A Simple Proof of the Riemann's Hypothesis

Charaf Ech-Chatbi

A Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

1 September 2022

Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.
MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_{1}^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

*One Raffles Quay, North Tower. 048583 Singapore. Email: charaf.chatbi@gmail.com. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\big(\frac{\pi s}{2}\big)\Gamma(1-s)\zeta(1-s) \tag{4}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{5}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{6}$$

Or in other terms:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}}dx \tag{7}$$

[1]This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\big(\frac{\pi s}{2}\big)\Gamma(s)\zeta(s) \tag{3}$$

Let's denote the following functions:

$$\begin{aligned}
\epsilon(x) &= \{x\} && (8)\\
K(\sigma,\tau) &= \frac{\tau}{(1-\sigma)^2+\tau^2} && (9)\\
\cos^+(x) &= \max\big(\cos(x),0\big) && (10)\\
\cos^-(x) &= \max\big(-\cos(x),0\big) && (11)\\
\sin^+(x) &= \max\big(\sin(x),0\big) && (12)\\
\sin^-(x) &= \max\big(-\sin(x),0\big) && (13)
\end{aligned}$$

To continue, we will prove the following lemmas.

Lemma 1.1. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define two integrals $I(a,\sigma,\tau)$ and $J(a,\sigma,\tau)$ as follows:*

$$\begin{aligned}
I(a,\sigma,\tau) &= \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx && (14)\\
J(a,\sigma,\tau) &= \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx && (15)
\end{aligned}$$

Therefore

$$\begin{aligned}
I(a,\sigma,\tau) &= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (16)\\
J(a,\sigma,\tau) &= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}+\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (17)
\end{aligned}$$

Proof. Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's take $a > 1$.

$$\begin{aligned}
I(a,\sigma,\tau) &= \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx && (18)\\
&= \int_0^{\ln(a)} \sin(\tau x)e^{(1-\sigma)x}dx && (19)\\
&= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (20)
\end{aligned}$$

And the same for $J(a,\sigma,\tau)$ for $a > 0$:

$$\begin{aligned}
J(a,\sigma,\tau) &= \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx && (21)\\
&= \int_0^{\ln(a)} \cos(\tau x)e^{(1-\sigma)x}dx && (22)\\
&= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}+\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) && (23)
\end{aligned}$$

□

Lemma 1.2. *Let's consider two variables σ and τ such that $0 < \sigma < 1$, $\tau > 0$ and $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions*

ϵ_n and $\overline{\epsilon}_n$ such that $\epsilon_0(x) = \overline{\epsilon}_0(x) = \{x\}$ and for each $n \geq 1$:

$$\epsilon_{n+1}(x) = \frac{1}{x}\int_0^x \epsilon_n(t) \tag{24}$$

$$\overline{\epsilon}_{n+1}(x) = \frac{1}{x}\int_1^x \overline{\epsilon}_n(t) \tag{25}$$

Therefore:

1. *For each $n \geq 1$:*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{1+s}} \quad = \quad -\frac{1}{(1-s)\,2^n} \tag{26}$$

2. *For each n:*

$$\int_1^{+\infty} dx \frac{\epsilon_n(x)}{x^{2-s}} \quad = \quad -\frac{1}{s\,2^n} \tag{27}$$

3. *For each $x \geq 1$:*

$$\epsilon_n(x) = \overline{\epsilon}_n(x) + \frac{1}{2^n\,x}\sum_{k=0}^{n-1}\frac{2^k \ln^k(x)}{k!} \tag{28}$$

4. *For each n and $x \geq 1$:*

$$\overline{\epsilon_n}(x) \quad = \quad \frac{1}{(n-1)!}\frac{1}{x}\int_1^x dt\,\overline{\epsilon_0}(t)\big(\ln(\frac{x}{t})\big)^{n-1} \tag{29}$$

5. *For each $x \geq 1$:*

$$\lim_{n\to+\infty} 2^n\,\epsilon_n(x) \quad = \quad x \tag{30}$$

Proof. Let's denote the integral $A(s)$ as following:

$$A(s) \quad = \quad \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} = -\frac{1}{(1-s)} \tag{31}$$

We use the integration by parts to write the following:

$$\begin{aligned}
A(s) \quad &= \quad \int_1^{+\infty} dx \frac{\epsilon_0(x)}{x^{1+s}} && (32)\\
&= \quad \left[\frac{1}{x^{1+s}}\int_0^x dx\,\epsilon_0(x)\right]_1^{+\infty} + (1+s)\int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} && (33)\\
&= \quad -\int_0^1 dx\,\epsilon_0(x) + (1+s)\int_1^{+\infty} dx \frac{\epsilon_1(x)}{x^{1+s}} && (34)\\
&= \quad -\int_0^1 dx\,\epsilon_0(x) - (1+s)\int_0^1 dx\,\epsilon_1(x) + (1+s)^2\int_1^{+\infty} dx \frac{\epsilon_2(x)}{x^{1+s}} && (35)\\
&\quad \ldots && (36)\\
&= \quad -\sum_{k=0}^{n}(1+s)^k\int_0^1 dx\,\epsilon_k(x) + (1+s)^{n+1}\int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{2+s}} && (37)
\end{aligned}$$

We could do the above integration by parts. Because the functions ϵ_k are continuous functions. We also can prove that the functions ψ_k are non-negative and bounded as we can prove that for each k for each $x > 0$ that:

$$0 < \epsilon_k(x) \quad \leq \quad 1 \tag{38}$$

This can be proven by recurrence using the fact that for each $x \geq 0$:

$$0 \leq \epsilon_0(x) = \{x\} \quad \leq \quad 1 \tag{39}$$

Now, we need to calculate the integrals I_k for $k \geq 0$:

$$I_k \quad = \quad \int_0^1 dx\, \epsilon_k(x) \tag{40}$$

For $x \in (0,1)$, we have:

$$\epsilon_0(x) = x \tag{41}$$

And

$$\psi_1(x) = \frac{x}{2} \tag{42}$$

Therefore, we can write for each k for $x \in (0,1)$:

$$\epsilon_k(x) = \frac{x}{2^k} \tag{43}$$

Therefore, for each $k \geq 1$:

$$I_k = \frac{1}{2^{k+1}} \tag{44}$$

Therefore, we can conclude:

$$A(s) \quad = \quad -\frac{1}{(1-s)} + (1+s)^{n+1}\left[\int_1^{+\infty} dx \frac{\epsilon_{n+1}(x)}{x^{1+s}} + \frac{1}{(1-s)\,2^{n+1}}\right] \tag{45}$$

