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**FREE RINGS AND THEIR GEOMETRIC REALIZATION AS
PLANAR CYCLE COMPLEXES.
EXTENSION OF J.J. ROTMAN'S FREE GROUP THEOREM.**

JAMES F. PETERS

Dedicated to W. Dyke, I.N. Herstein & S.V. Banavar

ABSTRACT. This paper introduces free rings, free path groups and free path rings. A free ring is a Herstein ring on a free group. A free path group G which is a J.H.C. Whitehead homotopy group free on a collection of continuous maps $h : [0, 1] \rightarrow X$ (called paths) in a path cycle on a space X . Each path cycle is a sequence of paths with no end path. A free path ring is a Herstein ring on a free path group. The main results in this paper are (1) Every path cycle is realizable as a free path ring (Extension of J.J. Rotman's free group theorem), (2) Every 1-cycle cell complex is realizable as a path cycle and (3) Every 1-cycle is realizable as a free ring (this reverbrates back to W. Dyke's 1882 work on the derivation of free groups from circuits on simply connected polygons). Inspired by Lupton-Oprea-Scoville digital fundamental groups, a number of conjectures about the geometric realization of free path rings as structures in digital images and as vector fields are given.

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1. INTRODUCTION

In this paper, Herstein rings [3] are extended to include free rings, defined in Def. 8 with respect to a pair of homomorphisms on a free group.

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From the beginning, geometry has played a key role in the introduction of free groups. A *free group* $G(\beta, +)$ is nonempty finite set G of n elements equipped with binary operation (typically written as a $+$) and a basis $\beta \subset G$ so that every member $a \in G$ can be written as a linear combination the generators $g \in \beta$, *i.e.*, $a = \sum_{g \in \beta} kg, k \bmod n$. Free groups were introduced by W. Dyck in 1882 as a natural biproduct of simply connected polygons P in which every vertex $a \in P$ can be reached by traversing the edges from a and a distinguished vertex $g \in P$ [1]. Later, J.H.C. Whitehead defined free groups in terms of circuits in graphs in which every vertex can be reached in series of moves from a generator vertex [15] and [16, Theorem 4, p. 799] in which circuits on connected surfaces also have corresponding free groups.

This paper also introduces free path rings $R(\beta, +, \cdot)$, defined in terms of a pair of homomorphisms on a free path group (see Def. 10). Briefly, a **free path group** is a free group $G(\beta, +)$ with a basis $\beta \subset G$ that is a collection of paths in a path cycle and there is a homomorphism $f : X \times \beta \rightarrow G$. A **path cycle** is a collection of paths that are continuous maps $h : [0, 1] \rightarrow X$ on a space X in which the vertexes $h(0), h(1)$ are simply connected via $h \in G$ and there is no end vertex [9],[10, App. A]. This leads to an extension of J.J. Rotman's Free Group Theorem. For the extension, see Theorem 4.

Theorem 1. [13, p. 237] *If X is a set, then there exists a group F that is free on X .*

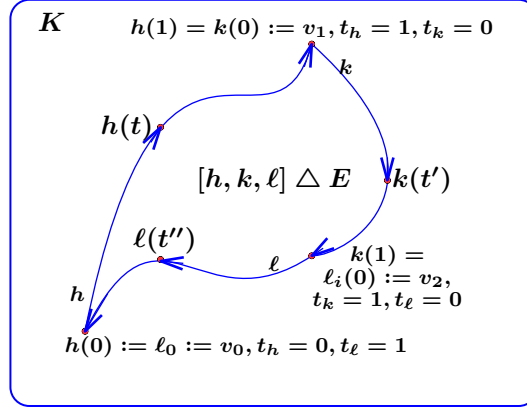


FIGURE 1. $[h, k, \ell] \triangle E$ (Curvilinear path triangle) \mapsto $[[h, k, \ell] \triangle E]$ (geometric realization of $[h, k, \ell] \triangle E$ as a curvilinear triangle) on a space K . During its construction over a temporal interval, $h(t), k(t'), \ell(t'')$ occur at times t, t', t'' , respectively.

