A Simple Proof of the Riemann's Hypothesis

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Abstract

We present a simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

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1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

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Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x. There is another way [2] to analytically continue $\zeta(s)$ to the region $\Re(s) > 0$. The idea is to observe that for $\Re(s) > 1$:

$$(1 - \frac{2}{2^s})\zeta(s) \quad = \quad \sum_{n=1}^{+\infty} \frac{1}{n^s} - \sum_{n=1}^{+\infty} \frac{2}{(2n)^s} \tag{3}$$

Thus,

$$\zeta(s) \quad = \quad (1 - \frac{2}{2^s})^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \tag{4}$$

$$= \quad (1 - \frac{2}{2^s})^{-1}\eta(s) \tag{5}$$

The Dirichlet eta function $\eta(s)$ converges conditionally when $\Re(s) > 0$ and $s \neq 1 + i\frac{2k\pi}{\ln(2)}$. $\eta(s)$ is used as analytical continuation of the Zeta function on the domain where $\Re(s) > 0$. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \tag{7}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0$, $\forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

[1] This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s) \tag{6}$$

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{8}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{9}$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \tag{10}$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma-1)^2 + \tau^2} = \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x)) - i \sin(\tau \ln(x))\Big)\{x\}}{x^{\sigma+1}} dx \tag{11}$$

The integral is absolutely convergent. We take the real part and the imaginary part of both sides of the above equation and define the functions F and G as following:

$$\begin{aligned} F(\sigma, \tau) &= \frac{\sigma - 1}{(\sigma-1)^2 + \tau^2} && (12) \\ &= \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x))\Big)\{x\}}{x^{\sigma+1}} dx && (13) \end{aligned}$$

And

$$\begin{aligned} G(\sigma, \tau) &= \frac{\tau}{(\sigma-1)^2 + \tau^2} && (14) \\ &= \int_1^{+\infty} \frac{\Big(\sin(\tau \ln(x))\Big)\{x\}}{x^{\sigma+1}} dx && (15) \end{aligned}$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$\begin{aligned} F(1-\sigma, \tau) &= \frac{\sigma_1 - 1}{(\sigma_1-1)^2 + \tau_1^2} && (16) \\ &= \frac{-\sigma}{\sigma^2 + \tau^2} && (17) \\ &= \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x))\Big)\{x\}}{x^{2-\sigma}} dx && (18) \end{aligned}$$

And

$$G(1-\sigma,\tau) = \frac{\tau_1}{(\sigma_1-1)^2+\tau_1^2} \tag{19}$$

$$= \frac{\tau}{\sigma^2+\tau^2} \tag{20}$$

$$= \int_1^{+\infty} \frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx \tag{21}$$

From now on, σ and τ are not necessarily connected to a ζ zero. Before we move forward, we need to define the following function $I(a,\sigma,\tau)$ for $a>0$:

$$I(a,\sigma,\tau) = \int_1^a \frac{\sin(\tau\ln(x))}{x^\sigma}dx \tag{22}$$

$$= \int_0^{\ln(a)} \sin(\tau x)e^{(1-\sigma)x}dx \tag{23}$$

$$= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{24}$$

Where

$$K(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{25}$$

Let's define the function $J(a,\sigma,\tau)$ for $a>0$:

$$J(a,\sigma,\tau) = \int_1^a \frac{\cos(\tau\ln(x))}{x^\sigma}dx \tag{26}$$

$$= \int_0^{\ln(a)} \cos(\tau x)e^{(1-\sigma)x}dx \tag{27}$$

$$= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(a))}{a^{\sigma-1}}+\frac{\sin(\tau\ln(a))}{a^{\sigma-1}}\Big) \tag{28}$$

Where $K(\sigma,\tau)$ is defined above in the equation (25).

Now, let's write $U(\sigma,\tau)$ as the limit of a sequence as follows:

$$U(\sigma,\tau) = \lim_{N\to+\infty}\int_1^N \frac{\{x\}\sin(\tau\ln(x))}{x^{1+\sigma}}dx \tag{29}$$

$$= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\ \sin(x)\frac{\epsilon(e^{\frac{x}{\tau}})e^{-\frac{\sigma}{\tau}x}}{\tau} \tag{30}$$

$$= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\, g(x)\sin(x) \tag{31}$$

$$= \lim_{N\to+\infty} U(N,\sigma) \tag{32}$$

Where

$$U(N,\sigma) = \int_0^{\tau\ln(N)} dx\, g(x)\sin(x) \tag{33}$$

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau}e^{-\frac{\sigma}{\tau}x} \tag{34}$$