Since $1 + s \neq 0$, we conculde the point 1. The point 2 can be proved the same way by replacing s by $1 - s$. ■

The point 3) can be proved by recurrence. For $n = 0$. We have $\epsilon_0(x) = \overline{\epsilon_0}(x)$. Let's assume that it is true till n and let's prove for $n+1$. We have:

$$\begin{aligned}
\epsilon_{n+1}(x) \quad &= \quad \frac{1}{x}\int_0^1 dx\, \epsilon_n(x) + \frac{1}{x}\int_1^x dx\, \epsilon_n(x) && (46)\\
&= \quad \frac{1}{x\,2^{n+1}} + \frac{1}{x\,2^n}\int_1^x \frac{dt}{t}\sum_{k=0}^{n-1}\frac{2^k \ln^k(t)}{k!} + \frac{1}{x}\int_1^x dx\, \overline{\epsilon_n}(x) && (47)\\
&= \quad \frac{1}{x\,2^{n+1}} + \frac{1}{x\,2^{n+1}}\sum_{k=0}^{n-1}\frac{2^{k+1}\ln^{k+1}(x)}{(k+1)!} + \overline{\epsilon_{n+1}}(x) && (48)\\
&= \quad \frac{1}{x\,2^{n+1}}\sum_{k=0}^{n}\frac{2^k \ln^k(x)}{(k)!} + \overline{\epsilon_{n+1}}(x) && (49)
\end{aligned}$$

$\blacksquare$

Let's prove the 4^{th} point. We proceed by recurrence. For $n = 1$, we retrieve the definition of $\overline{\epsilon_1}(x)$. Let's assume that it is true up to n and let's prove it for $n+1$. We have thanks to the integral order change:

$$\begin{aligned}
\overline{\epsilon}_{n+1}(x) &= \frac{1}{x}\int_1^x dt\,\overline{\epsilon}_n(t) && (50)\\
&= \frac{1}{(n-1)!}\frac{1}{x}\int_1^x \frac{dt}{t}\int_1^t ds\,\overline{\epsilon}_1(s)\big(\ln(\frac{t}{s})\big)^{n-1} && (51)\\
&= \frac{1}{(n-1)!}\frac{1}{x}\int_1^x ds\,\overline{\epsilon}_1(s)\int_s^x \frac{dt}{t}\big(\ln(\frac{t}{s})\big)^{n-1} && (52)\\
&= \frac{1}{n!}\frac{1}{x}\int_1^x ds\,\overline{\epsilon}_1(s)\big(\ln(\frac{x}{s})\big)^{n} && (53)
\end{aligned}$$

And this proves ou point of the lemma. $\blacksquare$

Let's now prove the last point. Let's take $x > 1$. From the point (4) above, we can apply the dominated convergence theorem to prove that:

$$\lim_{n\to+\infty} 2^n\,\overline{\epsilon}_n(x) = 0 \qquad (54)$$

From point (3), we can conclude that:

$$\begin{aligned}
\lim_{n\to+\infty} 2^n\,\epsilon_n(x) &= \lim_{n\to+\infty}\left(2^n\,\overline{\epsilon}_n(x) + \frac{1}{x}\sum_{k=0}^{n-1}\frac{2^k\ln^k(x)}{k!}\right) && (55)\\
&= \lim_{n\to+\infty}\frac{1}{x}\sum_{k=0}^{n-1}\frac{\ln^k(x^2)}{k!} && (56)\\
&= \frac{1}{x}\exp(ln(x^2)) = x && (57)
\end{aligned}$$

Hence the proof of the point (5) of the lemma. $\blacksquare$

$\square$

Lemma 1.3. *Let's consider two variables σ and τ such that $0 < \sigma \leq \frac{1}{2}$ and $\tau > 0$ such that $s = \sigma + i\tau$ is a zeta zero. Let's define the sequence of functions $E_{n,\sigma}(x)$, $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$ and $H_{n,\sigma}(x)$ over $[1,+\infty)$ for each $n \geq 1$ as follows:*

$$\begin{aligned}
E_{n,\sigma}(x) &= \int_1^x dt\,\frac{\sin(\tau\ln(t))}{x^{1+\sigma}}\,\epsilon_n(x) && (58)\\
F_{n,\sigma}(x) &= \int_1^x dt\,\frac{\cos(\tau\ln(t))}{x^{1+\sigma}}\,\epsilon_n(x) && (59)\\
G_{n,\sigma}(x) &= \big(E_{n,\sigma}(x)\big)^2 + \big(F_{n,\sigma}(x)\big)^2 && (60)\\
H_{n,\sigma}(x) &= G_{n,1-\sigma}(x) - G_{n,\sigma}(x) && (61)
\end{aligned}$$

Let's define the sequence of functions $I_{n,\sigma}(x)$ over $[0,+\infty)$ for each $n \geq 1$ as follows:

$$I_{n,\sigma}(x) = H_{n,\sigma}(x + e^{\frac{3}{2\tau}}) \qquad (62)$$

We denote $a_0 = e^{\frac{3}{2\tau}}$ and $b_0 = e^{\frac{\pi}{2\tau}}$.