2. PRELIMINARIES

This section briefly visits basic structures underlying free ring presentations of path cycles and path nerve complexes. Recall

Definition 1. Path.

Let $I = [0, 1]$, the unit interval. A **path** in a space (S, P) is a continuous map $h : I \rightarrow S$ with endpoints $h(0) = x_0, h(1) = x_1 \in P$ and $h(t) \in S$ for $t \in I$ [14, §2.1,p.11]. The geometric realization of a path h is denoted $|h|$. ■

Remark 1. Paths over Temporal Unit Intervals.

In application, it is often appropriate to consider paths over a temporal unit interval. In that case, the variable $t \in I$ would be conveniently thought of as a clock tick and a path $h(t) \in \mathbb{R}^k$ for $t \in [0, 1]$ would depend on not only the set of points passed through by h but also the time at which a point $h(t)$ is visited [6, §VII.1, p. 174], i.e.,

$$\begin{aligned} h[0, 1] &\rightarrow X. \\ t &\in [0, 1], \text{ with} \\ &\mathbf{0 = start time, } t = \text{elapsed time, } \mathbf{1 = end time} \\ h(t) &\in \overbrace{[0, \dots, t, \dots, 1]}, \end{aligned}$$

and $h(t)$ is an event that occurs at time t . For example, the trajectory of a moving billiard ball E traces a path $h(t)$ with an initial time at $t = 0$ when the billiard starts moving, visiting a sequence of points (0-cells, or vertexes) as it travels on some surface, and end time $t = 1$, when E reaches an end point. ■

Example 1. A sample path triangle (also called a **temporal path triangle**) constructed over a temporal interval is shown in Fig. 1. This path triangle models the paths h_1, h_2, h_3 traced by a moving object such as a billiard ball during its movements before it reaches a rest state $h_3(1)$. Temporal path triangles can be found in triangulated video frames [2] in which the changing triangles on triangulated video frame shapes result from the optical flow recorded in the video. ■

Definition 2. Path Cycle [10, App. A].

A **path cycle** $hCycE$ in a CW space K is a sequence of continuous maps $\{h_i\}_{i=0}^{(n-1)[n]}$,

$h_i : I \rightarrow K, [n] = \text{mod } n$, with

$$h_0(0) = h_{n[n]}(0) = v_0 \in K, h_{n-1[n]}(1) = v_{n-1[n]} \in K.$$

Given $h_i \in \{h_i\}_{i=0}^{(n-1)[n]}$,

$g = h_i(0)$, a basis element in $hCycE$, so that

$$\begin{aligned} & \mathbf{h_{i+k}(0) \text{ walks forward } k \text{ vertexes from } g \text{ to reach } h_{i+k}(0)} \\ v & \quad \quad \quad := \overbrace{kg = kh_i(0) = h_{i+k}(0)} \in K, k \in \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} & \mathbf{\bar{h}_{i+k}(0) \text{ walks back } k \text{ vertexes from } h_{i+k}(0) \text{ to reach } h_i(0)} \\ \bar{h}_{i+k}(0) & \quad \quad \quad := \overbrace{h_{(i+k)-k}(0) = h_i(0)}. \end{aligned}$$

$+$: $K \times K \rightarrow K$, is defined by

$$\begin{aligned} & \mathbf{move \ to \ v' \ via \ h_i(0) = v, h_{i+k}(v')} \\ +(v, v') & = \overbrace{+(kg, k'g) = kg + k'g}. \\ & \mathbf{zero \ moves \ from \ v} \\ +(v, 0) & = \overbrace{+(kg, 0) = kg + 0 = kg = v}. \\ & \mathbf{move \ back \ to \ v \ from \ v} \\ +(v, -v) & = \overbrace{+(kg, -kg) = (k - k)g = 0g = g + 0 = v}. \\ & \mathbf{move \ back \ to \ v'' \ from \ v \in hCycE} \\ +(v, -v') & = \overbrace{+(kg, -k'g) = (k - k')g = k''g}. \quad \blacksquare \end{aligned}$$

Remark 2. Order of a path cycle.

