Spectral analysis of a non-homogeneous rotating Timoshenko beam

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In this paper we examine the spectral analysis of a spatially non-homogeneous Timoshenko beam mounted on the periphery of a rigid root rotating about its axis at a constant angular speed. The junction between the beam and the root is assumed to be elastically restrained and damped. The unbounded operator associated to the physical problem in the associated Hilbert space is non-self-adjoint and with a compact resolvent. We show that under some hypotheses on the physical properties of the beam, there exists a Riesz basis of root vectors of this unbounded operator. Furthermore, the solution of the initial value problem has an expansion in terms of this Riesz basis, uniform with respect to the time in a bounded interval.

1 Introduction

Flexible beams are fundamental components in many areas of structural engineering. The most commonly used beam models are those of Euler-Bernoulli, Rayleigh and Timoshenko. Among these, the Timoshenko beam model is the most advanced, because it includes the effects of rotary inertia and transverse shear deformation and is suitable for thick beams.

The present work is devoted to the spectral analysis of a rotating Timoshenko beam model. We consider a spatially non-homogeneous Timoshenko beam mounted on the periphery of a rigid root rotating about its axis (fixed in space) at a constant angular speed [17]. We assume that the junction between the beam and the root is elastically restrained and damped [16, 18–20]. The unbounded operator associated to the physical problem has a compact resolvent but due to the damping it is non-self-adjoint. For that reason one must take into account not only the eigenvectors of this operator but also its generalized eigenvectors or root vectors. A core problem is to show that the set of root vectors of this operator is complete in the Hilbert space associated to the physical problem. An other key issue is to prove that there exists a Riesz basis of root vectors of this operator in the associated Hilbert space. In this paper we show that under some hypotheses on the physical properties of the beam (see hypotheses 2.1, 3.6, 3.8, 3.12), there exists a Riesz basis of root vectors of this operator in the related Hilbert space.

The spectral analysis of the Timoshenko beam model has been extensively studied in the past few years. In the paper [29] the spectral analysis of a (non-rotating) Timoshenko beam model with the same boundary damping as ours is analyzed. In this paper the completeness of the set of root vectors of the unbounded operator associated to the physical problem is shown by applying the Keldysh theorem [6], p. 257, Theorem 8.1, [29], Theorem 6.4. Then applying a theorem of Bari [6], p. 311, Theorem 2.1-5, and a theorem of Carleson related the the Riesz basis property of a set of nonharmonic exponentials [26], p. 261, Theorem on Exponentials, the Riesz basis property of a sequence of root vectors of the aforementioned operator is shown [29], Theorem 7.1.

In papers [36], [4], [37], the Riesz basis property of a sequence of root vectors of the unbounded operator associated to the (non-rotating) Timoshenko beam model with similar boundary damping is examined. In these papers the authors apply a theorem of Guo [7], Theorem 1 based on a theorem of Bari [6], p. 317, Theorem 2.3.

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All the aforementioned papers about the Riesz basis property are based on a precise asymptotic of the root vectors.

In papers [39], [9], [8], [38], the spectral analysis of the (non-rotating) Timoshenko beam model with various boundary conditions is studied by applying a theorem from Xu [35], Theorem 1.1 which provides sufficient conditions for a sequence of root vectors of an unbounded operator to form a Riesz basis. The advantage of this theorem is that it does not require to estimate the asymptotic of the root vectors. The idea of the method carried out in the above-mentioned articles is basically to find the asymptotic of the spectrum of the associated unbounded operator by reducing the problem to a first-order system asymptotically linear in the spectral parameter, then to apply the Phragmén-Lindelöf principle to show that the set of root vectors of the operator is complete, and finally to apply the theorem from Xu [35], Theorem 1.1. This latter method sounds simpler than the former ones and for that reason it has been used in the present work.

The paper [21] is concerned with the completeness, minimality and the Riesz basis (with parentheses) property of a sequence of root vectors of boundary value problems for first order systems of ordinary differential equations on a bounded interval. The spectral problem discussed in [21] is linear in the spectral parameter. The results of this article have been applied to the spectral analysis of the (non-rotating) Timoshenko beam model with boundary damping. But these results cannot be applied directly to the rotating Timoshenko beam model, because of the terms coming from the rotation which lead to a first order system of ordinary differential equations only asymptotically linear (and not linear) in the spectral parameter. Like paper [21], the paper [33] treats the spectral analysis of boundary value problems for first order systems of ordinary differential equations on a bounded interval but the problem discussed in this article is linear in the spectral parameter and for the same reasons the results of [33] cannot be applied directly to the rotating Timoshenko beam model.

As far as the (mathematical) spectral analysis of a rotating Timoshenko beam model is concerned, to the best of our knowledge, very few research papers deal with this problem. Let us quote the papers [12, 13, 31, 32] where a very simplified model (without centrifugal force) is considered [12], p. 147. In the present work we have chosen the most common formulation of the rotating Timoshenko beam model given for example in [5], p. 279 or [1], Appendix A. This model takes into account flexural and shear deformations of the beam. More advanced models taking into account the coupling between extensional and flexural deformations can be found for example in [30] or [17]. Let us stress that the equations of the rotating Timoshenko beam model in these two articles are not exactly the same (compare equation (26) of [30] and equations (37), (38), (39) of [17]).

This paper is organized as follows. In Section 2, we outline a variational formulation of the problem and the corresponding equations and boundary conditions. In order to deal with a coercive form we make a change of variables regarding the dependance of the solution with respect to time. We write the equations as a first order evolution equation and we prove that the (space) unbounded operator in this evolution equation (denoted $\mathcal{A}$) has a compact resolvent. As a result its spectrum consists of isolated eigenvalues of finite algebraic multiplicity. Moreover this operator is maximal-dissipative and thus is the generator of a contraction semigroup. In Section 3 we study the asymptotics of its spectrum. To this end we make a change of variable in order to reduce the problem to a first-order system asymptotically linear in the spectral parameter. Then we apply a result of R. Mennicken and R. Möller [25] to get an asymptotic fundamental matrix function of this system from which we obtain the asymptotics of its spectrum under some hypotheses on the physical properties of the beam (see hypotheses 2.1, 3.6, 3.8). It is shown that under an additional hypothesis on the physical properties of the beam (see hypothesis 3.12), the eigenvalues of the first-order system are asymptotically algebraically simple. In order to prove that the same property is true for the operator $\mathcal{A}$ we have used some results of [25] and [24] on root functions of holomorphic families of operators. Let us point out that the approach consisting in comparing carefully the algebraic multiplicities of the root vectors of $\mathcal{A}$ and those of the root functions associated to the first order system is not usual in the literature on the spectral analysis of the Timoshenko beam model and this is an original feature of the present paper (see for example [9], [8], [38]). In Section 4 using estimates of the resolvent in different parts of the complex plane and the Phragmén-Lindelöf principle we prove that under the aforementioned hypotheses on the physical properties of the beam the set of root vectors of $\mathcal{A}$ is complete and that there is a system of root vectors of this operator which is minimal (Theorem 4.3). Using the theorem from Xu [35], Theorem 1.1, we prove that under the same hypotheses there exists a sequence of root vectors of $\mathcal{A}$ which forms a Riesz basis (Theorem 4.5). As a consequence one can obtain an asymptotic expansion of the solution of the related (hyperbolic) initial valued problem. This expansion is uniform with respect to the time in a bounded interval (Theorem 4.7).

Comparisons of the hypotheses and results of the present paper and other papers in the literature are given in Remark 4.6.

2 Formulation of the rotating Timoshenko beam model

In what follows, R is the (constant) radius of the root, Ω its (constant) angular speed, l the (constant) length of the beam, S its cross-sectional area, E its Young modulus, ρ its mass density, J the moment of inertia of its cross-section, G its shear modulus, $\alpha_s > 0$ its (constant) shear correction factor, S, E, ρ, J and G depending on the abscissa x on the beam. The infinitesimal transverse displacement will be denoted by w, the infinitesimal rotation of the cross-section by φ, the corresponding (constant) spring constants at the junction by k_w and k_φ and the corresponding (constant) damping constants at the junction by c_w and c_φ ($k_w > 0$, $k_\varphi > 0$, $c_w > 0$, $c_\varphi > 0$). The centrigugal force $P(x)$ is given by (see [5], p. 279, [1], Appendix A)

$$P(x) = \int_x^l \rho S\Omega^2(R+y)dy\,(\geq 0),\ x \in [0,l]. \tag{2.1}$$

Hereafter, the derivative with respect to the time will be denoted by a dot and the derivative with respect the spatial variable by a prime.

The principle of virtual work for this rotating Timoshenko beam model is formally written as follows: find w, φ sufficiently smooth functions with values in $\mathbb{C}$ such that for all w_*, φ_* sufficiently smooth functions with values in $\mathbb{C}$,

$$\begin{aligned}&\int_0^l [Pw'\overline{w_*}' + EJ\varphi'\overline{\varphi_*}']dx + \int_0^l \alpha_s GS[(w'-\varphi)\overline{w_*}' - (w'-\varphi)\overline{\varphi_*}]dx + \\ &\int_0^l \rho J[\ddot{\varphi} - \Omega^2\varphi]\overline{\varphi_*}dx + \int_0^l \rho S\ddot{w}\overline{w_*}dx = \\ &-[k_w w + c_w\dot{w}](0)\overline{w_*}(0) - [k_\varphi\varphi + c_\varphi\dot{\varphi}](0)\overline{\varphi_*}(0).\end{aligned} \tag{2.2}$$

One gets formally the following equations (see [5], p. 279, [1], Appendix A)

$$(Pw')' + [\alpha_s GS(w'-\varphi)]' = \rho S\ddot{w}, \tag{2.3}$$

$$(EJ\varphi')' = \rho J(\ddot{\varphi} - \varphi\Omega^2) - \alpha_s GS(w'-\varphi), \tag{2.4}$$

and boundary conditions

$$(Pw')(0) + [\alpha_s GS(w'-\varphi)](0) = [k_w w + c_w\dot{w}](0), (EJ\varphi')(0) = [k_\varphi\varphi + c_\varphi\dot{\varphi}](0), \tag{2.5}$$

$$(Pw')(l) + [\alpha_s GS(w'-\varphi)](l) = 0, (EJ\varphi')(l) = 0. \tag{2.6}$$

Let us recall some definitions. For $m \in \mathbb{N}$, $H^m(0,l)$ is the space of functions u in $L^2(0,l)$ such that the derivatives of u in the sense of distributions up to order m are in $L^2(0,l)$ (in particular $H^0(0,l) = L^2(0,l)$). The space $W^{m,\infty}(0,l)$ is the space of functions $u \in L^\infty(0,l)$ such that the derivatives of u in the sense of distributions up to order m are in $L^\infty(0,l)$. A sesquilinear form a on a Hilbert space V is coercive if there exists a constant $C > 0$ such that $\mathrm{Re}(a(u,u)) \geq C||u||_V^2$, $\forall u \in V$. In the sequel we will make the following hypothesis

Hypothesis 2.1 $S, E, J, G \in W^{1,\infty}(0,l)$, $\rho \in L^\infty(0,l)$ and there exists a constant $C > 0$ such that $S \geq C$, $E \geq C$, $J \geq C$, $G \geq C$, $\rho \geq C$ on $(0,l)$.

For $m \in \mathbb{N}$, set $H^m = H^m(0,l) \otimes \mathbb{C}^2$ (in particular $H^0 = L^2(0,l) \otimes \mathbb{C}^2$). With the notations

$$\mathbf{u} = \begin{pmatrix} w \\ \varphi \end{pmatrix}, \mathbf{u}_* = \begin{pmatrix} w_* \\ \varphi_* \end{pmatrix}, \tag{2.7}$$

define the Hermitian forms a_0, a and b on H^1 by: for all $\mathbf{u}$, $\mathbf{u}_* \in H^1$,

$$a_0(\mathbf{u},\mathbf{u}_*) = \int_0^l [Pw'\overline{w_*}' + EJ\varphi'\overline{\varphi_*}']dx + \int_0^l \alpha_s GS[(w'-\varphi)\overline{w_*}' - (w'-\varphi)\overline{\varphi_*}]dx + k_w w(0)\overline{w_*}(0) + k_\varphi\varphi(0)\overline{\varphi_*}(0), \tag{2.8}$$

$$a(\mathbf{u},\mathbf{u}_*) = a_0(\mathbf{u},\mathbf{u}_*) - \Omega^2\int_0^l \rho J\varphi\overline{\varphi_*}dx, \tag{2.9}$$

$$b(\mathbf{u},\mathbf{u}_*) = c_w w(0)\overline{w_*}(0) + c_\varphi\varphi(0)\overline{\varphi_*}(0), \tag{2.10}$$

and the Hermitian form m on H^0 by: for all $\mathbf{u}$, $\mathbf{u}_* \in H^0$,

$$m(\mathbf{u},\mathbf{u}_*) = \int_0^l \rho J\varphi\overline{\varphi_*}dx + \int_0^l \rho Sw\overline{w_*}dx. \tag{2.11}$$

With the foregoing assumptions (Hypothesis 2.1) the Hermitian forms a_0, a, b, m are well defined and continuous on their respective domains of definition. The Hermitian forms a_0 and b are non-negative on H^1 and the Hermitian form m is coercive on H^0.

A variational formulation of the problem reads as follows: Given $\mathbf{u}_0$, $\mathbf{v}_0$ in subspaces of H^1 and H^0 to be precised later, find $\mathbf{u} \in \mathcal{C}^1([0,+\infty);H^1)\cap\mathcal{C}^2([0,+\infty);H^0)$ such that

$$m(\ddot{\mathbf{u}},\mathbf{u}_*) + b(\dot{\mathbf{u}},\mathbf{u}_*) + a(\mathbf{u},\mathbf{u}_*) = 0,\ \forall \mathbf{u}_* \in H^1, \tag{2.12}$$

and satisfying the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0;\ \dot{\mathbf{u}}(0) = \mathbf{v}_0. \tag{2.13}$$

In order to deal with coercive forms we need the following lemma

Lemma 2.2 *a_0 is coercive: there exists $C>0$ such that for all $\mathbf{u}\in H^1$,*

$$a_0(\mathbf{u},\mathbf{u}) \geq C||\mathbf{u}||^2_{H^1}. \tag{2.14}$$

Consequently for all $\mathbf{u}\in H^1$,

$$a(\mathbf{u},\mathbf{u}) + \Omega^2 m(\mathbf{u},\mathbf{u}) \geq C||\mathbf{u}||^2_{H^1}, \tag{2.15}$$

and there exists $\Omega_0>0$ such that for all Ω, $0\leq\Omega\leq\Omega_0$, there exists $C_\Omega>0$ such that for all $\mathbf{u}\in H^1$,

$$a(\mathbf{u},\mathbf{u}) \geq C_\Omega||\mathbf{u}||^2_{H^1}. \tag{2.16}$$

Proof. Suppose that (2.14) is not true. Hence one can find a sequence $(\mathbf{u}_n)\in H^1$ such that $\forall n\in\mathbb{N}$, $||\mathbf{u}_n||_{H^1}=1$ and $a_0(\mathbf{u}_n,\mathbf{u}_n)\to 0$ when $n\to+\infty$. One can extract a subsequence still denoted by $(\mathbf{u}_n)$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in H^1 (weak convergence) and $\mathbf{u}_n\to\mathbf{u}$ in H^0 when $n\to+\infty$. Since $a_0(\mathbf{u}_n,\mathbf{u}_n)\to 0$ when $n\to+\infty$, it follows that $\mathbf{u}_n(0)\to 0$, $\varphi_n'\to 0$ and $w_n'-\varphi_n\to 0$ in $L^2(0,l)$ when $n\to+\infty$. Consequently (φ_n) is a Cauchy sequence, hence this sequence converges in $H^1(0,l)$ towards an element $\varphi\in H^1(0,l)$ such that $\varphi'=0$ and $\varphi(0)=0$, so that $\varphi=0$. Hence $w_n'\to 0$ in $L^2(0,l)$ when $n\to+\infty$. In the same way as for the sequence (φ_n) it implies that $w_n\to 0$ in $H^1(0,l)$. Therefore $\mathbf{u}_n\to 0$ in H^1. This gives a contradiction since $\forall n\in\mathbb{N}$, $||\mathbf{u}_n||_{H^1}=1$. □

