A Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

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Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

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1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

*One Raffles Quay, North Tower. 048583 Singapore. Email: charaf.chatbi@gmail.com.
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Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\big(\frac{\pi s}{2}\big)\Gamma(1-s)\zeta(1-s) \tag{4}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0$, $\forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1 - s) = \zeta(\bar{s}) = \zeta(1 - \bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{5}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{6}$$

Or in other terms:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma + i\tau + 1}}dx \tag{7}$$

[1]This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\big(\frac{\pi s}{2}\big)\Gamma(s)\zeta(s) \tag{3}$$

Let's denote the following functions:

$$\epsilon(x) = \{x\} \tag{8}$$
$$K(\sigma,\tau) = \frac{\tau}{(1-\sigma)^2+\tau^2} \tag{9}$$

To continue, we will prove the following lemmas.

Lemma 1.1. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define two integrals $I(a,\sigma,\tau)$ and $J(a,\sigma,\tau)$ as follows:*

$$I(a,\sigma,\tau) = \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx \tag{10}$$
$$J(a,\sigma,\tau) = \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx \tag{11}$$

Therefore

$$I(a,\sigma,\tau) = K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{12}$$
$$J(a,\sigma,\tau) = K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{13}$$

Proof. Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's take $a > 1$.

$$\begin{aligned} I(a,\sigma,\tau) &= \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx && (14)\\ &= \int_0^{\ln(a)} \sin(\tau x)e^{(1-\sigma)x}dx && (15)\\ &= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (16) \end{aligned}$$

And the same for $J(a,\sigma,\tau)$ for $a > 0$:

$$\begin{aligned} J(a,\sigma,\tau) &= \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx && (17)\\ &= \int_0^{\ln(a)} \cos(\tau x)e^{(1-\sigma)x}dx && (18)\\ &= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (19) \end{aligned}$$

□

Lemma 1.2. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions ϵ_n and $\overline{\epsilon}_n$ such that $\epsilon_0(x) = \overline{\epsilon}_0(x) = \{x\}$ and for each $n \geq 1$:*

$$\epsilon_{n+1}(x) = \frac{1}{x}\int_0^x dt\, \epsilon_n(t) \tag{20}$$

$$\overline{\epsilon}_{n+1}(x) = \frac{1}{x}\int_1^x dt\, \overline{\epsilon}_n(t) \tag{21}$$

$$\epsilon(x) = \sum_{k=0}^{+\infty} \epsilon_k(x) \tag{22}$$

Therefore:

1. *For each $n \geq 1$:*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{1+s}} = -\frac{1}{(1-s)\,2^n} \tag{23}$$

2. *For each n:*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{2-s}} = -\frac{1}{s\,2^n} \tag{24}$$

3. *For each $x \geq 1$:*

$$\epsilon_n(x) = \overline{\epsilon}_n(x) + \frac{1}{2^n\, x}\sum_{k=0}^{n-1} \frac{2^k \ln^k(x)}{k!} \tag{25}$$

4. *For each n and $x \geq 1$:*

$$\overline{\epsilon_n}(x) = \frac{1}{(n-1)!}\frac{1}{x}\int_1^x dt\, \overline{\epsilon_0}(t)\big(\ln(\frac{x}{t})\big)^{n-1} \tag{26}$$

5. *For each $x \geq 1$:*

$$\lim_{n \to +\infty} 2^n\, \epsilon_n(x) = x \tag{27}$$

6. *For each $x \geq 1$:*

$$0 \leq 2^n \epsilon_n(x) \leq x \tag{28}$$

7. *For each $x \geq 0$:*

$$\epsilon(x) = \epsilon_0(x) + \int_0^x dt\, \frac{\epsilon_0(t)}{t} \tag{29}$$

Proof. We have $s = \sigma + i\tau$ a *zeta* zero. We denote $A(s)$ the integral of the equation (6):

$$A(s) = \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} = -\frac{1}{(1-s)} \tag{30}$$

We use the integration by parts to write the following:

$$\begin{aligned}
A(s) &= \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} && (31)\\
&= \left[\frac{1}{x^{1+s}} \int_0^x dt\, \epsilon_0(t)\right]_1^{+\infty} + (1+s)\int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} && (32)\\
&= -\int_0^1 dx\, \epsilon_0(x) + (1+s)\int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} && (33)\\
&= -\int_0^1 dx\, \epsilon_0(x) - (1+s)\int_0^1 dx\, \epsilon_1(x) + (1+s)^2 \int_1^{+\infty} dx \frac{\epsilon_2(x)}{x^{1+s}} && (34)\\
\ldots & && (35)\\
&= -\sum_{k=0}^{n} (1+s)^k \int_0^1 dx\, \epsilon_k(x) + (1+s)^{n+1}\int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{2+s}} && (36)
\end{aligned}$$

We could do the above integration by parts. Because the functions $x \to \frac{\epsilon_k(x)}{x^{1+s}}$ are piecewise continuous[2] and integrable over $[1, +\infty)$ as they are dominated by the function $x \to \frac{1}{x^{1+\sigma}}$ that is integrable over $[1, +\infty)$. In fact, we can prove that the functions ϵ_k are non-negative and bounded by 1 as we can prove by recurrence that for each k for each $x > 0$ that:

$$0 < \epsilon_{k+1}(x) = \frac{1}{x}\int_0^x dt\, \epsilon_k(t) \leq \frac{1}{x}\int_0^x 1\, dt = 1 \tag{37}$$

This can be proven by recurrence using the fact that it is true for the initial case as we have for each $x \geq 0$:

$$0 \leq \epsilon_0(x) = \{x\} \quad \leq \quad 1 \tag{38}$$

Now, we need to calculate the integrals I_k for $k \geq 0$:

$$I_k \quad = \quad \int_0^1 dx\, \epsilon_k(x) \tag{39}$$

For $x \in (0, 1)$, we have:

$$\epsilon_0(x) = x \tag{40}$$

And

$$\psi_1(x) = \frac{x}{2} \tag{41}$$

Therefore, we can write for each k for $x \in (0, 1)$:

$$\epsilon_k(x) = \frac{x}{2^k} \tag{42}$$

Therefore, for each $k \geq 1$:

$$I_k = \frac{1}{2^{k+1}} \tag{43}$$

[2] All the functions $x \to \frac{\epsilon_k(x)}{x^{1+s}}$ are continuous except when $k = 0$ as the function ϵ_0 is piecewise continuous.