$$\epsilon(x) = \{x\} \tag{35}$$

And the same for $V(\sigma,\tau)$:

$$\begin{aligned}
V(\sigma,\tau) &= \lim_{N\to+\infty}\int_1^N \frac{\{x\}\cos(\tau\ln(x))}{x^{1+\sigma}}dx && (36)\\
&= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\ \cos(x)\frac{\epsilon(e^{\frac{x}{\tau}})e^{-\frac{\sigma}{\tau}x}}{\tau} && (37)\\
&= \lim_{N\to+\infty}\int_0^{\tau\ln(N)} dx\, g(x)\cos(x) && (38)\\
&= \lim_{N\to+\infty} V(N,\sigma) && (39)
\end{aligned}$$

Where

$$V(N,\sigma) = \int_0^{\tau\ln(N)} dx\, g(x)\cos(x) \qquad (40)$$

Lemma 1.1. *The sequence* $\left(\sum_{n=1}^N \frac{\cos(\tau\ln(n))}{n^\sigma} - J(N,\sigma,\tau)\right)_{N\geq 1}$ *converges and its limit is as follows:*

$$\lim_{N\to+\infty}\sum_{n=1}^N \frac{\cos(\tau\ln(n))}{n^\sigma} - J(N,\sigma,\tau) = 1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau) \qquad (41)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\cos(\tau\ln(n))}{n^\sigma} = J(N,\sigma,\tau)+1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N(\sigma,\tau) \qquad (42)$$

Where (α_N) *is a bounded sequence with limit zero.*
The sequence $\left(\sum_{n=1}^N \frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau)\right)_{N\geq 1}$ *converges and its limit is as follows:*

$$\lim_{N\to+\infty}\sum_{n=1}^N \frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau) = \tau V(\sigma,\tau)-\sigma U(\sigma,\tau) \qquad (43)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\sin(\tau\ln(n))}{n^\sigma} = I(N,\sigma,\tau)+\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N(\sigma,\tau) \qquad (44)$$

Where (β_N) *is a bounded sequence with limit zero.* ■

Method One

Proof. Let's study the function g over $\mathbb{R}$. We have the function g piecewise continuous and and its derivatives are also peicewise continuous and:

$$\begin{aligned}
g(x) &= \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau}e^{-\frac{\sigma}{\tau}x} && (45)\\
g\prime(x) &= \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}})-\frac{\sigma}{\tau}g(x) && (46)
\end{aligned}$$

The interval $(0,\tau\ln(N))$ is the union of the intervals $(\tau\ln(n),\tau\ln(n+1))$ where $1\leq n<N$. We have $\epsilon\prime(x)=1$ for each $x\in(\tau\log(n),\tau\log(n+1))$ for each $1\leq n<N$. Therefore, thanks to the integration by parts we can write:

$$
\begin{aligned}
U(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, g(x)\sin(x) && (47)\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{N-1}\int_{\tau\ln(n)+\epsilon}^{\tau\ln(n+1)-\epsilon} dx\, g(x)\sin(x) && (48)\\
&= g(0^+)\cos(0)+\lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[g(\tau\ln(n)+\epsilon)\cos(\tau\ln(n)+\epsilon)-g(\tau\ln(n)-\epsilon)\cos(\tau\ln(n)-\epsilon)\Big] && (49)\\
&\quad -g(\tau\ln(N)^-)\cos(\tau\ln(N))+\int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) && (50)\\
&= \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[g(\tau\ln(n)+\epsilon)-g(\tau\ln(n)-\epsilon)\Big]\cos(\tau\ln(n)) && (51)\\
&\quad -\frac{\cos(\tau\ln(N))}{\tau N^\sigma}+\int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) && (52)\\
&= -\frac{1}{\tau}\sum_{n=2}^{N-1}\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{\cos(\tau\ln(N))}{\tau N^\sigma}+\int_0^{\tau\ln(N)} dx\, g\prime(x)\cos(x) && (53)\\
&= -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{\sigma}{\tau}\int_0^{\tau\ln(N)} dx\, g(x)\cos(x)+\int_0^{\tau\ln(N)} dx\left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}})\right)\cos(x) && (54)\\
&= -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{\sigma}{\tau}V(N,\sigma)+\int_0^{\tau\ln(N)} dx\,\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\cos(x) && (55)
\end{aligned}
$$

Therefore

$$
U(N,\sigma)+\frac{\sigma}{\tau}V(N,\sigma) = -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}+\frac{1}{\tau}J(N,\sigma,\tau) \qquad (56)
$$

