\Big(B_{3N,3N} - B_{N,3N}\Big) + \sum_{n=N+1}^{3N} V_{N,3N}^2 \quad (326)$$

Therefore

$$Z_{1,3N,3N} = \frac{1}{2}\Big(B_{3N,3N} + B_{N,3N}\Big)\Big(B_{3N,3N} - B_{N,3N}\Big) + ZZ_{1,N,3N} + \sum_{n=N+1}^{3N} V_{N,3N}^2 \quad (327)$$

We also have:

$$B_{3N,3N} = B_{N,3N} + B_{3N} - B_N \quad (328)$$

Therefore

$$Z_{1,3N,3N} = B_{N,3N}\Big(B_{3N} - B_N\Big) + \frac{1}{2}\Big(B_{3N} - B_N\Big)^2 + ZZ_{1,N,3N} + \sum_{n=N+1}^{3N} V_{N,3N}^2 \quad (329)$$

We also have:

$$Z_{1,N,3N} = \sum_{n=1}^{N} G_{N,3N} = \sum_{n=1}^{N} V_{N,3N}\, B_{N,3N} \tag{330}$$

For $n \leq N$, we have:

$$\begin{aligned} B_{N,3N} &= \sum_{k=1}^{n} V_{k,3N} & (331) \\ &= \sum_{k=1}^{n} \frac{\sin(\tau \ln(k)) - 3^{1-\sigma}\sin(\tau \ln(3k))}{k^{\sigma}} & (332) \\ &= \big(1 - 3^{1-\sigma}\cos(\tau \ln(3))\big) B_n - 3^{1-\sigma}\sin(\tau \ln(3)) A_n & (333) \\ &= c\, A_n + d\, B_n & (334) \end{aligned}$$

Where

$$c = -3^{1-\sigma}\sin(\tau \ln(3)), d = 1 - 3^{1-\sigma}\cos(\tau \ln(3)) \tag{335}$$

Remark. *We note here that c and d are slightly different from the terms c and d in the proof of the lemma 3.1.*

From lemma 1.1 we have:

$$\begin{aligned} A_n &= K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{\sin(\tau \ln(n))}{n^{\sigma-1}}\right) + \frac{\epsilon_n}{n^{\sigma}} & (336) \\ B_n &= K(\sigma,\tau)\left(-\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau \ln(n))}{n^{\sigma-1}}\right) + \frac{\gamma_n}{n^{\sigma}} & (337) \end{aligned}$$

$$\begin{aligned} B_{N,3N} &= c\,K(\sigma,\tau)\left(\frac{(1-\sigma)}{\tau}\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{\sin(\tau \ln(n))}{n^{\sigma-1}}\right) + \frac{\epsilon_n}{n^{\sigma}} & (338) \\ &\quad + d\,K(\sigma,\tau)\left(-\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau \ln(n))}{n^{\sigma-1}}\right) + \frac{\gamma_n}{n^{\sigma}} & (339) \\ &= a\,\frac{\sin(\tau \ln(n))}{n^{\sigma-1}} + b\,\frac{\cos(\tau \ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}} & (340) \end{aligned}$$

Where

$$a = K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big), b = K(\sigma,\tau)\Big(\frac{(1-\sigma)}{\tau}c - d\Big) \tag{341}$$

Remark. *We note here that a and b have the same expressions like in the case of the lemma 3.1 but they are also different because the definition of c and d are different.*

Therefore

$$Z_{1,N,3N} = \sum_{n=1}^{N} B_{N,3N}\Big(\big(1-3^{1-\sigma}\sin(\tau\ln(3))\big)\frac{\cos(\tau\ln(n))}{n^{\sigma}} - 3^{1-\sigma}\cos(\tau\ln(3))\frac{\sin(\tau\ln(n))}{n^{\sigma}}\Big) \quad (342)$$

$$= \sum_{n=1}^{N}\left(a\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) \quad (343)$$

