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Improved interior regularity for elliptic membrane shells

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Mathematical problems in mechanics

Abstract

In this Note we study the regularity of the solution of an obstacle problem for linearly elastic elliptic membrane shells, obtained as a result of a rigorous asymptotic analysis. Since the solution of this boundary value problem is uniquely determined, the problem in object is formulated as a set of variational inequalities posed over a non-empty, closed, and convex subset of a Sobolev space. We will show that, by imposing a higher regularity on the applied body force density acting on the linearly elastic elliptic membrane shell under consideration, the displacement vector field that solves the aforementioned variational inequalities actually enjoys, at least locally, a regularity higher by one order for each of its components.

Dans cette Note, nous étudions la régularité de la solution d'un problème d'obstacle pour des coques membranaires elliptiques linéairement élastiques, obtenu à la suite d'une analyse asymptotique rigoureuse. Puisque la solution de ce problème aux limites est déterminée de manière unique, le problème dans l'objet est formulé comme un ensemble d'inégalités variationnelles posées sur un sous-ensemble non vide, fermé et convexe d'un espace de Sobolev. Nous montrerons qu'en imposant une régularité plus élevée à la densité de force corporelle appliquée agissant sur la membrane elliptique linéairement élastique considérée, le champ de vecteurs de déplacement qui résout les inégalités variationnelles susmentionnées bénéficie en fait, au moins localement, d'une régularité supérieure d'un ordre pour chacun de ses composants.

1. Preliminaries

For definitions, notations and other preliminaries, we refer to [1]. The complete proofs of the results presented in this Note will be found in [2].

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2. Formulation of an obstacle problem for a linearly elastic elliptic membrane shell

Let ω be a domain in \mathbb{R}^2 , let $\gamma := \partial\omega$, and let γ_0 be a non-empty relatively open subset of γ . For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[\text{ and } \Gamma_0^\varepsilon := \gamma_0 \times]-\varepsilon, \varepsilon[,$$

we let $x^\varepsilon = (x_i^\varepsilon)$ designate a generic point in the set $\overline{\Omega^\varepsilon}$, and we let $\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon$. Hence we also have $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$.

Given an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ and $\varepsilon > 0$, consider a *shell* with *middle surface* $\boldsymbol{\theta}(\overline{\omega})$ and with *constant thickness* 2ε . This means that the *reference configuration* of the shell is the set $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$, where the mapping $\boldsymbol{\Theta} : \overline{\Omega^\varepsilon} \rightarrow \mathbb{E}^3$ is defined by

$$\boldsymbol{\Theta}(x^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \mathbf{a}^3(y) \text{ at each point } x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega^\varepsilon}.$$

One can then show (cf., e.g., Theorem 3.1-1 of [3]) that, if $\varepsilon > 0$ is small enough, such a mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega^\varepsilon}; \mathbb{E}^3)$ is an *immersion*, in the sense that the three vectors

$$\mathbf{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(x^\varepsilon)$$

are linearly independent at each point $x^\varepsilon \in \overline{\Omega^\varepsilon}$; these vectors then constitute the *covariant basis* at the point $\boldsymbol{\Theta}(x^\varepsilon)$, while the three vectors $\mathbf{g}^{j,\varepsilon}(x^\varepsilon)$ defined by the relations

$$\mathbf{g}^{j,\varepsilon}(x^\varepsilon) \cdot \mathbf{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j$$

constitute the *contravariant basis* at the same point. It will be implicitly assumed in the sequel that $\varepsilon > 0$ is *small enough* so that $\boldsymbol{\Theta} : \overline{\Omega^\varepsilon} \rightarrow \mathbb{E}^3$ is an *immersion*.

One then defines the *metric tensor associated with the immersion* $\boldsymbol{\Theta}$ by means of its *covariant components*

$$g_{ij}^\varepsilon := \mathbf{g}_i^\varepsilon \cdot \mathbf{g}_j^\varepsilon \in \mathcal{C}^1(\overline{\Omega^\varepsilon}),$$

or by means of its *contravariant components*

$$g^{ij,\varepsilon} := \mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}^{j,\varepsilon} \in \mathcal{C}^1(\overline{\Omega^\varepsilon}).$$

Note that the symmetric matrix field $(g^{ij,\varepsilon})$ is then the inverse of the matrix field (g_{ij}^ε) , that $\mathbf{g}^{j,\varepsilon} = g^{ij,\varepsilon} \mathbf{g}_i^\varepsilon$ and $\mathbf{g}_i^\varepsilon = g_{ij}^\varepsilon \mathbf{g}^{j,\varepsilon}$, and that the *volume element* in $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$ is given at each point $\boldsymbol{\Theta}(x^\varepsilon)$, $x^\varepsilon \in \overline{\Omega^\varepsilon}$, by $\sqrt{g^\varepsilon(x^\varepsilon)} dx^\varepsilon$, where

$$g^\varepsilon := \det(g_{ij}^\varepsilon) \in \mathcal{C}^1(\overline{\Omega^\varepsilon}).$$

