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# Controlling Electromagnetic Fields at Boundaries of Arbitrary Geometries

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## Abstract

Rapid developments in the emerging field of stretchable and conformable photonics necessitate analytical expressions for boundary conditions at metasurfaces of arbitrary geometries. Here, we introduce the concept of conformal boundary optics: a design theory that determines the optical response for designer input and output fields at such interfaces. Given any object, we can realise coatings to achieve exotic effects like optical illusions and anomalous diffraction behaviour. This approach is relevant to a broad range of applications from conventional refractive optics to the design of the next-generation of wearable optical components. This concept can be generalized to other fields of research where designer interfaces with nontrivial geometries are encountered.

## 1 Introduction

In a bulk medium, a wave (e.g., optical, sound, seismic) accumulates phase gradually and propagates without experiencing abrupt variations. At the boundary with another material, however, the wave can experience large -although physically admissible- discontinuities in its reflected and transmitted fields [1,2], as dictated by the boundary conditions of the system. The behavior of optical waves at interfaces plays a leading role in many industries: for example, in the hydrocarbon industry, where reverse seismic refraction is used to map petroleum reserves deep inside the

soil of the planet [3]. In optics, reflection and refraction at interfaces are central to the design of optical components (e.g., mirrors, windows, waveplates, and lenses) [7]. Recently, several unexpected interfacial optical effects of practical interests have been demonstrated. These include anomalous reflection and refraction [8-13], giant spin-Hall effect [14], flat lensing [15], controlled Cherenkov surface plasmon emission [16], holography [17-18] and surface cloaking [20]. Here, we introduce the concept of conformal boundary optics, an analytical method based on novel, first-principle derivations that allows us to engineer transmission ( $\mathbf{E}^t$ ) and reflection ( $\mathbf{E}^{ref}$ ) at will for any interface geometry and any given incident wave ( $\mathbf{E}^{inc}$ ). By resolving the boundary conditions between two materials at an interface of arbitrary geometry, this method addresses recent developments in nanophotonics with the general technique of differential geometry and coordinates transformation. Unlike transformation optics, our approach deals directly with abrupt changes in the fields, and therefore acts at the level of the boundary conditions of the electromagnetic fields. Whereas transformation optics determines bulk optical properties by exploiting the relationship between a given coordinate system and the coordinate system that conforms to the travel of light [21-32], the proposed concept determines the optical properties of a metasurface of arbitrary geometry by exploiting the relationship between a given ambient coordinate system and the coordinate system that conforms to the geometry of the boundary. While a powerful concept in itself, the mathematical derivation associated with its analytical formulation which we present in detail in the supplementary information is highly non-trivial since it cannot be generalized from existing boundary conditions for generic surface geometries. This concept provides a wide range of new design opportunities, for example, to hide objects behind an optical curtain, to create optical illusions by reflecting virtual images, or to suppress the diffraction generally occurring during light scattering at corrugated interfaces (Fig. 1a). This concept can address basic theoretical problems in electromagnetism such as conventional transmission and reflection at flat, arbitrary shaped interfaces and metasurfaces.

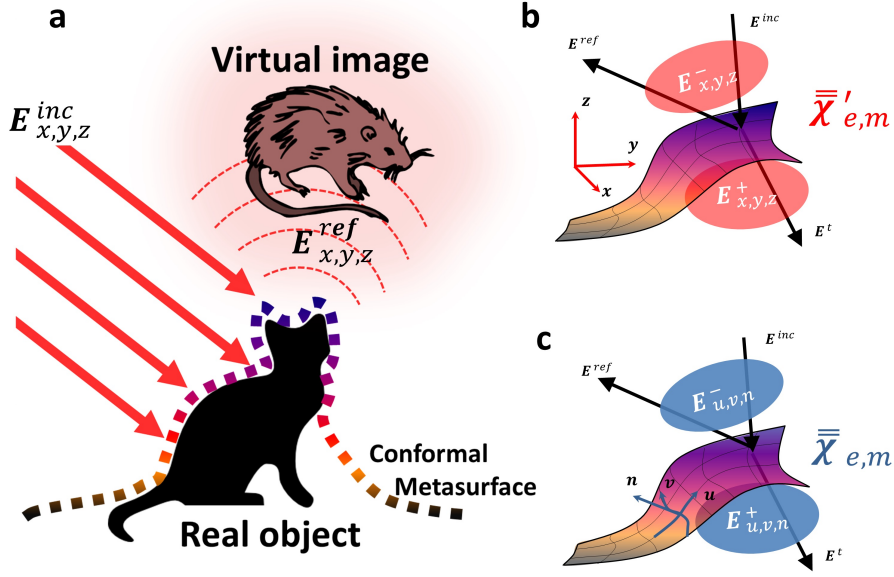


Figure 1: The surface susceptibility of an interface that conforms to an object can be designed to produce an arbitrary, user-specified response. In a, a cat is visually transmogrified into a rat as an incident light  $\mathbf{E}^{inc}_{x,y,z}$  is converted at the object interface to produce a user-specified virtual holographic image  $\mathbf{E}^{ref}_{x,y,z}$  in the far-field. This way, the actual geometry of the object no longer restricts the properties of the reflected light. Our proposed concept exploits the relationship between a given ambient coordinate system (b) and the coordinate system that conforms to the geometry of the boundary (c).  $\chi_{e,m}$  and  $\chi'_{e,m}$  denote the optical response of the interface quantities called surface susceptibilities tensors in these coordinate systems respectively.

The transmission and reflection of waves with ultrathin interfaces ( $\delta \ll \lambda$ ) has been recently demonstrated in various experiments reviewed by refs. [9,12, 13]. Abrupt modifications of the fields across an interface are engineered by depositing an array of sub-wavelength resonators specifically tailored to address local amplitude, phase and polarization changes in the light traversing the interface. Several examples have been reported on the control of the radiation patterns of thermal, acoustic, seismic and electromagnetic waves, indicating that these are probably the most suitable tools for engineering the discontinuities at will. This technique which has been dubbed metasurface physics and has given rise to a broad range of applications across physics, chemistry, biology and materials science is considered to be one of the most promising and disruptive emerging technologies of recent times. Metasurfaces introduce abrupt space-dependent optical responses which are generally not achievable using homogeneous surface treatment or traditional surface coating technology. The concept of conformal boundary optics pushes the limits of metasurface physics beyond the design of simple planar interfaces. This has interesting applications, accounting for phenomena as diverse as the engineering of modal field distributions of optical resonators with various shapes, and the creation of optical illusions, which are required in cloaking, virtual imaging, and kinoform holography. It is well-understood that the far-field image of an object is

the result of light reflected or transmitted from its surface. By choosing the appropriate boundary conditions, therefore, one can modify the reflection and transmission of light to produce various kinds of unexpected optical effects. For user-specified incident, transmitted and reflected fields, and a given surface geometry represented by coordinate system  $(u, v, n)$ , one can determine the required surface susceptibility tensors  $\chi_{e,d,m,b}$  using our concept. This technique accommodates any choice of ambient coordinate system, which happens to be Cartesian in Fig. 1b and 1c where the electromagnetic fields before and after the interface are denoted  $E_{(x,y,z)}^{\pm}$ . Notably, the technique yields surface susceptibilities in the coordinate system of the interface, which is necessary when the optical components are manufactured. To date, a rigorous expression of the electromagnetic boundary conditions at designer interfaces has been proposed only for planar interfaces where the electromagnetic fields are defined using a Cartesian coordinate system [35-45]. These equations are known as the generalized sheet transition conditions (GSTCs). From a physical point of view, discontinuities in electromagnetic fields across any regular surface depend upon the constitutive parameters of the interface: namely, surface charge density  $\rho$ , the current density  $\mathbf{j}$ , the induced dipole moments at the interface and the optical response of the surrounding media. This requires  $(\rho, \mathbf{j})$  and the fields  $\mathbf{E}, \mathbf{H}, \mathbf{p}$  and  $\mathbf{m}$  in the Maxwells equations to be expressed in the sense of distributions, where  $\mathbf{E}, \mathbf{H}$  are respectively the electric and magnetic fields, and  $\mathbf{p}$  and  $\mathbf{m}$  respectively represent the surface electric and magnetic induced currents derived by averaging the local fields of the electric and magnetic induced dipole moments in the plane  $z = 0$ . In a planar configuration, writing each variable as  $\zeta(z) = \{\zeta(z)\} + \sum_{(k=0)}^N \zeta_k \delta^{(k)}(z)$ , with the function  $\zeta(z)$  discontinuous at  $z = 0$ , it is possible to derive a set of generalized sheet transition conditions (GSTCs) for the electromagnetic fields [35-37]. Boundary conditions have also been derived for spherical and cylindrical geometries (see for example [46,47]) after adopting spherical or cylindrical coordinate systems. However, these derivation is possible only because the ambient coordinate system are chosen to conform to the interface (e.g., a spherical/cylindrical coordinate system for a spherical/cylindrical interface), a method that becomes very restrictive when the interface contains arbitrary contours. Obtaining the electromagnetic boundary conditions at a nonplanar interface (Fig. 2) is highly non-trivial because the GSTCs apply only to interfaces whose local coordinate systems are Cartesian.