Therefore, we can conclude:

$$A(s) = -\frac{1}{(1-s)} + (1+s)^{n+1}\left[\int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{1+s}} + \frac{1}{(1-s)\,2^{n+1}}\right] \quad (44)$$

Since $1+s \neq 0$, we conculde the point 1. The point 2 can be proved the same way by replacing s by $1-s$. ■

The point 3) can be proved by recurrence. For $n=0$. We have $\epsilon_0(x) = \overline{\epsilon_0}(x)$. Let's assume that it is true till n and let's prove for $n+1$. We have:

$$\begin{aligned}
\epsilon_{n+1}(x) &= \frac{1}{x}\int_0^1 dt\, \epsilon_n(t) + \frac{1}{x}\int_1^x dt\, \epsilon_n(t) && (45)\\
&= \frac{1}{x\,2^{n+1}} + \frac{1}{x\,2^n}\int_1^x \frac{dt}{t}\sum_{k=0}^{n-1}\frac{2^k \ln^k(t)}{k!} + \frac{1}{x}\int_1^x dt\, \overline{\epsilon_n}(t) && (46)\\
&= \frac{1}{x\,2^{n+1}} + \frac{1}{x\,2^{n+1}}\sum_{k=0}^{n-1}\frac{2^{k+1}\ln^{k+1}(x)}{(k+1)!} + \overline{\epsilon_{n+1}}(x) && (47)\\
&= \frac{1}{x\,2^{n+1}}\sum_{k=0}^{n}\frac{2^k \ln^k(x)}{(k)!} + \overline{\epsilon_{n+1}}(x) && (48)
\end{aligned}$$

■

Let's prove the 4^{th} point. We proceed by recurrence. For $n=1$, we retrieve the definition of $\overline{\epsilon_1}(x)$. Let's assume that it is true up to n and let's prove it for $n+1$. We have thanks to the integral order change:

$$\begin{aligned}
\overline{\epsilon}_{n+1}(x) &= \frac{1}{x}\int_1^x dt\, \overline{\epsilon}_n(t) && (49)\\
&= \frac{1}{(n-1)!}\frac{1}{x}\int_1^x \frac{dt}{t}\int_1^t ds\, \overline{\epsilon}_0(s)\big(\ln(\frac{t}{s})\big)^{n-1} && (50)\\
&= \frac{1}{(n-1)!}\frac{1}{x}\int_1^x ds\, \overline{\epsilon}_0(s)\int_s^x \frac{dt}{t}\big(\ln(\frac{t}{s})\big)^{n-1} && (51)\\
&= \frac{1}{n!}\frac{1}{x}\int_1^x ds\, \overline{\epsilon}_0(s)\big(\ln(\frac{x}{s})\big)^{n} && (52)
\end{aligned}$$

And this proves our point of the lemma. ■

Let's now prove the last point. Let's take $x > 1$. Thanks to d'Alembert's criterion, we have for each $s \in [1, x]$, $\lim_{n\to+\infty} \frac{2^n\left(\ln(\frac{x}{s})\right)^n}{n!} = 0$. From the point (4) of this lemma, we apply the dominated convergence theorem to prove that:

$$\lim_{n\to+\infty} 2^n\, \overline{\epsilon}_n(x) = \frac{1}{x}\int_1^x \lim_{n\to+\infty}\frac{2^n\big(\ln(\frac{x}{s})\big)^n}{n!}\overline{\epsilon}_0(s)ds = 0 \quad (53)$$

From point (3), we can conclude that:

$$\lim_{n\to+\infty} 2^n\,\epsilon_n(x) = \lim_{n\to+\infty}\left(2^n\,\overline{\epsilon}_n(x) + \frac{1}{x}\sum_{k=0}^{n-1}\frac{2^k\ln^k(x)}{k!}\right) \tag{54}$$

$$= \lim_{n\to+\infty}\frac{1}{x}\sum_{k=0}^{n-1}\frac{\ln^k(x^2)}{k!} \tag{55}$$

$$= \frac{1}{x}\exp(\ln(x^2)) = x \tag{56}$$

Hence the proof of the point (5) of the lemma. Let's now prove the point (6). We have for each $x > 0$:

$$0 \le \epsilon_0(x) \le x \tag{57}$$

Therefore by integrating the equation above we get:

$$0 \le \frac{1}{x}\int_0^x dt\,\epsilon_0(t) \le \frac{1}{x}\int_0^x dt\,t \tag{58}$$

Therefore

$$0 \le \epsilon_1(x) \le \frac{x}{2} \tag{59}$$

By recurrence, we easily conclude that for each $n \ge 1$:

$$0 \le \epsilon_n(x) \le \frac{x}{2^n} \tag{60}$$

$\blacksquare$

Let's now prove the last point. From point (6), we conclude that the function $\epsilon(x)$ is well defined over $[0, +\infty)$ thanks to the absolute convergence of the series $\sum_{n\ge 0}\epsilon_n(x)$ (thanks to the point 6 above). Moreover, one can remark that the function $\epsilon(x)$ verifies the following differential equation for $x > 0$:

$$\epsilon(x) = \epsilon_0(x) + \frac{1}{x}\int_0^x dt\,\epsilon(t) \tag{61}$$

We solve for this differential equation and conclude the point (7). $\blacksquare$ $\square$