Therefore

$$
\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau) = -\tau U(N,\sigma)-\sigma V(N,\sigma) \qquad (57)
$$

Therefore, the sequence $\left(\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau)\right)_{N\geq 1}$ converges and its limit is as follows:

$$
\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau) = 1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau) \qquad (58)
$$

Therefore we can write the following:

$$\sum_{n=1}^{N} \frac{\cos(\tau\ln(n))}{n^{\sigma}} = J(N,\sigma,\tau) + 1 - \tau U(\sigma,\tau) - \sigma V(\sigma,\tau) + \alpha_N \tag{59}$$

Where (α_N) is a bounded sequence with limit zero.
And the same for $V(N,\sigma)$ as follows:

$$\begin{aligned}
V(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, g(x)\cos(x) && (60)\\
&= -g(0^+)\sin(0) + \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-g(\tau\ln(n)+\epsilon)\sin(\tau\ln(n)+\epsilon) + g(\tau\ln(n)-\epsilon)\sin(\tau\ln(n)-\epsilon)\Big] && (61)\\
&\quad + g(\tau\ln(N)^-)\sin(\tau\ln(N)) - \int_0^{\tau\ln(N)} dx\, g\prime(x)\sin(x) && (62)\\
&= \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-g(\tau\ln(n)+\epsilon) + g(\tau\ln(n)-\epsilon)\Big]\sin(\tau\ln(n)) && (63)\\
&\quad + \frac{\sin(\tau\ln(N))}{\tau N^{\sigma}} - \int_0^{\tau\ln(N)} dx\, g\prime(x)\sin(x) && (64)\\
&= \frac{1}{\tau}\sum_{n=2}^{N-1}\frac{\sin(\tau\ln(n))}{n^{\sigma}} + \frac{\sin(\tau\ln(N))}{\tau N^{\sigma}} - \int_0^{\tau\ln(N)} dx\, g\prime(x)\sin(x) && (65)\\
&= \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} + \frac{\sigma}{\tau}\int_0^{\tau\ln(N)} dx\, g(x)\sin(x) - \int_0^{\tau\ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}})\right)\sin(x) && (66)\\
&= \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} + \frac{\sigma}{\tau}U(N,\sigma) - \int_0^{\tau\ln(N)} dx\, \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\sin(x) && (67)
\end{aligned}$$

Therefore

$$V(N,\sigma) - \frac{\sigma}{\tau}U(N,\sigma) = \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} - \frac{1}{\tau}I(N,\sigma,\tau) \tag{68}$$

Therefore

$$\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} - I(N,\sigma,\tau) = \tau V(N,\sigma) - \sigma U(N,\sigma) \tag{69}$$

Therefore, the sequence $\left(\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} - I(N,\sigma,\tau)\right)_{N\geq 1}$ converges and its limit is as follows:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} - I(N,\sigma,\tau) = \tau V(\sigma,\tau) - \sigma U(\sigma,\tau) \tag{70}$$

Therefore we can write the following:

$$\sum_{n=1}^{N} \frac{\sin(\tau \ln(n))}{n^{\sigma}} = I(N,\sigma,\tau) + \tau V(\sigma,\tau) - \sigma U(\sigma,\tau) + \beta_N \tag{71}$$

Where (β_N) is a bounded sequence with limit zero. ■

□

Let's now define the functions $F_{\sigma}(x)$ and $G_{\sigma}(x)$ over $[1,+\infty)$ as follows:

$$F_{\sigma}(x) = \int_1^x \frac{\{x\}\sin(\tau \ln(x))}{x^{\sigma}} dx \tag{72}$$

And

$$G_{\sigma}(x) = \int_1^x \frac{\sin(\tau \ln(x))}{x^{\sigma}} dx \tag{73}$$

We have the function $x \to \frac{\{x\}\sin(\tau \ln(x))}{x^{\sigma}}$ piecewise continuous. The primitive function F_{σ} is continuous and its first derivative is piecewise continuous. The function G_{σ} is continuous and its first derivative is continuous.