$$= \sum_{n=1}^{N} Y_{N,3N} \quad (344)$$

Where

$$Y_{N,3N} = \left(a\frac{\sin(\tau\ln(n))}{n^{\sigma-1}} + b\frac{\cos(\tau\ln(n))}{n^{\sigma-1}} + \frac{\gamma_n}{n^{\sigma}}\right)\left(c\frac{\cos(\tau\ln(n))}{n^{\sigma}} + d\frac{\sin(\tau\ln(n))}{n^{\sigma}}\right) \quad (345)$$

$$= \frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}} \quad (346)$$

Therefore

$$Z_{1,N,3N} = \sum_{n=1}^{N}\left[\frac{ad+bc}{2}\frac{1}{n^{2\sigma-1}} + \frac{bc-ad}{2}\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{ac+bd}{2}\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (347)$$

$$= \sum_{n=1}^{N}\left[\alpha\frac{1}{n^{2\sigma-1}} + \beta\frac{\cos(2\tau\ln(n))}{n^{2\sigma-1}} + \gamma\frac{\sin(2\tau\ln(n))}{n^{2\sigma-1}} + \frac{\gamma_n}{n^{2\sigma}}\right] \quad (348)$$

$$= \left(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau+\gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma)-\gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2+2(1-\sigma)^2}\right)N^{2-2\sigma} \quad (349)$$

$$+V_0 + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (350)$$

Where V_0 is a real number and:

$$\alpha = \frac{ad+bc}{2},\ \beta = \frac{bc-ad}{2},\ \gamma = \frac{ac+bd}{2} \quad (351)$$

$$\sum_{n=N+1}^{3N} V_{N,3N}^2 = \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (352)$$

$$B_{N,3N} = a\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{\gamma_N}{N^{\sigma}} \quad (353)$$

And

$$B_{3N} - B_N = K(\sigma,\tau)\Bigg(- \frac{\cos(\tau\ln(3N))}{(3N)^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(3N))}{(3N)^{\sigma-1}}\Bigg) + \mathcal{O}(\frac{1}{N^{\sigma}}) \quad (354)$$

$$-K(\sigma,\tau)\Bigg(- \frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \frac{(1-\sigma)}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\Bigg) + \mathcal{O}(\frac{1}{N^{\sigma}}) \quad (355)$$

$$= K(\sigma,\tau)\Bigg(\frac{1-\sigma}{\tau}3^{1-\sigma}\sin(\tau\ln(3)) + \Big(1 - 3^{1-\sigma}\cos(\tau\ln(3))\Big)\Bigg)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} \quad (356)$$

$$+K(\sigma,\tau)\Bigg(\frac{1-\sigma}{\tau}\Big(3^{1-\sigma}\cos(\tau\ln(3)) - 1\Big) + 3^{1-\sigma}\sin(\tau\ln(3))\Bigg)\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^{\sigma}}) \quad (357)$$

$$= e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^{\sigma}}) \quad (358)$$

Where

$$e = -K(\sigma,\tau)\Big(c + \frac{(1-\sigma)}{\tau}d\Big) = -a, f = K(\sigma,\tau)\Big(- \frac{(1-\sigma)}{\tau}c + d\Big) = -b \quad (359)$$

Therefore

$$\big(B_{3N} - B_N\big)^2 = \Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^{\sigma}})\Bigg]^2 \quad (360)$$

$$= \frac{e^2+f^2}{2}\frac{1}{N^{2\sigma-2}} + \frac{f^2-e^2}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + ef\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (361)$$

And

$$B_{N,3N}\big(B_{3N} - B_N\big) = \Bigg(a\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + b\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^{\sigma}})\Bigg)\Bigg[e\,\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} + f\,\frac{\cos(\tau\ln(N))}{N^{\sigma-1}} + \mathcal{O}(\frac{1}{N^{\sigma}})\Bigg] \quad (362)$$

$$= \frac{ae+bf}{2}\frac{1}{N^{2\sigma-2}} + \frac{bf-ae}{2}\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} + \frac{af+be}{2}\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (363)$$