Lemma 1.3. *Let's consider two variables σ and τ such that $0 < \sigma \le \frac{1}{2}$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions $E_{n,\sigma}(x)$, $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$ and $H_{n,\sigma}(x)$ over $[1, +\infty)$ for each $n \ge 0$ as follows:*

$$E_{n,\sigma}(x) = \int_1^x dt\,\frac{\sin(\tau\ln(t))}{t^{1+\sigma}}\,\epsilon_n(t) \tag{62}$$

$$F_{n,\sigma}(x) = \int_1^x dt\,\frac{\cos(\tau\ln(t))}{t^{1+\sigma}}\,\epsilon_n(t) \tag{63}$$

$$G_{n,\sigma}(x) = \big(E_{n,\sigma}(x)\big)^2 + \big(F_{n,\sigma}(x)\big)^2 \tag{64}$$

$$H_{n,\sigma}(x) = G_{n,1-\sigma}(x) - G_{n,\sigma}(x) \tag{65}$$

Let's denote $a_0 = e^{\frac{3}{2\tau}}$ and $b_0 = e^{\frac{\pi}{2\tau}}$.

1. *For each $n \geq 0$ there exists $x_n > b_0$ such as:*

$$H_{n,\sigma}(x_n) = 0 \tag{66}$$

2. *For each $n \geq 0$, there exists $x_n > b_0$ such as:*

$$\sum_{k=n+1}^{+\infty} H_{k,\sigma}(x_n) = 0 \tag{67}$$

In other terms:

$$\sum_{k=n+1}^{+\infty} G_{k,1-\sigma}(x_n) = \sum_{k=n+1}^{+\infty} G_{k,\sigma}(x_n) \tag{68}$$

Proof. Let's prove the first point. Let's take $n \geq 0$ an integer. Since $\sigma \leq \frac{1}{2}$. We have for each $x > 1$: $\frac{1}{x^{1+\sigma}} > \frac{1}{x^{2-\sigma}}$. Therefore

$$E_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_n(t) > 0 \tag{69}$$

$$F_{n,\sigma}(b_0) = \int_1^{e^{\frac{\pi}{2\tau}}} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_n(t) > 0 \tag{70}$$

$$E_{n,\sigma}(b_0) > E_{n,1-\sigma}(b_0) > 0 \tag{71}$$

$$F_{n,\sigma}(b_0) > F_{n,1-\sigma}(b_0) > 0 \tag{72}$$

Therefore

$$\big(E_{n,\sigma}(b_0)\big)^2 > \big(E_{n,1-\sigma}(b_0)\big)^2 \tag{73}$$

$$\big(F_{n,\sigma}(b_0)\big)^2 > \big(F_{n,1-\sigma}(b_0)\big)^2 \tag{74}$$

And therefore

$$G_{n,\sigma}(b_0) > G_{n,1-\sigma}(b_0) \tag{75}$$

$$H_{n,\sigma}(b_0) < 0 \tag{76}$$

We also have from lemma 1.2:

$$\lim_{x\to+\infty} G_{n,\sigma}(x) = \Big(\int_1^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_n(t)\Big)^2 + \Big(\int_1^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_n(t)\Big)^2 \tag{77}$$

$$= \frac{1}{2^{2n}\|1-s\|^2} \tag{78}$$

$$\lim_{x\to+\infty} G_{n,1-\sigma}(x) = \Big(\int_1^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{t^{2-\sigma}}\, \epsilon_n(t)\Big)^2 + \Big(\int_1^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{t^{2-\sigma}}\, \epsilon_n(t)\Big)^2 \tag{79}$$

$$= \frac{1}{2^{2n}\|s\|^2} \tag{80}$$

Therefore

$$\lim_{x\to+\infty} H_{n,\sigma}(x) = \frac{1}{2^{2n}}\Big(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2}\Big) > 0 \tag{81}$$

Since the function $H_{n,\sigma}$ is continuous over $[1,+\infty)$. From (73) and (78) and thanks to the Mean value theorem, we can conclude that there exists an $x_n > b_0$ such that:

$$H_{n,\sigma}(x_n) = 0 \tag{82}$$

■

For the point (2), we proceed the same way as above. We prove it for the case $n = 0$, and the proof remains the same for the general case. From the equation (76), we can deduce that:

$$\sum_{k=0}^{+\infty} H_{k,\sigma}(b_0) \leq 0 \tag{83}$$

Let's define now the sequence of functions $s(n,\sigma,x)$ such that for $n \geq 0$ and $x \geq 1$:

$$s(n,\sigma,x) = \sum_{k=0}^{n} G_{k,\sigma}(x) \tag{84}$$

In order to use the Mean Value theorem like in the point 1) of this lemma, 1) we will prove that the function $s(+\infty,\sigma,x)$ $\Big($resp. $s(+\infty, 1-\sigma, x)\Big)$ is continuous over $[1,+\infty)$. and 2) we will prove that:

$$\lim_{x\to+\infty} \sum_{k=0}^{+\infty} H_{k,\sigma}(x) = \frac{4}{3}\Big(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2}\Big) > 0 \tag{85}$$

To prove the two points we can use the term-by-term integration theorem (or the Dominated convergence theorem applied on integral series). But because of the square terms in $G_{k,\sigma}(x) = E_{k,\sigma}^2(x) + F_{k,\sigma}^2(x)$, we are going to prove the two results directly using the Double Sequences[8]. Let's take a sequence $(y_n \geq 1)$ such that it has a limit y. y can be finite or $+\infty$. The proof is similar for both cases. So, let's consider the double sequence $s(n,\sigma,m)$ such that:

$$s(n,\sigma,m) = s(n,\sigma,y_m) \tag{86}$$

We are going to prove that the double sequence $(s(n,\sigma,m))$ is a Cauchy sequence. So, let's take (p,q) and (n,m) and N such that $p > n \geq N$ and $q > m \geq N$.

$$s(p,\sigma,q) - s(n,\sigma,m) = \sum_{k=0}^{p} G_{k,\sigma}(y_q) - \sum_{k=0}^{n} G_{k,\sigma}(y_m) \tag{87}$$

$$= \sum_{k=0}^{n}\Big(G_{k,\sigma}(y_q) - G_{k,\sigma}(y_m)\Big) + \sum_{k=n+1}^{p} G_{k,\sigma}(y_q) \tag{88}$$