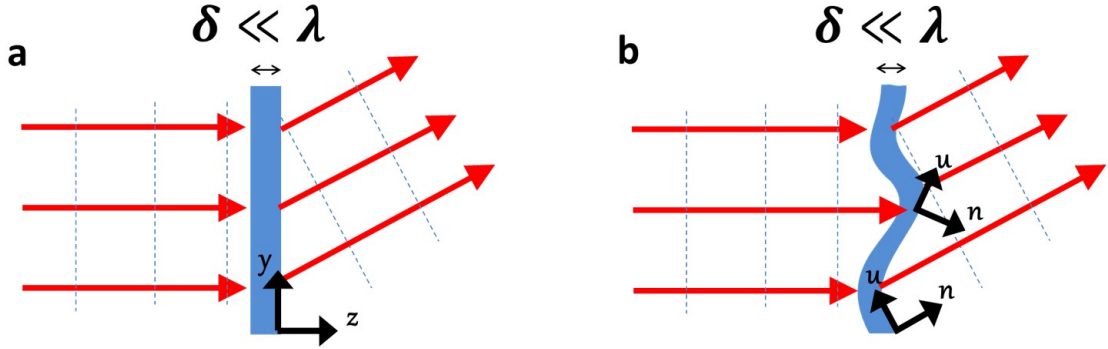


Figure 2: a), a 2D planar metasurface of sub-wavelength thickness  $\delta \ll \lambda$  can transmit any incident optical field at a specific angle by imposing a gradient of phase discontinuity. For planar interfaces, GSTC boundary conditions readily apply and the surface susceptibility tensors can be calculated. b) The local coordinate system of the surface follows its local curvature, and therefore it changes with the position along the interface. Boundary conditions of the fields are obtained in the coordinate system of the interface, and are therefore position dependent. To produce an effect equivalent to that in b, the surface susceptibilities of the optical interface have to be engineered to account for the effect of the physical distortion. The dashed blue lines denote the equiphase fronts of the electromagnetic fields.

One cannot generalize the existing GSTC boundary expressions to a non-Cartesian coordinate system because it is unclear if the components are covariant or contravariant. In the Cartesian coordinate system of GSTCs, the tangent-cotangent isomorphism is the identity, meaning that there is no difference between the covariant and contravariant components. This is not true for coordinate systems in general. Considering a volumetric boundary of sub-wavelength thickness as elucidated in the supplementary materials – we treat  $S$  as a free-standing interface (same media on both sides) and define  $(u, v)$  to be the coordinate system that conforms to it. This approximation is valid for optical metasurfaces of subwavelength thicknesses. As such, the treatment of general, non-planar surfaces necessitates novel first-principle derivations starting from the integral forms of Maxwells equations. In the following, we will use the language of differential geometry to derive these electromagnetic boundary conditions. Details of the mathematical procedure are given in section 2. There we will give the boundary conditions in the surface coordinates, and then derive the relations between the surface susceptibilities and the desired fields' coordinate system. We show that the electromagnetic boundary conditions we obtain reduce to the generalised sheet transition conditions (GSTCs) [35-42] in the case of planar surface. Finally, in section 3, we illustrate the power and robustness of this technique with two examples: a steering interface that reflects incident light as though from a planar surface, and the anomalous diffraction of light at a curved transmission interface.

## 2 Generalized boundary conditions for surfaces of arbitrary geometries

### 2.1 Definitions and notation

Conventional theories treat the metasurface  $\mathcal{S}$  as a 2-dimensional manifold (2-manifold). Here, however, we will model  $\mathcal{S}$  to be a 3-manifold (see figure 3(a)), with sub-wavelength thickness “ $\delta \ll \lambda$ ”, which may vary across  $\mathcal{S}$  [36]. Strictly speaking, all physical objects are 3-dimensional, and 2-manifolds represent their boundaries. The boundaries of  $\mathcal{S}$  are disjointed surfaces  $\partial\mathcal{S}^+$  and  $\partial\mathcal{S}^-$ , having normals pointing outwards of  $\mathcal{S}$ . Define  $\partial\mathcal{S} = \partial\mathcal{S}^+ \cup \partial\mathcal{S}^-$ . Next we define two disjointed 3-manifolds  $\mathcal{M}^+$  and  $\mathcal{M}^-$ , and let each of them have boundary  $\partial\mathcal{M}^+$  and  $\partial\mathcal{M}^-$  respectively. The boundaries  $\partial\mathcal{M}^\pm$  have normals pointing outwards of  $\mathcal{M}^\pm$  (throughout the text, the superscript  $\pm$ , for example  $\mathcal{M}^\pm$ , means for both  $\mathcal{M}^+$  and  $\mathcal{M}^-$  respectively). We then identify  $\partial\mathcal{M}^\pm$  with  $\partial\mathcal{S}^\pm$  and glue the three manifolds together to form a single manifold  $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{S} \cup \mathcal{M}^-$ . This way, for every point  $r$  on the boundaries,  $r$  can be thought of as a point on  $\partial\mathcal{M}^\pm$  or  $\partial\mathcal{S}^\pm$ .  $\mathcal{M}$  forms the entire space we are concerned with, with  $\mathcal{M}^\pm$  being the ambient medium. The normal of  $\partial\mathcal{M}^\pm$  is pointing in the opposite direction of the normal of  $\partial\mathcal{S}^\pm$ . Note that an attempt to write boundary conditions by assuming instead of a sharp surface a layer of finite thickness separating each domain by two surfaces has previously been discussed in [29].