A k -path cycle has order k and contains k paths. \blacksquare

Example 2. Path cycle $h \triangle E$ in Fig. 2 is a 3-path cycle (its order is 3) and path cycle $hCycE'$ in Fig. 5 is a 9-path cycle (its order is 9). \blacksquare

Example 3. Path cycle $h \triangle E$ in Fig. 3 is a 4-path cycle (its order is 3) and path cycle $hCycE'$ in Fig. 5 is a 9-path cycle (its order is 9). \blacksquare

Definition 3. 1-Cycle. In a CW space K [17], a 1-cycle E (denoted by $cycE$) in a CW space K is a collection of path-connected vertexes (0-cells) on edges (1-cells) attached to each other with no end vertex. \blacksquare

Definition 4. Path-Connected.

A pair of 0-cells v, v' in a cell complex K is **path-connected**, provided there is a sequence of paths h_1, \dots, h_k , starting with $h_1(0) := v$ and ending with $h_{k-1}(1) = h_k(0) := v'$. \blacksquare

Lemma 1. Every path has a geometric realization as an edge.

Proof. From Def. 1, a path $h : I \rightarrow X$ includes all points $h(t), t \in [0, 1]$, i.e., we have the closed set of points

$$\{h(0), \dots, h(t), \dots, h(1)\}, \text{ for } 0 \leq t \leq 1,$$

which is geometrically realized as an edge $|h| = \overline{h(0)h(1)}$ with $h(t) \in \text{int}(|h|), t \in (0, 1)$ (interior of edge $|h|$). \square

Theorem 2. Let $hCycE$ be a path cycle in a CW space.

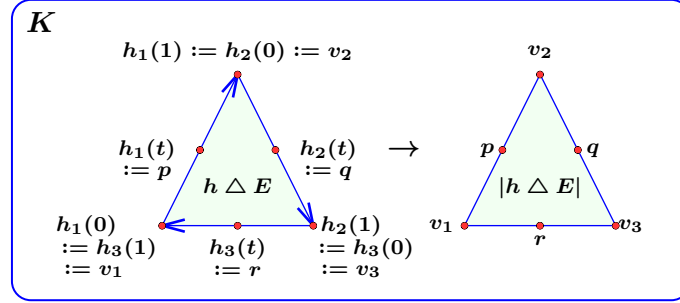


FIGURE 2. Three Paths that Construct the Sides of a Path Triangle $h \triangle E$ in a CW space \mathbf{K} , leading to its geometric realization $|h \triangle E|$ as a 2-cell (triangle)

- 1° Every 3-path cycle has a geometric realization as a 1-cycle triangle.
- 2° Every 4-path cycle has a geometric realization as a 1-cycle rectangle.
- 3° Every k -path cycle, $k \geq 1$ has a geometric realization as a 1-cycle polytope.

Proof.

1°: Let $h \triangle E$ be a 3-path cycle containing paths h, h', h'' . From Lemma 1, the geometric realization of a path h is a 1-cell (edge) $|h|$ in a cell complex. Attach the three paths h, h', h'' together so that

$$\begin{aligned} h'(0) &:= h(1) \text{ defines edge } \overline{h(1), h'(0)} \\ h''(0) &:= h'(1) \text{ defines edge } \overline{h'(1), h''(0)} \\ h(0) &:= h''(1) \text{ defines edge } \overline{h''(1), h(0)}. \end{aligned}$$

From Def. 4, the vertexes are path-connected in this sequence of three edges (call it $\triangle E$). Hence, $\triangle E$ is a 1-cycle that is a triangle, which is the geometric realization of $h \triangle E$ (denoted by $|h \triangle E|$).

2°: Replace $h \triangle E$ in 1° with a 4-path cycle containing paths h, h', h'', h''' and the desired result follows.