Let $(H^0)'$ (resp. $(H^1)'$) be the space of continuous antilinear forms on H^0 (resp. H^1). The form m being continuous on H^0, it defines a continuous linear operator M from H^0 to $(H^0)'$ by: for all $\mathbf{u},\mathbf{u}_*\in H^0$, $m(\mathbf{u},\mathbf{u}_*) = \langle M\mathbf{u},\mathbf{u}_*\rangle_{(H^0)',H^0}$. The forms a and b being continuous on H^1, they define continuous linear operators A and B from H^1 to $(H^1)'$ by: for all $\mathbf{u},\mathbf{u}_*\in H^1$, $a(\mathbf{u},\mathbf{u}_*) = \langle A\mathbf{u},\mathbf{u}_*\rangle_{(H^1)',H^1}$, $b(\mathbf{u},\mathbf{u}_*) = \langle B\mathbf{u},\mathbf{u}_*\rangle_{(H^1)',H^1}$. If H^0 and $(H^0)'$ are identified by the Riesz representation theorem, then $H^1\subset H^0\subset(H^1)'$

with continuous and dense injections and for all $\mathbf{u} \in H^0$, $\mathbf{u}_* \in H^1$, one can write $m(\mathbf{u}, \mathbf{u}_*) = \langle M\mathbf{u}, \mathbf{u}_* \rangle_{(H^1)', H^1}$. Consequently (2.12) can be replaced by: find $\mathbf{u} \in \mathcal{C}^1([0, +\infty); H^1) \cap \mathcal{C}^2([0, +\infty); H^0)$ such that

$$M\ddot{\mathbf{u}} + B\dot{\mathbf{u}} + A\mathbf{u} = 0 \text{ in } (H^1)'. \tag{2.17}$$

The Hermitian form a is not coercive. In order to deal with a coercive form let us replace $\mathbf{u}(t)$ with $e^{\Omega t}\mathbf{u}(t)$. Consequently the variational problem reduces to: find $\mathbf{u} \in \mathcal{C}^1([0, +\infty); H^1) \cap \mathcal{C}^2([0, +\infty); H^0)$ such that

$$M\ddot{\mathbf{u}} + B_\Omega\dot{\mathbf{u}} + A_\Omega\mathbf{u} = 0 \text{ in } (H^1)', \tag{2.18}$$

where

$$A_\Omega = A + \Omega B + \Omega^2 M, \tag{2.19}$$

$$B_\Omega = B + 2\Omega M, \tag{2.20}$$

or equivalently such that

$$m(\ddot{\mathbf{u}}, \mathbf{u}_*) + b_\Omega(\dot{\mathbf{u}}, \mathbf{u}_*) + a_\Omega(\mathbf{u}, \mathbf{u}_*) = 0, \ \forall \mathbf{u}_* \in H^1, \tag{2.21}$$

where

$$a_\Omega = a + \Omega b + \Omega^2 m, \tag{2.22}$$

$$b_\Omega = b + 2\Omega m, \tag{2.23}$$

and satisfying the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0; \dot{\mathbf{u}}(0) = \mathbf{v}_0 - \Omega\mathbf{u}_0. \tag{2.24}$$

The Hermitian form a_Ω is coercive on H^1 because of (2.15) and since b is non-negative. This implies in particular that A_Ω is an isomorphism from H^1 onto $(H^1)'$. Set $\mathcal{H}^{1,1} = H^1 \times H^1$, $\mathcal{H}^{1,0} = H^1 \times H^0$, $\mathcal{H}_*^{1,1} = H^1 \times (H^1)'$: $\mathcal{H}^{1,1} \subset \mathcal{H}^{1,0} \subset \mathcal{H}_*^{1,1}$ with continuous and dense injections. The operator

$$\tilde{\mathcal{M}} = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} \tag{2.25}$$

is an isomorphism from $\mathcal{H}^{1,0}$ onto $\mathcal{H}^{1,0}$. The operator

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & I \\ -A_\Omega & -B_\Omega \end{pmatrix} \tag{2.26}$$

is continuous from $\mathcal{H}^{1,1}$ on $\mathcal{H}_*^{1,1}$. Set

$$\mathbf{v} = \begin{pmatrix} z \end{pmatrix}, \mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \mathbf{U_0} = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 - \Omega\mathbf{u}_0 \end{pmatrix}. \tag{2.27}$$

The variational problem (2.21), (2.24) reduces to: find $\mathbf{U} \in \mathcal{C}^0([0, +\infty); \mathcal{H}^{1,1}) \cap \mathcal{C}^1([0, +\infty); \mathcal{H}^{1,0})$ such that

$$\tilde{\mathcal{M}}\dot{\mathbf{U}} = \tilde{\mathcal{A}}\mathbf{U}, \tag{2.28}$$

and satisfying the initial conditions

$$\mathbf{U}(0) = \mathbf{U}_0. \tag{2.29}$$

In order to solve (2.28), (2.29) in the space $\mathcal{H}^{1,0} = H^1 \times H^0$, let us define the unbounded operator $\mathcal{A}$ on $\mathcal{H}^{1,0}$ with domain

$$D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H}^{1,1}, \tilde{\mathcal{A}}\mathbf{U} \in \mathcal{H}^{1,0}\} = \{\mathbf{U} \in \mathcal{H}^{1,1}, A_\Omega \mathbf{u} + B_\Omega \mathbf{v} \in H^0\} \tag{2.30}$$

by: for all $\mathbf{U} \in D(\mathcal{A})$, $\mathcal{A}\mathbf{U} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{A}}\mathbf{U}$, that is

$$\mathcal{A}\mathbf{U} = \begin{pmatrix} \mathbf{v} \\ -M^{-1}(A_\Omega \mathbf{u} + B_\Omega \mathbf{v}) \end{pmatrix}. \tag{2.31}$$

In the definition of $D(\mathcal{A})$ one can replace "$\mathbf{U} \in \mathcal{H}^{1,1}$" with "$\mathbf{U} \in \mathcal{H}^{1,0}$". For smooth $\mathbf{u}$ and $\mathbf{u}_*$, Green's formula for the Hermitian form a is written as follows:

$$a(\mathbf{u}, \mathbf{u}_*) = \int_0^l \mathcal{L}_a \mathbf{u} \cdot \mathbf{u}_* + [\mathcal{B}_a \mathbf{u} \cdot \mathbf{u}_*]_0^l \tag{2.32}$$

where

$$\mathcal{L}_a \mathbf{u} = \begin{pmatrix} -\{(Pw')' + [\alpha_s GS(w' - \varphi)]'\} \\ -\{(EJ\varphi')' + \alpha_s GS(w' - \varphi)\} - \rho J \Omega^2 \varphi \end{pmatrix}, \tag{2.33}$$

$$\mathcal{B}_a \mathbf{u}(0) = \begin{pmatrix} (Pw')(0) + [\alpha_s GS(w' - \varphi)](0) - k_w w(0) \\ (EJ\varphi')(0) - k_\varphi \varphi(0) \end{pmatrix}, \tag{2.34}$$

and

$$\mathcal{B}_a \mathbf{u}(l) = \begin{pmatrix} (Pw')(l) + [\alpha_s GS(w' - \varphi)](l) \\ (EJ\varphi')(l) \end{pmatrix}. \tag{2.35}$$

Similarly for the Hermitian form b:

$$b(\mathbf{u}, \mathbf{u}_*) = [\mathcal{B}_b \mathbf{u} \cdot \mathbf{u}_*]_0^l \tag{2.36}$$

where

$$\mathcal{B}_b \mathbf{u}(0) = \begin{pmatrix} -c_w w(0) \\ -c_\varphi \varphi(0) \end{pmatrix} \tag{2.37}$$

and $\mathcal{B}_b \mathbf{u}(l) = 0$. Finally for the Hermitian form m we have:

$$m(\mathbf{u}, \mathbf{u}_*) = \int_0^l M\mathbf{u} \cdot \mathbf{u}_* \tag{2.38}$$

where

$$M = \begin{pmatrix} \rho S & 0 \\ 0 & \rho J \end{pmatrix} \tag{2.39}$$

(M is an operator of order 0). Using the previous formulas, we infer that

$$\begin{array}{l} D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H}^{1,1}, \\ \mathcal{L}_a \mathbf{u} + \Omega^2 M\mathbf{u} + 2\Omega M\mathbf{v} \in H^0, \\ [\mathcal{B}_a \mathbf{u} + \Omega \mathcal{B}_b \mathbf{u} + \mathcal{B}_b \mathbf{v}](0) = 0, \mathcal{B}_a \mathbf{u}(l) = 0\}. \end{array} \tag{2.40}$$

Set $\mathcal{H}^{2,1} = H^2 \times H^1$. One verifies that there exist constants $C_1 > 0$, $C_2 > 0$ such that for all $\mathbf{U} \in D(\mathcal{A})$,

$$C_1(||\mathbf{U}||_{\mathcal{H}^{1,1}} + ||\mathcal{A}\mathbf{U}||_{\mathcal{H}^{1,0}}) \leq ||\mathbf{U}||_{\mathcal{H}^{2,1}} \leq C_2(||\mathbf{U}||_{\mathcal{H}^{1,1}} + ||\mathcal{A}\mathbf{U}||_{\mathcal{H}^{1,0}}). \tag{2.41}$$

Hence on $D(\mathcal{A})$ the graph norm and the norm $||.||_{\mathcal{H}^{2,1}}$ are equivalent and $\mathcal{A}$ is a closed operator (that is to say its graph is closed). Equations (2.40) and (2.41) give

$$D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H}^{2,1}, [\mathcal{B}_a\mathbf{u} + \Omega\mathcal{B}_b\mathbf{u} + \mathcal{B}_b\mathbf{v}](0) = 0, \mathcal{B}_a\mathbf{u}(l) = 0\} \tag{2.42}$$

that is

$$\begin{array}{l} D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H}^{2,1}, \\ (Pw')(0) + [\alpha_s GS(w' - \varphi)](0) = [(k_w + \Omega c_w)w + c_w z](0), \\ (EJ\varphi')(0) = [(k_\varphi + \Omega c_\varphi)\varphi + c_\varphi\psi](0), \\ (Pw')(l) + [\alpha_s GS(w' - \varphi)](l) = 0, (EJ\varphi')(l) = 0\} \end{array} \tag{2.43}$$

and for $\mathbf{U} \in D(\mathcal{A})$,

$$A_\Omega\mathbf{u} + B_\Omega\mathbf{v} = \mathcal{L}_a\mathbf{u} + \Omega^2 M\mathbf{u} + 2\Omega M\mathbf{v}, \tag{2.44}$$

so that

$$\mathcal{A}\mathbf{U} = \begin{pmatrix} \mathbf{v} \\ -M^{-1}(\mathcal{L}_a\mathbf{u} + \Omega^2 M\mathbf{u} + 2\Omega M\mathbf{v}) \end{pmatrix}. \tag{2.45}$$

The adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is given by

$$\begin{array}{l} D(\mathcal{A}^*) = \{\mathbf{U} \in \mathcal{H}^{1,1}, \\ \mathcal{L}_a\mathbf{u} + \Omega^2 M\mathbf{u} + 2\Omega M\mathbf{v} \in H^0, \\ [\mathcal{B}_a\mathbf{u} + \Omega\mathcal{B}_b\mathbf{u} - \mathcal{B}_b\mathbf{v}](0) = 0, \mathcal{B}_a\mathbf{u}(l) = 0\} \end{array} \tag{2.46}$$

and

$$\mathcal{A}^*\mathbf{U} = \begin{pmatrix} \mathbf{v} \\ -M^{-1}(\mathcal{L}_a\mathbf{u} + \Omega^2 M\mathbf{u} + 2\Omega M\mathbf{v}) \end{pmatrix}, \forall \mathbf{U} \in D(\mathcal{A}^*) \tag{2.47}$$

(same formula as $\mathcal{A}\mathbf{U}$). Since $D(\mathcal{A}) \neq D(\mathcal{A}^*)$, $\mathcal{A} \neq \mathcal{A}^*$: $\mathcal{A}$ is not self-adjoint. One can write for $\mathbf{U} \in D(\mathcal{A})$

$$\mathcal{L}_a\mathbf{u} + \Omega^2 M\mathbf{u} = A_0\mathbf{u} + A_1\mathbf{u}' + A_2\mathbf{u}'' \tag{2.48}$$

where A_i, $i = 0, 1, 2$ are defined by

$$A_0 = \begin{bmatrix} \Omega^2\rho S & (\alpha_s GS)' \\ 0 & \alpha_s GS \end{bmatrix}, \tag{2.49}$$

$$A_1 = \begin{bmatrix} -(P + \alpha_s GS)' & \alpha_s GS \\ -\alpha_s GS & -(EJ)' \end{bmatrix}, \tag{2.50}$$

$$A_2 = \begin{bmatrix} -(P + \alpha_s GS) & 0 \\ 0 & -EJ \end{bmatrix}. \tag{2.51}$$

The boundary conditions in the characterization of $D(\mathcal{A})$ ((2.42)) may be written in the form:

$$\begin{array}{l} \left[A_0^b\mathbf{u} + A_1^b\mathbf{u}' + B^b\mathbf{v}\right](0) = 0, \\ \left[A_0^b\mathbf{u} + A_1^b\mathbf{u}' + B^b\mathbf{v}\right](l) = 0, \end{array} \tag{2.52}$$

where A_i^b $(i = 0, 1)$ and B^b are matrices defined on the set $\{0, l\}$ by

$$A_0^b(0) = \begin{bmatrix} k_w + \Omega c_w & \alpha_s GS \\ 0 & k_\varphi + \Omega c_\varphi \end{bmatrix}(0), A_0^b(l) = \begin{bmatrix} 0 & \alpha_s GS \\ 0 & 0 \end{bmatrix}(l), \tag{2.53}$$

$$A_1^b(0) = \begin{bmatrix} -(P + \alpha_s GS) & 0 \\ 0 & -EJ \end{bmatrix}(0), A_1^b(l) = \begin{bmatrix} -(P + \alpha_s GS) & 0 \\ 0 & -EJ \end{bmatrix}(l), \tag{2.54}$$

$$B^b(0) = \begin{bmatrix} c_w & 0 \\ 0 & c_\varphi \end{bmatrix}, B^b(l) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.55}$$

Let us recall some definitions, notations and results about unbounded operators in Banach spaces. If E and F are Banach spaces, $L(E,F)$ is the space of continuous linear operators from E to F, denoted by $L(E)$ if $E = F$. For an operator $T \in L(E,F)$, $N(T) = \{x \in E \,|\, Tx = 0\}$ denotes the null space and $R(T) = \{Tx \,|\, x \in E\}$ the range of T. An operator $T \in L(E,F)$ is called a Fredholm operator if both its nullity nul$T$ = dim$N(T)$ and its deficiency def$T$ = codim$R(T)$ are finite. The set of Fredholm operators from E to F is denoted by $\Phi(E,F)$. If $T \in \Phi(E,F)$, ind$T$ = nul$T$ - def$T$ is well-defined and is called the index of T. Let A be an unbounded closed operator in a Banach space E with domain $D(A)$ (by definition an unbounded operator A is closed if its graph is closed in $E \times E$). If $\lambda \in \mathbb{C}$ and $A - \lambda I$ is a bijection from $D(A)$ onto E, λ is said to belong to the resolvent set of A, denoted by $\rho(A)$. If $\lambda \in \rho(A)$, since A is closed, it follows from the closed graph theorem that $(A - \lambda I)^{-1} \in L(E)$ [15], p. 419. The spectrum of A is the complementary of $\rho(A)$ in $\mathbb{C}$, denoted by $\sigma(A)$ ($\sigma(A) = \mathbb{C} \setminus \rho(\mathcal{A})$). The point spectrum of A is the set of eigenvalues of A, denoted by $\sigma_p(A)$ ($\sigma_p(A) \subset \sigma(A)$). The geometric multiplicity of an eigenvalue $\lambda_0 \in \sigma_p(A)$ is nul$(A - \lambda_0 I)$. For $\lambda_0 \in \sigma_p(A)$ and all $k \in \mathbb{N}$ set

$$\mathcal{N}^k_{\lambda_0}(A) = \{u \,|\, u \in D(A^{k+1}), (A - \lambda_0 I)^{k+1} u = 0\}. \tag{2.56}$$

For all $k \in \mathbb{N}$, $N^k_{\lambda_0}(A) \subset N^{k+1}_{\lambda_0}(A)$. The smallest integer $k > 0$ such that $N^k_{\lambda_0}(A) = N^{k+1}_{\lambda_0}(A)$ is called the ascent of $A - \lambda_0 I$. The root subspace of A corresponding to $\lambda_0 \in \sigma_p(A)$ is defined by

$$\mathcal{R}_{\lambda_0}(A) = \bigcup_{k \in \mathbb{N}} \mathcal{N}^k_{\lambda_0}(A). \tag{2.57}$$

Elements of $\mathcal{R}_{\lambda_0}(A)$ are root vectors. The dimension of $\mathcal{R}_{\lambda_0}(A)$ is the algebraic multiplicity of the eigenvalue $\lambda_0 \in \sigma_p(A)$ of A [40], p. 26. If $\lambda_0 \in \sigma_p(\mathcal{A})$, the ascent of $\mathcal{A} - \lambda_0 I$ is less than or equal to the algebraic multiplicity of λ_0. If A is an unbounded closed operator in E with compact resolvent, the spectrum of A consists of isolated eigenvalues with finite algebraic multiplicity [10], p. 187, [3], p. 2292. The following lemma gives a sufficient condition for an unbounded operator to have a compact resolvent (consequence of [10], Chapter 4, Remark 1.4).