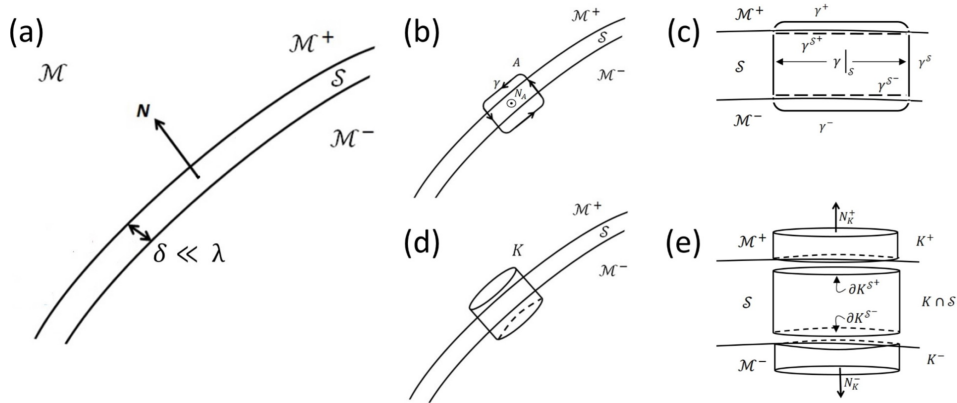


Figure 3: Schematic describing the mathematical method used to rewrite the electromagnetic field discontinuities in the language of differential geometry. (a) We define three disjoint 3-manifolds (3 dimensional manifolds)  $M^+$ ,  $M^-$  and  $\mathcal{S}$ . We identify the boundaries of  $M^\pm$  with the boundaries of  $\mathcal{S}$  to form a single manifold  $\mathcal{M} = M^+ \cup \mathcal{S} \cup M^-$ .  $\mathcal{S}$  has sub-wavelength thickness “ $\delta \ll \lambda$ ”.  $\mathbf{N}$  is the unit vector field normal to  $\mathcal{S}$ . To derive the boundary conditions from Maxwell’s integral equations, we consider an Amperian loop in (b) and a Gaussian pillbox in (d). In (c),  $\gamma^\pm$  and  $\gamma|_{\mathcal{S}}$  is defined such that  $\gamma = \gamma^\pm \cup \gamma|_{\mathcal{S}}$ , and  $\gamma^{\mathcal{S}\pm}$  is defined such that  $\gamma^{\mathcal{S}\pm} \cup \gamma|_{\mathcal{S}} = \gamma^{\mathcal{S}}$  forms a close loop inside  $\mathcal{S}$ . In (e),  $\partial K^\pm = \partial K \cap \mathcal{M}^\pm$ , each having outward pointing unit normals  $\mathbf{N}_K^\pm$ . Define  $\partial k^{\mathcal{S}\pm}$  such that together with  $\partial K \cap \mathcal{S}$  form a Gaussian pillbox inside  $\mathcal{S}$ ; let  $\mathbf{N}_K^{\mathcal{S}}$  be the outward pointing unit normal of this Gaussian pillbox.

For any field  $\mathbf{V}$  (throughout the text, field will always means the vector field, unless otherwise stated) defined in  $\mathcal{M}$ , denote  $\mathbf{V}^\pm = \mathbf{V}|_{\mathcal{M}^\pm}$ , the restriction of the field  $\mathbf{V}$  to  $\mathcal{M}^\pm$ , and  $\mathbf{V}^{\mathcal{S}} = \mathbf{V}|_{\mathcal{S}}$ , the restriction of the field  $\mathbf{V}$  to  $\mathcal{S}$ . For any  $r \in \mathcal{S}$ , let

$$\begin{aligned} \mathbf{V}|_{\mathcal{S}}^+(r) &= \lim_{r^+ \rightarrow \partial \mathcal{M}^+} \mathbf{V}^+(r^+) - \lim_{r^- \rightarrow \partial \mathcal{M}^-} \mathbf{V}^-(r^-), \\ \mathbf{V}^{\text{av}}(r) &= \frac{1}{2} \left[ \lim_{r^+ \rightarrow \partial \mathcal{M}^+} \mathbf{V}^+(r^+) + \lim_{r^- \rightarrow \partial \mathcal{M}^-} \mathbf{V}^-(r^-) \right], \end{aligned} \quad (1)$$

where  $r^+ \in \mathcal{M}^+$  and  $r^- \in \mathcal{M}^-$  are such that the line through  $r^+, r, r^-$  is always perpendicular to the boundary of  $\mathcal{S}$ . We assume that all the fields are smooth in each of the manifolds, but the fields are undefined at the boundaries.

Here we will assume that the field distribution across the region of the interface to be constant. There are two reasons for doing so. Firstly, because the optical metasurfaces is of subwavelength thickness  $\delta \ll \lambda$ , it allows us, in good approximation, to use the method of homogenisation and model the metasurface with uniform quantities, that is the effective (average) surface susceptibilities [43, 42]. Secondly, by taking the surface field distributions to be constant, the derivative of the fields in the direction normal to the surface will be zero, which allows us, in the special case of planar metasurface in free-space, to recover the usual expression of generalised sheet transition condition (GSTC) (see equations (42)-(45)). The homogenised fields in the metasurface is therefore



given by the relation.

$$\begin{aligned}
\mathbf{E}^{\mathcal{S}} &= \chi_e(\mathbf{E}^{\overline{av}}), \\
\mathbf{D}^{\mathcal{S}} &= \chi_d(\mathbf{D}^{\overline{av}}), \\
\mathbf{H}^{\mathcal{S}} &= \chi_m(\mathbf{H}^{\overline{av}}), \\
\mathbf{B}^{\mathcal{S}} &= \chi_b(\mathbf{B}^{\overline{av}}),
\end{aligned} \tag{2}$$

where  $\chi_e$ ,  $\chi_d$ ,  $\chi_m$ , and  $\chi_b$  are the susceptibilities accounting for the discontinuity in the fields. In the following, we define the surface susceptibilities in terms of the discontinuity of the fields across the surface by considering the proportionality between the homogenized electric field within the surface  $\mathbf{E}^{\mathcal{S}}$  and the averaged value of the fields on either side of the interface  $\mathbf{E}^{av}$ . In the case where the metasurface is in a medium on both sides shares the same permittivity and permeability, then  $\chi_e = \chi_d$ , and  $\chi_m = \chi_b$ . Note that in general  $\chi_a$ ,  $a = e, d, m, b$  are mixed tensor fields. In this paper, we will consider only metasurfaces with linear response, meaning that  $\chi_a$  are first tensor fields of type  $(1, 1)$ .

For any  $r \in \mathcal{S}$ , we say that a vector  $\mathbf{V}$  at  $r$ , i.e.  $\mathbf{V}(r) \in T_r\mathcal{M}$ , is tangent to  $\mathcal{S}$  if  $\mathbf{V}$  points in the direction parallel to  $\partial\mathcal{S}^\pm$  at  $r$ , and  $\mathbf{V}$  is normal to  $\mathcal{S}$  if it points in the same direction to the normal of  $\partial\mathcal{S}^+$ , which is in the opposite direction as the normal of  $\partial\mathcal{S}^-$ . Let  $\mathbf{N}$  be the unit smooth vector field defined on  $\mathcal{S}$  such that it is normal to  $\mathcal{S}$  everywhere (see figure 3(a)). We are now ready to derive the boundary conditions from Maxwell's equations.

## 2.2 Faraday's and Ampere-Maxwell's boundary conditions

In this section, we derive the boundary conditions for Faraday's and Ampere-Maxwell's law. Let  $A \subset \mathcal{M}$  be a compact smooth surface (2-submanifold) such that has non-empty intersections with all the three manifolds (see figure 3(b)). Assume that  $A$  is small enough to be approximated by a (flat) rectangle. Define  $\mathbf{N}_A$  to be a unit smooth vector field on  $A$  everywhere normal to the surface  $A$ . We position  $A$  such that  $\mathbf{N}_A$  is tangent to  $\mathcal{S}$ . Let  $\gamma$  (usually denoted as  $ds$ ) be a parameterised closed curve tracing out  $\partial A$  the boundary of  $A$ , in the positive direction induced by  $\mathbf{N}_A$  (the right-hand rule). Let  $\gamma^\pm = \gamma|_{\mathcal{M}^\pm}$  and define  $\gamma^{\mathcal{S}^\pm}$  on  $A \cap \partial\mathcal{S}^\pm$  such that  $\gamma^{\mathcal{S}^\pm}$  and  $\gamma|_{\mathcal{S}}$  form a close loop  $\gamma^{\mathcal{S}}$  inside  $\mathcal{S}$ . Note that  $\gamma^{\mathcal{S}^+}$  is in the opposite direction as  $\gamma^{\mathcal{S}^-}$ . Let  $\mathbf{l}_A$  be the unit vector field defined on  $\partial A$  such that it points in the direction of  $\gamma$ . Define  $\mathbf{l}_A^\pm$ ,  $\mathbf{l}_A^{\mathcal{S}^\pm}$  and  $\mathbf{l}_A^{\mathcal{S}}$  analogously. Finally, let  $\alpha$  (usually denoted as  $dA$ ) be the metric area element on  $A$ . Given any vector fields  $\mathbf{V}$  and  $\mathbf{W}$ , denote  $\mathbf{V} \cdot \mathbf{W}$  to be the (point-by-point) inner product of  $\mathbf{V}$  and  $\mathbf{W}$ .