1. *For each $n \geq 1$ there exists $x_n > b_0$ such as:*

$$H_{n,\sigma}(x_n) = 0 \tag{63}$$

2. *For each $n \geq 1$ there exists $x_n > b_0 - a_0 > 0$ such as:*

$$I_{n,\sigma}(x_n) = 0 \tag{64}$$

In other terms:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \tag{65}$$

Proof. Let's prove first point. Let's take $n \geq 1$ an integer. Since $\sigma \leq \frac{1}{2}$. We have for each $x > 1$: $\frac{1}{x^{1+\sigma}} > \frac{1}{x^{2-\sigma}}$. Therefore

$$\begin{aligned}
E_{n,\sigma}(b_0) &= \int_1^{e^{\frac{\pi}{2\tau}}} dt\, \frac{\sin(\tau \ln(t))}{x^{1+\sigma}}\, \epsilon_n(x) > 0 && (66)\\
F_{n,\sigma}(b_0) &= \int_1^{e^{\frac{\pi}{2\tau}}} dt\, \frac{\cos(\tau \ln(t))}{x^{1+\sigma}}\, \epsilon_n(x) > 0 && (67)\\
E_{n,\sigma}(b_0) &> E_{n,1-\sigma}(b_0) > 0 && (68)\\
F_{n,\sigma}(b_0) &> F_{n,1-\sigma}(b_0) > 0 && (69)
\end{aligned}$$

Therefore

$$\begin{aligned}
\big(E_{n,\sigma}(b_0)\big)^2 &> E_{n,1-\sigma}(b_0)\big)^2 && (70)\\
\big(F_{n,\sigma}(b_0) &> \big(F_{n,1-\sigma}(b_0)\big)^2 && (71)
\end{aligned}$$

And therefore

$$\begin{aligned}
G_{n,\sigma}(b_0) &> G_{n,1-\sigma}(b_0) && (72)\\
H_{n,\sigma}(b_0) &< 0 && (73)
\end{aligned}$$

We also have from the lemma 1.2:

$$\begin{aligned}
\lim_{x\to+\infty} G_{n,\sigma}(x) &= \Big(\int_1^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{x^{1+\sigma}}\, \epsilon_n(x)\Big)^2 + \Big(\int_1^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{x^{1+\sigma}}\, \epsilon_n(x)\Big)^2 && (74)\\
&= \frac{1}{2^{2n}\|1-s\|^2} && (75)\\
\lim_{x\to+\infty} G_{n,1-\sigma}(x) &= \Big(\int_1^{+\infty} dt\, \frac{\sin(\tau \ln(t))}{x^{2-\sigma}}\, \epsilon_n(x)\Big)^2 + \Big(\int_1^{+\infty} dt\, \frac{\cos(\tau \ln(t))}{x^{2-\sigma}}\, \epsilon_n(x)\Big)^2 && (76)\\
&= \frac{1}{2^{2n}\|s\|^2} && (77)
\end{aligned}$$

Therefore

$$\lim_{x\to+\infty} H_{n,\sigma}(x) = \frac{1}{2^{2n}}\Big(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2}\Big) > 0 \tag{78}$$

Since the function $H_{n,\sigma}$ is continuous over $[1, +\infty)$. From (73) and (78) and thanks to the Mean value theorem, we can conclude that there exists an $x_n > b_0$ such that:

$$H_{n,\sigma}(x_n) = 0 \tag{79}$$

■

To prove the second point, we proceed by the same approach as in the first point. We remark that since $b_0 - a_0 > 0$:

$$I_{n,\sigma}(b_0 - a_0) = H_{n,\sigma}(a_0 + b_0 - a_0) = H_{n,\sigma}(b_0) < 0 \quad (80)$$

And the same like above:

$$\lim_{x\to+\infty} I_{n,\sigma}(x) = \lim_{x\to+\infty} H_{n,\sigma}(a_0 + x) \quad (81)$$

$$= \frac{1}{2^{2n}}\Big(\frac{1}{\|s\|^2} - \frac{1}{\|1-s\|^2}\Big) > 0 \quad (82)$$

And the same we conclude thanks to the Mean value theorem. ■

□

Lemma 1.4. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau \geq 1$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore for each $n \geq 1$:*

$$\sigma = \frac{1}{2} \quad (83)$$

Proof. Let's suppose that $\sigma < \frac{1}{2}$. We use the previous lemma notations. From the lemma 1.3, we can construct a sequence $(x_n > b_0 - a_0)$ such that:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \quad (84)$$

We can distinguish between two cases:

1. The sequence(x_n) is bounded.
2. The sequence (x_n) is unbounded.

From the lemma 1.2, we have:

$$\lim_{n\to+\infty} 2^n\,\epsilon_n(x) = x \quad (85)$$

Therefore for each $x \geq 1$:

$$\lim_{n\to+\infty} \frac{\sin(\tau\ln(x))}{x^{1+\sigma}}\,2^n\,\epsilon_n(x) = \frac{\sin(\tau\ln(x))}{x^{\sigma}} \quad (86)$$

$$\lim_{n\to+\infty} \frac{\cos(\tau\ln(x))}{x^{1+\sigma}}\,2^n\,\epsilon_n(x) = \frac{\cos(\tau\ln(x))}{x^{\sigma}} \quad (87)$$

$$\lim_{n\to+\infty} \frac{\sin(\tau\ln(x))}{x^{2-\sigma}}\,2^n\,\epsilon_n(x) = \frac{\sin(\tau\ln(x))}{x^{1-\sigma}} \quad (88)$$

$$\lim_{n\to+\infty} \frac{\cos(\tau\ln(x))}{x^{2-\sigma}}\,2^n\,\epsilon_n(x) = \frac{\cos(\tau\ln(x))}{x^{1-\sigma}} \quad (89)$$

Case 1: The sequence (x_n) is bounded In such case, without loss of generality, we can assume that the limit of (x_n) is $a \geq b_0 - a_0 > 0$. Thanks to the dominated convergence theorem applied on the sequence of functions in $(86-89)$, we can write:

$$2^{2n}\, G_{n,\sigma}(x_n + a_0) \quad \sim_{n\to+\infty} \quad \Big(\int_1^{a+a_0} dt\, \frac{\sin(\tau \ln(t))}{t^{\sigma}}\Big)^2 + \Big(\int_1^{a+a_0} dt\, \frac{\cos(\tau \ln(t))}{t^{\sigma}}\Big)^2 \tag{90}$$

$$2^{2n}\, G_{n,1-\sigma}(x_n + a_0) \quad \sim_{n\to+\infty} \quad \Big(\int_1^{a+a_0} dt\, \frac{\sin(\tau \ln(t))}{t^{1-\sigma}}\Big)^2 + \Big(\int_1^{a+a_0} dt\, \frac{\cos(\tau \ln(t))}{t^{1-\sigma}}\Big)^2 \tag{91}$$

Thanks to lemma 1.1, we can write:

$$\frac{(a+a_0)^{2\sigma} - 2(a+a_0)^{\sigma}\cos(\tau\ln(a+a_0)) + 1}{\sigma^2+\tau^2} = \frac{(a+a_0)^{2(1-\sigma)} - 2(a+a_0)^{(1-\sigma)}\cos(\tau\ln(a+a_0)) + 1}{(1-\sigma)^2+\tau^2} \tag{92}$$