Therefore

$$B_{N,3N}\big(B_{3N} - B_N\big) + \frac{1}{2}\big(B_{3N} - B_N\big)^2 \quad (364)$$

$$= \Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big)\frac{1}{N^{2\sigma-2}} + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\frac{\cos(2\tau\ln(N))}{N^{2\sigma-2}} \quad (365)$$

$$+\Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\frac{\sin(2\tau\ln(N))}{N^{2\sigma-2}} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (366)$$

Therefore, like in the proof of the lemma 1.2, we have:

$$Z_{1,3N,3N} = V_0 + \Bigg(\frac{\alpha}{2(1-\sigma)} + \frac{\Big(\beta\tau + \gamma(1-\sigma)\Big)\sin(2\tau\ln(N)) + \Big(\beta(1-\sigma) - \gamma\tau\Big)\cos(2\tau\ln(N))}{2\tau^2 + 2(1-\sigma)^2}\Bigg)N^{2-2\sigma} \quad (367)$$

$$+ \Bigg[\Big(\frac{ae+bf}{2} + \frac{e^2+f^2}{4}\Big) + \Big(\frac{bf-ae}{2} + \frac{f^2-e^2}{4}\Big)\cos(2\tau\ln(N)) \quad (368)$$

$$+ \Big(\frac{af+be}{2} + \frac{ef}{2}\Big)\sin(2\tau\ln(N))\Bigg]N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (369)$$

$$= V_0 + \Big(U + V\sin(2\tau\ln(N)) + W\cos(2\tau\ln(N))\Big)N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (370)$$

Where

$$U = \frac{\alpha}{2(1-\sigma)} + \frac{e^2+f^2}{4} + \frac{ae+bf}{2} \quad (371)$$

$$V = \frac{\beta\tau+\gamma(1-\sigma)}{2\tau^2+2(1-\sigma)^2} + \frac{af+be}{2} + \frac{ef}{2} \quad (372)$$

$$W = \frac{\beta(1-\sigma)-\gamma\tau}{2\tau^2+2(1-\sigma)^2} + \frac{bf-ae}{2} + \frac{f^2-e^2}{4} \quad (373)$$

After simplification, we have:

$$U = 0, W = 0, V = -K(\sigma,\tau)\frac{dc}{2\tau} \quad (374)$$

And

$$Z_{1,3N,3N} = -K(\sigma,\tau)\frac{dc}{2\tau}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (375)$$

■ □

Lemma 3.3. *Let's σ be such that $0 < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

- *if $\sigma \neq \frac{1}{2}$, then $\tau = \frac{k\pi}{\ln(3)}$ with $k \in \mathbb{N}^*$.*
- *if $\tau \neq \frac{k\pi}{\ln(3)}$ with $k \in \mathbb{N}^*$, then $\sigma = \frac{1}{2}$.*

Proof. Let's suppose $\sigma > \frac{1}{2}$. From the lemma 3.1, we have:

$$\sum_{n=1}^{3N} F_{N,3N} = U_0 - \frac{3^{1-\sigma}\sin(\tau\ln(3))\Big(1-3^{1-\sigma}\cos(\tau\ln(3))\Big)}{2\big((1-\sigma)^2+\tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (376)$$

We have $s = \sigma + i\tau$ is a zeta zero. Therefore, from equation (110), we have:

$$\lim_{N\to+\infty} A^2_{N,N} = 2\sum_{n=1}^{N} F_{n,N} - \sum_{n=1}^{N} U^2_{n,N} = 0 \quad (377)$$

Therefore as $\sigma > \frac{1}{2}$

$$\lim_{N\to+\infty}\sum_{n=1}^{N} F_{n,N} = \lim_{N\to+\infty}\frac{1}{2}\sum_{n=1}^{N} U^2_{n,N} < +\infty \quad (378)$$