Using Faraday's induction law, we have

$$\begin{aligned}
-\int_A \mathbf{N}_A \cdot \frac{\partial \mathbf{B}}{\partial t} \alpha &= \int_{\partial A} \mathbf{l}_A \cdot \mathbf{E} \gamma \\
&= \int_{\partial A \cap \mathcal{M}^+} \mathbf{l}_A^+ \cdot \mathbf{E}^+ \gamma^+ + \int_{\partial A \cap \mathcal{M}^-} \mathbf{l}_A^- \cdot \mathbf{E}^- \gamma^- + \int_{\partial A \cap \mathcal{S}} \mathbf{l}_A^S \cdot \mathbf{E}^S \gamma^S.
\end{aligned} \tag{3}$$

By Eq. (2) and the fact that  $\gamma^{S^+}$  and  $\gamma^{S^-}$  are in the opposite direction,

$$\int_{A \cap \partial \mathcal{S}^+} \mathbf{l}_A^{S^+} \cdot \mathbf{E}^S \gamma^{S^+} = - \int_{A \cap \partial \mathcal{S}^-} \mathbf{l}_A^{S^-} \cdot \mathbf{E}^S \gamma^{S^-}. \tag{4}$$

Therefore, we have

$$\begin{aligned}
\int_{\partial A \cap \mathcal{S}} \mathbf{l}_A^S \cdot \mathbf{E}^S \gamma^S &= \int_{\partial A \cap \mathcal{S}} \mathbf{l}_A^S \cdot \mathbf{E}^S \gamma^S + \int_{A \cap \partial \mathcal{S}^+} \mathbf{l}_A^{S^+} \cdot \mathbf{E}^S \gamma^{S^+} + \int_{A \cap \partial \mathcal{S}^-} \mathbf{l}_A^{S^-} \cdot \mathbf{E}^S \gamma^{S^-} \\
&= \int_{\partial(A \cap \mathcal{S})} \mathbf{l}_A^S \cdot \mathbf{E}^S \gamma^S \\
&= \int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \text{curl } \mathbf{E}^S \alpha,
\end{aligned} \tag{5}$$

where we used Stoke's theorem in the last equality, noting that  $\mathbf{E}^S$  is smooth and compactly supported in  $A \cap \mathcal{S}$ .

Now shrink  $A$  such that  $\partial A \cap \mathcal{M}^\pm \rightarrow \partial \mathcal{M}^\pm$ . Then in the limit, Eq. (3) becomes

$$\begin{aligned}
-\int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \frac{\partial \mathbf{B}^S}{\partial t} \alpha &= - \int_A \mathbf{N}_A \cdot \frac{\partial \mathbf{B}}{\partial t} \alpha \\
&= \int_{A \cap \partial \mathcal{M}^+} \mathbf{l}_A^+ \cdot \mathbf{E}^+ \gamma^+ + \int_{A \cap \partial \mathcal{M}^-} \mathbf{l}_A^- \cdot \mathbf{E}^- \gamma^- + \int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \text{curl } \mathbf{E}^S \alpha \\
&= \int_{A \cap \partial \mathcal{S}^+} \mathbf{l}_A \cdot \mathbf{E}|_{\perp}^+ \gamma + \int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \text{curl } \mathbf{E}^S \alpha.
\end{aligned} \tag{6}$$

The last equality holds because we identified  $\partial \mathcal{M}^\pm$  with  $\partial \mathcal{S}^\pm$ , and because  $\mathbf{E}^S$  is constant across the depth of  $\mathcal{S}$ , and  $\gamma^+$  is just the opposite direction of  $\gamma^-$ .

Again using the assumption that the fields are constant across the surface, we obtain

$$\begin{aligned}
\int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \frac{\partial \mathbf{B}^S}{\partial t} \alpha &= \delta \int_{A \cap \partial \mathcal{S}^+} \mathbf{N}_A \cdot \frac{\partial \mathbf{B}^S}{\partial t} \gamma, \\
\int_{A \cap \mathcal{S}} \mathbf{N}_A \cdot \text{curl } \mathbf{E}^S \alpha &= \delta \int_{A \cap \partial \mathcal{S}^+} \mathbf{N}_A \cdot \text{curl } \mathbf{E}^S \gamma.
\end{aligned} \tag{7}$$

Finally, allowing the vector fields in  $\mathcal{S}$  absorb the length  $\delta$ , we obtain

$$\boldsymbol{l}_A \cdot \boldsymbol{E}|_{\pm}^+ + \boldsymbol{N}_A \cdot \text{curl} \boldsymbol{E}^{\mathcal{S}} = -\boldsymbol{N}_A \cdot \frac{\partial \boldsymbol{B}^{\mathcal{S}}}{\partial t}. \quad (8)$$

Note that after absorbing  $\delta$ , the field  $\boldsymbol{E}^{\mathcal{S}}$ , etc, no longer have the dimensions of the fields, but those of field multiplied by distance. Observe that for any vector field  $\boldsymbol{X}$  tangent to  $\mathcal{S}$ , one can check that  $\boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{N}$  is also tangent to  $\mathcal{S}$ , and satisfies the right hand rule with  $\boldsymbol{X}$  [34]. Then (8) can be written as

$$-\boldsymbol{Y} \cdot \boldsymbol{E}|_{\pm}^+ = \boldsymbol{X} \cdot \frac{\partial \boldsymbol{B}^{\mathcal{S}}}{\partial t} + \boldsymbol{X} \cdot \text{curl} \boldsymbol{E}^{\mathcal{S}}. \quad (9)$$

Using the same derivation for the Ampere-Maxwell law, assuming the absence of free surface current, we obtain

$$-\boldsymbol{Y} \cdot \boldsymbol{H}|_{\pm}^+ = \boldsymbol{X} \cdot \text{curl} \boldsymbol{H}^{\mathcal{S}} - \boldsymbol{X} \cdot \frac{\partial \boldsymbol{D}^{\mathcal{S}}}{\partial t}. \quad (10)$$

Note that  $\boldsymbol{X}$  and  $\boldsymbol{Y}$ , which are vector fields that are tangential to  $\mathcal{S}$  at every point, are not constant in general due to the curvature of  $\mathcal{S}$ .

### 2.3 Gauss's electric and magnetic boundary conditions

To derive the boundary conditions for Gauss's electric and magnetic law, consider a compact 3-submanifold  $K \subset \mathcal{M}$  such that it has non-empty intersection with all three manifolds (see figure 3(c)). Assume  $K$  is small enough such that it can be approximated by a pill box. Let  $\partial K^{\pm} = \partial K \cap \mathcal{M}^{\pm}$  and define  $\partial K^{\mathcal{S}^{\pm}}$  such that  $\partial K^{\mathcal{S}^{\pm}}$  and  $\partial K \cap \mathcal{S}$  form the boundary of  $K \cap \mathcal{S}$ . Let  $\boldsymbol{N}_K$  be a outward pointing unit normal vector field defined on  $\partial K$  the boundary of  $K$ . Likewise we will let  $\boldsymbol{N}_K^{\pm} = \boldsymbol{N}_K|_{\mathcal{M}^{\pm}}$ , and let  $\boldsymbol{N}_K^{\mathcal{S}}$  to be a unit normal outward pointing vector field defined on  $\partial(K \cap \mathcal{S})$ . Observe that  $\boldsymbol{N}_K^{\mathcal{S}}|_{\mathcal{S} \setminus \partial \mathcal{S}} = \boldsymbol{N}_K|_{\mathcal{S} \setminus \partial \mathcal{S}}$ . Let  $\alpha$  be the metric area element on  $\partial K$  induce by  $\boldsymbol{N}_K$ , and  $\alpha^{\mathcal{S}}$  be the metric area element defined on  $\partial(K \cap \mathcal{S})$  induced by  $\boldsymbol{N}_K^{\mathcal{S}}$ . Define  $\alpha^{\pm}$  and  $\alpha^{\mathcal{S}^{\pm}}$  analogously. Finally, let  $dV$  be the metric volume element.