Let's study now the function $f\colon x \to \frac{(a+a_0)^{2x} - 2(a+a_0)^{x}\cos(\tau\ln(a+a_0)) + 1}{x^2+\tau^2}$. Since $\tau > 0$, the function f is infinitely differentiable. Let's calculate its first derivative $f\prime(x)$:

$$(x^2+\tau^2)^2 f\prime(x) \;=\; 2\ln(a+a_0)(a+a_0)^x\Big((a+a_0)^x - \cos(\tau\ln(a+a_0))\Big)(x^2+\tau^2) \tag{93}$$

$$-2x\Big((a+a_0)^{2x} - 2(a+a_0)^x\cos(\tau\ln(a+a_0)) + 1\Big) \tag{94}$$

Since $a + a_0 \geq e^{\frac{\pi}{2\tau}} > e^{\frac{3}{2\tau}}$. For $x > 0$, we have:

$$\Big((a+a_0)^x - \cos(\tau\ln(a+a_0))\Big) > 0 \tag{95}$$

And we can write:

$$2(a+a_0)^x\Big((a+a_0)^x - \cos(\tau\ln(a+a_0))\Big) = (a+a_0)^{2x} - 2(a+a_0)^x\cos(\tau\ln(a+a_0)) + 1 + \underbrace{(a+a_0)^{2x} - 1}_{>0} \tag{96}$$

$$> \Big((a+a_0)^{2x} - 2(a+a_0)^x\cos(\tau\ln(a+a_0)) + 1\Big) > 0 \tag{97}$$

And we also have:

$$x^2+\tau^2 \;>\; 2x\tau \tag{98}$$

$$\ln(a+a_0) \;>\; \ln(a_0) = \frac{3}{2\tau} \tag{99}$$

Therefore by combining the inequations $(95-99)$, we conclude:

$$2\ln(a+a_0)(a+a_0)^x\Big((a+a_0)^x - \cos(\tau\ln(a+a_0))\Big)(x^2+\tau^2) \;>\; \frac{3}{2\tau}\Big((a+a_0)^{2x} - 2(a+a_0)^x\cos(\tau\ln(a+a_0)) + 1\Big)2x\tau \tag{100}$$

$$> 2x\Big((a+a_0)^{2x} - 2(a+a_0)^x\cos(\tau\ln(a+a_0)) + 1\Big) \tag{101}$$

Therefore for each $x > 0$:

$$(x^2+\tau^2)^2 f\prime(x) \;>\; 0 \tag{102}$$

Therefore, the function f is a strictly increasing function over $(0, +\infty)$. From the equation (92), we have $f(\sigma) = f(1-\sigma)$. Which is a contradiction if $\sigma \neq 1-\sigma$. Therefore σ must be equal $1-\sigma$. Hence

$$\sigma = 1 - \sigma = \frac{1}{2} \tag{103}$$

This ends the proof of the first case. ■.

Case 2: The sequence (x_n) is unbounded In this case, without loss of generality, we can assume that the limit of (x_n) is $+\infty$.

The full proof of this case is in lemma 1.6. Here, we will only give an intuition about the proof: Similarly like in the first case, we hope that we can write:

$$2^{2n}\, G_{n,\sigma}(x_n + a_0) \quad \sim_{n\to+\infty} \quad \Big(\int_1^{x_n+a_0} dt\, \frac{\sin(\tau \ln(t))}{t^\sigma}\Big)^2 + \Big(\int_1^{x_n+a_0} dt\, \frac{\cos(\tau \ln(t))}{t^\sigma}\Big)^2 \tag{104}$$

And then from the lemma 1.1, we could write:

$$2^{2n}\, G_{n,\sigma}(x_n + a_0) \quad \sim_{n\to+\infty} \quad \frac{(x_n+a_0)^{2(1-\sigma)} - 2(x_n+a_0)^{(1-\sigma)}\cos(\tau\ln(x_n+a_0)) + 1}{(1-\sigma)^2+\tau^2} \tag{105}$$

$$\sim_{n\to+\infty} \quad \frac{(x_n)^{2(1-\sigma)}}{(1-\sigma)^2+\tau^2} \tag{106}$$

And the same for $1-\sigma$:

$$2^{2n}\, G_{n,1-\sigma}(x_n + a_0) \quad \sim_{n\to+\infty} \quad \frac{(x_n+a_0)^{2\sigma} - 2(x_n+a_0)^{\sigma}\cos(\tau\ln(x_n+a_0)) + 1}{\sigma^2+\tau^2} \tag{107}$$

$$\sim_{n\to+\infty} \quad \frac{(x_n)^{2\sigma}}{\sigma^2+\tau^2} \tag{108}$$

And from the relation (84), we could conclude:

$$x_n^{2(1-2\sigma)} \quad \sim_{n\to+\infty} \quad \frac{(1-\sigma)^2+\tau^2}{\sigma^2+\tau^2} \tag{109}$$

Which would be a contradiction since the limit of (x_n) is $+\infty$. Therefore $1-2\sigma$ would have to be equal zero.
Unfortunately, the equivalence relations $(104-108)$ are not necessarily true since the functions sin and cos are not positive or of constant sign but it gives us the idea of the proof. See lemmas 1.5 and 1.6 below.

$\square$

Lemma 1.5. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define the following integrals:*

$$I(+,\sigma,\tau,n) \quad = \quad \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\sin^+(\tau\ln(x))}{x^\sigma}dx \tag{110}$$

$$I(-,\sigma,\tau,n) \quad = \quad \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\sin^-(\tau\ln(x))}{x^\sigma}dx \tag{111}$$

$$J(+,\sigma,\tau,n) \quad = \quad \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\cos^+(\tau\ln(x))}{x^\sigma}dx \tag{112}$$

$$J(-,\sigma,\tau,n) \quad = \quad \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\cos^-(\tau\ln(x))}{x^\sigma}dx \tag{113}$$