3°: Recall that a planar polytope is a convex hull that is a finite collection of intersecting half spaces [18]. Let $hCycE$ be a k -path cycle. Replace $h \triangle E$ in 1° with a 4-path cycle containing paths h_1, \dots, h_k . From Lemma 1, the geometric realization of a path $h \in hCycE$ is a 1-cell (edge). Assume that the geometric realization each path $h \in hCycE$ is a straight edge on a half plane. Since each $h_i(1) = h_{i+1}(0)$ for every $h_i, h_{i+1} \in hCycE$, the pairs of paths with geometrically-realized edges with common vertexes have geometric realizations as intersecting half planes. Replace $h \triangle E$ in 1° with the k -path cycle $hCycE$ and the desired result follows. \square

Example 4. A 3-path cycle $h \triangle E$ is shown in Fig. 2. From Theorem 2.1, $h \triangle E$ has a geometric realization as a 1-cycle triangle $|h \triangle E|$, also depicted in Fig. 2.

■

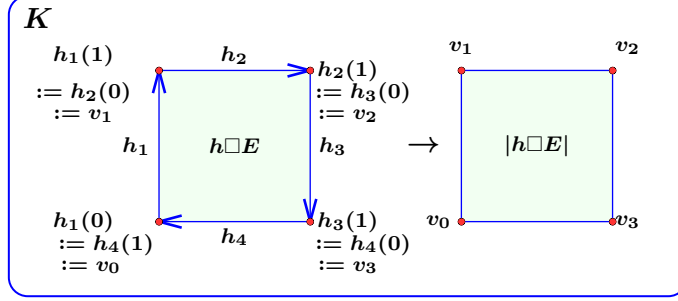


FIGURE 3. Four Paths h_1, h_2, h_3, h_4 geometrically realized as the sides of a Path Rectangle $h \square E$ in a CW space K , leading to its geometric realization $|h \square E|$ as a 1-cycle (rectangle)

Example 5. A 4-path cycle $h \square E$ is shown in Fig. 3. From Theorem 2.2, $h \square E$ has a geometric realization as a 1-cycle rectangle $|h \square E|$, also depicted in Fig. 3.

■

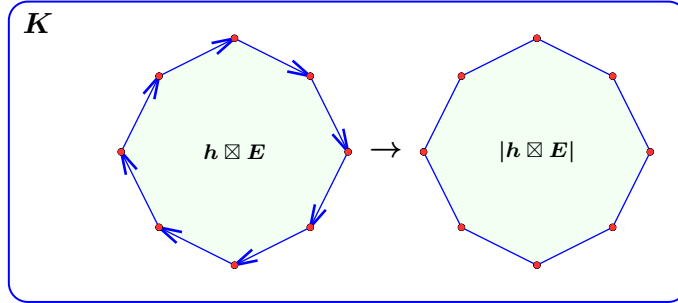


FIGURE 4. $k \geq 1$ paths h_1, h_2, \dots, h_8 geometrically realized as the sides of an 8-sided path polytope $h \boxtimes E$ in a CW space K , leading to its geometric realization $|h \boxtimes E|$ as a 8-sided 1-cycle polytope

Example 6. An 8-path polytope $h \boxtimes E$ is shown in Fig. 4. From Theorem 2.3, $h \boxtimes E$ has a geometric realization as an 8-sided 1-cycle polytope $|h \boxtimes E|$, also depicted in Fig. 4.

■

3. FREE RINGS

This section introduces free rings, which are an extension of free groups. Recall

Definition 5. Ring [3, §7,p. 46].

A ring is a nonempty set R , $a, b, c \in R$, with operations $+$ and \cdot defined on R such that

- (1) $a + b \in R$.
- (2) $(a + b) + c = a + (b + c)$.
- (3) There is an element $0 \in R$ so that $a + 0 \in R$ for all $a \in R$.
- (4) For every $a \in R$, there is an $-a \in R$ so that $a + (-a) = 0$.
- (5) $a \cdot b \in R$.
- (6) $a \cdot (b + c) = a \cdot b + a \cdot c$ and
- (7) $(b + c) \cdot a = b \cdot a + c \cdot a$. ■

Also recall

Definition 6. Homomorphism [3, §7,p. 46].