By Gauss's law,

$$\begin{aligned} \int_K \rho dV &= \int_{\partial K} \boldsymbol{N}_K \cdot \boldsymbol{D} \alpha \\ &= \int_{\partial K^+} \boldsymbol{N}_K^+ \cdot \boldsymbol{D}^+ \alpha^+ + \int_{\partial K^-} \boldsymbol{N}_K^- \cdot \boldsymbol{D}^- \alpha^- + \int_{\partial K \cap \mathcal{S}} \boldsymbol{N}_K^{\mathcal{S}} \cdot \boldsymbol{D}^{\mathcal{S}} \alpha^{\mathcal{S}}. \end{aligned} \quad (11)$$

Using the identities given in (2), and because we are approximating  $K$  by a pill box,  $\boldsymbol{N}_K^{\mathcal{S}^+} = -\boldsymbol{N}_K^{\mathcal{S}^-}$ ,

we get

$$\begin{aligned} \int_{\partial K \cap \mathcal{S}} \mathbf{N}_K^{\mathcal{S}} \cdot \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}} &= \int_{\partial K \cap \mathcal{S}} \mathbf{N}_K^{\mathcal{S}} \cdot \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}} + \int_{\partial K^{\mathcal{S}^+}} \mathbf{N}_K^{\mathcal{S}^+} \cdot \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}^+} + \int_{\partial K^{\mathcal{S}^-}} \mathbf{N}_K^{\mathcal{S}^-} \cdot \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}^-} \\ &= \int_{\partial(K \cap \mathcal{S})} \mathbf{N}_K^{\mathcal{S}} \cdot \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}} = \int_{K \cap \mathcal{S}} \operatorname{div} \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}}, \end{aligned} \quad (12)$$

where in the last equality we used the fact that  $\mathbf{D}^{\mathcal{S}}$  is smooth and compactly supported in  $K \cap \mathcal{S}$ .

Now we will shrink  $K$  such that  $\partial K^{\pm} \rightarrow \partial \mathcal{M}^{\pm}$ , then in the limit, Eq. (11) becomes

$$\begin{aligned} \int_{K \cap \mathcal{S}} \rho \, dV &= \int_k \rho \, dV = \int_{\partial K^+} \mathbf{N}_K^+ \cdot \mathbf{D}^+ \alpha^+ + \int_{\partial K^-} \mathbf{N}_K^- \cdot \mathbf{D}^- \alpha^- + \int_{K \cap \mathcal{S}} \operatorname{div} \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}} \\ &= \int_{K \cap \mathcal{M}^+} \mathbf{N}_K^+ \cdot \mathbf{D}^+ \alpha^+ + \int_{K \cap \mathcal{M}^-} \mathbf{N}_K^- \cdot \mathbf{D}^- \alpha^- + \int_{K \cap \mathcal{S}} \operatorname{div} \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}} \\ &= \int_{K \cap \partial \mathcal{S}^+} \mathbf{N}_K^+ \cdot \mathbf{D}|_{\mathcal{S}^+}^+ \alpha^+ + \int_{K \cap \mathcal{S}} \operatorname{div} \mathbf{D}^{\mathcal{S}} \alpha^{\mathcal{S}}, \end{aligned} \quad (13)$$

where the last equality follows from the argument used in Eq. (6).

Again using the identities in (2), we obtain

$$\mathbf{N}_K^+ \cdot \mathbf{D}|_{\mathcal{S}^+}^+ + \operatorname{div} \mathbf{D}^{\mathcal{S}} = \rho_{\mathcal{S}}. \quad (14)$$

Observe that  $\mathbf{N}_K^+ = \mathbf{N}|_K$ . Since  $K$  can be positioned anywhere along  $\mathcal{S}$ , we obtain

$$\mathbf{N} \cdot \mathbf{D}|_{\mathcal{S}^+}^+ + \operatorname{div} \mathbf{D}^{\mathcal{S}} = \rho_{\mathcal{S}}. \quad (15)$$

Using a similar argument for Gauss's law for magnetism, we have

$$\mathbf{N} \cdot \mathbf{B}|_{\mathcal{S}^+}^+ + \operatorname{div} \mathbf{B}^{\mathcal{S}} = 0. \quad (16)$$

Once again, because of the curvature of  $\mathcal{S}$ ,  $\mathbf{N}$  is not constant in general. In the next section, we will derive the expression of Eq. (9), (10), (15), and (16) in the coordinates of  $\mathcal{S}$ .

## 2.4 Electromagnetic boundary conditions in local coordinates

Section 2.2 and 2.3 was necessary to express an abstract version of the boundary conditions. To be useful, these equations have to be written in some local coordinates. Therefore we now think of  $\mathcal{M}$  as being a subset of a 3-dimensional Euclidean space, meaning  $\mathcal{M} \subset \mathbb{R}^3$ , equipped with the standard Cartesian coordinates  $(x, y, z)$ . Throughout this paper, we use the Cartesian coordinate

as the global ambient coordinate. It is possible to derive the expressions in other well known coordinate system like spherical or cylindrical. Because of (2), we may treat  $\mathcal{S}$  as a 2-submanifold of  $\mathcal{M}$ . Let  $n : \mathcal{M} \rightarrow \mathbb{R}$  be a smooth function such that the surface  $\mathcal{S}$  is a level set of  $n$ , meaning that there is a  $r_0 \in \mathbb{R}$  such that  $\mathcal{S} = n^{-1}(r_0)$ . We assume that  $n$  is injective with respect to  $z$ . Then by the implicit function theorem [33], there exists a smooth function  $f : \mathbb{R}^2 \cap \mathcal{M} \rightarrow \mathbb{R}$ ,  $f(x, y) = z$ , such that  $\mathcal{S}$  is the graph of  $f$  (for a closed surface, one may choose the ambient coordinates to be the spherical coordinate,  $(r, \phi, \theta)$ , and let  $n$  be injective with respect to the coordinate  $r$ . Then define  $f(\phi, \theta) = r$  to be the graph of  $\mathcal{S}$ ). We will now derive the expression of  $\mathbf{N}$ . Since  $\mathcal{S}$  is a level set of  $n$ , then the gradient of  $n$ ,  $\nabla n = \frac{\partial n}{\partial x} \mathbf{e}_x + \frac{\partial n}{\partial y} \mathbf{e}_y + \frac{\partial n}{\partial z} \mathbf{e}_z$ , is a vector field normal to the surface  $\mathcal{S}$  [33]. So by definition of  $\mathbf{N}$  being the unit vector field normal to  $\mathcal{S}$ ,  $\mathbf{N} = \frac{\nabla n}{|\nabla n|}$ , where  $|\cdot|$  is the Euclidean norm. Letting  $(u, v)$  be the coordinates system on  $\mathcal{S}$ , then the coordinates chart of  $\mathcal{S}$  is  $x(u, v) = u$ ,  $y(u, v) = v$ ,  $z(u, v) = f(u, v)$ . Therefore  $\mathbf{e}_u$ ,  $\mathbf{e}_v$ , and  $\mathbf{e}_n = \mathbf{N}$  forms a frame for the space of vector fields on  $\mathcal{S}$ , where

$$\begin{aligned} \mathbf{e}_u &= \frac{\partial x}{\partial u} \mathbf{e}_x + \frac{\partial y}{\partial u} \mathbf{e}_y + \frac{\partial z}{\partial u} \mathbf{e}_z = \mathbf{e}_x + \frac{\partial f}{\partial u} \mathbf{e}_z, \\ \mathbf{e}_v &= \frac{\partial x}{\partial v} \mathbf{e}_x + \frac{\partial y}{\partial v} \mathbf{e}_y + \frac{\partial z}{\partial v} \mathbf{e}_z = \mathbf{e}_y + \frac{\partial f}{\partial v} \mathbf{e}_z, \\ \mathbf{e}_n &= \frac{1}{|\nabla n|} \left( \frac{\partial n}{\partial x} \mathbf{e}_x + \frac{\partial n}{\partial y} \mathbf{e}_y + \frac{\partial n}{\partial z} \mathbf{e}_z \right). \end{aligned} \tag{17}$$