Therefore

$$I(+,\sigma,\tau,n) \quad \sim_{n\to+\infty} \quad K(\sigma,\tau)\frac{e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1} \tag{114}$$

$$I(-,\sigma,\tau,n) \quad \sim_{n\to+\infty} \quad K(\sigma,\tau)\left(\frac{e^{\frac{\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right)e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}} \tag{115}$$

$$J(+,\sigma,\tau,n) \quad \sim_{n\to+\infty} \quad K(\sigma,\tau)\left(\frac{1-\sigma}{\tau}+\frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right)e^{\frac{2\pi(1-\sigma)(n+1)}{\tau}} \tag{116}$$

$$J(-,\sigma,\tau,n) \quad \sim_{n\to+\infty} \quad K(\sigma,\tau)\frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}e^{\frac{2\pi(1-\sigma)(n+1)}{\tau}} \tag{117}$$

Proof. Let's take $n \geq 1$, $\sigma > 0$ and $\tau > 0$. Let's calculate $J(+,\sigma,\tau,n)$:

$$J(+,\sigma,\tau,n) = \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\cos^+(\tau\ln(x))}{x^\sigma}dx \tag{118}$$

$$= \int_1^{e^{\frac{\pi}{2\tau}}} dx\frac{\cos(\tau\ln(x))}{x^\sigma} + \sum_{k=0}^{n-1}\int_{e^{\frac{(4k+3)\pi}{2\tau}}}^{e^{\frac{(4k+5)\pi}{2\tau}}} dx\frac{\cos(\tau\ln x)}{x^\sigma} + \int_{e^{\frac{(4n+3)\pi}{2\tau}}}^{e^{\frac{(4n+4)\pi}{2\tau}}} dx\frac{\cos(\tau\ln x)}{x^\sigma} \tag{119}$$

From lemma 1.1, we can write the following:

$$\sum_{k=0}^{n-1}\int_{e^{\frac{(4k+3)\pi}{2\tau}}}^{e^{\frac{(4k+5)\pi}{2\tau}}} dx\frac{\cos(\tau\ln x)}{x^\sigma} = K(\sigma,\tau)\sum_{k=0}^{n-1}\left(e^{\frac{(4k+5)\pi(1-\sigma)}{2\tau}}+e^{\frac{(4k+3)\pi(1-\sigma)}{2\tau}}\right) \tag{120}$$

And

$$A(n) = \int_{e^{\frac{(4n+3)\pi}{2\tau}}}^{e^{\frac{(4n+4)\pi}{2\tau}}} dx\frac{\cos(\tau\ln x)}{x^\sigma} \tag{121}$$

$$= K(\sigma,\tau)e^{\frac{3\pi(1-\sigma)}{2\tau}}\left(1+\frac{(1-\sigma)}{\tau}e^{\frac{\pi(1-\sigma)}{2\tau}}\right)e^{\frac{2\pi(1-\sigma)(n)}{\tau}} \tag{122}$$

And therefore:

$$J(+,\sigma,\tau,n) = K(\sigma,\tau)\left(-\frac{(1-\sigma)}{\tau}+e^{\frac{\pi(1-\sigma)}{2\tau}}\right)+K(\sigma,\tau)\sum_{k=0}^{n-1}\left(e^{\frac{(4k+5)\pi(1-\sigma)}{2\tau}}+e^{\frac{(4k+3)\pi(1-\sigma)}{2\tau}}\right)+A(n) \tag{123}$$

$$= K(\sigma,\tau)\left(-\frac{(1-\sigma)}{\tau}+e^{\frac{\pi(1-\sigma)}{2\tau}}\right)+K(\sigma,\tau)\sum_{k=0}^{n-1}\left(e^{\frac{5\pi(1-\sigma)}{2\tau}}e^{\frac{2k\pi(1-\sigma)}{\tau}}+e^{\frac{3\pi(1-\sigma)}{2\tau}}e^{\frac{2k\pi(1-\sigma)}{\tau}}\right)+A(n) \tag{124}$$

$$= K(\sigma,\tau)\left(-\frac{(1-\sigma)}{\tau}+e^{\frac{\pi(1-\sigma)}{2\tau}}\right)+K(\sigma,\tau)\frac{\left(e^{\frac{5\pi(1-\sigma)}{2\tau}}+e^{\frac{3\pi(1-\sigma)}{2\tau}}\right)\left(e^{\frac{2\pi(1-\sigma)(n)}{\tau}}-1\right)}{e^{\frac{2\pi(1-\sigma)}{\tau}}-1}+A(n) \tag{125}$$

$$= K(\sigma,\tau)\left(-\frac{(1-\sigma)}{\tau}+e^{\frac{\pi(1-\sigma)}{2\tau}}+e^{\frac{3\pi(1-\sigma)}{2\tau}}\frac{e^{\frac{2\pi(1-\sigma)(n)}{\tau}}-1}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}+e^{\frac{(4n+3)\pi(1-\sigma)}{2\tau}}+\frac{(1-\sigma)}{\tau}e^{\frac{(4n+4)\pi(1-\sigma)}{2\tau}}\right) \tag{126}$$

Also we can write:

$$J(-,\sigma,\tau,n) = J(+,\sigma,\tau,n) - J(\sigma,\tau,e^{\frac{(4n+4)\pi}{2\tau}}) \tag{127}$$

$$= J(+,\sigma,\tau,n) + K(\sigma,\tau)\frac{(1-\sigma)}{\tau}\left(1 - e^{\frac{2\pi(1-\sigma)(n+1)}{\tau}}\right) \tag{128}$$

$$= K(\sigma,\tau)\left(e^{\frac{\pi(1-\sigma)}{2\tau}} + e^{\frac{\pi(1-\sigma)(4n+3)}{2\tau}} + e^{\frac{3\pi(1-\sigma)}{2\tau}}\frac{e^{\frac{2\pi(1-\sigma)(n)}{\tau}} - 1}{e^{\frac{\pi(1-\sigma)}{\tau}} - 1}\right) \tag{129}$$

Therefore:

$$J(+,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\left(\frac{1-\sigma}{\tau} + \frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}} - 1}\right)e^{\frac{2\pi(1-\sigma)(n+1)}{\tau}} \tag{130}$$

$$J(-,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}} - 1}e^{\frac{2\pi(1-\sigma)(n+1)}{\tau}} \tag{131}$$