Lemma 1.2. *Let's σ and τ be such that $\sigma > 0$ and $\tau > 0$. Therefore for each integer $N \geq 1$:*

$$F_{1+\sigma}(N) = \sum_{n=1}^{N} G_{1+\sigma}(n) - N\, G_{1+\sigma}(N) + G_{\sigma}(N) \tag{74}$$

Proof. Let's take an integer $N \geq 1$. We have

$$F_{1+\sigma}(N) = \int_1^N \frac{\{x\}\sin(\tau \ln(x))}{x^{1+\sigma}} dx \tag{75}$$

$$= \sum_{n=1}^{N-1} \int_n^{n+1} \frac{\{x\}\sin(\tau \ln(x))}{x^{1+\sigma}} dx \tag{76}$$

$$= \sum_{n=1}^{N-1} \int_n^{n+1} \frac{(x-n)\sin(\tau \ln(x))}{x^{1+\sigma}} dx \tag{77}$$

$$= \sum_{n=1}^{N-1} \int_n^{n+1} \frac{\sin(\tau \ln(x))}{x^{\sigma}} dx - n \int_n^{n+1} \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} dx \tag{78}$$

$$= \int_1^N \frac{\sin(\tau \ln(x))}{x^{\sigma}} dx - \sum_{n=1}^{N-1} n\Big(G_{1+\sigma}(n+1) - G_{1+\sigma}(n)\Big) \tag{79}$$

$$= G_{\sigma}(N) - \sum_{n=1}^{N-1} n\Big(G_{1+\sigma}(n+1) - G_{1+\sigma}(n)\Big) \tag{80}$$

$$= G_{\sigma}(N) - \sum_{n=1}^{N-1} n\, G_{1+\sigma}(n+1) + \sum_{n=1}^{N-1} n\, G_{1+\sigma}(n) \tag{81}$$

$$= \sum_{n=1}^{N} G_{1+\sigma}(n) - N\, G_{1+\sigma}(N) + G_{\sigma}(N) \tag{82}$$

■

□

Lemma 1.3. *Let's σ be such that $0 < \sigma < \frac{1}{2}$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$\lim_{N\to+\infty} F_{2-\sigma}(N) - F_{1+\sigma}(N) = 0 \tag{83}$$

Proof. Let's take an integer $N \geq 1$. We have from the lemma 1.2 that:

$$F_{2-\sigma}(N) - F_{1+\sigma}(N) = \sum_{n=1}^{N}\Big(G_{2-\sigma}(n) - G_{1+\sigma}(n)\Big) - N\left(G_{2-\sigma}(n) - G_{1+\sigma}(N)\right) \tag{84}$$

$$+\Big(G_{1-\sigma}(N) - G_{\sigma}(N)\Big) \tag{85}$$

Let's deal now with the first term in the equation(85-86) above:

$$\sum_{n=1}^{N}\Big(G_{2-\sigma}(n) - G_{1+\sigma}(n)\Big) = \sum_{n=1}^{N}\Big(I(n, 2-\sigma, \tau) - I(n, 1+\sigma, \tau)\Big) \tag{86}$$

$$= N\Big(K(\sigma,\tau) - K(1-\sigma,\tau)\Big) \tag{87}$$

$$+K(1-\sigma,\tau)\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^{\sigma}} + K(1-\sigma,\tau)\frac{\sigma}{\tau}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} \tag{88}$$

$$-K(\sigma,\tau)\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^{1-\sigma}} - K(\sigma,\tau)\frac{1-\sigma}{\tau}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{1-\sigma}} \tag{89}$$

From the lemma 1.1 we can write the following:

$$\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^{\sigma}} = J(N,\sigma,\tau) + 1 - \tau U(\sigma,\tau) - \sigma V(\sigma,\tau) + \alpha_N(\sigma,\tau) \tag{90}$$

$$\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{\sigma}} = I(N,\sigma,\tau) + \tau V(\sigma,\tau) - \sigma U(\sigma,\tau) + \beta_N(\sigma,\tau) \tag{91}$$

$$\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^{1-\sigma}} = J(N,1-\sigma,\tau) + 1 - \tau U(1-\sigma,\tau) - (1-\sigma)V(1-\sigma,\tau) + \alpha_N(1-\sigma,\tau) \tag{92}$$

$$\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^{1-\sigma}} = I(N,1-\sigma,\tau) + \tau V(1-\sigma,\tau) - (1-\sigma)U(1-\sigma,\tau) + \beta_N(1-\sigma,\tau) \tag{93}$$

Where $(\alpha_N(\sigma,\tau))$, $(\beta_N(\sigma,\tau))$, $(\alpha_N(1-\sigma,\tau))$, $(\beta_N(1-\sigma,\tau))$ are bounded sequences with limit zero.