One also defines the *Christoffel symbols* associated with the immersion $\boldsymbol{\Theta}$ by

$$\Gamma_{ij}^{p,\varepsilon} := \partial_i \mathbf{g}_j^\varepsilon \cdot \mathbf{g}^{p,\varepsilon} = \Gamma_{ji}^{p,\varepsilon} \in \mathcal{C}^0(\overline{\Omega^\varepsilon}).$$

Note that $\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0$.

Given a vector field $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathcal{C}^1(\overline{\Omega^\varepsilon}; \mathbb{R}^3)$, the associated vector field

$$\tilde{\mathbf{v}}^\varepsilon := v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$$

can be viewed as a *displacement field* of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$ of the shell, thus defined by means of its *covariant components* v_i^ε over the vectors $\mathbf{g}^{i,\varepsilon}$ of the contravariant bases in the reference configuration.

If the norms $\|v_i^\varepsilon\|_{\mathcal{C}^1(\overline{\Omega^\varepsilon})}$ are small enough, the mapping $(\boldsymbol{\Theta} + v_i^\varepsilon \mathbf{g}^{i,\varepsilon})$ is also an immersion, so that one can also define the metric tensor of the *deformed configuration* $(\boldsymbol{\Theta} + v_i^\varepsilon \mathbf{g}^{i,\varepsilon})(\overline{\Omega^\varepsilon})$ by means of its covariant components

$$\begin{aligned} g_{ij}^\varepsilon(v^\varepsilon) &:= (\mathbf{g}_i^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon) \cdot (\mathbf{g}_j^\varepsilon + \partial_j^\varepsilon \tilde{\mathbf{v}}^\varepsilon) \\ &= g_{ij}^\varepsilon + \mathbf{g}_i^\varepsilon \cdot \partial_j \tilde{\mathbf{v}}^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon \cdot \mathbf{g}_j^\varepsilon + \partial_i \tilde{\mathbf{v}}^\varepsilon \cdot \partial_j \tilde{\mathbf{v}}^\varepsilon. \end{aligned}$$

The linear part with respect to $\tilde{\mathbf{v}}^\varepsilon$ in the difference $(g_{ij}^\varepsilon(\mathbf{v}^\varepsilon) - g_{ij}^\varepsilon)/2$ is then called the *linearised strain tensor* associated with the displacement field $v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$, the covariant components of which are thus defined by

$$e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon) := \frac{1}{2} (\mathbf{g}_i^\varepsilon \cdot \partial_j \tilde{\mathbf{v}}^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon \cdot \mathbf{g}_j^\varepsilon) = \frac{1}{2} (\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) - \Gamma_{ij}^{p;\varepsilon} v_p^\varepsilon = e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon).$$

The functions $e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon)$ are called the *linearised strains in curvilinear coordinates* associated with the displacement field $v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$.

We assume throughout this Note that, for each $\varepsilon > 0$, the reference configuration $\Theta(\overline{\Omega^\varepsilon})$ of the shell is a *natural state* (i.e., stress-free) and that the material constituting the shell is *homogeneous, isotropic, and linearly elastic*. The behaviour of such an elastic material is thus entirely governed by its two *Lamé constants* $\lambda \geq 0$ and $\mu > 0$ (for details, see, e.g., Section 3.8 of [4]).

We will also assume that the shell is subjected to *applied body forces* whose density per unit volume is defined by means of its covariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$, and to a *homogeneous boundary condition of place* along the portion Γ_0^ε of its lateral face (i.e., the displacement vanishes on Γ_0^ε).

In this Note, we consider a specific *obstacle problem* for such a shell, in the sense that the shell is also subjected to a *confinement condition*, expressing that any *admissible* deformed configuration remains in a *half-space* of the form

$$\mathbb{H} := \{\mathbf{Ox} \in \mathbb{E}^3; \mathbf{Ox} \cdot \mathbf{q} \geq 0\},$$

where $\mathbf{q} \in \mathbb{E}^3$ is a *non-zero vector* given once and for all. In other words, any admissible displacement field must satisfy

$$(\Theta(x^\varepsilon) + v_i^\varepsilon(x^\varepsilon) \mathbf{g}^{i,\varepsilon}(x^\varepsilon)) \cdot \mathbf{q} \geq 0$$

for all $x^\varepsilon \in \overline{\Omega^\varepsilon}$, or possibly only for almost all (a.a. in what follows) $x^\varepsilon \in \Omega^\varepsilon$ when the covariant components v_i^ε are required to belong to the Sobolev space $H^1(\Omega^\varepsilon)$ as in Theorem 2.1 below.