And the same for I:

$$I(+,\sigma,\tau,n) = \int_1^{e^{\frac{(4n+4)\pi}{2\tau}}} \frac{\sin^+(\tau\ln(x))}{x^\sigma}dx \tag{132}$$

$$= \sum_{k=0}^{n}\int_{e^{\frac{(4k)\pi}{2\tau}}}^{e^{\frac{(4k+2)\pi}{2\tau}}} dx\frac{\sin(\tau\ln x)}{x^\sigma} \tag{133}$$

From lemma 1.1, we can write the following:

$$\sum_{k=0}^{n}\int_{e^{\frac{(4k)\pi}{2\tau}}}^{e^{\frac{(4k+2)\pi}{2\tau}}} dx\frac{\sin(\tau\ln x)}{x^\sigma} = K(\sigma,\tau)\sum_{k=0}^{n}\left(e^{\frac{(4k+2)\pi(1-\sigma)}{2\tau}} + e^{\frac{(4k)\pi(1-\sigma)}{2\tau}}\right) \tag{134}$$

And therefore:

$$I(+,\sigma,\tau,n) = K(\sigma,\tau)\sum_{k=0}^{n}\left(e^{\frac{(4k+2)\pi(1-\sigma)}{2\tau}} + e^{\frac{(4k)\pi(1-\sigma)}{2\tau}}\right) \tag{135}$$

$$= K(\sigma,\tau)\sum_{k=0}^{n}\left(e^{\frac{(2)\pi(1-\sigma)}{2\tau}} + 1\right)e^{\frac{(4k)\pi(1-\sigma)}{2\tau}} \tag{136}$$

$$= K(\sigma,\tau)\frac{\left(e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}} - 1\right)}{e^{\frac{\pi(1-\sigma)}{\tau}} - 1} \tag{137}$$

Also we can write:

$$I(-,\sigma,\tau,n) = I(+,\sigma,\tau,n) - I(\sigma,\tau,e^{\frac{(4n+4)\pi}{2\tau}}) \tag{138}$$

$$= I(+,\sigma,\tau,n) - K(\sigma,\tau)\left(1 - e^{\frac{\pi(1-\sigma)(4n+4)}{2\tau}}\right) \tag{139}$$

$$= K(\sigma,\tau)\frac{e^{\frac{\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}} - 1}\left(e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}} - 1\right) \tag{140}$$

Therefore:

$$
\begin{aligned}
I(+,\sigma,\tau,n) &\sim_{n\to+\infty} K(\sigma,\tau)\frac{e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1} && (141)\\
I(-,\sigma,\tau,n) &\sim_{n\to+\infty} K(\sigma,\tau)\left(\frac{e^{\frac{\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right)e^{\frac{2(n+1)\pi(1-\sigma)}{\tau}} && (142)
\end{aligned}
$$

■

□

Lemma 1.6. *Let's consider two variables σ and τ such that $0 < \sigma < 1$ and $\tau \geq 1$ such that $s = \sigma + i\tau$ is a zeta zero. Therefore for each $n \geq 1$:*

$$\sigma = \frac{1}{2} \quad (143)$$

Proof. Here we will proceed like in the proof of the lemma 1.4. Let's suppose that $\sigma < \frac{1}{2}$. Like in lemma 1.4, let's (x_n) be the sequence $(x_n > b_0 - a_0)$ such that:

$$G_{n,1-\sigma}(x_n + a_0) = G_{n,\sigma}(x_n + a_0) \quad (144)$$

Where a_0 and b_0 are as defined in the lemma 1.4. We can distinguish between two cases:

1. The sequence (x_n) is unbounded.
2. The sequence(x_n) is bounded.

Here we will focus only on the unbounded case. The reader can refer to lemma 1.4 for the bounded case.
So, let's assume that the sequence (x_n) is unbounded. Without loss of generality, we can assume that the limit of (x_n) is $+\infty$. We also can construct a strictly increasing sequence of integers (k_n) such as for each $n \geq 1$:

$$e^{\frac{(2k_n)\pi}{\tau}} \leq x_n < e^{\frac{2(1+k_n)\pi}{\tau}} \quad (145)$$

By construction, the sequence $\left(y_n = (x_n + a_0)e^{-\frac{(2k_n)\pi}{\tau}}\right)$ is bounded and its elements are in the interval $(1, e^{\frac{2\pi}{\tau}})$. Without loss of generality we can assume further that (y_n) converges to $y \in [1, e^{\frac{2\pi}{\tau}}]$.

From the equation (144), we can write the following:

$$\begin{aligned}
2^{2n}G_{n,\sigma}(x_n+a_0) &= 2^{2n}G_{n,1-\sigma}(x_n+a_0) && (146)\\
&= \Big(\int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\sin(\tau\ln(t))}{t^{1+\sigma}}2^n\epsilon_n(t)+\int_{e^{\frac{(2k_n)\pi}{\tau}}}^{x_n+a_0} dt\,\frac{\sin(\tau\ln(t))}{t^{1+\sigma}}2^n\epsilon_n(t)\Big)^2 && (147)\\
&+ \Big(\int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\cos(\tau\ln(t))}{t^{1+\sigma}}2^n\epsilon_n(t)+\int_{e^{\frac{(2k_n)\pi}{\tau}}}^{x_n+a_0} dt\,\frac{\cos(\tau\ln(t))}{t^{1+\sigma}}2^n\epsilon_n(t)\Big)^2 && (148)\\
&= \Big(\int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\sin(\tau\ln(t))}{t^{2-\sigma}}2^n\epsilon_n(t)+\int_{e^{\frac{(2k_n)\pi}{\tau}}}^{x_n+a_0} dt\,\frac{\sin(\tau\ln(t))}{t^{2-\sigma}}2^n\epsilon_n(t)\Big)^2 && (149)\\
&+ \Big(\int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\cos(\tau\ln(t))}{t^{2-\sigma}}2^n\epsilon_n(t)+\int_{e^{\frac{(2k_n)\pi}{\tau}}}^{x_n+a_0} dt\,\frac{\cos(\tau\ln(t))}{t^{2-\sigma}}2^n\epsilon_n(t)\Big)^2 && (150)
\end{aligned}$$