Therefore

$$\begin{aligned}
\sum_{n=1}^{N}\Big(G_{2-\sigma}(n)-G_{1+\sigma}(n)\Big) &= N\Big(K(\sigma,\tau)-K(1-\sigma,\tau)\Big) && (94)\\
&+K(1-\sigma,\tau)\Bigg[J(N,\sigma,\tau)+1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N(\sigma,\tau)\Bigg] && (95)\\
&+K(1-\sigma,\tau)\frac{\sigma}{\tau}\Bigg[I(N,\sigma,\tau)+\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N(\sigma,\tau)\Bigg] && (96)\\
&-K(\sigma,\tau)\Bigg[J(N,1-\sigma,\tau)+1-\tau U(1-\sigma,\tau)-(1-\sigma)V(1-\sigma,\tau)+\alpha_N(1-\sigma,\tau)\Bigg] && (97)\\
&-K(\sigma,\tau)\frac{1-\sigma}{\tau}\Bigg[I(N,1-\sigma,\tau)+\tau V(1-\sigma,\tau)-(1-\sigma)U(1-\sigma,\tau)+\beta_N(1-\sigma,\tau)\Bigg] && (98)
\end{aligned}$$

Where

$$\begin{aligned}
I(N,\sigma,\tau) &= K(\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}-\frac{(\sigma-1)}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\Big) && (99)\\
J(N,\sigma,\tau) &= K(\sigma,\tau)\Big(\frac{(\sigma-1)}{\tau}-\frac{(\sigma-1)}{\tau}\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+\frac{\sin(\tau\ln(N))}{N^{\sigma-1}}\Big) && (100)\\
I(N,1-\sigma,\tau) &= K(1-\sigma,\tau)\Big(1-\frac{\cos(\tau\ln(N))}{N^{-\sigma}}+\frac{(\sigma)}{\tau}\frac{\sin(\tau\ln(N))}{N^{-\sigma}}\Big) && (101)\\
J(N,1-\sigma,\tau) &= K(1-\sigma,\tau)\Big(\frac{(-\sigma)}{\tau}+\frac{(\sigma)}{\tau}\frac{\cos(\tau\ln(N))}{N^{-\sigma}}+\frac{\sin(\tau\ln(N))}{N^{-\sigma}}\Big) && (102)
\end{aligned}$$

$$K(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{103}$$

Here we have $s=\sigma+i\tau$ is a zeta zero, therefore:

$$U(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2}=K(\sigma,\tau) \tag{104}$$

And

$$V(\sigma,\tau) = \frac{\sigma-1}{(\sigma-1)^2+\tau^2}=K(\sigma,\tau)\frac{\sigma-1}{\tau} \tag{105}$$

And the second term in the equation(85-86):

$$\begin{aligned}
N\Big(G_{2-\sigma}(n)-G_{1+\sigma}(N)\Big) &= N\Big(K(\sigma,\tau)-K(1-\sigma,\tau)\Big) && (106)\\
&+K(1-\sigma,\tau)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+K(1-\sigma,\tau)\frac{\sigma}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} && (107)\\
&-K(\sigma,\tau)\frac{\cos(\tau\ln(N))}{N^{-\sigma}}-K(\sigma,\tau)\frac{1-\sigma}{\tau}\frac{\sin(\tau\ln(N))}{N^{-\sigma}} && (108)
\end{aligned}$$

And the third term in the equation(85-86):

$$\begin{aligned}
\Big(G_{1-\sigma}(n)-G_{\sigma}(N)\Big) &= \Big(K(1-\sigma,\tau)-K(\sigma,\tau)\Big) && (109)\\
&+K(\sigma,\tau)\frac{\cos(\tau\ln(N))}{N^{\sigma-1}}+K(\sigma,\tau)\frac{\sigma-1}{\tau}\frac{\sin(\tau\ln(N))}{N^{\sigma-1}} && (110)\\
&-K(1-\sigma,\tau)\frac{\cos(\tau\ln(N))}{N^{-\sigma}}-K(1-\sigma,\tau)\frac{-\sigma}{\tau}\frac{\sin(\tau\ln(N))}{N^{-\sigma}} && (111)
\end{aligned}$$

We inject the equations (99-102) into the equation (95-98). After simplification, and taking into accounts the equations (106-108) and (109-111), we simplify the equations(85-86) into the following:

$$
\begin{aligned}
F_{2-\sigma}(N) - F_{1+\sigma}(N) &= \frac{3(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) + K(1-\sigma,\tau)\alpha_N(\sigma,\tau) & (112)\\
&+K(1-\sigma,\tau)\frac{\sigma}{\tau}\beta_N(\sigma,\tau) - K(\sigma,\tau)\alpha_N(1-\sigma,\tau) - K(\sigma,\tau)\frac{1-\sigma}{\tau}\beta_N(1-\sigma,\tau) & (113)
\end{aligned}
$$

Thanks to lemma 1.1, the limit of the sequences $(\beta_N(\sigma,\tau))$, $(\beta_N(1-\sigma,\tau))$, $(\alpha_N\sigma,\tau)$ and $(1-\alpha_N\sigma,\tau)$ is zero. Therefore:

$$
\lim_{N\to+\infty} F_{2-\sigma}(N) - F_{1+\sigma}(N) = \frac{3(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) \quad (114)
$$

■ □

Lemma 1.4. *Let's σ be such that $0 < \sigma < \frac{1}{2}$ and $s = \sigma + i\tau$ is a zeta zero. Therefore:*

$$
\sigma = \frac{1}{2} \quad (115)
$$

And the Riemann's Hypothesis is true.