Let X be a nonempty set and let G, G' be groups on X . A mapping $f : G \rightarrow G'$ is a homomorphism, provided $f(ab) = f(a)f(b)$ for all $a, b \in G$. ■

In this work, we assume that $f : G \rightarrow G'$ is a homomorphism for a free group $G = G'$.

Example 7. Let group G equal group G' in Def. 6 and let G be a free group $G(\beta, +)$, with basis $\beta \subset G$ and move operation $+$ in the presentation of a path cycle complex $hCycE$ (see, e.g., [11, Appendix B, p. 393]).

Definition 7. Free Group.

A group $G(\beta, \circ)$ containing a basis set $\beta \subset G$ is a **free group** on β , provided there is a homomorphism $f : X \times \beta \rightarrow G$ defined by $f(a \circ b) = ka \circ k'b$, $a, b \in X$ for every subset X in G . ■

Theorem 3. [8](Path Cycle Free Group Presentation).

Every path cycle in a CW space has a free group presentation.

For the structure of a path cycle, see Def. 2. Let $hCycE$ be a path cycle. Each $a \in G$ can be written as kg for $g \in \beta, k \in \mathbb{Z}$. Also, let $f : G \rightarrow G$, a homomorphism defined by

$$f(a + b) = f(a) + f(b) = kg + k'g = a + b = (k + k')g = k''g, \text{ for}$$

$a, b \in G$. Hence, from Def. 7, $hCycE$ has a free ring presentation. ■

A free ring $R(\beta, +, \cdot)$, introduced in this paper, is the counterpart of a free group $G(\beta, +)$ with basis β (see, e.g., [13, p. 237, 1955 edition]). Unlike a free group, a free ring R is defined in terms of a pair of homomorphisms on a free group.

Definition 8. Free Ring.

Let $G(\beta, +)$ be a free group with basis $\beta \subset G$ and X be a nonempty set $X \subseteq G$, generator $g \in \beta, a, b \in X$. Then a ring $R(\beta, +, \cdot)$ is free on G , provided every map $f : X \times \beta \rightarrow R$ has a corresponding pair of unique homomorphisms $\alpha : X \times \beta \rightarrow R$

with respect to $+$ and $\gamma : X \times \beta \rightarrow R$ with respect to \cdot , $k \bmod n \in \mathbb{Z}^+$, defined by

$$\begin{aligned} X &\subseteq R, \beta \subset R. \\ \alpha(a, g) &= kg \in R, \gamma(a, b) = k'g \in R, k \in \mathbb{Z}. \\ \alpha(a + b, g) &= \alpha(a, g) + \alpha(b, g) \in R \text{ for all } a, b \in X. \\ \gamma(a, g) &= kg, k \in \mathbb{Z}. \\ \gamma(a \cdot b) &= \gamma(a, g) \cdot \gamma(b, g) \in R \text{ for all } a, b \in X. \quad \blacksquare \end{aligned}$$

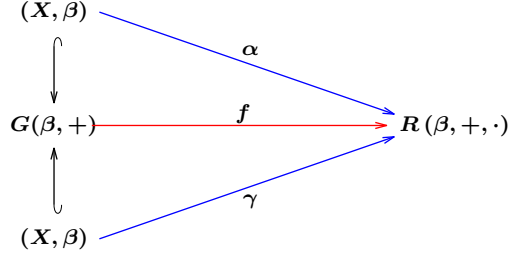


FIGURE 5. Free path ring $R(\beta, +, \cdot)$ on a free group $G(\beta, +)$ with basis β . In this diagram, black arrows represent inclusion maps, blue arrows are homomorphisms and $\beta = \{v_5 = v'_0 = g\}$.