Lemma 2.3 *Let A be an unbounded closed operator in a Banach space E with domain $D(A)$. $D(A)$ equipped with the graph norm is a Banach space continuously embedded in E and A is continuous from $D(A)$ equipped with the graph norm into E. Assume moreover that the embedding from $D(A)$ equipped with the graph norm into E is compact. Then, if $\rho(A) \neq \emptyset$, for all $\lambda \in \rho(A)$, the resolvent $(A - \lambda I)^{-1}$ is compact.*

Applying the previous lemma we obtain

Lemma 2.4 *$0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is a compact operator of $\mathcal{H}^{1,0}$. Consequently $\mathcal{A}$ has a compact resolvent and the spectrum of $\mathcal{A}$ consists of isolated eigenvalues of finite algebraic multiplicity. In particular $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \subset \mathbb{C}^*$.*

Proof. Let $\mathbf{F} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} \in \mathcal{H}^{1,0}$. We must find $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathbf{U} = \mathbf{F}$. This gives: $\mathbf{v} = \mathbf{f}$ and $A_\Omega \mathbf{u} = -M\mathbf{g} - B_\Omega \mathbf{f}$. Since A_Ω is an isomorphism from H^1 onto $(H^1)'$ and since $-M\mathbf{g} - B_\Omega \mathbf{f} \in (H^1)'$, the latter equation has a unique solution $\mathbf{u} \in H^1$. By construction $\mathbf{v} \in H^1$ and $A_\Omega \mathbf{u} + B_\Omega \mathbf{v} = -M\mathbf{g} \in H^0$, so that $\mathbf{U} \in D(\mathcal{A})$. It follows that $0 \in \rho(\mathcal{A})$. □

Remark 2.5 Quite frequently in the literature results like that of Lemma 2.4 are shown by explicit and rather lengthy calculations [29], [9], [8], [38]. But these explicit calculations are not possible here because of the terms coming from the rotation. Moreover, the proof of Lemma 2.4 is far simpler and more general.

The state space $\mathcal{H}^{1,0}$ will be equiped with the scalar product: for $\mathbf{U}_1, \mathbf{U}_2 \in \mathcal{H}^{1,0}$,

$$(\mathbf{U}_1, \mathbf{U}_2)_{\mathcal{H}^{1,0}} = a_\Omega(\mathbf{u}_1, \mathbf{u}_2) + m(\mathbf{v}_1, \mathbf{v}_2) \tag{2.58}$$

which is equivalent to the natural scalar product.

Lemma 2.6 *$\mathcal{A}$ is maximal-dissipative, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}^{1,0}$ and $\mathcal{A}$ is the generator of a contraction semi-group on $\mathcal{H}^{1,0}$.*

P r o o f. For all $\mathbf{U} \in D(\mathcal{A})$,

$$(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}^{1,0}} = a_\Omega(\mathbf{v}, \mathbf{u}) - (A_\Omega \mathbf{u} + B_\Omega \mathbf{v}, \mathbf{v})_{H^0} = a_\Omega(\mathbf{v}, \mathbf{u}) - \langle A_\Omega \mathbf{u} + B_\Omega \mathbf{v}, \mathbf{v}\rangle_{(H^1)', H^1} = a_\Omega(\mathbf{v}, \mathbf{u}) - a_\Omega(\mathbf{u}, \mathbf{v}) - b_\Omega(\mathbf{v}, \mathbf{v}), \tag{2.59}$$

so that

$$\mathrm{Re}(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}^{1,0}} = -b_\Omega(\mathbf{v}, \mathbf{v}) \leq 0. \tag{2.60}$$

It follows that $\mathcal{A}$ is dissipative. On the other hand, since $0 \in \rho(\mathcal{A})$, and since $\rho(\mathcal{A})$ is open, there exists $\lambda \in \mathbb{C}$ such that Re $\lambda > 0$ and $R(\mathcal{A} - \lambda I) = \mathcal{H}^{1,0}$. Consequently $\mathcal{A}$ is maximal dissipative and $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}^{1,0}$ [34], Theorem 3.1.7. The Lumer-Phillips theorem (see [34], Theorem 3.8.4) shows that $\mathcal{A}$ is the generator of a contraction semigroup on $\mathcal{H}^{1,0}$. □

From Lemma 2.6 and Proposition 2.3.5 of [34] we obtain the following proposition (by replacing $\mathbf{u}(t)$ with $e^{-\Omega t}\mathbf{u}(t)$)

Proposition 2.7 *i) With the notation* (2.27) *the condition* $\mathbf{U}_0 \in D(\mathcal{A})$ *is equivalent to the conditions* $\mathbf{u}_0 \in H^1$, $\mathbf{v}_0 \in H^1$, $A\mathbf{u}_0 + B\mathbf{v}_0 \in H^0$ *or to the conditions* $\mathbf{u}_0 \in H^2$, $\mathbf{v}_0 \in H^1$, $[\mathcal{B}_a\mathbf{u}_0 + \mathcal{B}_b\mathbf{v}_0](0) = 0$, $[\mathcal{B}_a\mathbf{u}](l) = 0$.

ii) Let $t \in [0, +\infty) \to T(t)$ *be the contraction semigroup generated by* $\mathcal{A}$. *If* $\mathbf{U}_0 \in D(\mathcal{A})$ *there exists a unique solution* $\in \mathcal{C}^1([0, +\infty); \mathcal{H}^{1,0}) \cap \mathcal{C}^0([0, +\infty); D(\mathcal{A}))$ *to the initial value problem on* $[0, +\infty)$

$$\dot{\mathbf{U}} = \mathcal{A}\mathbf{U},\ \mathbf{U}(0) = \mathbf{U}_0 \tag{2.61}$$

given by $\mathbf{U}(t) = T(t)\mathbf{U}_0$, *(*$t \in [0, +\infty)$*).*

iii) If $\mathbf{u}_0 \in H^1$, $\mathbf{v}_0 \in H^1$, $A\mathbf{u}_0 + B\mathbf{v}_0 \in H^0$, *there exists a unique function* $\mathbf{u} \in \mathcal{C}^1([0, +\infty); H^1) \cap \mathcal{C}^2([0, +\infty); H^0)$ *satisfying* $A\mathbf{u} + B\dot{\mathbf{u}} \in \mathcal{C}^0([0, +\infty); H^0)$ *and* (2.12), (2.13).

P r o o f. i) The condition $\mathbf{U}_0 \in D(\mathcal{A})$ is equivalent to the conditions $\mathbf{u}_0 \in H^1$, $\mathbf{v}_0 - \Omega\mathbf{u}_0 \in H^1$, $A_\Omega\mathbf{u}_0 + B_\Omega(\mathbf{v}_0 - \Omega\mathbf{u}_0) \in H^0$. One verifies that these conditions are equivalent to the stated ones.

ii) This part is a consequence of Proposition 2.3.5 of [34]

iii) Setting $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$ we see that $\mathbf{u}$ is the unique function $\in \mathcal{C}^1([0, +\infty); H^1) \cap \mathcal{C}^2([0, +\infty); H^0)$ satisfying $A_\Omega\mathbf{u} + B_\Omega\dot{\mathbf{u}} \in \mathcal{C}^0([0, +\infty); H^0)$ and (2.21), (2.24). Replacing $\mathbf{u}(t)$ by $e^{-\Omega t}\mathbf{u}(t)$ we see that there is a unique function $\mathbf{u} \in \mathcal{C}^1([0, +\infty); H^1) \cap \mathcal{C}^2([0, +\infty); H^0)$ satisfying $A\mathbf{u} + B\dot{\mathbf{u}} \in \mathcal{C}^0([0, +\infty); H^0)$ and (2.12), (2.13). □

3 Asymptotics of the spectrum

Let us recall some definitions and results about holomorphic families of operators [25].

If Ω is an open nonempty subset of $\mathbb{C}$, E is a Banach space, we will denote by $H(\Omega, E)$ the space of holomorphic functions from Ω to E. If F is a Banach space and $T \in H(\Omega, L(E, F))$, $\rho(T) = \{\lambda \in \Omega, T(\lambda)$ is invertible$\}$ is the resolvent set of T, $\sigma(T) = \Omega \setminus \rho(T)$ its spectrum, $\sigma_p(T) = \{\lambda \in \Omega, \exists\, x \in E, x \neq 0, T(\lambda)x = 0\}$ its point spectrum (or the set of eigenvalues of T) [25], p. 6. If $\mu \in \Omega$, $x \in H(\Omega, E)$ is called a root function of T at μ if $x(\mu) \neq 0$ and $(Tx)(\mu) = 0$. $\nu(x)$ denotes the order of the zero of Tx at μ and is called the multiplicity of x with respect to T at μ. If $T \in H(\Omega, \Phi(E, F))$ and $\rho(T) \neq \emptyset$, $\sigma(T)$ is a discrete subset of Ω and T^{-1} is a meromorphic operator function in Ω. If $\mu \in \sigma(T)$, T^{-1} has a pole at μ [25], Theorem 1.3.1. Let $T \in H(\Omega, \Phi(E, F))$ be such that $\rho(T) \neq \emptyset$. For all $\mu \in \sigma(T)$ and $n \in N^*$ set

$$\tilde{L}_n = \{y_0 \in N(T(\mu));\ \text{there is a root function } y \text{ of } T \text{ at } \mu \text{ with } y(\mu) = y_0 \text{ and } \nu(y) \geq n\} \tag{3.1}$$

and

$$L_n = \tilde{L}_n \cup \{0\} \tag{3.2}$$

which is a subspace of $N(T(\mu))$. For $j \in \mathbb{N}$ with $0 < j \leq \text{nul } T(\mu)$, let us define

$$m_j = \max \{n \in \mathbb{N}^* | \dim L_n \geq j\}. \tag{3.3}$$

The numbers m_j are called the partial multiplicities of T at μ. They are well-defined since $L_1 = N(T(\mu))$ and $L_n = \{0\}$ if n is larger than the pole order of T^{-1} at μ. Obviously, $m_j \geq m_{j+1}$. The number $r = \dim N(T(\mu))$ is called the geometric multiplicity of T at μ, and the number

$$m = \sum_{j=1}^{r} m_j \tag{3.4}$$

is called the algebraic multiplicity of T at μ [25], p. 14. In the sequel we will need the following lemma.

Lemma 3.1 *Let E_A, F_A, E_B, F_B be Banach spaces, Ω an open nonempty subset of $\mathbb{C}$, $\lambda \mapsto A(\lambda) \in H(\Omega, \Phi(E_A, F_A))$ and $\lambda \mapsto B(\lambda) \in H(\Omega, \Phi(E_B, F_B))$ holomorphic families of Fredholm operators. Let $\mu \in \Omega$ be such that $0 <$ nul$(A(\mu))$ = nul$(B(\mu)) < +\infty$. For all $n \in \mathbb{N}^*$, let us denote by $L_n(A)$ and $L_n(B)$ the spaces defined by* (3.1) *and* (3.2) *corresponding to the holomorphic families A and B. Assume that for all $n \in \mathbb{N}^*$ there exists an injection i_n from $L_n(A)$ into $L_n(B)$. If we denote by $m_j(A)$ and $m_j(B)$ the partial multiplicities of A and B at μ defined by* (3.3) *then for all $j = 1, \ldots,$ nul$(A(\mu))$ (= nul$(B(\mu))$), $m_j(A) \leq m_j(B)$. If we denote by $m(A)$ and $m(B)$ the algebraic multiplicities of A and B at μ then $m(A) \leq m(B)$.*

Proof. Since for all $n \in \mathbb{N}^*$, dim $L_n(A)$ = dim $i_n(L_n(A)) \leq$ dim $L_n(B)$ definitions (3.1), (3.2), (3.3) and (3.4) give the result. □

Let A be an unbounded closed operator in a Banach space E with domain $D(A)$ and $\lambda_0 \in \sigma_p(A)$. The algebraic multiplicity of $\lambda_0 \in \sigma_p(A)$ has been defined as the dimension of the root subspace of A corresponding to λ_0. Since A is closed, $D(A)$ equipped with the graph norm is a Banach space and A is a continuous operator from $D(A)$ equipped with the graph norm to E. $D(A)$ will be equipped with the graph norm. If Ω is an open nonempty subset of $\mathbb{C}$ such that $\lambda_0 \in \Omega$, we can associate to the operator A the holomorphic family $\lambda \in \Omega \mapsto T(\lambda) = A - \lambda I \in H(\Omega, L(D(A), E))$. If we assume that the embedding from $D(A)$ into E is compact and $\rho(A) \neq \emptyset$, from Lemma 2.3, A has a compact resolvent consequently the spectrum of A consists entirely of isolated eigenvalues (of finite algebraic multiplicity). Therefore $\rho(T) \neq \emptyset$ and from [10], Theorem IV.5.26 or [25], p. 41, $T \in H(\Omega, \Phi(D(A), E))$ and Ind $T(\lambda) = 0$, $\forall \lambda \in \Omega$. The following lemma asserts that the two definitions of algebraic multiplicity (for an operator and for a holomorphic family) are consistent (its proof is very simple and is omitted).