The Riemannian metric tensor and inverse Riemannian metric tensor on  $\mathcal{S}$  induced by the Euclidean norm is given by

$$\begin{aligned} (g_{ij}^{\mathcal{S}}) &= \begin{pmatrix} \mathbf{e}_u \cdot \mathbf{e}_u & \mathbf{e}_u \cdot \mathbf{e}_v & \mathbf{e}_u \cdot \mathbf{e}_n \\ \mathbf{e}_v \cdot \mathbf{e}_u & \mathbf{e}_v \cdot \mathbf{e}_v & \mathbf{e}_v \cdot \mathbf{e}_n \\ \mathbf{e}_n \cdot \mathbf{e}_u & \mathbf{e}_n \cdot \mathbf{e}_v & \mathbf{e}_n \cdot \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 1 + \left( \frac{\partial f}{\partial u} \right)^2 & \frac{\partial f}{\partial u} \frac{\partial f}{\partial v} & 0 \\ \frac{\partial f}{\partial v} \frac{\partial f}{\partial u} & 1 + \left( \frac{\partial f}{\partial v} \right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ (g^{\mathcal{S}})^{ij} &= (g_{ij}^{\mathcal{S}})^{-1} = \frac{1}{g^{\mathcal{S}}} \begin{pmatrix} 1 + \left( \frac{\partial f}{\partial v} \right)^2 & -\frac{\partial f}{\partial u} \frac{\partial f}{\partial v} & 0 \\ -\frac{\partial f}{\partial v} \frac{\partial f}{\partial u} & 1 + \left( \frac{\partial f}{\partial u} \right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{18}$$

respectively, where  $i, j = u, v, n$  and  $g^{\mathcal{S}} := \det(g_{ij}^{\mathcal{S}}) = 1 + \left( \frac{\partial f}{\partial u} \right)^2 + \left( \frac{\partial f}{\partial v} \right)^2$ .

Defining  $\mathbf{X}^u$  to be  $g^{\mathcal{S}}{}^{ui} \mathbf{e}_i$ ,  $i = u, v, n$ , one can check that  $\mathbf{Y}^u = \mathbf{X}^u \times \mathbf{N} = \frac{-1}{\sqrt{g^{\mathcal{S}}}} \mathbf{e}_v$ . Similarly, letting  $\mathbf{X}^v = g^{\mathcal{S}}{}^{vi} \mathbf{e}_i$ ,  $i = u, v, n$ , then  $\mathbf{Y}^v = \frac{1}{\sqrt{g^{\mathcal{S}}}} \mathbf{e}_u$ . Given any vector field  $\mathbf{V}$ , we have  $\mathbf{X}^u \cdot \mathbf{V} = V^u$ ,  $\mathbf{Y}^u \cdot \mathbf{V} = \frac{-1}{\sqrt{g^{\mathcal{S}}}} V_v$ ,  $\mathbf{X}^v \cdot \mathbf{V} = V^v$ , and  $\mathbf{Y}^v \cdot \mathbf{V} = \frac{1}{\sqrt{g^{\mathcal{S}}}} V_u$ .

We will now obtain the expression of *curl* in local coordinates. In general, it is given by [33, 34]

$$\mathit{curl} \mathbf{V} = \frac{[ijk]}{\sqrt{g^S}} \partial_j V_k \mathbf{e}_i, \quad (19)$$

where  $\partial_j$  denotes differentiating with respect to the  $j = u, v, n$  coordinate, and

$$[ijk] = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation of } (u, v, n), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (u, v, n), \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Now because we assume the fields are constant across the depth of the metasurface, differentiating the fields in  $\mathcal{S}$  in the direction normal to  $\mathcal{S}$  will be zero. So

$$\mathbf{X}^u \cdot \mathit{curl} \mathbf{V}^S = \frac{1}{\sqrt{g^S}} \frac{\partial V_n^S}{\partial v}, \quad \text{and} \quad \mathbf{X}^v \cdot \mathit{curl} \mathbf{V}^S = -\frac{1}{\sqrt{g^S}} \frac{\partial V_n^S}{\partial u}. \quad (21)$$

Applying all the discussion thus far to (9) and (10), using the fact that for a metasurface with linear response we have  $E^{S^i} = \chi_e^{ij} E_j^{\bar{a}\bar{v}}$ ,  $D^{S^i} = \chi_d^{ij} D_j^{\bar{a}\bar{v}}$ ,  $H^{S^i} = \chi_m^{ij} H_j^{\bar{a}\bar{v}}$ , and  $B^{S^i} = \chi_b^{ij} B_j^{\bar{a}\bar{v}}$  for  $i, j = u, v, n$ , we obtain

$$\begin{aligned} \frac{1}{\sqrt{g^S}} E_v|_{\pm}^{\pm} &= \frac{1}{\sqrt{g^S}} \frac{\partial}{\partial v} (\chi_e^{nk} E_k^{\bar{a}\bar{v}}) + \frac{\partial}{\partial t} (\chi_b^{uk} B_k^{\bar{a}\bar{v}}), \\ \frac{-1}{\sqrt{g^S}} E_u|_{\pm}^{\pm} &= \frac{-1}{\sqrt{g^S}} \frac{\partial}{\partial u} (\chi_e^{nk} E_k^{\bar{a}\bar{v}}) + \frac{\partial}{\partial t} (\chi_b^{vk} B_k^{\bar{a}\bar{v}}), \end{aligned} \quad (22)$$

and

$$\begin{aligned} \frac{1}{\sqrt{g^S}} H_v|_{\pm}^{\pm} &= \frac{1}{\sqrt{g^S}} \frac{\partial}{\partial v} (\chi_m^{nk} H_k^{\bar{a}\bar{v}}) - \frac{\partial}{\partial t} (\chi_d^{uk} D_k^{\bar{a}\bar{v}}), \\ \frac{-1}{\sqrt{g^S}} H_u|_{\pm}^{\pm} &= \frac{-1}{\sqrt{g^S}} \frac{\partial}{\partial u} (\chi_m^{nk} H_k^{\bar{a}\bar{v}}) - \frac{\partial}{\partial t} (\chi_d^{vk} D_k^{\bar{a}\bar{v}}), \end{aligned} \quad (23)$$

for  $k = u, v, n$ , and using Einstein's summation notation.

Now using (15) and (16), the definition of *div* [33], and the same arguments as above, assuming the absence of free electron, we have

$$D_n|_{\pm}^{\pm} + \frac{1}{\sqrt{g^S}} \partial_i (\sqrt{g^S} \chi_d^{ik} D_k^{\bar{a}\bar{v}}) = 0, \quad (24)$$

$$B_n|_{\pm}^{\pm} + \frac{1}{\sqrt{g^S}} \partial_i (\sqrt{g^S} \chi_b^{ik} B_k^{\bar{a}\bar{v}}) = 0, \quad (25)$$

for  $i = u, v, k = u, v, n$ .

Summarising all the results above, and denoting  $\partial_t = \frac{\partial}{\partial t}$ ,

$$\frac{[ij]}{\sqrt{g^{\mathcal{S}}}} E_j|_{\pm}^{\pm} \mathbf{e}_i = \frac{[ij]}{\sqrt{g^{\mathcal{S}}}} \partial_j (\chi_e^{nk} E_k^{\overline{\text{av}}}) \mathbf{e}_i + \partial_t (\chi_b^{ik} B_k^{\overline{\text{av}}}) \mathbf{e}_i, \quad (26)$$

$$\frac{[ij]}{\sqrt{g^{\mathcal{S}}}} H_j|_{\pm}^{\pm} \mathbf{e}_i = \frac{[ij]}{\sqrt{g^{\mathcal{S}}}} \partial_j (\chi_m^{nk} H_k^{\overline{\text{av}}}) \mathbf{e}_i - \partial_t (\chi_d^{ik} D_k^{\overline{\text{av}}}) \mathbf{e}_i, \quad (27)$$

$$D_n|_{\pm}^{\pm} + \frac{1}{\sqrt{g^{\mathcal{S}}}} \partial_i (\sqrt{g^{\mathcal{S}}} \chi_d^{ik} D_k^{\overline{\text{av}}}) = 0, \quad (28)$$

$$B_n|_{\pm}^{\pm} + \frac{1}{\sqrt{g^{\mathcal{S}}}} \partial_i (\sqrt{g^{\mathcal{S}}} \chi_b^{ik} B_k^{\overline{\text{av}}}) = 0, \quad (29)$$

for  $i, j = u, v, k = u, v, n$ , and

$$[ij] = \begin{cases} 1 & \text{if } i = u \text{ and } j = v, \\ -1 & \text{if } i = v \text{ and } j = u, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

## 2.5 Electromagnetic boundary conditions in global coordinates

The formulation in the previous section gives us the relations between the susceptibilities and the specified fields on both sides of the surface in the surface coordinates. However, the fields are usually given in a global coordinate system, say the Cartesian. Thus we need to find the relation between the susceptibilities expressed in surface coordinates, to the fields expressed in the global coordinates. However in manufacturing the metasurface, the susceptibilities need to be in terms of the coordinates system on  $\mathcal{S}$ . Therefore, after computing the expressions in the global frame decomposition, the susceptibilities will need to be transformed back.