Proof. From the lemma 1.3, we have:

$$
\lim_{N\to+\infty} F_{2-\sigma}(N) - F_{1+\sigma}(N) = \frac{3(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) \quad (116)
$$

Also, we have $s = \sigma + i\tau$ is a zeta zero, therefore:

$$
\begin{aligned}
\lim_{N\to+\infty} F_{2-\sigma}(N) &= K(1-\sigma,\tau) & (117)\\
\lim_{N\to+\infty} F_{1+\sigma}(N) &= K(\sigma,\tau) & (118)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\lim_{N\to+\infty} F_{2-\sigma}(N) - F_{1+\sigma}(N) &= K(1-\sigma,\tau) - K(\sigma,\tau) & (119)\\
&= \frac{(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) & (120)
\end{aligned}
$$

From the equations (116) and (120) we have:

$$
\frac{3(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) = \frac{(1-2\sigma)}{\tau} K(\sigma,\tau)K(1-\sigma,\tau) \quad (121)
$$

Since $\tau > 0$, therefore:

$$
1 - 2\sigma = 0 \quad (122)
$$

Therefore

$$
\sigma = \frac{1}{2} \quad (123)
$$

Hence, the Riemann's Hypothesis is true. ■

□

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$*. In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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Appendix: Second Proof of Lemma 1.1

Method Two

Proof. Now, let's write $U(N,\sigma)$ using a function f instead of g as follows:

$$\begin{aligned} U(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, f(x)\epsilon(e^{\frac{x}{\tau}}) && (124)\\ f(x) &= \frac{\sin(x)}{\tau}e^{-\frac{\sigma}{\tau}x} && (125)\\ \epsilon(x) &= \{x\} && (126) \end{aligned}$$

And the same for $V(\sigma,\tau)$:

$$\begin{aligned} V(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, h(x)\epsilon(e^{\frac{x}{\tau}}) && (127)\\ h(x) &= \frac{\cos(x)}{\tau}e^{-\frac{\sigma}{\tau}x} && (128)\\ \epsilon(x) &= \{x\} && (129) \end{aligned}$$

We have the function $e(x) = \epsilon(e^{\frac{x}{\tau}})$ piecewise continuous and and its derivatives are also peicewise continuous and:

$$\begin{aligned} e(x) &= \epsilon(e^{\frac{x}{\tau}}) && (130)\\ e\prime(x) &= \frac{e^{\frac{x}{\tau}}}{\tau}\epsilon\prime(e^{\frac{x}{\tau}}) && (131) \end{aligned}$$

The interval $(0,\tau\ln(N))$ is the union of the intervals $(\tau\ln(n),\tau\ln(n+1))$ where $1 \leq n < N$. We have $\epsilon\prime(x) = 1$ for each $x \in (\tau\log(n),\tau\log(n+1))$ for each $1 \leq n < N$. Therefore, thanks to the integration by parts we can write:

$$\begin{aligned}
U(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, f(x)e(x) && (132)\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{N-1}\int_{\tau\ln(n)+\epsilon}^{\tau\ln(n+1)-\epsilon} dx\, f(x)e(x) && (133)\\
&= -F(0^+)e(0) + \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-F(\tau\ln(n)+\epsilon)e(\tau\ln(n)+\epsilon) + F(\tau\ln(n)-\epsilon)e(\tau\ln(n)-\epsilon)\Big] && (134)\\
&\quad +F(\tau\ln(N)^-)e(\tau\ln(N)^-) - \int_0^{\tau\ln(N)} dx\, F(x)e\prime(x) && (135)\\
&= \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-e(\tau\ln(n)+\epsilon) + e(\tau\ln(n)-\epsilon)\Big]F(\tau\ln(n)) && (136)\\
&\quad +F(\tau\ln(N)) + \int_0^{\tau\ln(N)} dx\, F(x)e\prime(x) && (137)\\
&= \sum_{n=2}^{N-1} F(\tau\ln(n)) + F(\tau\ln(N)) - \int_0^{\tau\ln(N)} dx\, F(x)e\prime(x) && (138)\\
&= \sum_{n=2}^{N} F(\tau\ln(n)) - \int_0^{\tau\ln(N)} dx\, \left(\frac{e^{\frac{x}{\tau}}}{\tau}e\prime(e^{\frac{x}{\tau}})\right)F(x) && (139)\\
&= \sum_{n=2}^{N} F(\tau\ln(n)) - \int_0^{\tau\ln(N)} dx\, \frac{e^{\frac{x}{\tau}}}{\tau}F(x) && (140)\\
&= \sum_{n=2}^{N} F(\tau\ln(n)) - \int_1^{N} dx\, F(\tau\ln(x)) && (141)
\end{aligned}$$