Lemma 3.2 *With the previous definitions and hypotheses the algebraic multiplicity of the eigenvalue $\lambda_0 \in \sigma_p(A)$ is equal to the algebraic multiplicity of T at λ_0.*

To the unbounded operator $\mathcal{A}$ we will associate the holomorphic family of operators on $\mathbb{C}^*$:

$$\lambda \in \mathbb{C}^* \mapsto A(\lambda) = \mathcal{A} - \lambda I : D(\mathcal{A}) \to \mathcal{H}^{1,0}. \tag{3.5}$$

$\mathcal{A}$ is closed, the embedding from $D(\mathcal{A})$ in $\mathcal{H}^{1,0}$ is compact and $\rho(\mathcal{A}) \neq \emptyset$. Consequently $\rho(A) \neq \emptyset$, $A \in H(\mathbb{C}^*, \Phi(D(\mathcal{A}), \mathcal{H}^{1,0}))$ and ind$A(\lambda) = 0$, $\forall \lambda \in \mathbb{C}$. Let $\lambda_0 \in \sigma_p(\mathcal{A})$ and let $\lambda \in \mathbb{C}^* \mapsto \mathbf{U}(\lambda) \in D(\mathcal{A})$ be a root function of the holomorphic (polynomial) family $\lambda \in \mathbb{C}^* \mapsto A(\lambda)$ at λ_0 of multiplicity $\nu(\mathbf{U})$. Write $\mathbf{U}(\lambda) = \begin{pmatrix} \mathbf{u}(\lambda) \\ \mathbf{v}(\lambda) \end{pmatrix}$. For $\lambda \in \mathbb{C}^*$ we have $\mathbf{u}(\lambda) \in H^2$, $\mathbf{v}(\lambda) \in H^1$, and from (2.52)

$$\begin{array}{l} \left[A_0^b \mathbf{u}(\lambda) + A_1^b \mathbf{u}(\lambda)' + B^b \mathbf{v}(\lambda)\right](0) = 0, \\ \left[A_0^b \mathbf{u}(\lambda) + A_1^b \mathbf{u}(\lambda)' + B^b \mathbf{v}(\lambda)\right](l) = 0. \end{array} \tag{3.6}$$

Moreover from (2.45) and (2.48) the relation $[(\mathcal{A} - \lambda I)\mathbf{U}]^{(i)}(\lambda_0) = 0$ $(0 \leq i \leq \nu(\mathbf{U}) - 1)$ can be written

$$[\mathbf{v} - \lambda \mathbf{u}]^{(i)}(\lambda_0) = 0,\ 0 \leq i \leq \nu(\mathbf{U}) - 1, \tag{3.7}$$

$$[A_0 \mathbf{u} + A_1 \mathbf{u}' + A_2 \mathbf{u}'' + 2\Omega M \mathbf{v} + \lambda M \mathbf{v}]^{(i)}(\lambda_0) = 0,\ 0 \leq i \leq \nu(\mathbf{U}) - 1. \tag{3.8}$$

If we set $\mathbf{w}(\lambda) = \mathbf{u}(\lambda)'/\lambda$, we obtain

$$\begin{array}{l} \{\left[A_0^b\mathbf{u}(\lambda) + \lambda A_1^b\mathbf{w}(\lambda) + \lambda B^b\mathbf{u}(\lambda)\right]^{(i)}(\lambda_0)\}(0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1, \\ \{\left[A_0^b\mathbf{u}(\lambda) + \lambda A_1^b\mathbf{w}(\lambda) + \lambda B^b\mathbf{u}(\lambda)\right]^{(i)}(\lambda_0)\}(l) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1, \end{array} \tag{3.9}$$

and

$$\left[A_0\mathbf{u} + \lambda A_1\mathbf{w} + \lambda A_2\mathbf{w}' + 2\Omega\lambda M\mathbf{u} + \lambda^2 M\mathbf{u}\right]^{(i)}(\lambda_0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1. \tag{3.10}$$

It is easily seen that if $\lambda_0 \in \mathbb{C}^*$, E is a Banach space, $\lambda \in \mathbb{C}^* \mapsto x(\lambda) \in E$ is holomorphic and $n \in \mathbb{N}$, the conditions "$x^{(i)}(\lambda_0) = 0$, $i = 0, \ldots, n$" and "$(x/\lambda)^{(i)}(\lambda_0) = 0$, $i = 0, \ldots, n$" are equivalent. Hence (3.9), (3.10) give:

$$\begin{array}{l} \left\{\left[A_0^b\frac{\mathbf{u}(\lambda)}{\lambda} + A_1^b\mathbf{w}(\lambda) + B^b\mathbf{u}(\lambda)\right]^{(i)}(\lambda_0)\right\}(0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1, \\ \left\{\left[A_0^b\frac{\mathbf{u}(\lambda)}{\lambda} + A_1^b\mathbf{w}(\lambda) + B^b\mathbf{u}(\lambda)\right]^{(i)}(\lambda_0)\right\}(l) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1, \end{array} \tag{3.11}$$

$$\left[A_2^{-1}A_0\frac{\mathbf{u}}{\lambda} + A_2^{-1}A_1\mathbf{w} + \mathbf{w}' + 2\Omega A_2^{-1}M\mathbf{u} + \lambda A_2^{-1}M\mathbf{u}\right]^{(i)}(\lambda_0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1. \tag{3.12}$$

Setting

$$Y = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ \mathbf{u}'/\lambda \end{pmatrix}, \tag{3.13}$$

we obtain

$$\left[Y' - (\lambda\tilde{A}_1 + \tilde{A}_0 + \frac{1}{\lambda}\tilde{A}_{-1})Y\right]^{(i)}(\lambda_0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1, \tag{3.14}$$

with

$$\tilde{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\rho S}{P+\alpha_s GS} & 0 & 0 & 0 \\ 0 & \frac{\rho}{E} & 0 & 0 \end{pmatrix}, \tag{3.15}$$

$$\tilde{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2\Omega\rho S}{P+\alpha_s GS} & 0 & -\frac{(P+\alpha_s GS)'}{P+\alpha_s GS} & \frac{\alpha_s GS}{P+\alpha_s GS} \\ 0 & \frac{2\Omega\rho}{E} & -\frac{\alpha_s GS}{EJ} & -\frac{(EJ)'}{EJ} \end{pmatrix}, \tag{3.16}$$

$$\tilde{A}_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\Omega^2\rho S}{P+\alpha_s GS} & \frac{(\alpha_s GS)'}{P+\alpha_s GS} & 0 & 0 \\ 0 & \frac{\alpha_s GS}{EJ} & 0 & 0 \end{pmatrix}. \tag{3.17}$$

The boundary conditions (3.11) can be written under the form:

$$\left[(\tilde{B}_1 + \frac{1}{\lambda}\tilde{B}_0)Y(0) + (\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0)Y(l)\right]^{(i)}(\lambda_0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1 \tag{3.18}$$

where

$$\tilde{B}_1 = \begin{pmatrix} B^b(0) & A_1^b(0) \\ 0_2 & 0_2 \end{pmatrix}, \tilde{B}_0 = \begin{pmatrix} A_0^b(0) & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \tag{3.19}$$

$$\tilde{C}_1 = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & A_1^b(l) \end{pmatrix}, \tilde{C}_0 = \begin{pmatrix} 0_2 & 0_2 \\ A_0^b(l) & 0_2 \end{pmatrix}. \tag{3.20}$$

Set

$$\alpha_1 = \sqrt{\frac{\rho S}{P + \alpha_s G S}} \tag{3.21}$$

and

$$\alpha_2 = \sqrt{\frac{\rho}{E}}. \tag{3.22}$$

The eigenvalues of $\tilde{A}_1$ are $\pm\alpha_1$ and $\pm\alpha_2$. A matrix of eigenvectors of $\tilde{A}_1$ associated to the eigenvalues α_1, α_2, -α_1, -α_2 is

$$T_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \alpha_1 & 0 & -\alpha_1 & 0 \\ 0 & \alpha_2 & 0 & -\alpha_2 \end{pmatrix}. \tag{3.23}$$

Setting

$$Z(\lambda) = T_0^{-1} Y(\lambda) \in H^1(0,l) \otimes \mathbb{C}^4, \tag{3.24}$$

we get the following equation

$$\begin{aligned} (Z')^{(i)}(\lambda_0) &= \left[T_0^{-1}(\lambda \tilde{A}_1 + \tilde{A}_0 + \tfrac{1}{\lambda}\tilde{A}_{-1}) T_0 Z - T_0^{-1} T_0' Z\right]^{(i)}(\lambda_0) \\ &= \left[(\lambda \tilde{\Lambda}_1 + \tilde{\Lambda}_0 + \tfrac{1}{\lambda}\tilde{\Lambda}_{-1}) Z\right]^{(i)}(\lambda_0),\ 0 \le i \le \nu(\mathbf{U}) - 1, \end{aligned} \tag{3.25}$$

where

$$\tilde{\Lambda}_1 = T_0^{-1} \tilde{A}_1 T_0 = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & -\alpha_1 & 0 \\ 0 & 0 & 0 & -\alpha_2 \end{pmatrix}, \tag{3.26}$$

$$\tilde{\Lambda}_0 = T_0^{-1} \tilde{A}_0 T_0 - T_0^{-1} T_0' = \begin{pmatrix} \Omega\alpha_1 - f_1 & & & * \\ & \Omega\alpha_2 - f_2 & & \\ & & -\Omega\alpha_1 - f_1 & \\ * & & & -\Omega\alpha_2 - f_2 \end{pmatrix}, \tag{3.27}$$

where f_1 and f_2 are the functions defined on $[0,l]$ by

$$f_1 = \frac{1}{4}\left(\frac{(P + \alpha_s G S)'}{P + \alpha_s G S} + \frac{(\rho S)'}{\rho S}\right) \tag{3.28}$$

and

$$f_2 = \frac{1}{4}\left(\frac{(EJ)'}{EJ} + \frac{(\rho J)'}{\rho J}\right), \tag{3.29}$$

and the boundary conditions

$$\left[(\tilde{B}_1 + \frac{1}{\lambda}\tilde{B}_0) T_0 Z(0) + (\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0) T_0 Z(l)\right]^{(i)}(\lambda_0) = 0,\ 0 \le i \le \nu(\mathbf{U}) - 1. \tag{3.30}$$

With the boundary eigenvalue problem (3.25), (3.30) we associate the holomorphic family of operators on $\mathbb{C}^*$:

$$\lambda \in \mathbb{C}^* \mapsto T(\lambda) = \begin{pmatrix} T^D(\lambda) \\ T^R(\lambda) \end{pmatrix} : H^1(0,l) \otimes \mathbb{C}^4 \to (L^2(0,l) \otimes \mathbb{C}^4) \times \mathbb{C}^4 \tag{3.31}$$

defined by

$$T^D(\lambda)Z = Z' - (\lambda\tilde{\Lambda}_1 + \tilde{\Lambda}_0 + \frac{1}{\lambda}\tilde{\Lambda}_{-1})Z, \tag{3.32}$$

$$T^R(\lambda)Z = (\tilde{B}_1 + \frac{1}{\lambda}\tilde{B}_0)T_0Z(0) + (\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0)T_0Z(l). \tag{3.33}$$

By Corollary 3.1.3 of [25], $T \in H(\mathbb{C}^*, \Phi(H^1(0,l) \otimes \mathbb{C}^4, (L^2(0,l) \otimes \mathbb{C}^4) \times \mathbb{C}^4))$ and Ind $T(\lambda) = 0$, $\forall \lambda \in \mathbb{C}^*$, thus $\sigma(T) = \sigma_p(T)$.

The previous calculations show that $\mathbf{U} \in D(\mathcal{A})$, $\mathbf{U} \neq 0$ meets the condition $\mathcal{A}\mathbf{U} = \lambda_0\mathbf{U}$ iff Z defined by $Z = T_0^{-1}Y$, where Y and T_0 are defined by (3.13), (3.23) satisfies $Z \in H^1(0,l) \otimes \mathbb{C}^4$, $Z \neq 0$ and $T(\lambda_0)Z = 0$. We have proved

Lemma 3.3 *$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ = $\sigma(T) = \sigma_p(T)$. Let $\lambda_0 \in \sigma_p(\mathcal{A})$ and let $\lambda \in \mathbb{C}^* \mapsto \mathbf{U}(\lambda) \in D(\mathcal{A})$ be a root function of the holomorphic (polynomial) family $\lambda \in \mathbb{C}^* \mapsto A(\lambda)$ at λ_0 of multiplicity $\nu(\mathbf{U})$. Then $\lambda \in \mathbb{C}^* \mapsto Z(\lambda) \in H^1(0,l) \otimes \mathbb{C}^4$ is a root function of the holomorphic family $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ at λ_0 of multiplicity $\geq \nu(\mathbf{U})$.*

A consequence of Lemma 3.3 is

Lemma 3.4 *Let $\lambda_0 \in \sigma_p(\mathcal{A})$. The holomorphic families $\lambda \in \mathbb{C}^* \mapsto A(\lambda)$ and $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ have the same geometric multiplicities at λ_0 (= r). Let us denote by $m_j(A, \lambda_0)$ (resp. $m_j(T, \lambda_0)$) the partial multiplicities of the holomorphic families $\lambda \in \mathbb{C}^* \mapsto A(\lambda)$ (resp. $T(\lambda)$) at λ_0. Then $m_j(A, \lambda_0) \leq m_j(T, \lambda_0)$, $\forall j$, $0 < j \leq r$. As a consequence, if we denote by $m(A, \lambda_0)$ (resp. $m(T, \lambda_0)$) the algebraic multiplicities of the holomorphic families $\lambda \in \mathbb{C}^* \mapsto A(\lambda)$ (resp. $T(\lambda)$) at λ_0, then $m(A, \lambda_0) \leq m(T, \lambda_0)$.*

Proof. The map $J : \mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \in N(A(\lambda_0)) \mapsto Z = T_0^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}'/\lambda_0 \end{pmatrix} \in N(T(\lambda_0))$ is an isomorphism. Therefore the two holomorphic families A and T have the same geometric multiplicities at λ_0. For all $n \in \mathbb{N}^*$ let us denote by $L_n(A, \lambda_0)$ and $L_n(T, \lambda_0)$ the spaces defined by (3.1) and (3.2) corresponding to the holomorphic families A and T. Lemma 3.3 shows that the restriction of J to $L_n(A, \lambda_0)$ is an injection from $L_n(A, \lambda_0)$ into $L_n(T, \lambda_0)$. Applying Lemma 3.1 the result follows. □

Recall that a matrix function $\lambda \in \mathbb{C}^* \to Z(\lambda) \in H^1(0,l) \otimes \mathbb{C}^4$ is a fundamental matrix function of $T^D(\lambda)Z = 0$ if for all $\lambda \in \mathbb{C}^*$ and for each $y \in N(T^D(\lambda))$ there is a $c(\lambda) \in \mathbb{C}^4$ such that $y = Z(\lambda)c(\lambda)$ (see [25], Definition 2.5.2 and Theorem 2.5.3 for the existence of a fundamental matrix function). In order to derive asymptotics of the eigenvalues of T, we use the following proposition which guarantees the existence of an asymptotic fundamental matrix function of $T^D(\lambda)Z = 0$. This proposition holds for systems which are asymptotically linear in λ and for which the coefficient of λ is diagonal. The proof of the proposition can be found in [25], Theorem 2.8.2 (see also [33], Theorem 2.2). In the following statement, if $m \in \mathbb{N}^*$ and S is a set, $M_m(S)$ denotes the set of $m \times m$ matrices with coefficients in S.

Proposition 3.5 *Let $a, b \in \mathbb{R}$, $a < b$, $k \in \mathbb{N}$, $\gamma > 0$ and let $\hat{A}(., \rho)$, $\rho \in \mathbb{C}$ be such that*

$$\hat{A}(., \rho) = \rho\hat{A}_1 + \hat{A}_0 + \sum_{j=1}^{k} \rho^{-j}\hat{A}_{-j} + \rho^{-k-1}\hat{A}_{-k-1}(., \rho), \ |\rho| > \gamma, \tag{3.34}$$

with the following properties:

1. *$\hat{A}_1 \in M_m(W^{k,\infty}(a,b))$ and $\hat{A}_{-j} \in M_m(W^{k-j,\infty}(a,b))$, $j = 0, \ldots, k$;*
2. *$\hat{A}_{-k-1}(., \rho) \in M_m(L^\infty(a,b))$ for $|\rho| > \gamma$ and is bounded in $M_m(L^\infty(a,b))$ as $\rho \to +\infty$;*

3. $\hat{A}_1$ *has a diagonal form:*

$$\hat{A}_1 = \begin{pmatrix} r_0 I_{n_0} & & & 0 \\ & r_1 I_{n_1} & & \\ & & \ddots & \\ 0 & & & r_l I_{n_l} \end{pmatrix} \tag{3.35}$$

where I_{n_ν} denotes the n_ν-dimensional unit matrix, $\sum_{\nu=0}^{l} n_\nu = m$, and the functions r_ν are such that $r_\nu = 0$ for some $\nu \in \{0, \dots, l\}$, without loss of generality $r_0 = 0$ (where we allow n_0 to be 0, whereas $n_\nu > 0$ for $\nu > 0$),

$$(r_\nu - r_\mu)^{-1} \in L^\infty(a,b),\ \nu, \mu = 0, \dots, l,\ \nu \neq \mu, \tag{3.36}$$

$$r_\nu - r_\mu = |r_\nu - r_\mu| e^{i\varphi_{\nu\mu}}\ \nu, \mu = 0, \dots, l,\ \textit{on}\ [a,b], \tag{3.37}$$

with some constants $\varphi_{\nu\mu} \in \mathbb{R}$. In particular, for $\mu = 0$, we obtain

$$r_\nu^{-1} \in L^\infty(a,b),\ \nu = 1, \dots, l, \tag{3.38}$$

$$r_\nu = |r_\nu| e^{i\varphi_\nu},\ \nu = 1, \dots, l,\ \textit{on}\ [a,b], \tag{3.39}$$

with $\varphi_\nu = \varphi_{\nu 0} = \varphi_{0\nu} \pm \pi$.