And

$$
\begin{aligned}
G_{k,\sigma}(y_q) - G_{k,\sigma}(y_m) &= E_{k,\sigma}^2(y_q) - E_{k,\sigma}^2(y_m) + F_{k,\sigma}^2(y_q) - F_{k,\sigma}^2(y_m) && (89)\\
&= \Big(\int_{y_m}^{y_q} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t)\Big)\Big(E_{k,\sigma}(y_q) + E_{k,\sigma}(y_m)\Big) && (90)\\
&+ \Big(\int_{y_m}^{y_q} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t)\Big)\Big(F_{k,\sigma}(y_q) + F_{k,\sigma}(y_m)\Big) && (91)
\end{aligned}
$$

We simplify the notations as follows:

$$
\begin{aligned}
a_k &= G_{k,\sigma}(y_q) && (92)\\
b_k &= G_{k,\sigma}(y_m) && (93)\\
u_k &= \int_{y_m}^{y_q} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t) && (94)\\
v_k &= E_{k,\sigma}(y_q) + E_{k,\sigma}(y_m) && (95)\\
w_k &= \int_{y_m}^{y_q} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t) && (96)\\
z_k &= F_{k,\sigma}(y_q) + F_{k,\sigma}(y_m) && (97)
\end{aligned}
$$

Therefore

$$
a_k - b_k = u_k v_k + w_k z_k \tag{98}
$$

Therefore

$$
s(p,\sigma,q) - s(n,\sigma,m) = \sum_{k=0}^{n}\Big(u_k v_k + w_k z_k\Big) + \sum_{k=n+1}^{p} G_{k,\sigma}(y_q) \tag{99}
$$

And thanks to Cauchy–Schwarz inequality, we can write:

$$
|s(p,\sigma,q) - s(n,\sigma,m)| \leq \sum_{k=0}^{n} |u_k v_k| + \sum_{k=0}^{n} |w_k z_k| + \sum_{k=n+1}^{p} G_{k,\sigma}(y_q) \tag{100}
$$

$$
\sum_{k=0}^{n} |u_k v_k| + \sum_{k=0}^{n} |w_k z_k| \leq \left(\sum_{k=0}^{n} u_k^2\right)\left(\sum_{k=0}^{n} v_k^2\right) + \left(\sum_{k=0}^{n} w_k^2\right)\left(\sum_{k=0}^{n} z_k^2\right) \tag{101}
$$

And

$$
\begin{aligned}
\sum_{k=0}^{n} u_k^2 &\leq \Big(\sum_{k=0}^{n} |u_k|\Big)^2 && (102)\\
&\leq \Big(\sum_{k=0}^{n} \int_{y_m}^{y_q} dt\, \frac{\epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (103)\\
&\leq \Big(\int_{y_m}^{y_q} dt\, \frac{\sum_{k=0}^{n} \epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (104)\\
&\leq \Big(\int_{y_m}^{y_q} dt\, \frac{\sum_{k=0}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (105)\\
&\leq \gamma_{q,m}^2 = \Big(\int_{y_m}^{y_q} dt\, \frac{\epsilon(t)}{t^{1+\sigma}}\Big)^2 && (106)
\end{aligned}
$$

Where

$$\epsilon(x) = \sum_{k=0}^{+\infty} \epsilon_k(x) \tag{107}$$

And the same for (w_k):

$$\sum_{k=0}^{n} w_k^2 \leq \gamma_{q,m}^2 = \Big(\int_{y_m}^{y_q} dt \, \frac{\epsilon(t)}{t^{1+\sigma}} \Big)^2 \tag{108}$$

Therefore

$$\begin{aligned} |s(p,\sigma,q) - s(n,\sigma,m)| &\leq \gamma_{q,m}^2 \sum_{k=0}^{n} \left(v_k^2 + z_k^2\right) + \sum_{k=n+1}^{p} G_{k,\sigma}(y_q) && (109) \\ &\leq 2\gamma_{q,m}^2 \sum_{k=0}^{n} \Big(G_{k,\sigma}(y_q) + G_{k,\sigma}(y_m) \Big) + \sum_{k=n+1}^{p} G_{k,\sigma}(y_q) && (110) \end{aligned}$$

For any $x \geq 1$, we can write:

$$\begin{aligned} \sum_{k=0}^{n} G_{k,\sigma}(x) &= \sum_{k=0}^{n} \Big(E_{k,\sigma}^2(x) + F_{k,\sigma}^2(x) \Big) && (111) \\ &\leq \sum_{k=0}^{n} \Big(\int_1^x dt \, \frac{|\sin(\tau \ln(t))|}{t^{1+\sigma}} \, \epsilon_k(t) + \int_1^x dt \, \frac{|\cos(\tau \ln(t))|}{t^{1+\sigma}} \, \epsilon_k(t) \Big)^2 && (112) \\ &\leq \sum_{k=0}^{n} \Big(2 \int_1^x dt \, \frac{\epsilon_k(t)}{t^{1+\sigma}} \Big)^2 && (113) \\ &\leq 4 \Big(\int_1^x dt \, \frac{\sum_{k=0}^{n} \epsilon_k(t)}{t^{1+\sigma}} \Big)^2 && (114) \\ &\leq 4\gamma^2 = 4 \Big(\int_1^{+\infty} dt \, \frac{\epsilon(t)}{t^{1+\sigma}} \Big)^2 && (115) \end{aligned}$$