And the same for $V(N,\sigma)$:

$$\begin{aligned}
V(N,\sigma) &= \int_0^{\tau\ln(N)} dx\, h(x)e(x) && (142)\\
&= \sum_{n=2}^{N} H(\tau\ln(n)) - \int_1^{N} dx\, H(\tau\ln(x)) && (143)
\end{aligned}$$

Where F and H can calculated using the equations (24) and (28) as follows.

$$\begin{aligned}
F(\tau\ln(x)) &= \int_0^{\tau\ln(x)} dx\, f(x) && (144)\\
&= K(\sigma,\tau)\Big(1 - \frac{\cos(\tau\ln(x))}{x^\sigma} - \frac{(\sigma)}{\tau}\frac{\sin(\tau\ln(x))}{x^\sigma}\Big) && (145)
\end{aligned}$$

Where

$$K(\sigma,\tau) = \frac{\tau}{\sigma^2+\tau^2} \qquad (146)$$

And

$$\begin{aligned}
\sum_{n=2}^{N} F(\tau\ln(n)) &= K(\sigma,\tau)\sum_{n=2}^{N}\Big(1-\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{(\sigma)}{\tau}\frac{\sin(\tau\ln(n))}{n^\sigma}\Big) && (147)\\
&= K(\sigma,\tau)\Big(N-1-\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{(\sigma)}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma}\Big) && (148)\\
&= K(\sigma,\tau)\Big(N-\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-\frac{(\sigma)}{\tau}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma}\Big) && (149)\\
&= K(\sigma,\tau)\Big(N-\Big(J(N,\sigma,\tau)+1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N\Big) && (150)\\
&\quad -\frac{(\sigma)}{\tau}\Big(I(N,\sigma,\tau)+\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N\Big)\Big) && (151)\\
&= K(\sigma,\tau)\Bigg(N-J(N,\sigma,\tau)-\frac{(\sigma)}{\tau}I(N,\sigma,\tau) && (152)\\
&\quad -\Big(1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N\Big)-\frac{(\sigma)}{\tau}\Big(\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N\Big)\Bigg) && (153)
\end{aligned}$$

And

$$\begin{aligned}
\int_1^N dx\, F(\tau\ln(x)) &= K(\sigma,\tau)\int_1^N dx\,\Big(1-\frac{\cos(\tau\ln(x))}{x^\sigma}-\frac{(\sigma)}{\tau}\frac{\sin(\tau\ln(x))}{x^\sigma}\Big) && (154)\\
&= K(\sigma,\tau)\Big(N-1-\int_1^N dx\,\frac{\cos(\tau\ln(x))}{x^\sigma}-\frac{(\sigma)}{\tau}\int_1^N dx\,\frac{\sin(\tau\ln(x))}{x^\sigma}\Big) && (155)\\
&= K(\sigma,\tau)\Big(N-1-J(N,\sigma,\tau)-\frac{(\sigma)}{\tau}I(N,\sigma,\tau)\Big) && (156)
\end{aligned}$$

Therefore

$$\begin{aligned}
U(N,\sigma) &= \sum_{n=2}^{N} F(\tau\ln(n))-\int_1^N dx\, F(\tau\ln(x)) && (157)\\
&= K(\sigma,\tau)\Bigg(-\Big(0-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N\Big)-\frac{(\sigma)}{\tau}\Big(\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N\Big)\Bigg) && (158)\\
&= -K(\sigma,\tau)\Bigg(\Big(0-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N\Big)+\frac{(\sigma)}{\tau}\Big(\tau V(\sigma,\tau)-\sigma U(\sigma,\tau)+\beta_N\Big)\Bigg) && (159)\\
&= -K(\sigma,\tau)\Bigg(0-\Big(\tau+\frac{\sigma^2}{\tau}\Big)U(\sigma,\tau)-\alpha_N-\frac{(\sigma)}{\tau}\beta_N\Bigg) && (160)\\
&= U(\sigma,\tau)-\Bigg(\alpha_N+\frac{(\sigma)}{\tau}\beta_N\Bigg) && (161)\\
& && (162)
\end{aligned}$$