We will of course assume that the reference configuration satisfies the confinement condition, i.e., that

$$\Theta(\overline{\Omega^\varepsilon}) \subset \mathbb{H}.$$

Such a confinement condition renders the study of this problem considerably more difficult, however, as the constraint now bears on a vector field, the displacement vector field of the reference configuration, instead of on only a single component of this field.

The mathematical modelling of such an *obstacle problem for a linearly elastic shell* is then clear; since, *apart from* the confinement condition, the rest, i.e., the *function space* and the expression of the quadratic energy J^ε , is classical (see, e.g. [3]). More specifically, let

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) = A^{jikl,\varepsilon} = A^{klij,\varepsilon}$$

denote the contravariant components of the *elasticity tensor* of the elastic material constituting the shell. Then the unknown of the problem, which is the vector field $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ where the functions $u_i^\varepsilon : \overline{\Omega^\varepsilon} \rightarrow \mathbb{R}$ are the three covariant components of the unknown “three-dimensional” displacement vector field $u_i^\varepsilon \mathbf{g}^{i,\varepsilon}$ of the reference configuration of the shell, should minimise the energy $J^\varepsilon : \mathbf{H}^1(\Omega^\varepsilon) \rightarrow \mathbb{R}$ defined by

$$J^\varepsilon(\mathbf{v}^\varepsilon) := \frac{1}{2} \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k||l}^\varepsilon(\mathbf{v}^\varepsilon) e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon - \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon$$

for each $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon)$ over the *set of admissible displacements* defined by:

$$\begin{aligned} \mathbf{U}(\Omega^\varepsilon) := & \{ \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon); \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon, \\ & (\Theta(x^\varepsilon) + v_i^\varepsilon(x^\varepsilon) \mathbf{g}^{i,\varepsilon}(x^\varepsilon)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } x^\varepsilon \in \Omega^\varepsilon \}. \end{aligned}$$

The solution to this *minimisation problem* exists and is unique, and it can be also characterised as the unique solution of a set of appropriate variational inequalities (cf., Theorem 2.1 of [5]).

Theorem 2.1 *The quadratic minimisation problem: Find a vector field $\mathbf{u}^\varepsilon \in \mathbf{U}(\Omega^\varepsilon)$ such that*

$$J^\varepsilon(\mathbf{u}^\varepsilon) = \inf_{\mathbf{v}^\varepsilon \in \mathbf{U}(\Omega^\varepsilon)} J^\varepsilon(\mathbf{v}^\varepsilon)$$

has one and only one solution. Besides, the vector field \mathbf{u}^ε is also the unique solution of the variational problem $\mathcal{P}(\Omega^\varepsilon)$: Find $\mathbf{u}^\varepsilon \in \mathbf{U}(\Omega^\varepsilon)$ that satisfies the following variational inequalities:

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k||l}^\varepsilon(\mathbf{u}^\varepsilon) \left(e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon) - e_{i||j}^\varepsilon(\mathbf{u}^\varepsilon) \right) \sqrt{g^\varepsilon} dx^\varepsilon \geq \int_{\Omega^\varepsilon} f^{i,\varepsilon} (v_i^\varepsilon - u_i^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon$$

for all $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{U}(\Omega^\varepsilon)$. □

The limit problem we are interested in is derived as a result of the rigorous asymptotic analysis conducted in Theorem 4.1 of [5].

Theorem 2.2 *Let ω be a domain in \mathbb{R}^2 , let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$ be an immersion such that the surface $\boldsymbol{\theta}(\bar{\omega})$ is elliptic, and let $\mathbf{q} \in \mathbb{E}^3$ be a non-zero vector. Define the space and sets*

$$\begin{aligned} \mathbf{V}_M(\omega) &:= H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \\ \mathbf{U}_M(\omega) &:= \{ \boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega); \\ &\quad (\boldsymbol{\theta}(y) + \eta_i(y) \mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } y \in \omega \}, \\ \tilde{\mathbf{U}}_M(\omega) &:= \{ \boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^1(\omega); \\ &\quad (\boldsymbol{\theta}(y) + \eta_i(y) \mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } y \in \omega \}, \end{aligned}$$

and assume that the immersion $\boldsymbol{\theta}$ is such that

$$\tilde{d} := \min_{y \in \bar{\omega}} (\boldsymbol{\theta}(y) \cdot \mathbf{q}) > 0,$$

is independent of ε , and assume that the following “density property” holds:

$$\tilde{\mathbf{U}}_M(\omega) \text{ is dense in } \mathbf{U}_M(\omega) \text{ with respect to the norm of } \|\cdot\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}.$$