And

$$\begin{aligned} \sum_{k=n+1}^{p} G_{k,\sigma}(x) &= \sum_{k=n+1}^{p} \Big(E_{k,\sigma}^2(x) + F_{k,\sigma}^2(x) \Big) && (116) \\ &\leq \sum_{k=n+1}^{p} \Big(\int_1^x dt \, \frac{|\sin(\tau \ln(t))|}{t^{1+\sigma}} \, \epsilon_k(t) + \int_1^x dt \, \frac{|\cos(\tau \ln(t))|}{t^{1+\sigma}} \, \epsilon_k(t) \Big)^2 && (117) \\ &\leq \sum_{k=n+1}^{p} \Big(2 \int_1^x dt \, \frac{\epsilon_k(t)}{t^{1+\sigma}} \Big)^2 && (118) \\ &\leq 4 \Big(\int_1^x dt \, \frac{\sum_{k=n+1}^{p} \epsilon_k(t)}{t^{1+\sigma}} \Big)^2 && (119) \\ &\leq 4\gamma_n^2 = 4 \Big(\int_1^{+\infty} dt \, \frac{\sum_{k=n+1}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}} \Big)^2 && (120) \end{aligned}$$

Therefore

$$|s(p,\sigma,q) - s(n,\sigma,m)| \leq 16\, \gamma^2 \, \gamma_{q,m}^2 + 4\, \gamma_n^2 \tag{121}$$

From lemma 1.2, we have for each $x \geq 1$:

$$0 < \epsilon(x) \leq 2 + \ln(x) \tag{122}$$

And also, for each $x \geq 1$:

$$\lim_{n \to +\infty} \sum_{k=n+1}^{+\infty} \epsilon_k(x) = 0 \tag{123}$$

Since we have

$$\int_1^{+\infty} dt \, \frac{\epsilon(t)}{t^{1+\sigma}} < \int_1^{+\infty} dt \, \frac{2 + \ln(t)}{t^{1+\sigma}} < +\infty \tag{124}$$

Therefore, thanks to the Dominated Convergence theorem, we conclude that:

$$\lim_{n \to +\infty} \gamma_n = 0 \tag{125}$$

And also

$$\lim_{q,m \to +\infty} \gamma_{q,m} = 0 \tag{126}$$

Therefore, from (121), we conclude that the double sequence $(\mathrm{s}(n, \sigma, m))$ is a Cauchy sequence. And therefore the double sequence $(\mathrm{s}(n, \sigma, m))$ converges and the iterated limits are equal[8]. Hence the continuity of the function $s(+\infty, \sigma, x)$ $\Big(\text{resp. } s(+\infty, 1 - \sigma, x)\Big)$ over $[1, +\infty)$ and moreover we have for each $n \geq 0$:

$$\lim_{x \to +\infty} \sum_{k=n}^{+\infty} G_{k,\sigma}(x) = \sum_{k=n}^{+\infty} G_{k,\sigma}(+\infty) \tag{127}$$

$$\lim_{x \to +\infty} \sum_{k=n}^{+\infty} G_{k,1-\sigma}(x) = \sum_{k=n}^{+\infty} G_{k,1-\sigma}(+\infty) \tag{128}$$

Also, from lemma 1.2, we have for each $k \geq 0$:

$$G_{k,\sigma}(+\infty) = \frac{1}{4^k \, \|1 - s\|^2} \tag{129}$$

$$G_{k,1-\sigma}(+\infty) = \frac{1}{4^k \, \|s\|^2} \tag{130}$$

Therefore

$$\lim_{x \to +\infty} \sum_{k=n}^{+\infty} G_{k,\sigma}(x) = \frac{1}{3} \frac{1}{4^{n-1} \, \|1 - s\|^2} \tag{131}$$

$$\lim_{x \to +\infty} \sum_{k=n}^{+\infty} G_{k,1-\sigma}(x) = \frac{1}{3} \frac{1}{4^{n-1} \, \|s\|^2} \tag{132}$$

Therefore

$$\lim_{x \to +\infty} \sum_{k=n}^{+\infty} H_{k,\sigma}(x) = \frac{1}{3} \frac{1}{4^{n-1}} \Big(\frac{1}{\|s\|^2} - \frac{1}{\|1 - s\|^2} \Big) > 0 \tag{133}$$

Since the function $\sum_{k=n+1}^{+\infty} H_{k,\sigma}(x)$ is continuous over $[1, +\infty)$. From (83) and (85) and thanks to the Mean value theorem, we can conclude that there exists an $x_n > b_0$ such that:

$$\sum_{k=n+1}^{+\infty} H_{n,\sigma}(x_n) \quad = \quad 0 \tag{134}$$

■

□

Lemma 1.4. *Let's consider two variables σ and τ such that $0 < \sigma < \frac{1}{2}$ and $\tau \geq 1$ such that $s = \sigma + i\tau$. Let's consider the sequence of functions $E_{n,\sigma}(x)$, $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$ and $H_{n,\sigma}(x)$ defined in the previous lemma 1.3*

1. *For each $a > 1$:*

$$4^n \, G_{n,\sigma}(a) \quad \sim_{n\to+\infty} \quad f_a(1-\sigma) \tag{135}$$

Where f_a is a strictly increasing function defined as follows:

$$f_a(\sigma) \quad = \quad \frac{a^{2\sigma} - 2a^{\sigma}\cos(\tau \ln(a)) + 1}{\sigma^2 + \tau^2} \tag{136}$$

2. *For each $a > 1$:*

$$4^n \, H_{n,\sigma}(a) \quad \sim_{n\to+\infty} \quad f_a(\sigma) - f_a(1-\sigma) \tag{137}$$