Example 8. The structure of a free ring $R(+, \cdot)$ is represented in the diagram¹ in Fig. 5. Let $\alpha(a, g) = kg \in R, \gamma(a, b) = k'g \in R, k \in \mathbb{Z}$ and let $\alpha : X \times \beta \rightarrow R$ and $\gamma(a \cdot b) = \gamma(a, g) \cdot \gamma(b, g) \in R$ for all $a, b \in X$ for $X \subseteq G, g \in \beta, a, b \in X$ are homomorphisms for $+$ and \cdot , respectively, defined by

$$\begin{aligned} \alpha(a + b, g) &= \alpha(a, g) + \alpha(b, g) \in R \text{ from Def. 8} \\ &= kg + k'g \text{ from Def. 8, for all } a, b \in X, k, k' \in \mathbb{Z} \\ &= (k + k')g \\ &= k''g \in R, k'' \in \mathbb{Z}. \\ \gamma(a \cdot b) &= \gamma(a, g) \cdot \gamma(b, g) \text{ from Def. 8,} \\ &= kg \cdot k'g \text{ from Def. 8, for all } a, b \in X, k, k' \in \mathbb{Z} \\ &= (k \cdot k')g \\ &= k'''g \in R, k''' \in \mathbb{Z}. \quad \blacksquare \end{aligned}$$

4. FREE PATH RINGS

Free path rings are free ring structures built on free path groups.

Definition 9. Free Path Group.

A group $G(\beta, +)$ with basis set $\beta \subseteq G$ (a collection of paths in a path cycle $hCycE$) is a **free path group** on G , provided there is a homomorphism $f : X \times \beta \rightarrow G$ defined by $f(h + h') = kg + k'g \in G, X \subset hCycE \subset G \subseteq hCycE, h, h' \in X$ for every nonvoid set of paths $X \subseteq G$. \blacksquare

¹This diagram is a modified version of the gluing diagram introduced by Tane Vergili in [12, p. 3].

Free path groups lead to free path rings, each with a structure that is similar to an ordinary free ring.

Definition 10. Free Path Ring.

A ring $R(\beta, +, \cdot)$ on a free path group $G(\beta, +)$ with basis set $\beta \subseteq G$ is a **path free ring** on G , provided there is a pair of homomorphisms $\alpha : X \times \beta \rightarrow R$ defined for $+$ by $\alpha(h + h') = kg + k'g \in R$ and $\gamma : X \times \beta \rightarrow R$ defined for \cdot by $\alpha(h \cdot h') = kg \cdot k'g \in R, X \subset hCycE \subset R$, for every pair of paths $h, h' \in X$ for $X \subseteq G, g \in \beta$.

In either case, a reverse move is possible, i.e.,

$$\begin{aligned} \alpha(h - h') &= \overbrace{kg + (-k')g = (k - k')g = k''g = k''h(0) = h''(0)}^{k'' \text{ reverse moves from } h(0) \text{ to reach } h''(0)}. \\ \alpha(h \cdot h') &= \overbrace{kg \cdot (-k')g = (k \cdot k')g = k'''g = k'''h(0) = h'''(0)}^{k''' \text{ reverse moves from } h(0) \text{ to reach } h'''(0)}. \end{aligned}$$

It is also the case that a free path ring has a zero move (also called the null move), serving as an identity in a free path Ring, which is possible for both $+$ and \cdot . ■

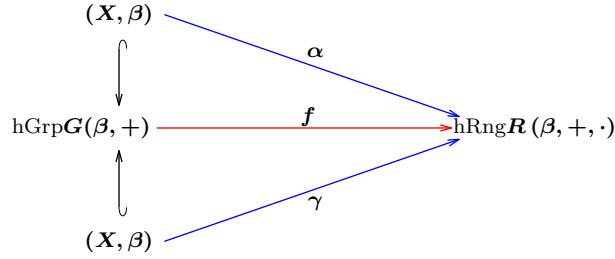


FIGURE 6. Free path ring $hRngR(\beta, +, \cdot)$ on a free path group $hGrpG(\beta, +)$ with basis β . In this diagram, black arrows represent inclusion maps, blue arrows are homomorphisms, and $hRngR$ is defined on a free path group $hGrpG(\beta, +)$.