And

$$\begin{aligned}
H(\tau\ln(x)) &= \int_0^{\tau\ln(x)} dx\, h(x) && (163)\\
&= K(\sigma,\tau)\Big(\frac{(\sigma)}{\tau}-\frac{(\sigma)}{\tau}\frac{\cos(\tau\ln(x))}{x^\sigma}+\frac{\sin(\tau\ln(x))}{x^\sigma}\Big) && (164)
\end{aligned}$$

Therefore

$$U(N,\sigma)+\frac{\sigma}{\tau}V(N,\sigma) = -\frac{1}{\tau}\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}+\frac{1}{\tau}J(N,\sigma,\tau) \tag{165}$$

Therefore

$$\sum_{n=2}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau) = -\tau U(N,\sigma)-\sigma V(N,\sigma) \tag{166}$$

Therefore, the sequence $\Big(\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau)\Big)_{N\geq 1}$ converges and its limit is as follows:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma}-J(N,\sigma,\tau) = 1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau) \tag{167}$$

Therefore we can write the following:

$$\sum_{n=1}^{N}\frac{\cos(\tau\ln(n))}{n^\sigma} = J(N,\sigma,\tau)+1-\tau U(\sigma,\tau)-\sigma V(\sigma,\tau)+\alpha_N \tag{168}$$

Where (α_N) is a bounded sequence with limit zero.
And the same for $V(N,\sigma)$ as follows:

$$V(N,\sigma) = \int_0^{\tau\ln(N)}dx\, g(x)\cos(x) \tag{169}$$

$$= -g(0^+)\sin(0)+\lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-g(\tau\ln(n)+\epsilon)\sin(\tau\ln(n)+\epsilon)+g(\tau\ln(n)-\epsilon)\sin(\tau\ln(n)-\epsilon)\Big] \tag{170}$$

$$+g(\tau\ln(N)^-)\sin(\tau\ln(N))-\int_0^{\tau\ln(N)}dx\, g\prime(x)\sin(x) \tag{171}$$

$$= \lim_{\epsilon\to 0}\sum_{n=2}^{N-1}\Big[-g(\tau\ln(n)+\epsilon)+g(\tau\ln(n)-\epsilon)\Big]\sin(\tau\ln(n)) \tag{172}$$

$$+\frac{\sin(\tau\ln(N))}{\tau N^\sigma}-\int_0^{\tau\ln(N)}dx\, g\prime(x)\sin(x) \tag{173}$$

$$= \frac{1}{\tau}\sum_{n=2}^{N-1}\frac{\sin(\tau\ln(n))}{n^\sigma}+\frac{\sin(\tau\ln(N))}{\tau N^\sigma}-\int_0^{\tau\ln(N)}dx\, g\prime(x)\sin(x) \tag{174}$$

$$= \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma}+\frac{\sigma}{\tau}\int_0^{\tau\ln(N)}dx\, g(x)\sin(x)-\int_0^{\tau\ln(N)}dx\left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\epsilon\prime(e^{\frac{x}{\tau}})\right)\sin(x) \tag{175}$$

$$= \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma}+\frac{\sigma}{\tau}U(N,\sigma)-\int_0^{\tau\ln(N)}dx\,\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2}\sin(x) \tag{176}$$

Therefore

$$V(N,\sigma) - \frac{\sigma}{\tau}U(N,\sigma) \quad = \quad \frac{1}{\tau}\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - \frac{1}{\tau}I(N,\sigma,\tau) \tag{177}$$

Therefore

$$\sum_{n=2}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau) \quad = \quad \tau V(N,\sigma) - \sigma U(N,\sigma) \tag{178}$$

Therefore, the sequence $\left(\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau)\right)_{N\geq 1}$ converges and its limit is as follows:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} - I(N,\sigma,\tau) \quad = \quad \tau V(\sigma,\tau) - \sigma U(\sigma,\tau) \tag{179}$$

Therefore we can write the following:

$$\sum_{n=1}^{N}\frac{\sin(\tau\ln(n))}{n^\sigma} \quad = \quad I(N,\sigma,\tau) + \tau V(\sigma,\tau) - \sigma U(\sigma,\tau) + \beta_N \tag{180}$$

Where (β_N) is a bounded sequence with limit zero. ■

□

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