3. *For $a > 1$, for large enough integer n:*

$$4^n \, H_{n,\sigma}(a) \quad < \quad 0 \tag{138}$$

In other terms:

$$G_{n,\sigma}(a) \quad > \quad G_{n,1-\sigma}(a) \tag{139}$$

Proof. Like in lemma 1.3, let's denote $a_0 = e^{\frac{3}{2\tau}}$ and $b_0 = e^{\frac{\pi}{2\tau}}$. We have $\sigma < \frac{1}{2}$. From lemma 1.2, we have:

$$\lim_{n\to+\infty} 2^n \, \epsilon_n(x) \quad = \quad x \tag{140}$$

Therefore for each $x \geq 1$:

$$\lim_{n\to+\infty} \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \, 2^n \, \epsilon_n(x) \quad = \quad \frac{\sin(\tau \ln(x))}{x^{\sigma}} \tag{141}$$

$$\lim_{n\to+\infty} \frac{\cos(\tau \ln(x))}{x^{1+\sigma}} \, 2^n \, \epsilon_n(x) \quad = \quad \frac{\cos(\tau \ln(x))}{x^{\sigma}} \tag{142}$$

$$\lim_{n\to+\infty} \frac{\sin(\tau \ln(x))}{x^{2-\sigma}} \, 2^n \, \epsilon_n(x) \quad = \quad \frac{\sin(\tau \ln(x))}{x^{1-\sigma}} \tag{143}$$

$$\lim_{n\to+\infty} \frac{\cos(\tau \ln(x))}{x^{2-\sigma}} \, 2^n \, \epsilon_n(x) \quad = \quad \frac{\cos(\tau \ln(x))}{x^{1-\sigma}} \tag{144}$$

Let's take $a \geq a_0$. Thanks to the Dominated Convergence Theorem applied on the sequence of functions in $(141-144)$ over the bounded interval $[1, a]$, we can write:

$$\lim_{n\to+\infty} 2^{2n}\, G_{n,\sigma}(a) = \Big(\int_1^a dt\, \frac{\sin(\tau\ln(t))}{t^\sigma}\Big)^2 + \Big(\int_1^a dt\, \frac{\cos(\tau\ln(t))}{t^\sigma}\Big)^2 \quad (145)$$

$$\lim_{n\to+\infty} 2^{2n}\, G_{n,1-\sigma}(a) = \Big(\int_1^a dt\, \frac{\sin(\tau\ln(t))}{t^{1-\sigma}}\Big)^2 + \Big(\int_1^a dt\, \frac{\cos(\tau\ln(t))}{t^{1-\sigma}}\Big)^2 \quad (146)$$

Thanks to lemma 1.1, we can calculate the integrals in the equations $(145-146)$, we conclude:

$$2^{2n}\, G_{n,\sigma}(a) \sim_{n\to+\infty} f_a(1-\sigma) = \frac{a^{2(1-\sigma)} - 2a^{(1-\sigma)}\cos(\tau\ln(a)) + 1}{(1-\sigma)^2+\tau^2} \quad (147)$$

$$2^{2n}\, G_{n,1-\sigma}(a) \sim_{n\to+\infty} f_a(\sigma) = \frac{a^{2\sigma} - 2a^{\sigma}\cos(\tau\ln(a)) + 1}{\sigma^2+\tau^2} \quad (148)$$

Therefore

$$\lim_{n\to+\infty} 2^{2n}\, H_{n,\sigma}(a) = f_a(\sigma) - f_a(1-\sigma) \quad (149)$$

Let's now study the function $f_a\colon\ x \to \frac{a^{2x}-2a^x\cos(\tau\ln(a))+1}{x^2+\tau^2}$. Since $\tau > 0$, the function f_a is infinitely differentiable. Let's calculate its first derivative $f_a\prime(x)$:

$$(x^2+\tau^2)^2 f_a\prime(x) = 2\ln(a)\, a^x\Big(a^x - \cos(\tau\ln(a))\Big)(x^2+\tau^2) \quad (150)$$

$$-2x\Big(a^{2x} - 2a^x\cos(\tau\ln(a)) + 1\Big) \quad (151)$$

Since $a \geq a_0 = e^{\frac{\pi}{2\tau}} > e^{\frac{3}{2\tau}}$. For $x > 0$, we have:

$$\Big(a^x - \cos(\tau\ln(a))\Big) > 0 \quad (152)$$

And we can write:

$$2a^x\Big(a^x - \cos(\tau\ln(a))\Big) = a^{2x} - 2a^x\cos(\tau\ln(a)) + 1 + \underbrace{a^{2x}-1}_{>0} \quad (153)$$

$$> \Big(a^{2x} - 2a^x\cos(\tau\ln(a)) + 1\Big) > 0 \quad (154)$$

And we also have:

$$x^2+\tau^2 > 2x\tau \quad (155)$$

$$\ln(a) \geq \ln(a_0) = \frac{3}{2\tau} \quad (156)$$

Therefore by combining the inequations $(152-156)$, we conclude:

$$2\ln(a)\, a^x\Big(a^x - \cos(\tau\ln(a))\Big)(x^2+\tau^2) > \frac{3}{2\tau}\Big(a^{2x} - 2a^x\cos(\tau\ln(a)) + 1\Big)\, 2x\tau \quad (157)$$

$$> 2x\Big(a^{2x} - 2a^x\cos(\tau\ln(a)) + 1\Big) \quad (158)$$

Therefore for each $x > 0$:

$$(x^2 + \tau^2)^2 f_a\prime(x) \quad > \quad 0 \tag{159}$$

Therefore, the function f_a is a strictly increasing function over $(0, +\infty)$. Therefore since $1 - \sigma > \sigma$, $f_a(1-\sigma) > f_a(\sigma)$. Therefore, from equations $(147 - 148)$, we can conclude that for large enough integer n, we have:

$$2^{2n}\, G_{n,\sigma}(a) \sim f_a(1-\sigma) \quad > \quad 2^{2n}\, G_{n,1-\sigma}(a) \sim f_a(\sigma) \tag{160}$$

This ends the proof of the case where $a \geq a_0$.
For $a < a_0$, as we saw in the proof of lemma 1.3, $G_{n,\sigma}(a) > G_{n,1-\sigma}(a)$ because $\sigma < \frac{1}{2}$ and for each $x \in [1, a]$: $\frac{\cos(\tau \ln(x))}{x^{1+\sigma}} > \frac{\cos(\tau \ln(x))}{x^{2-\sigma}} > 0$ and $\frac{\sin(\tau \ln(x))}{x^{1+\sigma}} > \frac{\sin(\tau \ln(x))}{x^{2-\sigma}} > 0$. This ends the proof of the lemma. ■