Example 9. The structure of a free path ring $hRngR(+, \cdot)$ is represented in the diagram in Fig. 5. Let $\alpha(a, g) = kg, k \in \mathbb{Z}$ and let $\alpha : X \times \beta \rightarrow R$ and $\gamma : X \times \beta \rightarrow R$ for $X \subseteq G, g \in \beta, a, b \in X$ are homomorphisms for $+$ and \cdot , respectively, defined by

$$\begin{aligned} \alpha(a + b, g) &= \alpha(a, g) + \alpha(b, g) \in hRngR \text{ from Def. 8} \\ &= kg + k'g \text{ from Def. 8, for all } a, b \in X, k, k' \in \mathbb{Z} \\ &= (k + k')g \\ &= k''g \in R, k'' \text{ mod } n \in \mathbb{Z}^+. \\ \gamma(a \cdot b, g) &= \gamma(a, g) \cdot \gamma(b, g) \in hRngR \text{ from Def. 8,} \\ &= kg \cdot k'g \text{ from Def. 8, for all } a, b \in X, k, k' \in \mathbb{Z} \\ &= (k \cdot k')g \\ &= k'''g \in R, k''' \text{ mod } n \in \mathbb{Z}^+. \quad \blacksquare \end{aligned}$$

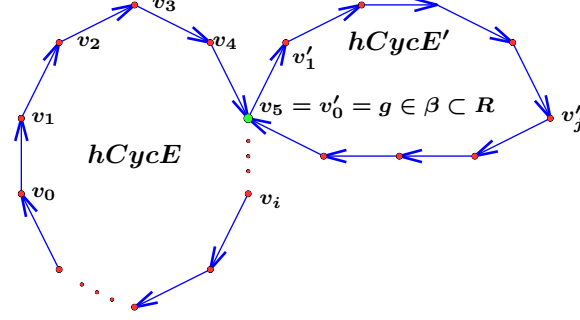


FIGURE 7. Free path ring $\mathbf{R}(\{hCycE, hCycE' : hCycE \cap hCycE' \neq \emptyset\}, +, \cdot)$ on a pair of intersecting path cycles $hCycE, hCycE'$.

Example 10. The pair of path cycles $hCycE, hCycE'$ with $\beta = hCycE \cap hCycE' = \{h_5(0) = v'_0 = g\}$ represent a free path ring $\mathbf{R}(\beta, +, \cdot)$ in Fig. 7. In this ring, the $+, \cdot$ operations are defined by

$$\begin{aligned}
 &h, h' \in hCycE \cup hCycE', \\
 &\beta = hCycE \cap hCycE' = \{h_5(0) = v'_0 = g\} \text{ basis} \\
 &\quad \mathbf{k \text{ moves from } h(0) \text{ to reach } h'(0)} \\
 kh(0) = &\quad \overbrace{kg = h'(0)}. \\
 &\quad \mathbf{k'' \text{ moves from } h(0) \text{ to reach } h''(0)} \\
 +(h, h') = &\quad \overbrace{kg + k'g = (k + k')g = k''g = k''h(0) = h''(0)}. \\
 &\quad \mathbf{k''' \text{ moves from } h(0) \text{ to reach } h'''(0)} \\
 \cdot(h, h') = &\quad \overbrace{kg \cdot k'g = (k \cdot k')g = k'''g = k'''h(0) : k'' \leq k''', h''(0) \leq h'''(0)}.
 \end{aligned}$$

In effect, $\cdot(h, h')$ results in a longer walk that starts at $h(0)$ than the walk $+(h, h')$ that starts at $h(0)$. This is a distinguishing property of the \cdot ring homomorphism.

■

Lemma 2. Every path cycle is realizable as a free path group.

Proof. Let $hCycE$ be a path cycle, paths $h, h' \in hCycE$, $\beta \subset hCycE, g = h(0) \in \beta$. From Def. 9, we can define a homomorphism $+(h, h') = kg + k'g = (k + k')g = k''g = k''h(0) \in hCycE$. In addition, every move $+(h, h')$ has a reverse move $-kg$, i.e., $+(h, h') = kg + (-k'g) = kg - k'g = (k - k')g = k''g = k'''h(0) \in hCycE$. With $k = 0$, we obtain a zero move, which is the identity or null move in $hCycE$, i.e., $+(h, h') = kg + (0g) = kg + 0 = kg = kh(0) \in hCycE$. Hence, we obtain $G(\beta, +)$, a free path group on $hCycE$. \square

Lemma 3. Every path cycle is realizable as a free path ring.