Let there be given a family of membrane shells with the same middle surface $\boldsymbol{\theta}(\bar{\omega})$ and thickness $2\varepsilon > 0$, and let

$$\begin{aligned} \mathbf{u}(\varepsilon) = (u_i(\varepsilon)) \in \mathbf{U}(\varepsilon; \Omega) &:= \{ \mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \gamma \times]-1, 1[, \\ &\quad (\boldsymbol{\theta}(y) + \varepsilon x_3 \mathbf{a}_3(y) + v_i(x) \mathbf{g}^i(\varepsilon)(x)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } x = (y, x_3) \in \Omega \} \end{aligned}$$

denote for each $\varepsilon > 0$ the unique solution of the corresponding scaled problem $\mathcal{P}(\varepsilon; \Omega)$ introduced in Theorem 3 of [2].

Then there exist functions $u_\alpha \in H^1(\Omega)$ independent of the variable x_3 and satisfying

$$u_\alpha = 0 \text{ on } \gamma \times]-1, 1[,$$

and there exists a function $u_3 \in L^2(\Omega)$ independent of the variable x_3 , such that

$$u_\alpha(\varepsilon) \rightarrow u_\alpha \text{ in } H^1(\Omega) \text{ and } u_3(\varepsilon) \rightarrow u_3 \text{ in } L^2(\Omega).$$

Define the average

$$\bar{\mathbf{u}} = (\bar{u}_i) := \frac{1}{2} \int_{-1}^1 \mathbf{u} dx_3 \in \mathbf{V}_M(\omega).$$

Then

$$\bar{\mathbf{u}} = \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta}$ is the unique solution of the two-dimensional variational problem $\mathcal{P}_M(\omega)$: Find $\boldsymbol{\zeta} \in \mathbf{U}_M(\omega)$ that satisfies the following variational inequalities

$$\int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta} - \boldsymbol{\zeta}) \sqrt{a} dy \geq \int_{\omega} p^i (\eta_i - \zeta_i) \sqrt{a} dy \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{U}_M(\omega),$$

where

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \quad \text{and} \quad p^i := \int_{-1}^1 f^i dx_3.$$

□

Critical to establish the convergence of the family $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ is the “density property” assumed there, which asserts that *the set $\tilde{\mathbf{U}}_M(\omega)$ is dense in the set $\mathbf{U}_M(\omega)$ with respect to the norm $\|\cdot\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}$* . The same “density property” is used to provide a justification, via a rigorous asymptotic analysis, of Koiter’s model for membrane shells subject to an obstacle (cf. [6], [7]). We hereby recall a sufficient *geometric* condition ensuring the assumed “density property” (cf. Theorem 5.1 of [5]).

Theorem 2.3 *Let $\boldsymbol{\theta} \in C^2(\bar{\omega}; \mathbb{E}^3)$ be an immersion with the following property: There exists a non-zero vector $\mathbf{q} \in \mathbb{E}^3$ such that*

$$\min_{y \in \bar{\omega}} (\boldsymbol{\theta}(y) \cdot \mathbf{q}) > 0 \quad \text{and} \quad \min_{y \in \bar{\omega}} (\mathbf{a}_3(y) \cdot \mathbf{q}) > 0.$$

Define the sets

$$\begin{aligned} \mathbf{U}_M(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega); \\ &\quad (\boldsymbol{\theta}(y) + \eta_i(y) \mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } y \in \omega\}, \\ \mathbf{U}_M(\omega) \cap \mathcal{D}(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in \mathcal{D}(\omega) \times \mathcal{D}(\omega) \times \mathcal{D}(\omega); \\ &\quad (\boldsymbol{\theta}(y) + \eta_i(y) \mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0 \text{ for a.a. } y \in \omega\}. \end{aligned}$$

Then the set $\mathbf{U}_M(\omega) \cap \mathcal{D}(\omega)$ is dense in the set $\mathbf{U}_M(\omega)$ with respect to the norm $\|\cdot\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}$. □

Examples of elliptic membrane shells satisfying the “density property” thus include those whose middle surface is a portion of an ellipsoid that is strictly contained in one of the open half-spaces that contain two of its main axes (see, e.g., that drawn in Figure 4.1-1 in [3]), the boundary of the half-space coinciding with the obstacle in this case.

As a final step, we de-scale Problem $\mathcal{P}_M(\omega)$ and we get the following variational formulation (cf. Theorem 4.2 of [5]).