□

Lemma 1.5. *Let's consider two variables σ and τ such that $0 < \sigma < \frac{1}{2}$ and $\tau \geq 1$ such that $s = \sigma + i\tau$ is a zeta zero. Let's consider the sequence of functions $E_{n,\sigma}(x)$, $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$ and $H_{n,\sigma}(x)$ defined in the previous lemma 1.3. Let's consider the sequence of numbers $x_n > b_0$ as defined in point 2) of lemma 1.3 such as:*

$$\sum_{k=n+1}^{+\infty} G_{k,1-\sigma}(x_n) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,\sigma}(x_n) \tag{161}$$

Let's define the sequences of sums $(s(n, \sigma))$ such as:

$$s(n,\sigma) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,\sigma}(x_n) \tag{162}$$

$$s(n,\sigma,x) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,\sigma}(x) \tag{163}$$

$$s^*(n,\sigma) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,\sigma}(+\infty) \tag{164}$$

Therefore:

1. *The sequence $(s(n, \sigma))$ converges to zero.*
2. *If the sequence (x_n) converges to $+\infty$, then*

$$s(n,\sigma) \quad \sim_{n \to +\infty} \quad s^*(n,\sigma) \tag{165}$$

3. *If the sequence (x_n) converges to a finite limit $x < +\infty$, then*

$$s(n,\sigma) \quad \sim_{n \to +\infty} \quad s(n,\sigma,x) \tag{166}$$

Proof. Let's prove the first point. Let's take $n \geq 1$. Like in the equations (110-114) of lemma 1.3 proof, we can write:

$$\begin{aligned}
\sum_{k=n+1}^{+\infty} G_{k,\sigma}(x_n) &\leq \sum_{k=n+1}^{+\infty} \Big(E_{k,\sigma}^2(x_n) + F_{k,\sigma}^2(x_n)\Big) && (167)\\
&\leq \sum_{k=n+1}^{+\infty} \Big(\int_1^{x_n} dt\, \frac{|\sin(\tau \ln(t))|}{t^{1+\sigma}}\, \epsilon_k(t) + \int_1^{x_n} dt\, \frac{|\cos(\tau \ln(t))|}{t^{1+\sigma}}\, \epsilon_k(t)\Big)^2 && (168)\\
&\leq \sum_{k=n+1}^{+\infty} \Big(2\int_1^{x_n} dt\, \frac{\epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (169)\\
&\leq 4\Big(\int_1^{x_n} dt\, \frac{\sum_{k=n+1}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (170)\\
&\leq 4\Big(\int_1^{+\infty} dt\, \frac{\sum_{k=n+1}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}}\Big)^2 && (171)\\
&\leq 4\gamma_n^2 && (172)
\end{aligned}$$

Where γ_n is defined in the equation (120). Therefore, the sequence $(\mathrm{s}(n,\sigma))$ converges to zero. ■

To prove the point 2, like in lemma 1.3, we will use the Cauchy–Schwarz inequality:

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \tag{173}$$

Let's take $n \geq 1$. Let's calculate the following:

$$s(n,\sigma) - s^*(n,\sigma) = \sum_{k=n+1}^{+\infty} \Big(G_{k,\sigma}(x_n) - G_{k,\sigma}(+\infty)\Big) \tag{174}$$

And

$$\begin{aligned}
G_{k,\sigma}(x_n) - G_{k,\sigma}(+\infty) &= -\Big(\int_{x_n}^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t)\Big)\Big(E_{k,\sigma}(x_n) + E_{k,\sigma}(+\infty)\Big) && (175)\\
&- \Big(\int_{x_n}^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t)\Big)\Big(F_{k,\sigma}(x_n) + F_{k,\sigma}(+\infty)\Big) && (176)
\end{aligned}$$

We simplify the notations as follows:

$$\begin{aligned}
a_k &= G_{k,\sigma}(x_n) && (177)\\
b_k &= G_{k,\sigma}(+\infty) && (178)\\
u_k &= -\int_{x_n}^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t) && (179)\\
v_k &= E_{k,\sigma}(x_n) + E_{k,\sigma}(+\infty) && (180)\\
w_k &= -\int_{x_n}^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{t^{1+\sigma}}\, \epsilon_k(t) && (181)\\
z_k &= F_{k,\sigma}(x_n) + F_{k,\sigma}(+\infty) && (182)
\end{aligned}$$

Therefore

$$a_k - b_k \quad = \quad u_k v_k + w_k z_k \tag{183}$$

Therefore

$$s(n,\sigma) - s^*(n,\sigma) \quad = \quad \sum_{k=n+1}^{+\infty} \Big(u_k v_k + w_k z_k \Big) \tag{184}$$

And thanks to Cauchy–Schwarz inequality, we can write:

$$\begin{aligned} |s(n,\sigma) - s^*(n,\sigma)| \quad &\leq \quad \sum_{k=n+1}^{+\infty} |u_k v_k| + \sum_{k=n+1}^{+\infty} |w_k z_k| & (185) \\ &\leq \quad \Bigg(\sum_{k=n+1}^{+\infty} u_k^2 \Bigg) \Bigg(\sum_{k=n+1}^{+\infty} v_k^2 \Bigg) + \Bigg(\sum_{k=n+1}^{+\infty} w_k^2 \Bigg) \Bigg(\sum_{k=n+1}^{+\infty} z_k^2 \Bigg) & (186) \end{aligned}$$