Given a vector  $\mathbf{V}$  on  $\mathcal{S}$ , the change in the components of the covector field is given by

$$\begin{aligned} V_u &= \mathbf{V} \cdot \mathbf{e}_u = \mathbf{V} \cdot (e_x + \frac{\partial f}{\partial u} e_z) = V_x + \frac{\partial f}{\partial u} V_z, \\ V_v &= \mathbf{V} \cdot \mathbf{e}_v = \mathbf{V} \cdot (e_y + \frac{\partial f}{\partial v} e_z) = V_y + \frac{\partial f}{\partial v} V_z, \\ V_n &= \mathbf{V} \cdot \mathbf{e}_n = \frac{1}{|\nabla n|} (\frac{\partial n}{\partial x} V_x + \frac{\partial n}{\partial y} V_y + \frac{\partial n}{\partial z} V_z). \end{aligned} \quad (31)$$

The above can be succinctly expressed as  $V_i = \Lambda_i^{i'} V_{i'}$ , where  $i = u, v, n$ ,  $i' = x, y, z$ , and

$$(\Lambda_i^{i'}) = \begin{pmatrix} 1 & 0 & \frac{1}{|\nabla n|} \frac{\partial n}{\partial x} \\ 0 & 1 & \frac{1}{|\nabla n|} \frac{\partial n}{\partial y} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{1}{|\nabla n|} \frac{\partial n}{\partial z} \end{pmatrix}. \quad (32)$$

Making the transformation, (26), (27), (28), and (29) become

$$\frac{[ij]}{\sqrt{g^S}} \Lambda_j^{k'} E_{k'}|_{\pm} \mathbf{e}_i = \frac{[ij]}{\sqrt{g^S}} \partial_j (\chi_e^{nk} \Lambda_k^{k'} E_{k'}^{\overline{av}}) \mathbf{e}_i + \partial_t (\chi_b^{ik} \Lambda_k^{k'} B_{k'}^{\overline{av}}) \mathbf{e}_i, \quad (33)$$

$$\frac{[ij]}{\sqrt{g^S}} \Lambda_j^{k'} H_{k'}|_{\pm} \mathbf{e}_i = \frac{[ij]}{\sqrt{g^S}} \partial_j (\chi_m^{nk} \Lambda_k^{k'} H_{k'}^{\overline{av}}) \mathbf{e}_i - \partial_t (\chi_d^{ik} \Lambda_k^{k'} D_{k'}^{\overline{av}}) \mathbf{e}_i, \quad (34)$$

$$\Lambda_n^{k'} D_{k'}|_{\pm} + \frac{1}{\sqrt{g^S}} \partial_i (\sqrt{g^S} \chi_d^{ik} \Lambda_k^{k'} D_{k'}^{\overline{av}}) = 0, \quad (35)$$

$$\Lambda_n^{k'} B_{k'}|_{\pm} + \frac{1}{\sqrt{g^S}} \partial_i (\sqrt{g^S} \chi_b^{ik} \Lambda_k^{k'} B_{k'}^{\overline{av}}) = 0, \quad (36)$$

where  $i, j = u, v$ ,  $k = u, v, n$ , and  $k' = x, y, z$ . To aid in the calculation, we shall define the virtual susceptibilities to be

$$\chi_a^{ik'} = \begin{cases} \sqrt{g^S} \chi_a^{ik} \Lambda_k^{k'} & \text{for } i = u, v, k' = x, y, z, \\ \chi_a^{ik} \Lambda_k^{k'} & \text{for } i = n, k' = x, y, z, \end{cases} \quad (37)$$

for  $a = e, d, m, b$ ,  $k = u, v, n$ . Multiplying (33) and (34) throughout by  $\sqrt{g^S}$ , using (37), we get

$$[ij] \Lambda_j^{k'} E_{k'}|_{\pm} \mathbf{e}_i = [ij] \partial_j (\chi_e^{nk'} E_{k'}^{\overline{av}}) \mathbf{e}_i + \partial_t (\chi_b^{ik'} B_{k'}^{\overline{av}}) \mathbf{e}_i, \quad (38)$$

$$[ij] \Lambda_j^{k'} H_{k'}|_{\pm} \mathbf{e}_i = [ij] \partial_j (\chi_m^{nk'} H_{k'}^{\overline{av}}) \mathbf{e}_i - \partial_t (\chi_d^{ik'} D_{k'}^{\overline{av}}) \mathbf{e}_i, \quad (39)$$

$$\Lambda_n^{k'} D_{k'}|_{\pm} + \sqrt{g^S} \partial_i (\chi_d^{ik'} D_{k'}^{\overline{av}}) = 0, \quad (40)$$

$$\Lambda_n^{k'} B_{k'}|_{\pm} + \sqrt{g^S} \partial_i (\chi_b^{ik'} B_{k'}^{\overline{av}}) = 0. \quad (41)$$

The set of equations (37 and 38-41) is thus a powerful means of obtaining the analytical susceptibilities of metasurfaces that are non-planar or are required to be wearable, conformable and/or stretchable. To the best of our knowledge, the findings summarized in (38-41) represent the first time that electromagnetic boundary conditions have been obtained for interfaces of arbitrary geometries.

Observe that for a flat surface  $\mathcal{S}$ , with the permittivity and permeability equal on both sides, we have  $x = u$ ,  $y = v$ ,  $z = n = 0$ ,  $\Lambda_j^{j'} = \delta_j^{j'}$ ,  $\sqrt{g^S} = 1$ ,  $\chi_e = \chi_d$  and  $\chi_m = \chi_b$ . Then (38), (39), (40),



and (41) reduce to

$$[ij]E_j|_{\pm}^{\pm} \mathbf{e}_i = [ij]\partial_j(\chi_e^{zk} \mathbf{E}_k^{\overline{av}}) \mathbf{e}_i + \mu \partial_t(\chi_m^{ik} \mathbf{H}_k^{\overline{av}}) \mathbf{e}_i, \quad (42)$$

$$[ij]H_j|_{\pm}^{\pm} \mathbf{e}_i = [ij]\partial_j(\chi_m^{zk} \mathbf{H}_k^{\overline{av}}) \mathbf{e}_i - \epsilon \partial_t(\chi_e^{ik} \mathbf{E}_k^{\overline{av}}) \mathbf{e}_i, \quad (43)$$

$$D_z|_{\pm}^{\pm} + \partial_i(\chi_e^{ik} D_k^{\overline{av}}) = 0, \quad (44)$$

$$B_z|_{\pm}^{\pm} + \partial_i(\chi_m^{ik} B_k^{\overline{av}}) = 0, \quad (45)$$

for  $i, j = x, y, k = x, y, z$ ,  $\epsilon$  and  $\mu$  is the permittivity and permeability of the ambient medium respectively. Note that for this case, our definitions of susceptibilities do reduce to the more conventional definition of the proportionality between the polarization/magnetization and the average local field, as previously reported in [36]. In the following section, we illustrate the versatility of this technique with two examples. In the first example, we analytically obtain the surface susceptibility of a any reflecting interface. In the second example, we analytically obtain surface susceptibilities that suppress the effect of diffraction at corrugated interfaces, thereby redirecting light along unconventional directions.