Problem $\mathcal{P}_M^\varepsilon(\omega)$ Find $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon) \in \mathbf{U}_M(\omega)$ satisfying the following variational inequalities:

$$\varepsilon \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon) \gamma_{\alpha\beta}(\boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon) \sqrt{a} \, dy \geq \int_{\omega} p^{i,\varepsilon} (\eta_i - \zeta_i^\varepsilon) \sqrt{a} \, dy,$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{U}_M(\omega)$, where $p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^\varepsilon$. ■

3. Augmentation of interior regularity of the solution of Problem $\mathcal{P}_M^\varepsilon(\omega)$

Before proving the main result of this section, we briefly outline the main steps we are going to implement for showing that the solution $\boldsymbol{\zeta}^\varepsilon$ of Problem $\mathcal{P}_M^\varepsilon(\omega)$ is also of class $H_{\text{loc}}^2(\omega) \times H_{\text{loc}}^2(\omega) \times H_{\text{loc}}^1(\omega)$. The main difficulty in doing so amounts to constructing an admissible test vector field from which it is possible to infer the desired augmented regularity property.

Our strategy consists in *locally perturbing* the solution $\boldsymbol{\zeta}^\varepsilon$ by a vector field with compact support in a neighbourhood of ω , compact support which is sufficiently far away from the boundary γ .

In order to implement this *local perturbation* strategy, we have to first fix a neighbourhood $\mathcal{U}_1 \subset \omega$ sufficiently far away from the boundary γ , and consider a smooth function φ whose support is compact in \mathcal{U}_1 .

Second, we multiply the smooth function φ by a suitable coefficient $\varrho > 0$, $\varrho = O(h^2)$ and a suitable second order finite difference quotient. The purpose of the positive coefficient ϱ is to ensure that the geometrical constraint appearing in the set $\mathbf{U}_M(\omega)$ is satisfied in the selected neighbourhood \mathcal{U}_1 . The

argument of the second order difference quotient is the product between the smooth function φ and a *smooth* approximation of the solution. The approximation by smooth functions requirement cannot be dropped since the transverse component of the solution, namely ζ_3^ε , is *a priori* only of class $L^2(\omega)$. This regularity is in general too low for working with a second order difference quotient.

Thanks to the *assumed* “density property”, we are in a position to consider an approximation by smooth functions of the solution ζ^ε , approximations which all belong to the non-empty, closed and convex subset $U_M(\omega)$.

To begin with, we establish two abstract preparatory lemmas. The complete proof of the first lemma can be found in [2], viz. Lemma 2.

Lemma 3.1 *Let ω , ω_0 and ω_1 be subsets of \mathbb{R}^2 as in Section 5 of [2]. For each $y \in \bar{\omega}$, let $\mathbf{a}^i(y)$ denote a vector of the contravariant basis at the point $\boldsymbol{\theta}(y)$ of the surface $\boldsymbol{\theta}(\bar{\omega})$.*

Assume that $\boldsymbol{\vartheta} \in C^1(\bar{\omega}; \mathbb{E}^3)$ is such that $\boldsymbol{\vartheta} \cdot \mathbf{q}$ concave on ω_0 . Let $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ be such that $(\boldsymbol{\vartheta}(y) + \eta_i(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for a.a. $y \in \omega_0$.

Then, for each $0 < h < d$ and all $0 < \varrho < h^2/2$, the vector field $\boldsymbol{\eta}_\varrho = (\eta_{\varrho,i}) \in \mathbf{L}^2(\omega_1)$ defined in a way such that

$$\eta_{\varrho,j}\mathbf{a}^j := \eta_i\mathbf{a}^i + \varrho\varphi_1\delta_{\rho h}(\eta_i\mathbf{a}^i),$$

is such that $(\boldsymbol{\vartheta}(y) + \eta_{\varrho,i}(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for a.a. $y \in \omega_1$.

Sketch of the proof. For a.a. $y \in \omega_1$ we have that

$$\begin{aligned} \eta_{\varrho,j}(y)\mathbf{a}^j(y) &= \eta_i(y)\mathbf{a}^i(y) + \varrho\varphi_1(y)\delta_{\rho h}(\eta_i\mathbf{a}^i)(y) \\ &= \frac{\varrho}{h^2}\varphi_1(y)(\eta_i\mathbf{a}^i)(y + h\mathbf{e}_\rho) + \left(1 - \frac{2\varrho}{h^2}\varphi_1(y)\right)(\eta_i\mathbf{a}^i)(y) + \frac{\varrho}{h^2}\varphi_1(y)(\eta_i\mathbf{a}^i)(y - h\mathbf{e}_\rho). \end{aligned}$$