And

$$\begin{aligned} \sum_{k=n+1}^{+\infty} u_k^2 \quad &\leq \quad \Big(\sum_{k=n+1}^{+\infty} |u_k| \Big)^2 & (187) \\ &\leq \quad \Big(\sum_{k=n+1}^{+\infty} \int_{x_n}^{+\infty} dt \, \frac{\epsilon_k(t)}{t^{1+\sigma}} \Big)^2 & (188) \\ &\leq \quad \Big(\int_{x_n}^{+\infty} dt \, \frac{\sum_{k=n+1}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}} \Big)^2 & (189) \\ &\leq \quad \Big(\int_{x_n}^{+\infty} dt \, \frac{\sum_{k=0}^{+\infty} \epsilon_k(t)}{t^{1+\sigma}} \Big)^2 & (190) \\ &\leq \quad \gamma_n = \Big(\int_{x_n}^{+\infty} dt \, \frac{\epsilon(t)}{t^{1+\sigma}} \Big)^2 & (191) \end{aligned}$$

And the same for (w_k):

$$\sum_{k=n+1}^{+\infty} w_k^2 \quad \leq \quad \gamma_n = \Big(\int_{x_n}^{+\infty} dt \, \frac{\epsilon(t)}{t^{1+\sigma}} \Big)^2 \tag{192}$$

Therefore

$$\begin{aligned} |s(n,\sigma) - s^*(n,\sigma)| \quad &\leq \quad \gamma_n \sum_{k=n+1}^{+\infty} \big(v_k^2 + z_k^2 \big) & (193) \\ &\leq \quad 2\gamma_n \sum_{k=n+1}^{+\infty} \Big(G_{k,\sigma}(x_n) + G_{k,\sigma}(+\infty) \Big) & (194) \\ &\leq \quad 2\gamma_n \Big(s(n,\sigma) + s^*(n,\sigma) \Big) & (195) \end{aligned}$$

The sequence (γ_n) converges to zero. Therefore there exist n_0 such that for each $n \geq n_0$ we have: $0 < \gamma_n < \frac{1}{10}$. Therefore for $n \geq n_0$ we have:

$$\frac{1 - 2\gamma_n}{1 + 2\gamma_n} \leq \frac{s(n,\sigma)}{s^*(n,\sigma)} \leq \frac{1 + 2\gamma_n}{1 - 2\gamma_n} \tag{196}$$

Therefore

$$\lim_{n\to+\infty}\frac{s(n,\sigma)}{s^*(n,\sigma)} \quad = \quad 1 \tag{197}$$

And hence

$$s(n,\sigma) \quad \sim_{n\to+\infty} \quad s^*(n,\sigma) \tag{198}$$

■

The point 3 of the lemma can be proved the same way as the point 2 above by replacing the integral over $(x_n,+\infty)$ by the integral over (x_n,x) and keep everything else the same. We conclude that:

$$s(n,\sigma) \quad \sim_{n\to+\infty} \quad s(n,\sigma,x) \tag{199}$$

■

□

Lemma 1.6. *Let's consider two variables σ and τ such that $0<\sigma<1$ and $\tau\geq 1$ such that $s=\sigma+i\tau$ is a zeta zero. Therefore for each $n\geq 1$:*

$$\sigma \quad = \quad \frac{1}{2} \tag{200}$$

Proof. This is a direct consequence of lemma 1.5. Let's suppose that $\sigma<\frac{1}{2}$. We consider the sequence (x_n) used in lemma 1.5 and defined in lemma 1.3. We can distinguish between two cases:

1. The sequence(x_n) is bounded.
2. The sequence (x_n) is unbounded.

Case 1: The sequence (x_n) is bounded In such case, without loss of generality, we can assume that the limit of (x_n) is $x\geq b_0>0$. From lemma 1.5, we have:

$$s(n,\sigma) \quad \sim_{n\to+\infty} \quad s(n,\sigma,x) \tag{201}$$

$$s(n,1-\sigma) \quad \sim_{n\to+\infty} \quad s(n,1-\sigma,x) \tag{202}$$

And from lemma 1.3, we have for each n:

$$s(n,\sigma)=s(n,1-\sigma) \tag{203}$$

Therefore:

$$s(n,\sigma,x) \quad \sim_{n\to+\infty} \quad s(n,1-\sigma,x) \tag{204}$$

Therefore, from lemma 1.4, we can conclude that:

$$\frac{1}{3}\frac{f_x(1-\sigma)}{4^n} \quad \sim_{n\to+\infty} \quad \frac{1}{3}\frac{f_x(\sigma)}{4^n} \tag{205}$$

Where the function f_x is defined in lemma 1.4. Therefore

$$f_x(1-\sigma) \quad = \quad f_x(\sigma) \tag{206}$$

Which is a contradiction, since the function f_x is increasing and we have $\sigma<1-\sigma$. Therefore σ can only be $\frac{1}{2}$.

Case 2: The sequence (x_n) is unbounded In this case, without loss of generality, we can assume that the limit of (x_n) is $+\infty$. From lemma 1.5, we have:

$$s(n,\sigma) \quad \sim_{n\to+\infty} \quad s^*(n,\sigma) \tag{207}$$
$$s(n,1-\sigma) \quad \sim_{n\to+\infty} \quad s^*(n,1-\sigma) \tag{208}$$

From equations (131-132):

$$s^*(n,\sigma) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,\sigma}(+\infty) = \frac{1}{3}\frac{1}{4^n\,\|1-s\|^2} \tag{209}$$
$$s^*(n,1-\sigma) \quad = \quad \sum_{k=n+1}^{+\infty} G_{k,1-\sigma}(+\infty) = \frac{1}{3}\frac{1}{4^n\,\|s\|^2} \tag{210}$$

And from (67) and (207-208), we conclude:

$$\frac{1}{3}\frac{1}{4^n\,\|s\|^2} \quad \sim_{n\to+\infty} \quad \frac{1}{3}\frac{1}{4^n\,\|1-s\|^2} \tag{211}$$

Therefore

$$\frac{1}{\|s\|^2} \quad = \quad \frac{1}{\|1-s\|^2} \tag{212}$$

Which is a contradiction since we have $\sigma < \frac{1}{2}$. Hence σ must be equal to $1-\sigma$. Therefore $\sigma = \frac{1}{2}$. ■

This ends the proof of the Riemann Hypothesis.

□

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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