For $x \in [a,b]$ and $\rho \in \mathbb{C}$ we set

$$R_\nu(x) = \int_a^x r_\nu(\xi) d\xi,\ \nu = 0, \dots l, \tag{3.40}$$

$$E_\nu(x,\rho) = \exp(\rho R_\nu(x)) I_{n_\nu},\ \nu = 0, \dots l, \tag{3.41}$$

$$\hat{E}(x,\rho) = \begin{pmatrix} E_0(x,\rho) & & & 0 \\ & E_1(x,\rho) & & \\ & & \ddots & \\ 0 & & & E_l(x,\rho) \end{pmatrix}. \tag{3.42}$$

Then there exists a fundamental matrix $\hat{Y}(.,\rho)$ of

$$y' - \hat{A}(.,\rho) y = 0 \tag{3.43}$$

such that for $|\rho| > \gamma$,

$$\hat{Y}(.,\rho) = (\sum_{r=0}^{k} \frac{1}{\rho^r} P^{[r]} + \frac{1}{\rho^k} B_k(.,\rho)) \hat{E}(.,\rho) \tag{3.44}$$

where $B_k(.,\rho) \in M_m(W^{1,\infty}(a,b))$, $|\rho| > \gamma$, $B_k(.,\rho) = o(1)$, $\frac{1}{\rho} B_k'(.,\rho) = o(1)$ with respect to the norm in $M_m(L^\infty(a,b))$, and where $P^{[r]} \in M_m(W^{k+1-r,\infty}(a,b))$ are determined by

$$P^{[0]} \hat{A}_1 - \hat{A}_1 P^{[0]} = 0,\ P^{[0]}(a) = I_m, \tag{3.45}$$

$$P^{[r]\prime} - \sum_{j=0}^{r} \hat{A}_{-j} P^{[r-j]} + P^{[r+1]} \hat{A}_1 - \hat{A}_1 P^{[r+1]} = 0,\ r = 0, \dots, k-1, \tag{3.46}$$

$$P_{\nu\nu}^{[k]\prime} - \hat{A}_{0,\nu\nu} P_{\nu\nu}^{[k]} = \sum_{q=0, q\neq\nu}^{l} \hat{A}_{0,\nu q} P_{q\nu}^{[k]} + \sum_{j=1}^{k} \sum_{q=0}^{l} \hat{A}_{-j,\nu q} P_{q\nu}^{[k-j]},\ \nu = 0, \dots, l, \tag{3.47}$$

with

$$\hat{A}_j = (\hat{A}_{j,\nu\mu})_{\nu,\mu=0}^{l},\ P^{[r]} = (P_{\nu\mu}^{[r]})_{\nu,\mu=0}^{l} \tag{3.48}$$

defined according to the block structure of $\hat{A}_1$. Moreover $P^{[0]}$ is invertible in $M_m(W^{k+1,\infty}(a,b))$.

From now on, we will make the following hypothesis

Hypothesis 3.6 With the notations (3.21), (3.22), $1/(\alpha_1-\alpha_2)\in L^\infty(0,l)$ ($\Leftrightarrow \exists C>0$ such that $|\alpha_1-\alpha_2|\geq C$ on $[0,l]$)

Taking into account Hypothesis 2.1, Hypothesis 3.6 is equivalent to

$$\exists C>0 \text{ such that } |P-S\cdot(E-\alpha_s G)|\geq C \text{ on } [0,l]. \tag{3.49}$$

With the hypotheses 2.1 and 3.6 the assumptions of Proposition 3.5 are fulfilled with $a=0$, $b=l$, $k=0$ and $\hat{A}(.,\rho)=\rho\tilde{\Lambda}_1+\tilde{\Lambda}_0+\frac{1}{\rho}\tilde{\Lambda}_{-1}$. Since $\alpha_1\neq\alpha_2$, from (3.45), the matrix $P^{[0]}$ is diagonal. Equation (3.47) gives $P_{ii}^{[0]\prime}-\tilde{\Lambda}_{0,ii}P_{ii}^{[0]}=0$ ($i=1,\dots,4$). Because of the initial condition $P^{[0]}(0)=I_4$ ((3.45)) and the form of $\tilde{\Lambda}_0$ ((3.27)) we get

$$P^{[0]}(x)= \begin{pmatrix} g_1(x)e^{\Omega\int_0^x\alpha_1(t)dt} & 0 & 0 & 0\\ 0 & g_2(x)e^{\Omega\int_0^x\alpha_2(t)dt} & 0 & 0\\ 0 & 0 & g_1(x)e^{-\Omega\int_0^x\alpha_1(t)dt} & 0\\ 0 & 0 & 0 & g_2(x)e^{-\Omega\int_0^x\alpha_2(t)dt} \end{pmatrix} \tag{3.50}$$

where g_1 and g_2 are the functions defined on $[0,l]$ by

$$g_1(x)=\left(\frac{(P+\alpha_sGS)(0)}{(P+\alpha_sGS)(x)}\frac{(\rho S)(0)}{(\rho S)(x)}\right)^{\frac{1}{4}},\ x\in[0,l] \tag{3.51}$$

and

$$g_2(x)=\left(\frac{(EJ)(0)}{(EJ)(x)}\frac{(\rho J)(0)}{(\rho J)(x)}\right)^{\frac{1}{4}},\ x\in[0,l]. \tag{3.52}$$

Set

$$\hat{E}(x,\lambda)=\exp\left(\lambda\int_0^x\tilde{\Lambda}_1(t)dt\right)= \begin{pmatrix} e^{\lambda\int_0^x\alpha_1(t)dt} & 0 & 0 & 0\\ 0 & e^{\lambda\int_0^x\alpha_2(t)dt} & 0 & 0\\ 0 & 0 & e^{-\lambda\int_0^x\alpha_1(t)dt} & 0\\ 0 & 0 & 0 & e^{-\lambda\int_0^x\alpha_2(t)dt} \end{pmatrix}. \tag{3.53}$$

Then the asymptotic fundamental matrix function of $T^D(\lambda)Z=0$ is given by

$$\tilde{E}(x,\lambda)=(P^{[0]}(x)+B_0(x,\lambda))\hat{E}(x,\lambda) \tag{3.54}$$

where $B_0(.,\lambda)\in M_4(W^{1,\infty}(0,l))$, $|\lambda|>\gamma$, $B_0(.,\lambda)=o(1)$, $\frac{1}{\lambda}B_0'(.,\lambda)=o(1)$ with respect to the norm in $M_4(L^\infty(0,l))$. Actually, due to the Sobolev embedding theorems, $B_0(.,\lambda)\in M_4(\mathcal{C}^0[0,l])$, $|\lambda|>\gamma$, $B_0(.,\lambda)=o(1)$ with respect to the norm in $M_4(\mathcal{C}^0[0,l])$. For $\lambda\in\mathbb{C}^*$ set

$$M(\lambda)=T^R(\lambda)\tilde{E}(.,\lambda)=\left(\tilde{B}_1+\frac{1}{\lambda}\tilde{B}_0\right)T_0\tilde{E}(0,\lambda)+\left(\tilde{C}_1+\frac{1}{\lambda}\tilde{C}_0\right)T_0\tilde{E}(l,\lambda) \tag{3.55}$$

and

$$\Delta(\lambda)=\ \det M(\lambda). \tag{3.56}$$

The map $\lambda\in\mathbb{C}^*\mapsto M(\lambda)$ belongs to $H(\mathbb{C}^*,M_4(\mathbb{C}))$ (see [25], p. 103) and also to $H(\mathbb{C}^*,\Phi(\mathbb{C}^4,\mathbb{C}^4))$. Applying Theorem 3.1.2 and Corollary 3.1.3 of [25] we deduce

Lemma 3.7 $\sigma(T) = \sigma_p(T) = \sigma(M) = \sigma_p(M) = \{\lambda \in \mathbb{C}^* \mid \Delta(\lambda) = 0\}$.

If $a \in \mathbb{C}$ or $a \in M_n(\mathbb{C})$ we will use the notation

$$[a]_1 = a + o(1) \tag{3.57}$$

if $[a]_1$ depends on λ and satisfies (3.57) when $\lambda \to +\infty$. We have

$$\tilde{E}(0,\lambda) = \begin{pmatrix} [1]_1 & [0]_1 & [0]_1 & [0]_1 \\ [0]_1 & [1]_1 & [0]_1 & [0]_1 \\ [0]_1 & [0]_1 & [1]_1 & [0]_1 \\ [0]_1 & [0]_1 & [0]_1 & [1]_1 \end{pmatrix} = [I]_1. \tag{3.58}$$

With the notations

$$\beta_1 = \int_0^l \alpha_1(t)dt, \beta_2 = \int_0^l \alpha_2(t)dt \tag{3.59}$$

and

$$\gamma_1 = g_1(l), \gamma_2 = g_2(l) \tag{3.60}$$

$\tilde{E}(l,\lambda)$ can be rewritten in the following form

$$\tilde{E}(l,\lambda) = \begin{pmatrix} [\gamma_1]_1 e^{\beta_1(\lambda+\Omega)} & [0]_1 e^{\beta_2((\lambda+\Omega)]} & [0]_1 e^{-\beta_1(\lambda+\Omega)} & [0]_1 e^{-\beta_2(\lambda+\Omega)} \\ [0]_1 e^{\beta_1(\lambda+\Omega)} & [\gamma_2]_1 e^{\beta_2(\lambda+\Omega)} & [0]_1 e^{-\beta_1(\lambda+\Omega)} & [0]_1 e^{-\beta_2(\lambda+\Omega)} \\ [0]_1 e^{\beta_1(\lambda+\Omega)} & [0]_1 e^{\beta_2(\lambda+\Omega)} & [\gamma_1]_1 e^{-\beta_1(\lambda+\Omega)} & [0]_1 e^{-\beta_2(\lambda+\Omega)} \\ [0]_1 e^{\beta_1(\lambda+\Omega)} & [0]_1 e^{\beta_2(\lambda+\Omega)} & [0]_1 e^{-\beta_1(\lambda+\Omega)} & [\gamma_2]_1 e^{-\beta_2(\lambda+\Omega)} \end{pmatrix}, \tag{3.61}$$

so that

$$M(\lambda) = \begin{pmatrix} [c_w - (\alpha_1(P+\alpha_s GS))(0)]_1 & [0]_1 & [c_w + (\alpha_1(P+\alpha_s GS))(0)]_1 & [0]_1 \\ [0]_1 & [c_\varphi - (\alpha_2 EJ)(0)]_1 & [0]_1 & [c_\varphi + (\alpha_2 EJ)(0)]_1 \\ -[\gamma_1(\alpha_1(P+\alpha_s GS))(l)]_1 e^{\beta_1(\lambda+\Omega)} & [0]_1 e^{\beta_2(\lambda+\Omega)} & [\gamma_1(\alpha_1(P+\alpha_s GS))(l)]_1 e^{-\beta_1(\lambda+\Omega)} & [0]_1 e^{-\beta_2(\lambda+\Omega)} \\ [0]_1 e^{\beta_1(\lambda+\Omega)} & -[\gamma_2(\alpha_2 EJ)(l)]_1 e^{\beta_2(\lambda+\Omega)} & [0]_1 e^{-\beta_1(\lambda+\Omega)} & [\gamma_2(\alpha_2 EJ)(l)]_1 e^{-\beta_2(\lambda+\Omega)} \end{pmatrix}. \tag{3.62}$$

Let us study the zeros of $\Delta(\lambda) = \det M(\lambda)$. We have

$$\lim_{\mathrm{Re}\lambda \to +\infty} \frac{\Delta(\lambda)}{e^{(\beta_1+\beta_2)(\lambda+\Omega)}} = (\gamma_1(\alpha_1(P+\alpha_s GS))(l))(\gamma_2(\alpha_2 EJ(l)))\times$$
$$(c_w + (\alpha_1(P+\alpha_s GS))(0))(c_\varphi + (\alpha_2 EJ)(0)) \tag{3.63}$$

and

$$\lim_{\mathrm{Re}\lambda \to -\infty} \frac{\Delta(\lambda)}{e^{-(\beta_1+\beta_2)(\lambda+\Omega)}} = (\gamma_1(\alpha_1(P+\alpha_s GS))(l))(\gamma_2(\alpha_2 EJ(l)))\times$$
$$(c_w - (\alpha_1(P+\alpha_s GS))(0))(c_\varphi - (\alpha_2 EJ)(0)). \tag{3.64}$$

In what follows, we will make the following hypothesis

Hypothesis 3.8 With the notations (3.21), (3.22), $c_w \neq (\alpha_1(P+\alpha_s GS))(0)$, $c_\varphi \neq (\alpha_2 EJ)(0)$ or equivalently $c_w \neq \sqrt{\rho S}(0)\sqrt{P+\alpha_s GS}(0)$, $c_\varphi \neq \sqrt{\rho J}(0)\sqrt{EJ}(0)$.

Lemmas 3.3 and 3.7, equations (2.60), (3.63), (3.64) and Hypothesis 3.8 yield the following lemma

Lemma 3.9 *$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ is contained in a strip parallel to the imaginary axis included in the left half-plane, that is there exists a constant $h > 0$ such that:*

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C}^* | \Delta(\lambda) = 0\} \subset \{\lambda \in \mathbb{C}^* | -h \leq Re\lambda \leq 0\}. \tag{3.65}$$

The next lemma gives formulas for the asymptotic of the eigenvalues of $\mathcal{A}$.