By virtue of the properties of φ_1 and ϱ , the functions $c_1(y) = c_{-1}(y) := \varrho h^{-2}\varphi_1(y)$ and $c_0(y) := 1 - 2\varrho h^{-2}\varphi_1(y)$ are non-negative in $\bar{\omega}$. Combining these properties with the assumption that $(\boldsymbol{\vartheta} + \eta_i\mathbf{a}^i) \cdot \mathbf{q} \geq 0$ almost everywhere in ω_0 gives:

$$\eta_{\varrho,j}(y)\mathbf{a}^j(y) \cdot \mathbf{q} \geq - \left(\boldsymbol{\vartheta}(y) + \varrho\varphi_1(y) \frac{\boldsymbol{\vartheta}(y + h\mathbf{e}_\rho) - 2\boldsymbol{\vartheta}(y) + \boldsymbol{\vartheta}(y - h\mathbf{e}_\rho)}{h^2} \right) \cdot \mathbf{q},$$

for a.a. $y \in \omega_1$.

The assumed concavity of $\boldsymbol{\vartheta} \cdot \mathbf{q}$ in ω_0 gives

$$\delta_{\rho h}\boldsymbol{\vartheta}(y) \cdot \mathbf{q} = \left(\frac{\boldsymbol{\vartheta}(y + h\mathbf{e}_\rho) - 2\boldsymbol{\vartheta}(y) + \boldsymbol{\vartheta}(y - h\mathbf{e}_\rho)}{h^2} \right) \cdot \mathbf{q} < 0,$$

for all $0 < h < d$ and all $y \in \omega_1$. Recalling that $\text{supp } \varphi_1 \subset\subset \omega_1$, we derive $(\boldsymbol{\vartheta}(y) + \eta_{\varrho,i}(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for a.a. $y \in \omega_1$, as it was to be proved. \square

The next result is a direct application of Lemma 3.1 and of the “density property” (see Corollary 1 in [2]).

Corollary 3.2 *Let ω , ω_0 and ω_1 be subsets of \mathbb{R}^2 as in Section 5 of [2]. For each $y \in \bar{\omega}$, let $\mathbf{a}^i(y)$ denote a vector of the contravariant basis at the point $\boldsymbol{\theta}(y)$ of the surface $\boldsymbol{\theta}(\bar{\omega})$.*

Assume that $\boldsymbol{\vartheta} \in C^1(\bar{\omega}; \mathbb{E}^3)$ is such that $\boldsymbol{\vartheta} \cdot \mathbf{q}$ concave on ω_0 . Let $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ be such that $(\boldsymbol{\vartheta}(y) + \eta_i(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for a.a. $y \in \omega_0$. Let $0 < h < d$ be given, and let $0 < \varrho < h^2/2$. In correspondence of one such number h , define the number $\epsilon = \epsilon(h) > 0$ by

$$0 < \epsilon \leq h^2 \left(\frac{\inf_{\rho \in \{1,2\}} \min_{y \in \bar{\omega}_1} (\delta_{\rho h}(-\boldsymbol{\vartheta} \cdot \mathbf{q}))}{2 \max\{\|\mathbf{a}^i \cdot \mathbf{q}\|_{C^0(\bar{\omega})}; 1 \leq i \leq 3\}} \right). \quad (1)$$

Assume that in correspondence of such a number ϵ there exists an element $\boldsymbol{\xi} = (\xi_i) \in \mathcal{C}^1(\bar{\omega})$ such that $(\boldsymbol{\vartheta}(y) + \xi_i(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for all $y \in \bar{\omega}$ and such that

$$|\xi_i(y) - \eta_i(y)| \leq \frac{\epsilon}{3} \quad \text{for a.a. } y \in \omega_1 \text{ and all } 1 \leq i \leq 3. \quad (2)$$

Then, the vector field $\hat{\boldsymbol{\eta}}_\rho$ defined in a way such that

$$\hat{\boldsymbol{\eta}}_{\rho,j} \mathbf{a}^j := \eta_i \mathbf{a}^i + \rho \varphi_1 \delta_{\rho h}(\xi_i \mathbf{a}^i),$$

is such that $\hat{\boldsymbol{\eta}}_\rho = (\hat{\boldsymbol{\eta}}_{\rho,i}) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ and $(\boldsymbol{\vartheta}(y) + \hat{\boldsymbol{\eta}}_{\rho,i}(y)\mathbf{a}^i(y)) \cdot \mathbf{q} \geq 0$ for a.a. $y \in \omega_1$. \square

For treating the case where the concavity assumption fails, we need the following auxiliary result (cf. Lemma 3 of [2]).