Therefore, in the asymptotic expansion in the equation(230), we must have:

$$\sin(\tau\ln(3))\Big(1-3^{1-\sigma}\cos(\tau\ln(3))\Big) = 0 \quad (379)$$

Otherwise the sequences $(A^2_{N,N})_{N\geq1}$ and $(B^2_{N,N})_{N\geq1}$ will diverge. Therefore, from lemma 3.2, similarly we have:

$$\sum_{n=1}^{3N} G_{N,3N} = V_0 + \frac{3^{1-\sigma}\sin(\tau\ln(3))\Big(1-3^{1-\sigma}\cos(\tau\ln(3))\Big)}{2\big((1-\sigma)^2+\tau^2\big)}\sin(2\tau\ln(N))N^{2-2\sigma} + \mathcal{O}(\frac{1}{N^{2\sigma-1}}) \quad (380)$$

Case: $\tau \neq \frac{k\pi}{\ln(3)}$**,** $k \in \mathbb{N}^*$ In this case we have:

$$\cos(\tau \ln(3)) = \frac{1}{3^{1-\sigma}} \tag{381}$$

We also have $1 - \overline{s} = 1 - \sigma + i\tau = \tilde{\sigma} + i\tau$ also a zeta zero. Therefore, as $\tilde{\sigma} > \frac{1}{2}$, by the same method, we have:

$$\sin(\tau \ln(3))\Big(1 - 3^{1-\tilde{\sigma}} \cos(\tau \ln(3))\Big) = 0 \tag{382}$$

Therefore

$$\cos(\tau \ln(3)) = \frac{1}{3^{\sigma}} \tag{383}$$

Therefore by combining (381) and (383), we get:

$$3^{1-\sigma} = 3^{\sigma} \tag{384}$$

Hence a contradiction with $\sigma > \frac{1}{2}$. Therefore we conclude the following:

- if $\sigma \neq \frac{1}{2}$, then we must have $\tau = \frac{k\pi}{\ln(3)}$ with $k \in \mathbb{N}^*$.
- if $\tau \neq \frac{k\pi}{\ln(3)}$ with $k \in \mathbb{N}^*$, then we must have $\sigma = \frac{1}{2}$.

■

□

Lemma 3.4. *Let's σ be such that $0 < \sigma \leq 1$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\sigma = \frac{1}{2} \tag{385}$$

And the Riemann's Hypothesis is true.

Proof. Let's suppose $\sigma \neq \frac{1}{2}$. From the lemma 2.3, there exists a $k \in \mathbb{N}^*$ such that:

$$\tau = \frac{k\pi}{\ln(2)} \tag{386}$$

From the lemma 3.3, there exists a $k\prime \in \mathbb{N}^*$ such that:

$$\tau = \frac{k\prime\pi}{\ln(3)} \tag{387}$$

Therefore, by combining the equations (386) and (387), we get:

$$\frac{\ln(3)}{\ln(2)} = \frac{k\prime}{k} \tag{388}$$

We have $\frac{\ln(3)}{\ln(2)}$ is irrational. Because otherwise we will have:

$$2^{k\prime} = 3^k \tag{389}$$

Which means that 2 divides 3^k. And this is not possible as 2 and 3 are two different prime numbers.

Therefore τ cannot be of the form $\frac{k\pi}{\ln(2)}$ with $k \in \mathbb{N}^*$ or of the form $\frac{k\prime\pi}{\ln(3)}$ with $k\prime \in \mathbb{N}^*$. Therefore from the lemmas 2.3 and 3.3, we have:

$$\sigma = \frac{1}{2} \tag{390}$$

Hence, the Riemann's Hypothesis is true. ■

□

Remark. *We can use similar functions λ_p like Dirichlet eta function or Widder lambda function. For a prime p we define λ_p:*

$$\lambda_p(s) = \sum_{n=0}^{+\infty}\left(\sum_{i=1}^{p-1}\frac{1}{(pn+i)^s} - \frac{p-1}{(pn+p)^s}\right) \tag{391}$$

We have the following relationship between the zeta ζ function and the λ_p function:

$$\zeta(s) = (1 - \frac{p}{p^s})^{-1}\lambda_p(s) \tag{392}$$

3.1 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$*. In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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