Lemma 3.10 *With the notations*

$$\mu^1 = \frac{1}{2\beta_1}\ln\left|\frac{\alpha_1(P+\alpha_s GS)(0)-c_w}{\alpha_1(P+\alpha_s GS)(0)+c_w}\right| = \frac{1}{2\beta_1}\ln\left|\frac{\sqrt{\rho S}(0)\sqrt{P+\alpha_s G\overline{S}}(0)-c_w}{\sqrt{\rho S}(0)\sqrt{P+\alpha_s G\overline{S}}(0)+c_w}\right|, \tag{3.66}$$

$$\nu_n^1 = \begin{cases} \frac{in\pi}{\beta_1} & \text{if } \alpha_1(P+\alpha_s GS)(0)-c_w > 0 \\ \frac{i(2n+1)\pi}{2\beta_1} & \text{if } \alpha_1(P+\alpha_s GS)(0)-c_w < 0, \end{cases} , n \in \mathbb{Z}, \tag{3.67}$$

$$\mu^2 = \frac{1}{2\beta_2}\ln\left|\frac{\alpha_2 EJ(0)-c_\varphi}{\alpha_2 EJ(0)+c_\varphi}\right| = \frac{1}{2\beta_2}\ln\left|\frac{\sqrt{\rho J}(0)\sqrt{E\overline{J}}(0)-c_\varphi}{\sqrt{\rho J}(0)\sqrt{E\overline{J}}(0)+c_\varphi}\right|, \tag{3.68}$$

$$\nu_n^2 = \begin{cases} \frac{in\pi}{\beta_2} & \text{if } \alpha_2 EJ(0)-c_\varphi > 0 \\ \frac{i(2n+1)\pi}{2\beta_2} & \text{if } \alpha_2 EJ(0)-c_\varphi < 0, \end{cases} , n \in \mathbb{Z}, \tag{3.69}$$

the eigenvalues of $\mathcal{A}$ are given asymptotically by the formulas

$$\lambda_n^1 = \mu^1 - \Omega + \nu_n^1 + \varepsilon_n^1 \ (\varepsilon_n^1 \to 0 \text{ when } |n| \to +\infty) \tag{3.70}$$

and

$$\lambda_n^2 = \mu^2 - \Omega + \nu_n^2 + \varepsilon_n^2 \ (\varepsilon_n^2 \to 0 \text{ when } |n| \to +\infty). \tag{3.71}$$

Proof. When $-h \le \mathrm{Re}\lambda \le 0$ and $|\lambda|$ (thus $|\mathrm{Im}\lambda|$) $\to +\infty$,

$$\Delta(\lambda) - \Delta_1(\lambda)\Delta_2(\lambda) \to 0 \tag{3.72}$$

where

$$\Delta_1(\lambda) = [\gamma_1\alpha_1(P+\alpha_s GS)(l)]^2[(c_w - \alpha_1(P+\alpha_s GS)(0))e^{-\beta_1(\lambda+\Omega)} + (c_w + \alpha_1(P+\alpha_s GS)(0))e^{\beta_1(\lambda+\Omega)}], \tag{3.73}$$

$$\Delta_2(\lambda) = [\gamma_2\alpha_2 EJ(l)]^2[(c_\varphi - \alpha_2 EJ(0))e^{-\beta_2(\lambda+\Omega)} + (c_\varphi + \alpha_2 EJ(0))e^{\beta_2(\lambda+\Omega)}]. \tag{3.74}$$

By Theorem 4 of [14] the zeros of Δ are given asymptotically by the zeros of Δ_1 and Δ_2, that is the solutions of the following equations

$$e^{2\beta_1(\lambda+\Omega)} = \frac{\alpha_1(P+\alpha_s GS)(0)-c_w}{\alpha_1(P+\alpha_s GS)(0)+c_w} \tag{3.75}$$

and

$$e^{2\beta_2(\lambda+\Omega)} = \frac{\alpha_2 EJ(0)-c_\varphi}{\alpha_2 EJ(0)+c_\varphi} \tag{3.76}$$

which proves the lemma. □

Recall the following Proposition of [25] (Proposition 1.8.5)

Proposition 3.11 *Let E and F be finite-dimensional spaces such that dimE = dimF. Let $T \in H(\Omega, L(E,F))$ and assume that $\rho(T) \neq \emptyset$. For $\mu \in \sigma(T)$ (= $\sigma_p(T)$) the algebraic multiplicity of T at μ is equal to the multiplicity of the zero of $\det T$ at μ.*

Hereafter we will make the following hypothesis

Hypothesis 3.12 With the notations (3.66) and (3.68), $\mu^1 \neq \mu^2$.

Lemma 3.13 *The eigenvalues of M are asymptotically algebraically simple.*

P r o o f. Lemmas 3.3, 3.7 and 3.9 show that $\rho(M) = \rho(\mathcal{A}) \neq \emptyset$. When $-h \leq \mathrm{Re}\lambda \leq 0$ and $|\lambda|$ (thus $|\mathrm{Im}\lambda|$) $\to +\infty$, from Cauchy's estimates of a holomorphic function (see [28], p. 213) we get

$$\Delta'(\lambda) - (\Delta_1'(\lambda)\Delta_2(\lambda) + \Delta_1(\lambda)\Delta_2'(\lambda)) \to 0, \tag{3.77}$$

consequently when $n \to +\infty$

$$\Delta'(\lambda_n^1) - \Delta_1'(\lambda_n^1)\Delta_2(\lambda_n^1) \to 0. \tag{3.78}$$

It is easily seen that since $\mu^1 \neq \mu^2$ there exists a constant $C > 0$ such that for all $n \in \mathbb{Z}$,

$$|\Delta_1'(\lambda_n^1)\Delta_2(\lambda_n^1)| \geq C, \tag{3.79}$$

so that there exists a constant $C > 0$ and $N \in \mathbb{N}^*$ such that for all $n \in \mathbb{Z}$, $|n| \geq N$,

$$|\Delta'(\lambda_n^1)| \geq C. \tag{3.80}$$

Similarly there exists a constant $C > 0$ and $N \in \mathbb{N}^*$ such that for all $n \in \mathbb{Z}$, $|n| \geq N$,

$$|\Delta'(\lambda_n^2)| \geq C. \tag{3.81}$$

Thus the zeros of Δ are asymptotically simple. From proposition 3.11 the conclusion follows. □

The following lemma asserts that the holomorphic families M and T have the same partial and algebraic multiplicities at an eigenvalue.

Lemma 3.14 *Let $\lambda_0 \in \sigma_p(\mathcal{A})$. The holomorphic families $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ and $\lambda \in \mathbb{C}^* \mapsto M(\lambda)$ have the same geometric multiplicities at λ_0 (= r). Let us denote by $m_j(T, \lambda_0)$ (resp. $m_j(M, \lambda_0)$) the partial multiplicities of the holomorphic families $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ (resp. $M(\lambda)$) at λ_0. Then $m_j(T, \lambda_0) = m_j(M, \lambda_0)$, $\forall j$, $0 < j \leq r$. Consequently if we denote by $m(T, \lambda_0)$ (resp. $m(M, \lambda_0)$) the algebraic multiplicities of the holomorphic families $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ (resp. $M(\lambda)$) at λ_0 then $m(T, \lambda_0) = m(M, \lambda_0)$.*

P r o o f. Proposition 3.4 of [24] implies that for $\lambda \in \mathbb{C}^*$, nul $T(\lambda)$ = nul $M(\lambda)$. The map $J : c \in N(M(\lambda_0)) \mapsto \tilde{E}(., \lambda_0)c \in N(T(\lambda_0))$ is an isomorphism. For all $n \in \mathbb{N}^*$ let us denote by $L_n(M, \lambda_0)$ and $L_n(T, \lambda_0)$ the spaces defined by (3.1) and (3.2) corresponding to the holomorphic families M and T. From [24], Proposition 3.2, if $\mu \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}^* \mapsto Z(\lambda)$ is a root function of T at μ of multiplicity p it follows that there exists a root function $\lambda \in \mathbb{C}^* \mapsto c(\lambda)$ of $\lambda \in \mathbb{C}^* \mapsto M(\lambda)$ at μ of multiplicity $\geq p$ such that $\lambda \in \mathbb{C}^* \mapsto Z(\lambda) - \tilde{E}(., \lambda)c(\lambda)$ has a zero of order $\geq p$ at μ and if $\lambda \in \mathbb{C}^* \mapsto c(\lambda)$ is a root function of M at μ of multiplicity p then there exists a root function of $\lambda \in \mathbb{C}^* \mapsto T(\lambda)$ at μ of multiplicity p (= $\lambda \in \mathbb{C}^* \mapsto \tilde{E}(., \lambda)c(\lambda)$). We deduce that for all $n \in \mathbb{N}^*$ the restriction of J to $L_n(M, \lambda_0)$ is an isomorphism from $L_n(M, \lambda_0)$ onto $L_n(T, \lambda_0)$. An application of Lemma 3.1 gives the result. □

Proposition 3.15 *Under hypotheses 2.1, 3.6, 3.8 and 3.12, the eigenvalues of the operator $\mathcal{A}$ are asymptotically algebraically simple. Moreover if $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda_k,\ k \in \mathbb{N}\}$ then $\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$.*

P r o o f. Lemmas 3.4, 3.14 and 3.13 show that the eigenvalues of the operator $\mathcal{A}$ are asymptotically algebraically simple. The second property of the eigenvalues of $\mathcal{A}$ is a consequence of the formulas (3.66), ..., (3.71) and Hypothesis 3.12. □

4 Riesz basis property and asymptotic expansion of the solution

In this section we show the completeness of the set of root vectors of $\mathcal{A}$ (Theorem 4.3), then the Riesz basis property of a sequence of root vectors of $\mathcal{A}$ (Theorem 4.5). Finally from the latter theorem we obtain an expansion of the solution of (2.12), (2.13) (Theorem 4.7).

Let $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda_k,\ k \in \mathbb{N}\}$. In order to prove the completeness of the set of root vectors of $\mathcal{A}$, one must show that the space spanned by the root subspaces $\mathcal{R}_{\lambda_k}(\mathcal{A})$, $k \in \mathbb{N}$, is dense in $\mathcal{H}^{1,0}$. For $k \in \mathbb{N}$, let $E(\lambda_k, \mathcal{A})$

be the Riesz projector associated to $\mathcal{A}$ and λ_k [3], p. 2255. According to [3], p. 2292, Lemma 2, for $k \in \mathbb{N}$, $E(\lambda_k, \mathcal{A})\mathcal{H}^{1,0} = \mathcal{R}_{\lambda_k}(\mathcal{A})$. Let $\mathbf{U} \in \mathcal{H}^{1,0}$ be such that $\mathbf{U} \perp \mathcal{R}_{\lambda_k}(\mathcal{A})$, $\forall k \in \mathbb{N}$. One must show that $\mathbf{U} = 0$. For all $\mathbf{F} \in \mathcal{H}^{1,0}$, for all $k \in \mathbb{N}$, $(\mathbf{U}, E(\lambda_k, \mathcal{A})\mathbf{F})_{\mathcal{H}^{1,0}} = 0$. By [10], p. 184, (6.52), $(E(\lambda_k, \mathcal{A}))^* = E(\overline{\lambda}_k, \mathcal{A}^*)$. Thus $E(\overline{\lambda}_k, \mathcal{A}^*)\mathbf{U} = 0$, $\forall k \in \mathbb{N}$ so that $\mathbf{U} \in S_\infty(\mathcal{A}^*) = \{\mathbf{U} \in \mathcal{H}^{1,0},\ \forall k \in \mathbb{N}, E(\overline{\lambda}_k, \mathcal{A}^*)\mathbf{U} = 0\}$. On account of [10], p. 184, $\mathcal{A}^*$ has a compact resolvent and $\sigma(\mathcal{A}^*) = \sigma_p(\mathcal{A}^*) = \{\overline{\lambda}_k,\ k \in \mathbb{N}\}$. From [3], p. 2296, Lemma 6, $S_\infty(\mathcal{A}^*)$ is the set of $\mathbf{U} \in \mathcal{H}^{1,0}$ such that $\lambda \in \mathbb{C} \mapsto (\mathcal{A}^* - \lambda I)^{-1}\mathbf{U}$ is an entire function of λ. It follows that for any $\mathbf{F} \in \mathcal{H}^{1,0}$, the function $\lambda \in \mathbb{C} \mapsto G(\lambda) = ((\mathcal{A}^* - \lambda I)^{-1}\mathbf{U}, \mathbf{F})_{\mathcal{H}^{1,0}}$ is an entire function of λ. Since $\mathcal{A}$ is the generator of a contraction semigroup on $\mathcal{H}^{1,0}$ (Lemma 2.6), the Hille-Yosida theorem (see [27], p. 11, Corollary 3.6) implies that for all $\lambda \in \mathbb{C}$, if Re $\lambda > 0$ then $\lambda \in \rho(\mathcal{A})$ and $||(\mathcal{A} - \lambda I)^{-1}|| \leq 1/\mathrm{Re}\lambda$. One deduces that for all $\lambda \in \mathbb{C}$, if Re $\lambda > 0$ then $\lambda \in \rho(\mathcal{A}^*)$ and $||(\mathcal{A}^* - \lambda I)^{-1}|| \leq 1/\mathrm{Re}\lambda$. Consequently there exists $\alpha > 0$ such that $G(\lambda)$ is bounded on the closed half-plane Re $\lambda \geq \alpha$.

The following proposition gives an estimate of the resolvent for $\lambda \in \mathbb{R}$, $\lambda \to -\infty$. The idea of its proof is to compare $\mathcal{A}$ with the self-adjoint operator $\mathcal{A}_0$ corresponding to the case where $c_w = c_\varphi = 0$ (no damping).

Proposition 4.1 *There exist constants $C_1 > 0$ and $C_2 > 0$ such that $(-\infty, -C_1] \subset \rho(\mathcal{A})$ and such that for $\lambda \in (-\infty, -C_1]$,*

$$||(\mathcal{A} - \lambda I)^{-1}|| \leq C_2/\sqrt{|\lambda|}. \tag{4.1}$$

Proof. In the proof below the constants C_1 and C_2 are generic constants.

Let $\mathcal{A}_0$ be the unbounded operator defined by (2.30), (2.31) with $c_w = c_\varphi = 0$. By Lemma 3.9 there exists $C > 0$ such that $(-\infty, -C] \subset \rho(\mathcal{A})$ and $(-\infty, -C] \subset \rho(\mathcal{A}_0)$.

Let $\mathbf{F} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} \in \mathcal{H}^{1,0}$, $\lambda \in (-\infty, -C]$, $\mathbf{U}_0 = (\mathcal{A}_0 - \lambda I)^{-1}\mathbf{F} = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{pmatrix}$, $\mathbf{U} = (\mathcal{A} - \lambda I)^{-1}\mathbf{F} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$. We have $\mathcal{A}_0\mathbf{U}_0 = \lambda\mathbf{U}_0 + \mathbf{F}$ so that with (2.60)

$$\lambda(\mathbf{U}_0, \mathbf{U}_0)_{\mathcal{H}^{1,0}} + Re(\mathbf{F}, \mathbf{U}_0)_{\mathcal{H}^{1,0}} = \lambda(a_\Omega(\mathbf{u}_0, \mathbf{u}_0) + m(\mathbf{v}_0, \mathbf{v}_0)) + Re(\mathbf{F}, \mathbf{U}_0)_{\mathcal{H}^{1,0}} = -2\Omega m(\mathbf{v}_0, \mathbf{v}_0). \tag{4.2}$$

From (4.2) we see that there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\lambda|(a_\Omega(\mathbf{u}_0, \mathbf{u}_0) + m(\mathbf{v}_0, \mathbf{v}_0)) \leq C_2|(\mathbf{F}, \mathbf{U}_0)_{\mathcal{H}^{1,0}}| \tag{4.3}$$

therefore

$$||(\mathcal{A}_0 - \lambda I)^{-1}|| \leq C_2/|\lambda|. \tag{4.4}$$

On the other hand we have $\mathcal{A}\mathbf{U} = \lambda\mathbf{U} + \mathbf{F}$ so that with (2.60)

$$\lambda(\mathbf{U}, \mathbf{U})_{\mathcal{H}^{1,0}} + Re(\mathbf{F}, \mathbf{U})_{\mathcal{H}^{1,0}} = \lambda(a_\Omega(\mathbf{u}, \mathbf{u}) + m(\mathbf{v}, \mathbf{v})) + Re(\mathbf{F}, \mathbf{U})_{\mathcal{H}^{1,0}} = -b_\Omega(\mathbf{v}, \mathbf{v}). \tag{4.5}$$

Equation (4.5) shows that there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\lambda|(a_\Omega(\mathbf{u}, \mathbf{u}) + m(\mathbf{v}, \mathbf{v})) \leq C_2(|(\mathbf{F}, \mathbf{U})_{\mathcal{H}^{1,0}}| + |\mathbf{v}(0)|^2) \tag{4.6}$$

and since $\mathbf{v}(0) = \lambda\mathbf{u}(0) + \mathbf{f}(0)$, from the trace theorem in $H^1(0, l)$ there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\lambda|||\mathbf{U}||^2_{\mathcal{H}^{1,0}} \leq C_2(||\mathbf{F}||_{\mathcal{H}^{1,0}}||\mathbf{U}||_{\mathcal{H}^{1,0}} + |\lambda\mathbf{u}(0)|^2 + ||\mathbf{F}||^2_{\mathcal{H}^{1,0}}). \tag{4.7}$$