Further, we want to work with positive functions, so we will develop the integrals above as follows:

$$\begin{aligned}
\int_1^{x_n+a_0} dt\,\frac{\sin(\tau\ln(t))}{t^{1+\sigma}}\epsilon_n(t) &= \int_1^{x_n+a_0} dt\,\frac{\sin^+(\tau\ln(t))-\sin^-(\tau\ln(t))}{t^{1+\sigma}}\epsilon_n(t) && (151)\\
\int_1^{x_n+a_0} dt\,\frac{\cos(\tau\ln(t))}{t^{1+\sigma}}\epsilon_n(t) &= \int_1^{x_n+a_0} dt\,\frac{\cos^+(\tau\ln(t))-\cos^-(\tau\ln(t))}{t^{1+\sigma}}\epsilon_n(t) && (152)\\
\int_1^{x_n+a_0} dt\,\frac{\sin(\tau\ln(t))}{t^{2-\sigma}}\epsilon_n(t) &= \int_1^{x_n+a_0} dt\,\frac{\sin^+(\tau\ln(t))-\sin^-(\tau\ln(t))}{t^{2-\sigma}}\epsilon_n(t) && (153)\\
\int_1^{x_n+a_0} dt\,\frac{\cos(\tau\ln(t))}{t^{2-\sigma}}\epsilon_n(t) &= \int_1^{x_n+a_0} dt\,\frac{\cos^+(\tau\ln(t))-\cos^-(\tau\ln(t))}{t^{2-\sigma}}\epsilon_n(t) && (154)
\end{aligned}$$

Let's denote the following integrals:

$$\begin{aligned}
I(+,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\sin^+(\tau\ln(x))}{x^{1+\sigma}}2^n\epsilon_n(x)dx && (155)\\
I(-,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\sin^-(\tau\ln(x))}{x^{1+\sigma}}2^n\epsilon_n(x)dx && (156)\\
J(+,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\cos^+(\tau\ln(x))}{x^{1+\sigma}}2^n\epsilon_n(x)dx && (157)\\
J(-,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\cos^-(\tau\ln(x))}{x^{1+\sigma}}2^n\epsilon_n(x)dx && (158)\\
II(+,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\sin^+(\tau\ln(x))}{x^{\sigma}}dx && (159)\\
II(-,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\sin^-(\tau\ln(x))}{x^{\sigma}}dx && (160)\\
JJ(+,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\cos^+(\tau\ln(x))}{x^{\sigma}}dx && (161)\\
JJ(-,\sigma,\tau,n) &= \int_1^{x_n+a_0} \frac{\cos^-(\tau\ln(x))}{x^{\sigma}}dx && (162)
\end{aligned}$$

Therefore from (147-150), we can write:

$$\Big(I(+,\sigma,\tau,n) - I(-,\sigma,\tau,n)\Big)^2 + \Big(J(+,\sigma,\tau,n) - J(-,\sigma,\tau,n)\Big)^2 = \tag{163}$$

$$\Big(I(+,1-\sigma,\tau,n) - I(-,1-\sigma,\tau,n)\Big)^2 + \Big(J(+,1-\sigma,\tau,n) - J(-,1-\sigma,\tau,n)\Big)^2 \tag{164}$$

Therefore

$$\begin{aligned}
& I^2(+,\sigma,\tau,n) + I^2(-,\sigma,\tau,n) + J^2(+,\sigma,\tau,n) + J^2(-,\sigma,\tau,n) && (165)\\
& +2I(+,1-\sigma,\tau,n)I(-,1-\sigma,\tau,n) + 2J(+,1-\sigma,\tau,n)J(-,1-\sigma,\tau,n) && (166)\\
= \;& I^2(+,1-\sigma,\tau,n) + I^2(-,1-\sigma,\tau,n) + J^2(+,1-\sigma,\tau,n) + J^2(-,1-\sigma,\tau,n) && (167)\\
& +2I(+,\sigma,\tau,n)I(-,\sigma,\tau,n) + 2J(+,\sigma,\tau,n)J(-,\sigma,\tau,n) && (168)
\end{aligned}$$

From the lemma 1.2, we have:

$$\lim_{n\to+\infty} 2^n\,\epsilon_n(x) = x \tag{169}$$

Therefore

$$\frac{\sin^+(\tau\ln(x))}{x^{1+\sigma}} 2^n\,\epsilon_n(x) \quad \sim_{n\to+\infty} \quad \frac{\sin^+(\tau\ln(x))}{x^{\sigma}} \tag{170}$$

The functions $x \to \frac{\sin^+(\tau\ln(x))}{x^{1+\sigma}} 2^n\,\epsilon_n(x)$ and $x \to \frac{\sin^+(\tau\ln(x))}{x^{\sigma}}$ and are non-negative and continuous over $[1,+\infty)$. Further, they are not integrable over $[1,+\infty)$. Therefore, thanks to the integral equivalence theorem, we can write:

$$\begin{aligned}
I(+,\sigma,\tau,n) &\sim_{n\to+\infty} II(+,\sigma,\tau,n) && (171)\\
I(-,\sigma,\tau,n) &\sim_{n\to+\infty} II(-,\sigma,\tau,n) && (172)\\
J(+,\sigma,\tau,n) &\sim_{n\to+\infty} JJ(+,\sigma,\tau,n) && (173)\\
J(+,\sigma,\tau,n) &\sim_{n\to+\infty} JJ(+,\sigma,\tau,n) && (174)
\end{aligned}$$

Further, using a variable change, we can write:

$$\begin{aligned}
II(+,\sigma,\tau,n) &= \int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\sin^+(\tau\ln(t))}{t^\sigma} + \int_{e^{\frac{(2k_n)\pi}{\tau}}}^{x_n+a_0} dt\,\frac{\sin^+(\tau\ln(t))}{t^\sigma} && (175)\\
&= \int_1^{e^{\frac{(2k_n)\pi}{\tau}}} dt\,\frac{\sin^+(\tau\ln(t))}{t^\sigma} + e^{\frac{(2k_n)\pi(1-\sigma)}{\tau}} \int_1^{y_n} dt\,\frac{\sin^+(\tau\ln(t))}{t^\sigma} && (176)
\end{aligned}$$

From lemma 1.5, and since the sequence (y_n) converges to y, we can conclude that:

$$II(+,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\left(\frac{1}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right) e^{\frac{2(k_n)\pi(1-\sigma)}{\tau}} + e^{\frac{(2k_n)\pi(1-\sigma)}{\tau}} \int_1^{y} dt\,\frac{\sin^+(\tau\ln(t))}{t^\sigma} \tag{177}$$