Proof. Let $\text{hCyc}E$ be a path cycle. From Lemma 2, $\text{hCyc}E$ is realizable as a free path group $G(\beta, +)$ with basis β and homomorphism $+$. Similarly, $\text{hCyc}E$ is realizable as a free path group $G(\beta, \cdot)$ with basis β and homomorphism \cdot .

every $h \in G$ has a reverse move $-h$ and there is a zero move $+(h, 0) = kg + 0 = kg = h(0) \in \text{hCyc}E$. Hence, G is a free path ring. \square

It is a straightforward step from free path groups to free path rings.

Theorem 4. *If X is a path cycle, then there exists a free path ring R that is free on X .*

Proof. Let X be a path cycle. From Lemma 2, X is realizable as a free path group. In addition, from Lemma 3, X is realizable as a free path ring. Replace $G(\beta, +)$ in Def. 3 with X and the desired result follows. \square

Theorem 5. *Every path cycle is realizable as a 1-cycle.*

Proof. From Lemma 1, the geometric realization of a path h is a 1-cell (edge) $|h|$ in a cell complex. Then each of the paths in a path cycle has a geometric realization as an edge. Hence, from Def. 3 (1-cycle), a path cycle has a geometric realization as a 1-cycle. \square

The converse of Theorem 5 also holds.

Lemma 4. *Every 1-cycle cell complex X in a CW space is realizable as a path cycle.*

Proof. Let $\text{cyc}E$ be a 1-cycle on a space X . Let a, b be vertices for an edge \widehat{ab} in $\text{cyc}E$. Then define the path

$$\begin{aligned} h &: I \rightarrow X. \\ h(0) &= a. \\ h(1) &= b. \end{aligned}$$

Hence, every 1-cycle in space X is realizable as a path cycle. \square

Theorem 6. *Every 1-cycle cell complex X in a CW space is realizable as a ring that is free on X .*

Proof. Let $\text{cyc}E$ be a 1-cycle on a space X . From Lemma 4, $\text{cyc}E$ is realizable as a path cycle. Then the desired result is an immediate consequence of Theorem 4. \square

Remark 3. *Theorem 4 is a fundamental theorem in homotopy theory, since free path rings have their counterpart in the free groups introduced by W. Dyke citeDyck1882freiGruppe. Both free path rings and free groups are a result of geometry. That is, free groups result from cycles in polygons and free path rings result are derivable from 1-cycles in cell complexes. \blacksquare*

5. CONJECTURES

This section briefly introduces conjectures, which lead to a transition from the results in this paper to closely related results in digital topology from G. Lupton, J. Oprea and N.A. Scoville [5], [4], *e.g.*,

Theorem 7. Theorem 5.16 [5, p.22].

Every finitely presented group occurs as the (digital) fundamental group of some digital image.

Recall that in every triangulated digital image, there is a least one maximal nucleus cluster (MNC) of triangles on a single vertex [7, §1.4, p. 9]. By attaching the edges to the barycenters of the MNC triangles, we obtain a 1-cycle. This leads to Conjecture 1.

Conjecture 1. *Every free path ring has a geometric realization as a 1-cycle in a triangulated digital image.*

One way to prove Conj. 1 is to define a path over each edge between consecutive barycenters. The end result will be a path cycle, which provides a basis for a free path ring. From Conj. 1, we also have the following three Conjectures.

Conjecture 2. *Every free path ring has a geometric realization in a collection of intersecting 1-cycles in a triangulated digital image.*

Conjecture 3. *Every free path ring has a geometric realization in some triangulated digital image.*

Conjecture 4. *Every free path ring has a realization as a vector field.*

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