Let $Y = \begin{pmatrix} \mathbf{u} \\ \mathbf{u}'/\lambda \end{pmatrix}$ and $Y_0 = \begin{pmatrix} \mathbf{u_0} \\ \mathbf{u_0}'/\lambda \end{pmatrix}$ be associated to $\mathbf{u}$ and $\mathbf{u}_0$ by (3.13) and set $Z = T_0^{-1}Y$, $Z_0 = T_0^{-1}Y_0$, where T_0 is defined by (3.23). If T is the holomorphic family of operators on $\mathbb{C}^*$ associated to the unbounded operator $\mathcal{A}$ by (3.31), (3.32), (3.33), $Z - Z_0$ satisfies the following equations

$$T^D(\lambda)(Z - Z_0) = 0 \tag{4.8}$$

and

$$T^R(\lambda)(Z-Z_0) = \begin{pmatrix} -\left(B^b(0)+\frac{1}{\lambda}\Omega B^b(0)\right)\mathbf{u}_0(0) \\ 0 \end{pmatrix}. \tag{4.9}$$

Since $Z - Z_0$ can be written under the form

$$Z - Z_0 = \tilde{E}(.,\lambda)c(\lambda) \tag{4.10}$$

where $c(\lambda) \in \mathbb{C}^4$ and since from (3.54) we have $\tilde{E}(0,\lambda) = [I]_1$, this gives

$$M(\lambda)[I]_1(Z(0)-Z_0(0)) = \begin{pmatrix} -\left(B^b(0)+\frac{1}{\lambda}\Omega B^b(0)\right)\mathbf{u}_0(0) \\ 0 \end{pmatrix} \tag{4.11}$$

consequently

$$Y(0) - Y_0(0) = T_0[I]_1 M^{-1}(\lambda) \begin{pmatrix} -\left(B^b(0)+\frac{1}{\lambda}\Omega B^b(0)\right)\mathbf{u}_0(0) \\ 0 \end{pmatrix}. \tag{4.12}$$

Equation (3.62) implies that when $\lambda \in \mathbb{R}$, $\lambda \to -\infty$

$$M^{-1}(\lambda) = D(\lambda)N(\lambda) \tag{4.13}$$

where

$$D(\lambda) = \text{diag}(1, 1, e^{\beta_1(\lambda+\Omega)}, e^{\beta_2(\lambda+\Omega)}) \tag{4.14}$$

and

$$N(\lambda) = \begin{pmatrix} [1/(c_w-(\alpha_1(P+\alpha_s GS))(0))]_1 & [0]_1 & [0]_1 & [0]_1 \\ [0]_1 & [1/(c_\varphi-(\alpha_2 EJ)(0))]_1 & [0]_1 & [0]_1 \\ [0]_1 & [0]_1 & [1/(\gamma_1(\alpha_1(P+\alpha_s GS))(l))]_1 & [0]_1 \\ [0]_1 & [0]_1 & [0]_1 & [1/(\gamma_2(\alpha_2 EJ)(l))]_1 \end{pmatrix}. \tag{4.15}$$

Equations (4.12), (4.13), (4.14) and (4.15) show that there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\mathbf{u}(0)| \leq C_2|\mathbf{u}_0(0)|. \tag{4.16}$$

According to (4.4) and from the trace theorem in $H^1(0,l)$ there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\lambda||\mathbf{u}_0(0)| \leq C_2||\mathbf{F}||_{\mathcal{H}^{1,0}}. \tag{4.17}$$

On account of (4.7), (4.16) and (4.17) there exist constants $C_1 \geq C > 0$ and $C_2 > 0$ such that for $\lambda \in (-\infty, -C_1]$,

$$|\lambda|||\mathbf{U}||^2_{\mathcal{H}^{1,0}} \leq C_2(||\mathbf{F}||_{\mathcal{H}^{1,0}}||\mathbf{U}||_{\mathcal{H}^{1,0}} + ||\mathbf{F}||^2_{\mathcal{H}^{1,0}}). \tag{4.18}$$

From (4.18), the result follows. □

Proposition 4.1 yields $(-\infty, -C_1] \subset \rho(\mathcal{A})$ thus $(-\infty, -C_1] \subset \rho(\mathcal{A}^*)$ and $G(\lambda) = ((\mathcal{A}^* - \lambda I)^{-1}\mathbf{U}, \mathbf{F})_{\mathcal{H}^{1,0}}$ $= (\mathbf{U}, (\mathcal{A} - \lambda I)^{-1}\mathbf{F})_{\mathcal{H}^{1,0}} \to 0$ when $\lambda \to -\infty$. Hence $G(\lambda)$ is bounded on the set $\text{Im}\lambda = 0$, $\text{Re}\,\lambda \leq \alpha$ (since G is an entire function, thus continuous).

Recall that an entire function f is said to be of exponential type if there is a constant $A \in \mathbb{R}$ and a constant $C > 0$ such that $|f(z)| \leq Ce^{A|z|}$, $\forall z \in \mathbb{C}$ [41], p. 53. This property is one of the assumptions of the Phragmén-Lindelöf principle.

Proposition 4.2 *The entire function $\lambda \in \mathbb{C} \mapsto G(\lambda)$ is of exponential type.*

P r o o f. Let $\mathbf{F} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} \in \mathcal{H}^{1,0}$, $\lambda \in \rho(\mathcal{A})$ and $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = (\mathcal{A} - \lambda I)^{-1}\mathbf{F}$. It follows that $\mathcal{A}\mathbf{U} = \lambda\mathbf{U} + \mathbf{F}$. Let $Y = \begin{pmatrix} \mathbf{u} \\ \mathbf{u}'/\lambda \end{pmatrix}$ be associated to $\mathbf{u}$ by (3.13) and set $Z = T_0^{-1}Y$, where T_0 is defined by (3.23). We get

$$Z' - \left(\lambda\tilde{\Lambda}_1 + \tilde{\Lambda}_0 + \frac{1}{\lambda}\tilde{\Lambda}_{-1}\right) Z = T_0^{-1}F_\lambda, \tag{4.19}$$

$$\left(\tilde{B}_1 + \frac{1}{\lambda}\tilde{B}_0\right) T_0 Z(0) + \left(\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0\right) T_0 Z(l) = \frac{1}{\lambda}\tilde{D} \tag{4.20}$$

where

$$F_\lambda = \begin{pmatrix} 0 \\ -A_2^{-1}(Mg + +2\Omega Mf + \lambda Mf)/\lambda \end{pmatrix} \tag{4.21}$$

and

$$\tilde{D} = \begin{pmatrix} -B^b(0)f(0) \\ 0 \end{pmatrix}. \tag{4.22}$$

By seeking a solution of (4.19) under the form $Z(x,\lambda) = \tilde{E}(x,\lambda)A(x,\lambda)$, where $A(x,\lambda) \in \mathbb{C}^4$, we get

$$A(x,\lambda) = \int_0^x \tilde{E}^{-1}(t,\lambda)T_0^{-1}F_\lambda(t)dt + K_\lambda \tag{4.23}$$

where $K_\lambda \in \mathbb{R}^4$ depends only on λ. An easy computation shows that

$$K_\lambda = M^{-1}(\lambda)\left(\frac{1}{\lambda}\tilde{D} - \left(\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0\right) T_0\tilde{E}(l,\lambda)\int_0^l \tilde{E}^{-1}(t,\lambda)T_0^{-1}F_\lambda(t)dt\right) \tag{4.24}$$

and

$$Y(x,\lambda) = T_0\tilde{E}(x,\lambda)M^{-1}(\lambda)N(x,\lambda) \tag{4.25}$$

where

$$N(x,\lambda) = M(\lambda)\int_0^x \tilde{E}^{-1}(t,\lambda)T_0^{-1}F_\lambda(t)dt + \frac{1}{\lambda}\tilde{D} -$$
$$\left(\tilde{C}_1 + \frac{1}{\lambda}\tilde{C}_0\right) T_0\tilde{E}(l,\lambda)\int_0^l \tilde{E}^{-1}(t,\lambda)T_0^{-1}F_\lambda(t)dt. \tag{4.26}$$

It is easily seen that there exist $\beta > 0$ and $C_1 > 0$, $C_2 > 0$ such that for all $\lambda \in \mathbb{C}$, $|\lambda| \geq C_1$, $||N(.,\lambda)||_{L^\infty(0,l)} \leq C_2 e^{\beta|\lambda|}||\mathbf{F}||_{\mathcal{H}^{1,0}}$. On the other hand, one can write

$$M^{-1}(\lambda)N(x,\lambda) = \frac{P(x,\lambda)}{\Delta(\lambda)} \tag{4.27}$$

where $P(x,\lambda)$ satisfies the same type of inequality than $N(x,\lambda)$. Hence Y can be written under the form

$$Y(x,\lambda) = \frac{Q(x,\lambda)}{\Delta(\lambda)} \tag{4.28}$$

where $Q(x,\lambda)$ satisfies the same type of inequality than $N(x,\lambda)$. Consequently there exist $\beta > 0$ and $C_1 > 0$, $C_2 > 0$ such that for all $\lambda \in \rho(\mathcal{A})$ and $|\lambda| \geq C_1$, $|\Delta(\lambda)|\ ||(\mathcal{A} - \lambda I)^{-1}\mathbf{F}||_{\mathcal{H}^{1,0}} \leq C_2 e^{\beta|\lambda|}||\mathbf{F}||_{\mathcal{H}^{1,0}}$ so that $|\Delta(\lambda)|\ ||(\mathcal{A} - \lambda I)^{-1}|| \leq C_2 e^{\beta|\lambda|}$. For $\lambda \in \rho(\mathcal{A}^*)$ (thus $\bar{\lambda} \in \rho(\mathcal{A})$), $G(\lambda) = ((\mathcal{A}^* - \lambda I)^{-1}\mathbf{U}, \mathbf{F})_{\mathcal{H}^{1,0}} = (\mathbf{U}, (\mathcal{A} - \bar{\lambda}I)^{-1}\mathbf{F})_{\mathcal{H}^{1,0}}$. Thus for all $\lambda \in \rho(\mathcal{A}^*)$ and $|\lambda| \geq C_1$, $|\Delta(\bar{\lambda})G(\lambda)| \leq C_2\ e^{\beta|\lambda|}||\mathbf{U}||_{\mathcal{H}^{1,0}}||\mathbf{F}||_{\mathcal{H}^{1,0}}$.

Since $\Delta(\bar{\lambda}) = 0$ for $\lambda \notin \rho(\mathcal{A}^*)$, we see that there exist $\beta > 0$ and $C > 0$ such that for all $\lambda \in \mathbb{C}$, $|\Delta(\bar{\lambda})G(\lambda)| \leq C\, e^{\beta|\lambda|}$.

Let us show that when $|\lambda| \to +\infty$, $|\Delta(\lambda)| \sim |\Delta(\bar{\lambda})|$. Let us remark that since the matrices $\tilde{\Lambda}_i$, $i = -1, 0, 1$ have real coefficients, if for all $\lambda \in \mathbb{C}^*$, $Z(.,\lambda)$ is a fundamental matrix of $T^D(\lambda)Z = 0$ it follows that for all $\lambda \in \mathbb{C}^*$, $\bar{Z}(.,\bar{\lambda})$ is also a fundamental matrix of $T^D(\lambda)Z = 0$. By the definition of a fundamental matrix there exists a λ-dependant invertible matrix $A(\lambda) \in M_4(\mathbb{C})$ ($\lambda \in \mathbb{C}^*$) such that for all $\lambda \in \mathbb{C}^*$, $\bar{Z}(.,\bar{\lambda}) = Z(.,\lambda)\ A(\lambda)$. Choosing the asymptotic fundamental matrix function given by Proposition 3.5, namely $\tilde{E}(x,\lambda)$ given by (3.54), we obtain $\bar{\tilde{E}}(x,\bar{\lambda}) = \tilde{E}(x,\lambda)\ A(\lambda)$. Taking $x = 0$ in this equation and letting $|\lambda|$ tend to infinity, we deduce that $A(\lambda) \to I$ when $|\lambda| \to +\infty$. As a consequence for all $\lambda \in \mathbb{C}^*$, $\bar{M}(\bar{\lambda}) = \bar{T}^R(\bar{\lambda})\bar{\tilde{E}}(x,\bar{\lambda}) = T^R(\lambda)\tilde{E}(x,\lambda)A(\lambda) = M(\lambda)A(\lambda)$ and $|\Delta(\bar{\lambda})| = |\Delta(\lambda)||\det A(\lambda)|$ thus $|\Delta(\bar{\lambda})| \sim |\Delta(\lambda)|$ when $|\lambda| \to +\infty$.

One deduces that there exist $\beta > 0$ and $C > 0$ such that for all $\lambda \in \mathbb{C}$, $|\Delta(\lambda)G(\lambda)| \leq C\, e^{\beta|\lambda|}$. Therefore $\Delta(\lambda)G(\lambda)$ is of exponential type. It follows that $G(\lambda)$ is the ratio of two entire functions of exponential type and thus is of exponential type [11], p. 22. □

Let us recall that if H is a Hilbert space, a sequence $(h_n)_{n\in\mathbb{N}}$ of vectors of H is called minimal if no vector h_{n_0} belongs to the closed linear span of the remaining vectors h_n, $n \neq n_0$ [23], p. 187. Lemma 2.4 of [22] asserts that if T is a compact operator of a Hilbert space such that $N(T) = \{0\}$, then there is a system of root vectors of T which is minimal. Since $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact, this lemma applies to $\mathcal{A}^{-1}$. We are in position to state and prove one of the main theorems of this paper

Theorem 4.3 *Under hypotheses 2.1, 3.6, 3.8 and 3.12, the set of root vectors of $\mathcal{A}$ is complete in $\mathcal{H}^{1,0}$ and there is a system of root vectors of $\mathcal{A}$ which is minimal in $\mathcal{H}^{1,0}$.*

Proof. We have proved that $G(\lambda)$ is bounded on the half-plane $\mathrm{Re}\lambda \geq \alpha$ and on the line $\mathrm{Re}\lambda \leq \alpha$, $\mathrm{Im}\lambda = 0$ and that $G(\lambda)$ is of exponential type. By the Phragmén-Lindelöf principle [41], p. 80, $G(\lambda)$ is bounded on $\mathbb{C}$. Since $G(\lambda) \to 0$ when $\mathrm{Re}\lambda \to +\infty$, the Liouville theorem shows that $G \equiv 0$ hence $(\mathcal{A}^* - \lambda I)^{-1}\mathbf{U} \equiv 0$ and $\mathbf{U} = 0$. The proof of Theorem 4.3 is complete. □

If H is a separable Hilbert space, a sequence $(h_n)_{n\in\mathbb{N}}$ of vectors of H is a Riesz basis of H if by definition there exists an invertible operator D such that the sequence (Dh_n) is an orthonormal basis of H [23], p. 168. Theorem 2.1, p. 310 of [6] states that a sequence $(h_n)_{n\in\mathbb{N}}$ of vectors of H is a Riesz basis of H iff it is complete in H and there exist two constants $a_1, a_2 > 0$ such that for all $n \in \mathbb{N}$, for all $\gamma_1, \cdots\ \gamma_n \in \mathbb{C}$,

$$a_2(\sum_{j=1}^{n} |\gamma_j|^2) \leq |\sum_{j=1}^{n} \gamma_j h_j|^2 \leq a_1(\sum_{j=1}^{n} |\gamma_j|^2). \tag{4.29}$$

Recall that two sequences $(h_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ of vectors of H are biorthogonal if $(h_n, k_m)_H = \delta_{nm}$ $(n, m \in \mathbb{N})$ and that if $(h_n)_{n\in\mathbb{N}}$ is a Riesz basis of H, there exists a Riesz basis $(k_n)_{n\in\mathbb{N}}$ of H such that $(h_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ are biorthogonal [6], p. 310. Let us recall the following theorem from Xu [35], Theorem 1.1:

Theorem 4.4 *Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{A}$ be the generator of a C^0 semigroup $T(t)$ on $\mathcal{H}$. Assume that*

(1) $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, where $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=0}^{+\infty}$ consists of isolated eigenvalues of $\mathcal{A}$ with finite algebraic multiplicity,

(2) $\sup_{k\geq 0} \dim E(\lambda_k, \mathcal{A})\mathcal{H} < +\infty$, where $E(\lambda_k, \mathcal{A})$ is the Riesz projector associated with λ_k,

(3) There is a constant α such that $\sup\{Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{Re\lambda, \lambda \in \sigma_2(\mathcal{A})\}$ and $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$.