Lemma 3.3 *Let the function $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3) \in \mathcal{C}^1(\bar{\omega}; \mathbb{E}^3)$ be such that $\boldsymbol{\vartheta} \cdot \mathbf{q} > 0$ in $\bar{\omega}$. Then, for every $y_0 = (y_{0,1}, y_{0,2}) \in \omega$, there exists a neighbourhood \mathcal{U} of y_0 and numbers $B \in \mathbb{R}$, $B_0 > 0$, $r > 0$ such that the function $(-\boldsymbol{\vartheta} \cdot \mathbf{q} + B)\tilde{g}$ is convex in \mathcal{U} , where*

$$\tilde{g}(y) := 1 - \frac{1}{2} \prod_{\rho=1}^2 \exp(ry_\rho - ry_{0,\rho}) \quad \text{for all } y = (y_1, y_2) \in \bar{\omega}.$$

Moreover, it results $\tilde{g}(y) \geq B_0$, for all $y \in \mathcal{U}$.

Sketch of the proof. Fix $y_0 \in \omega$. Owing to the fact that $\boldsymbol{\vartheta} \in \mathcal{C}^1(\bar{\omega}; \mathbb{E}^3)$ and $\boldsymbol{\vartheta} \cdot \mathbf{q} > 0$ in $\bar{\omega}$, we can find numbers $B \in \mathbb{R}$ and $T > 0$, and a neighbourhood \mathcal{U}_0 of y_0 such that

$$-\boldsymbol{\vartheta}(y) \cdot \mathbf{q} + B \leq -T < 0 \quad \text{for all } y \in \mathcal{U}_0.$$

For all $y = (y_1, y_2) \in \bar{\omega}$, define the function:

$$\Pi(y) := \prod_{\rho=1}^2 \exp(ry_\rho - ry_{0,\rho}).$$

We recall that a function is convex if and only if it is convex along any lines that intersect the function domain (cf., e.g., page 67 of [8]). In other words, checking the convexity of the function $(-\boldsymbol{\vartheta} \cdot \mathbf{q} + B)\tilde{g}$ in \mathcal{U} amounts to checking that, for all $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and all $y = (y_1, y_2) \in \mathcal{U}$, the function

$$H(t) := (-\boldsymbol{\vartheta}(y + t\mathbf{v}) \cdot \mathbf{q} + B)\tilde{g}(y + t\mathbf{v}), \quad t \in \mathbb{R}, (y + t\mathbf{v}) = (y_1 + tv_1, y_2 + tv_2) \in \mathcal{U},$$

is convex.

Thanks to the global uniform boundedness of the first and second derivative of $\boldsymbol{\vartheta} \cdot \mathbf{q}$ in $\bar{\omega}$, the properties of the numbers B and T , and the positiveness of the function \tilde{g} and Π , we derive that there exists a positive number M such that:

$$\frac{1}{\Pi} \frac{d^2 H}{dt^2}(t) \geq -M \left| \frac{1}{\Pi} - \frac{1}{2} \right| - rM + \frac{T}{2}(v_1 + v_2)^2 r^2.$$

We observe that, for r sufficiently large,

$$p_1(r) := \frac{T}{2}(v_1 + v_2)^2 r^2 - Mr - M > 0.$$

Let us choose a neighbourhood \mathcal{U} of y_0 such that $\mathcal{U} \subset \subset \mathcal{U}_0$ and

$$|y_\rho - y_{0,\rho}| \leq \frac{1}{2r} \ln \frac{3}{2} \quad \text{for all } y \in \mathcal{U} \text{ and all } \rho \in \{1, 2\}. \quad (3)$$

On the one hand, it is immediate to see that, by the monotonicity of the exponential and (3),

$$0 < \Pi(y) \leq \frac{3}{2} \quad \text{for all } y \in \mathcal{U}. \quad (4)$$

On the other hand, again by (3), we have that

$$\frac{1}{\Pi(y)} - \frac{1}{2} = \prod_{\rho=1}^2 \exp(-r(y_\rho - y_{0,\rho})) - \frac{1}{2} \leq 1 \quad \text{for all } y \in \mathcal{U}, \quad (5)$$

and, by virtue of (4), we have that

$$\frac{1}{\Pi(y)} - \frac{1}{2} \geq \frac{2}{3} - \frac{1}{2} > 0 \quad \text{for all } y \in \mathcal{U}. \quad (6)$$

In conclusion, putting (5) and (6) together, we have that

$$\left| \frac{1}{\Pi(y)} - \frac{1}{2} \right| \leq 1 \quad \text{for all } y \in \mathcal{U}. \quad (7)$$

By virtue of (4), we also obtain $\tilde{g}(y) \geq 1 - \Pi/2 \geq 1/4$, for all $y \in \mathcal{U}$. Letting $B_0 := 1/4$ completes the proof. \square

In the next preparatory lemma, we prove some convergence properties of the operator $D_{\rho h}$ for functions. Since its proof is technical, we directly refer to Lemma 4 of [2] for the proof.