And the same for:

$$II(-,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\left(\frac{e^{\frac{\pi(1-\sigma)}{\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right) e^{\frac{2(k_n)\pi(1-\sigma)}{\tau}} + e^{\frac{(2k_n)\pi(1-\sigma)}{\tau}} \int_1^{y} dt\,\frac{\sin^-(\tau\ln(t))}{t^\sigma} \tag{178}$$

$$JJ(+,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\left(\frac{1-\sigma}{\tau} + \frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}\right)e^{\frac{2\pi(1-\sigma)(k_n)}{\tau}} + e^{\frac{(2k_n)\pi(1-\sigma)}{\tau}}\int_1^y dt\,\frac{\cos^+(\tau\ln(t))}{t^\sigma} \quad (179)$$

$$JJ(-,\sigma,\tau,n) \sim_{n\to+\infty} K(\sigma,\tau)\frac{e^{\frac{\pi(1-\sigma)}{2\tau}}}{e^{\frac{\pi(1-\sigma)}{\tau}}-1}e^{\frac{2\pi(1-\sigma)(k_n)}{\tau}} + e^{\frac{(2k_n)\pi(1-\sigma)}{\tau}}\int_1^y dt\,\frac{\cos^-(\tau\ln(t))}{t^\sigma} \quad (180)$$

And the same for $1-\sigma$:

$$II(+,1-\sigma,\tau,n) \sim_{n\to+\infty} K(1-\sigma,\tau)\left(\frac{1}{e^{\frac{\pi(\sigma)}{\tau}}-1}\right)e^{\frac{2(k_n)\pi(\sigma)}{\tau}} + e^{\frac{(2k_n)\pi(\sigma)}{\tau}}\int_1^y dt\,\frac{\sin^+(\tau\ln(t))}{t^{1-\sigma}} \quad (181)$$

$$II(-,1-\sigma,\tau,n) \sim_{n\to+\infty} K(1-\sigma,\tau)\left(\frac{e^{\frac{\pi(\sigma)}{\tau}}}{e^{\frac{\pi(\sigma)}{\tau}}-1}\right)e^{\frac{2(k_n)\pi(\sigma)}{\tau}} + e^{\frac{(2k_n)\pi(\sigma)}{\tau}}\int_1^y dt\,\frac{\sin^-(\tau\ln(t))}{t^{1-\sigma}} \quad (182)$$

$$JJ(+,1-\sigma,\tau,n) \sim_{n\to+\infty} K(1-\sigma,\tau)\left(\frac{\sigma}{\tau} + \frac{e^{\frac{\pi(\sigma)}{2\tau}}}{e^{\frac{\pi(\sigma)}{\tau}}-1}\right)e^{\frac{2\pi(\sigma)(k_n)}{\tau}} + e^{\frac{(2k_n)\pi(\sigma)}{\tau}}\int_1^y dt\,\frac{\cos^+(\tau\ln(t))}{t^{1-\sigma}} \quad (183)$$

$$JJ(-,1-\sigma,\tau,n) \sim_{n\to+\infty} K(1-\sigma,\tau)\frac{e^{\frac{\pi(\sigma)}{2\tau}}}{e^{\frac{\pi(\sigma)}{\tau}}-1}e^{\frac{2\pi(\sigma)(k_n)}{\tau}} + e^{\frac{(2k_n)\pi(\sigma)}{\tau}}\int_1^y dt\,\frac{\cos^-(\tau\ln(t))}{t^{1-\sigma}} \quad (184)$$

Since $\sigma < 1-\sigma$, we keep only the terms of $e^{\frac{2\pi(1-\sigma)(k_n)}{\tau}}$ in the equation (165-168). Therefore, we have:

$$II^2(+,\sigma,\tau,n) + II^2(-,\sigma,\tau,n) + JJ^2(+,\sigma,\tau,n) + JJ^2(-,\sigma,\tau,n) \quad (185)$$
$$\sim_{n\to+\infty} 2\,II(+,\sigma,\tau,n)\,II(-,\sigma,\tau,n) + 2\,JJ(+,\sigma,\tau,n)\,JJ(-,\sigma,\tau,n) \quad (186)$$

From the equations (177-180), we conclude:

$$\left(-K(\sigma,\tau) + \int_1^y dt\,\frac{\sin(\tau\ln(t))}{t^\sigma}\right)^2 + \left(\frac{(1-\sigma)}{\tau}K(\sigma,\tau) + \int_1^y dt\,\frac{\cos(\tau\ln(t))}{t^\sigma}\right)^2 = 0 \quad (187)$$

From lemma 1.1, we calculate the two integrals, and after simplification, we get:

$$\underbrace{\sin^2(\tau\ln(y)) + \cos^2(\tau\ln(y))}_{=1} = 0 \quad (188)$$

Which is a contradiction. Hence σ must be equal to $1-\sigma$.

Therefore $\sigma = \frac{1}{2}$. This ends the proof of the Riemann Hypothesis. ■.

□

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$*. In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

Acknowledgments

I would like to thank Farhat Latrach, Giampiero Esposito, Jacques Gélinas, Michael Milgram, Léo Agélas, Ronald F. Fox, Kim Y.G, Masumi Nakajima, Maksym Radziwill and Shekhar Suman for thoughtful comments and discussions on my paper versions on the RH. All errors are mine.

References

[1] Bernhard Riemann. *On the Number of Prime Numbers less than a Given Quantity*
https://www.claymath.org/sites/default/files/ezeta.pdf

[2] Aleksandar Ivic. *The Riemann Zeta-Function: Theory and Applications*

[3] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*
http://wayback.cecm.sfu.ca/~pborwein/TEMP_PROTECTED/book.pdf

[4] Jørgen Veisdal. *The Riemann Hypothesis, explained*
https://medium.com/cantors-paradise/the-riemann-hypothesis-explained-fa01c1f75d3f

[5] Thai Pham. *Dirichlet's Theorem on Arithmetic Progressions*
https://web.stanford.edu/ thaipham/papers/MIT_18.104_Review_Paper.pdf

[6] G. H. Hardy. *The general theory of dirichlet series.*
https://archive.org/details/generaltheoryofd029816mbp/page/n9

[7] Garrett, Paul. *Primes in arithmetic progressions, 2011.*
http : //www.math.umn.edu/ garrett/m/mfms/notes_c/dirichlet.pdf