With these hypotheses the following assertions are true

(i) There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ with the property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, $\{E(\lambda_k, \mathcal{A})\mathcal{H}_2\}_{k=0}^{+\infty}$ forms a subspace Riesz basis for $\mathcal{H}_2$ (see [6], p. 332 for the definition) and $\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}$,

(ii) If $\sup_{k\geq 0} ||E(\lambda_k, \mathcal{A})|| < +\infty$, then $D(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}$,

(iii) $\mathcal{H}$ has the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (topological direct sum) if and only if

$$\sup_{n\geq 0} \left\| \sum_{k=0}^{n} E(\lambda_k, \mathcal{A}) \right\| < +\infty. \tag{4.30}$$

The other main result of this paper reads as follows

Theorem 4.5 *Under hypotheses 2.1, 3.6, 3.8 and 3.12, there exists a sequence of root vectors of $\mathcal{A}$ (defined by* (2.30)*,* (2.31)*) which forms a Riesz basis of $\mathcal{H}^{1,0}$.*

P r o o f. Lemmas 2.4, 2.6, 3.9 and Proposition 3.15 show that the hypotheses of theorem 4.4 are fulfilled with $\mathcal{H} = \mathcal{H}^{1,0}$, $\mathcal{A}$ defined by (2.30), (2.31), $\sigma_1(\mathcal{A}) = \emptyset$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. Therefore, since the eigenvalues of $\mathcal{A}$ are asymptotically algebraically simple, there exists a sequence of root vectors of $\mathcal{A}$ which forms a Riesz basis of $\mathcal{H}_2$. Theorem 4.3 gives $\mathcal{H}_2 = \mathcal{H} = \mathcal{H}^{1,0}$ and the result follows. □

Remark 4.6 Let us compare Theorem 4.5 with other results in the literature on the spectral analysis of the Timoshenko beam model.

In [29], the spectral analysis of the (non rotating) Timoshenko beam model with the same boundary damping as ours on one end is addressed. Under some hypotheses on the physical properties of the beam which are similar to ours, the Riesz basis property of a sequence of root vectors of the unbounded operator associated to the problem (corresponding to $\mathcal{A}$) is shown. More precisely our hypotheses 3.6 and 3.8 are equivalent to hypotheses (3.17) and (4.8) of [29]. But in [29] the hypothesis corresponding to our hypothesis 3.12 does not appear. In [29], Theorem 4.2, it is written "The entire set of eigenvalues asymptotically splits into two disjoint sets", but as it is well explained in [21], Remark 6.6, it is not the case in general, but this assertion is true if our hypothesis 3.12 is satisfied.

In [39], the spectral analysis of the (non rotating) Timoshenko beam model with a more general boundary damping than ours is examined. The Riesz basis property of a sequence of root vectors of the unbounded operator associated to the problem (corresponding to $\mathcal{A}$) is shown under hypotheses equivalent to our hypotheses 3.8 and 3.12 but with no hypothesis equivalent to our hypothesis 3.6 (Theorem 4.1 of [39]).

In [21], the spectral analysis of the (non rotating) Timoshenko beam model with the same boundary damping as in [39] is analysed. Under hypotheses on the regularity of the physical parameters, an hypothesis similar to our hypothesis 3.8 (conditions (6.29a) and (6.29b) of [21]), and an hypothesis which, with our notations, can be written $E/G = const.$ (condition (6.9) of [21]), it is shown that there is a system of root vectors of the unbounded operator associated to the problem (corresponding to $\mathcal{A}$) which is complete and minimal (Theorem 6.3 (i) of [21]). In the case of the same boundary damping as ours on one end, and with additional hypotheses on the regularity of the physical parameters, the Riesz basis property with parentheses of a sequence of root vectors of the unbounded operator associated to the problem is shown (Theorem 6.3 (ii) of [21]).

From Theorem 4.5 we get an expansion of the solution of (2.12), (2.13) as a function of a sequence of root vectors of $\mathcal{A}$.

Theorem 4.7 *Assume that hypotheses 2.1, 3.6, 3.8 and 3.12 are satisfied. Assume moreover that $\mathbf{u}_0 \in H^1$, $\mathbf{v}_0 \in H^1$, $A\mathbf{u}_0 + B\mathbf{v}_0 \in H^0$ or (which is equivalent) $\mathbf{u}_0 \in H^2$, $\mathbf{v}_0 \in H^1$, $[\mathcal{B}_a\mathbf{u}_0 + \mathcal{B}_b\mathbf{v}_0](0) = 0$, $[\mathcal{B}_a\mathbf{u}](l) = 0$. Let $\mathbf{u}$ be the solution to* (2.12)*,* (2.13) *given by Proposition 2.7, iii). Let $\sigma_p(\mathcal{A}) = \{\lambda_k, k \in \mathbb{N}\}$ be the (point) spectrum of $\mathcal{A}$. For all $k \in \mathbb{N}$, let r_k be the geometric multiplicity of λ_k, $m_{k,i}$, $i = 1, \dots, r_k$ the partial multiplicities of the holomorphic family $A(\lambda) = \mathcal{A} - \lambda I$ at λ_k, $(\mathbf{U}_{k,i,j})$, $i = 1, \dots r_k$, $j = 0, \dots, m_{k,i} - 1$ a family of root vectors corresponding to λ_k such that $(\mathcal{A} - \lambda_k I)\mathbf{U}_{k,i,j} = \mathbf{U}_{k,i,j-1}$, $i = 1, \dots r_k$, $j = 0, \dots, m_{k,i} - 1$ (with $\mathbf{U}_{k,i,-1} = 0$) and set $\mathbf{U}_{k,i,j} = \begin{pmatrix} \mathbf{u}_{k,i,j} \\ \mathbf{v}_{k,i,j} \end{pmatrix}$. Then there exists a family of polynomial functions $(P_{k,i,j})$ of degree $\leq m_{k,i} - 1 - j$, $k \in \mathbb{N}$, $i = 1, \dots r_k$, $j = 0, \dots, m_{k,i} - 1$ such that setting $a_{k,i,j}(t) = e^{\lambda_k t} P_{k,i,j}(t)$ ($t \in \mathbb{R}$) and*

$$\mathbf{u}^n(t) = \sum_{k=0}^{n} \sum_{i=1}^{r_k} \sum_{j=0}^{m_{k,i}-1} a_{k,i,j}(t)\mathbf{u}_{k,i,j} \; (n \in \mathbb{N},\, t \in \mathbb{R}), \tag{4.31}$$

we have

$$\mathbf{u}^n(t) \to e^{-\Omega t}\mathbf{u}(t) \text{ in } H^2 \text{ uniformly on } [0,+\infty), \tag{4.32}$$

$$\dot{\mathbf{u}}^n(t) \to (e^{-\Omega t}\mathbf{u})^{\cdot}(t) \text{ in } H^1 \text{ uniformly on } [0,+\infty), \tag{4.33}$$

$$\ddot{\mathbf{u}}^n(t) \to (e^{-\Omega t}\mathbf{u})^{\cdot\cdot}(t) \text{ in } H^0 \text{ uniformly on } [0,+\infty), \tag{4.34}$$

so that for all $T > 0$

$$e^{\Omega t}\mathbf{u}^n(t) \to \mathbf{u}(t) \text{ in } H^2 \text{ uniformly on } [0,T], \tag{4.35}$$

$$(e^{\Omega t}\mathbf{u}^n)^{\cdot}(t) \to \dot{\mathbf{u}}(t) \text{ in } H^1 \text{ uniformly on } [0,T], \tag{4.36}$$

$$(e^{\Omega t}\mathbf{u}^n)^{\cdot\cdot}(t) \to \ddot{\mathbf{u}}(t) \text{ in } H^0 \text{ uniformly on } [0,T]. \tag{4.37}$$

P r o o f. Let $t \in [0,+\infty) \to T(t)$ be the contraction semigroup generated by $\mathcal{A}$. Under the hypotheses of the theorem, from Proposition 2.7-i), $\mathbf{U}_0 = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 - \Omega\mathbf{u}_0 \end{pmatrix} \in D(\mathcal{A})$. From Proposition 2.7-ii) there exists a unique solution $\in \mathcal{C}^1([0,+\infty);\mathcal{H}^{1,0}) \cap \mathcal{C}^0([0,+\infty);D(\mathcal{A}))$ to the initial value problem (2.61) given by $\mathbf{U}(t) = T(t)\mathbf{U}_0$, $(t \in [0,+\infty))$.

Theorem 4.5 implies that the family $(\mathbf{U}_{k,i,j})$, $k \in \mathbb{N}$, $i = 1,\dots r_k$, $j = 0,\dots,m_{k,i}-1$ is a Riesz basis of $\mathcal{H}^{1,0}$. Let $(\mathbf{V}_{k,i,j})$, $k \in \mathbb{N}$, $i = 1,\dots r_k$, $j = 0,\dots,m_{k,i}-1$ be the Riesz basis of $\mathcal{H}^{1,0}$ which is biorthogonal to the Riesz basis $(\mathbf{U}_{k,i,j})$. Every element $\mathbf{U} \in \mathcal{H}^{1,0}$ can be written in a unique way $\mathbf{U} = \sum_{k,i,j}(\mathbf{U},\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}\mathbf{U}_{k,i,j}$, the convergence being in $\mathcal{H}^{1,0}$. Equation (4.29) implies $\sum_{k,i,j}|(\mathbf{U},\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}|^2 < +\infty$. By a proof similar to that of [2], Theorem 2.3.5-b, one can show that

$$D(\mathcal{A}) = \left\{ \mathbf{U} \in \mathcal{H}^{1,0};\ \sum_{k,i,j}|\lambda_k|^2|(\mathbf{U},\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}|^2 < +\infty \right\} \tag{4.38}$$

and for all $\mathbf{U} \in D(\mathcal{A})$,

$$\begin{array}{rcl} \mathcal{A}\mathbf{U} & = & \sum_{k,i,j}(\mathbf{U},\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}\{\lambda_k\mathbf{U}_{k,i,j} + \mathbf{U}_{k,i,j-1}\} \\ & = & \sum_{k,i,j}\{\lambda_k(\mathbf{U},\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}} + (\mathbf{U},\mathbf{V}_{k,i,j+1})_{\mathcal{H}^{1,0}}\}\mathbf{U}_{k,i,j} \end{array} \tag{4.39}$$

(with $\mathbf{V}_{k,i,m_{k,i}} = 0$), the convergence being in $\mathcal{H}^{1,0}$. Setting for all $n \in \mathbb{N}$

$$\mathbf{U}_0^n = \sum_{k=0}^{n}\left(\sum_{i,j}(\mathbf{U}_0,\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}\mathbf{U}_{k,i,j}\right), \tag{4.40}$$

we have: $\mathbf{U}_0^n \to \mathbf{U}_0$ in $\mathcal{H}^{1,0}$ when $n \to +\infty$. Since $\mathbf{U}_0 \in D(\mathcal{A})$, from (4.38) and (4.39), $\mathcal{A}\mathbf{U}_0^n \to \mathcal{A}\mathbf{U}_0$ in $\mathcal{H}^{1,0}$ when $n \to +\infty$. Since $T(t)$ is a contraction semigroup in $\mathcal{H}^{1,0}$ (for all $t \in [0,+\infty)$, $||T(t)||_{\mathcal{H}^{1,0}\to\mathcal{H}^{1,0}} \leq 1$) we infer that $\mathbf{U}^n(t) = T(t)\mathbf{U}_0^n \to T(t)\mathbf{U}_0 = \mathbf{U}(t)$ in $\mathcal{H}^{1,0}$ uniformly on $[0,+\infty)$. Moreover $\dot{\mathbf{U}}^n(t) = \mathcal{A}\mathbf{U}^n(t) = \mathcal{A}T(t)\mathbf{U}_0^n = T(t)\mathcal{A}\mathbf{U}_0^n \to T(t)\mathcal{A}\mathbf{U}_0 = \mathcal{A}T(t)\mathbf{U}_0 = \mathcal{A}\mathbf{U}(t) = \dot{\mathbf{U}}(t)$ in $\mathcal{H}^{1,0}$ uniformly on $[0,+\infty)$. Since from (2.41), on $D(\mathcal{A})$ the graph norm is equivalent to the norm $||.||_{\mathcal{H}^{2,1}}$ it follows that $\mathbf{U}^n(t) \to \mathbf{U}(t)$ in $\mathcal{H}^{2,1}$ uniformly on $[0,+\infty)$. If we set $\mathbf{W}_{k,i,j}(t) = T(t)\mathbf{U}_{k,i,j}$ then $\dot{\mathbf{W}}_{k,i,j}(t) = \mathcal{A}\mathbf{W}_{k,i,j}(t) = \mathcal{A}T(t)\mathbf{U}_{k,i,j} = T(t)\mathcal{A}\mathbf{U}_{k,i,j} = T(t)(\lambda_k\mathbf{U}_{k,i,j} + \mathbf{U}_{k,i,j-1}) = \lambda_k\mathbf{W}_{k,i,j} + \mathbf{W}_{k,i,j-1}$. This differential system and the initial conditions $\mathbf{W}_{k,i,j}(0) = \mathbf{U}_{k,i,j}$ give

$$\mathbf{W}_{k,i,j}(t) = e^{\lambda_k t}\left\{\mathbf{U}_{k,i,j} + \mathbf{U}_{k,i,j-1}t + \cdots + \mathbf{U}_{k,i,0}\frac{t^j}{j!}\right\}. \tag{4.41}$$

Therefore

$$\sum_{j=0}^{m_{k,i}-1}(\mathbf{U}_0,\mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}}\mathbf{W}_{k,i,j}(t) = e^{\lambda_k t}\left\{\sum_{j=0}^{m_{k,i}-1}P_{k,i,j}(t)\mathbf{U}_{k,i,j}\right\} \tag{4.42}$$

where

$$P_{k,i,j}(t) = (\mathbf{U}_0, \mathbf{V}_{k,i,j})_{\mathcal{H}^{1,0}} + (\mathbf{U}_0, \mathbf{V}_{k,i,j+1})_{\mathcal{H}^{1,0}} t + \cdots + (\mathbf{U}_0, \mathbf{V}_{k,i,m_{k,i}-1})_{\mathcal{H}^{1,0}} \frac{t^{m_{k,i}-1-j}}{(m_{k,i}-1-j)!} \quad (4.43)$$

so that

$$\mathbf{U}^n(t) = T(t)\mathbf{U}_0^n = \sum_{k=0}^{n} e^{\lambda_k t} \sum_{i=1}^{r_k} \left\{ \sum_{j=0}^{m_{k,i}-1} P_{k,i,j}(t)\mathbf{U}_{k,i,j} \right\}. \quad (4.44)$$

Recall that the solution to (2.61) is written under the form $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$. For $n \in \mathbb{N}$ set $\mathbf{U}^n = \begin{pmatrix} \mathbf{u}^n \\ \mathbf{v}^n \end{pmatrix}$. We have $\mathbf{v} = \dot{\mathbf{u}}$ and $\mathbf{v}^n = \dot{\mathbf{u}}^n$. Since $\mathbf{U}^n(t) \to \mathbf{U}(t)$ in $\mathcal{H}^{2,1}$ and $\dot{\mathbf{U}}^n(t) \to \dot{\mathbf{U}}(t)$ in $\mathcal{H}^{1,0}$ uniformly on $[0, +\infty)$ we deduce that $\mathbf{u}^n(t) \to \mathbf{u}(t)$ in H^2, $\dot{\mathbf{u}}^n(t) \to \dot{\mathbf{u}}(t)$ in H^1, $\ddot{\mathbf{u}}^n(t) \to \ddot{\mathbf{u}}(t)$ in H^0 uniformly on $[0, +\infty)$. Replacing $\mathbf{u}(t)$ by $e^{\Omega t}\mathbf{u}(t)$ the result follows. □

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