Lemma 3.4 *Let $\{v_k\}_{k \geq 1}$ be a sequence in $\mathcal{C}^1(\bar{\omega})$ that converges to a certain element $v \in H^1(\omega)$ with respect to the norm $\|\cdot\|_{H^1(\omega)}$. Then, we have that for all $0 < h < d$ and all $\rho \in \{1, 2\}$,*

$$D_{\rho h} v \in H^1(\omega_0) \quad \text{and} \quad D_{\rho h} v_k \rightarrow D_{\rho h} v \text{ in } H^1(\omega_0) \text{ as } k \rightarrow \infty. \quad (8)$$

\square

We are now ready to prove the main result of this Note.

Theorem 3.5 *Assume that the “density property” stated in Theorem 2.2, namely,*

$$\tilde{\mathcal{U}}_M(\omega) \text{ is dense in } \mathcal{U}_M(\omega) \text{ with respect to the norm of } \|\cdot\|_{H^1(\omega) \times H^1(\omega) \times L^2(\omega)}$$

holds. Assume also that the vector $\mathbf{f}^\varepsilon = (f^{i,\varepsilon})$ defining the applied body force density is of class $\mathbf{H}^1(\Omega^\varepsilon)$.

Then, the solution $\boldsymbol{\zeta}^\varepsilon$ of Problem $\mathcal{P}_M^\varepsilon(\omega)$ is of class $\mathbf{V}_M(\omega) \cap H_{\text{loc}}^2(\omega) \times H_{\text{loc}}^2(\omega) \times H_{\text{loc}}^1(\omega)$.

Sketch of the proof. For the complete proof, we refer to Theorem 6 of [2].

The idea consists in showing that $\|D_{\rho h}(\boldsymbol{\zeta}^\varepsilon \varphi)\|_{H^2(\mathcal{U}_1) \times H^2(\mathcal{U}_1) \times H^1(\mathcal{U}_1)}$ is uniformly bounded by a constant independent of h . The function φ is assumed to be of class \mathcal{C}^∞ with support compact in \mathcal{U}_1 .

To begin with, thanks to Lemmas 3.1–3.4 and the assumed “density property” it is possible to establish that the vector field $\tilde{\boldsymbol{\zeta}}_{\rho}^{\varepsilon,k}$ defined by

$$\tilde{\boldsymbol{\zeta}}_{\rho,j}^{\varepsilon,k}(y) \mathbf{a}^j(y) = \zeta_i^\varepsilon(y) \mathbf{a}^i(y) + \varrho(\tilde{g}(y))^{-1} \varphi^2(y) \delta_{\rho h} \left[\left(\zeta_i^{\varepsilon,k}(y) \mathbf{a}^i(y) + \frac{B\mathbf{q}}{|\mathbf{q}|^2} \right) \tilde{g}(y) \right] \quad \text{for a.a. } y \in \mathcal{U}_1, \quad (9)$$

satisfies $\boldsymbol{\theta}(y) \cdot \mathbf{q} + \tilde{\boldsymbol{\zeta}}_{\rho,i}^{\varepsilon,k}(y) \mathbf{a}^i(y) \cdot \mathbf{q} \geq 0$, for a.a. $y \in \mathcal{U}_1$ and is such that $\tilde{\boldsymbol{\zeta}}_{\rho}^{\varepsilon,k} \in H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)$.

By resorting to the assumed “density property”, and the previous lemmata, we can show that there exists a constant $\hat{C} > 0$ independent of h such that

$$\varepsilon \int_{\mathcal{U}_1} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)) \gamma_{\alpha\beta}(D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)) \sqrt{a} \, dy \leq \hat{C} (1 + \|D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)\|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}).$$

Finally, Korn’s inequality on a surface (cf., e.g., [9] and [10]) gives the existence of a constant $c > 0$ such that

$$\varepsilon \int_{\mathcal{U}_1} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)) \gamma_{\alpha\beta}(D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)) \sqrt{a} \, dy \geq \frac{1}{c} \|D_{\rho h}(\varphi \boldsymbol{\zeta}^\varepsilon)\|_{H^1(\mathcal{U}_1) \times H^1(\mathcal{U}_1) \times L^2(\mathcal{U}_1)}^2,$$

completing the proof. \square

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