### 3 Examples and Applications

In all the examples we are considering, we take the ambient medium to be free-space. This gives us  $\chi_d = \chi_e$ ,  $\chi_b = \chi_m$ ,  $\mathbf{D}^{\overline{av}} = \epsilon_0 \mathbf{E}^{\overline{av}}$ , and  $\mathbf{B}^{\overline{av}} = \mu_0 \mathbf{H}^{\overline{av}}$ . Expressing the transformation (37) in matrix form, we have for  $a = e, m$ ,

$$(\chi_a^{ik'}) = \sqrt{g^S} \begin{pmatrix} \chi_a^{uu} & \chi_a^{uv} & \chi_a^{un} \\ \chi_a^{vu} & \chi_a^{vv} & \chi_a^{vn} \\ \frac{\chi_a^{nu}}{\sqrt{g^S}} & \frac{\chi_a^{nv}}{\sqrt{g^S}} & \frac{\chi_a^{nn}}{\sqrt{g^S}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \\ \frac{1}{|\nabla n|} \frac{\partial n}{\partial x} & \frac{1}{|\nabla n|} \frac{\partial n}{\partial y} & \frac{1}{|\nabla n|} \frac{\partial n}{\partial z} \end{pmatrix}. \quad (46)$$

Then we can obtain  $\chi_a^{ij}$ ,  $a = e, m$ ,  $i, j = u, v, n$  via

$$\begin{pmatrix} \chi_a^{uu} & \chi_a^{uv} & \chi_a^{un} \\ \chi_a^{vu} & \chi_a^{vv} & \chi_a^{vn} \\ \frac{\chi_a^{nu}}{\sqrt{g^S}} & \frac{\chi_a^{nv}}{\sqrt{g^S}} & \frac{\chi_a^{nn}}{\sqrt{g^S}} \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \chi_a^{ux} & \chi_a^{uy} & \chi_a^{uz} \\ \chi_a^{vx} & \chi_a^{vy} & \chi_a^{vz} \\ \chi_a^{nx} & \chi_a^{ny} & \chi_a^{nz} \end{pmatrix} \begin{pmatrix} \frac{\partial n}{\partial z} - \frac{\partial f}{\partial v} \frac{\partial n}{\partial y} & \frac{\partial f}{\partial u} \frac{\partial n}{\partial y} & -|\nabla n| \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \frac{\partial n}{\partial x} & \frac{\partial n}{\partial z} - \frac{\partial f}{\partial u} \frac{\partial n}{\partial x} & -|\nabla n| \frac{\partial f}{\partial v} \\ -\frac{\partial n}{\partial x} & -\frac{\partial n}{\partial y} & |\nabla n| \end{pmatrix}, \quad (47)$$

where  $\gamma = \sqrt{g^S} \left( \frac{\partial n}{\partial z} - \frac{\partial n}{\partial x} \frac{\partial f}{\partial u} - \frac{\partial n}{\partial y} \frac{\partial f}{\partial v} \right)$ .

In all the examples we consider, we assume that  $\chi_a^{nx} = \chi_a^{ny} = \chi_a^{nz} = 0$  for  $a = e, m$ . This implies that for  $a = e, m$ ,  $\chi_a^{nu} = \chi_a^{nv} = \chi_a^{nn} = 0$ . Assuming an ambient medium of free space, and a time

dependent given by  $e^{i\omega t}$ , (38) and (39) are given in matrix form as

$$\begin{aligned} \begin{pmatrix} 0 & 1 & \frac{\partial f}{\partial v} \\ -1 & 0 & -\frac{\partial f}{\partial u} \end{pmatrix} \begin{pmatrix} E_x|_-^+ \\ E_y|_-^+ \\ E_z|_-^+ \end{pmatrix} &= i\omega\mu_0 \begin{pmatrix} \chi_m^{ux} & \chi_m^{uy} & \chi_m^{uz} \\ \chi_m^{vx} & \chi_m^{vy} & \chi_m^{vz} \end{pmatrix} \begin{pmatrix} H_x^{\bar{av}} \\ H_y^{\bar{av}} \\ H_z^{\bar{av}} \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & \frac{\partial f}{\partial v} \\ -1 & 0 & -\frac{\partial f}{\partial u} \end{pmatrix} \begin{pmatrix} H_x|_-^+ \\ H_y|_-^+ \\ H_z|_-^+ \end{pmatrix} &= -i\omega\epsilon_0 \begin{pmatrix} \chi_e^{ux} & \chi_e^{uy} & \chi_e^{uz} \\ \chi_e^{vx} & \chi_e^{vy} & \chi_e^{vz} \end{pmatrix} \begin{pmatrix} E_x^{\bar{av}} \\ E_y^{\bar{av}} \\ E_z^{\bar{av}} \end{pmatrix}, \end{aligned} \tag{48}$$

where  $i$  is the square root of  $-1$ .

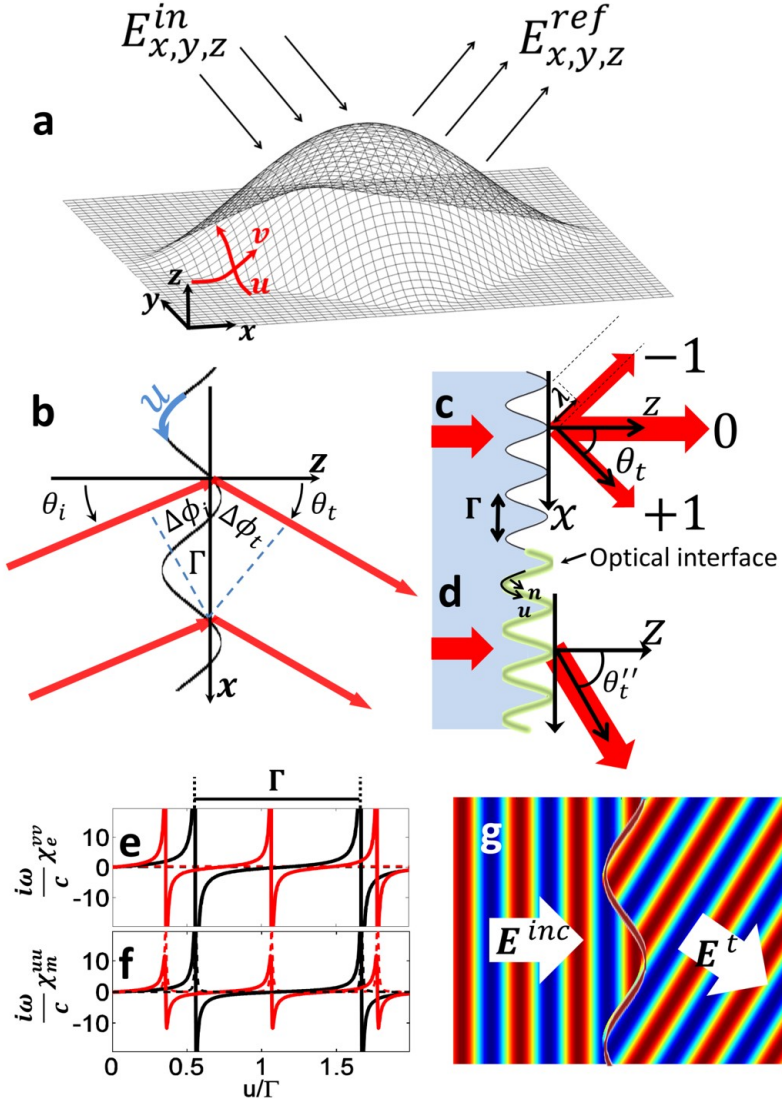


Figure 4: Mirror surface cloak described by  $f(u, v) = \cos u + \cos v$  reflects an incident plane wave at an angle of  $\theta$  with respect to the  $z$ -axis to form a plane wave propagating at the same angle  $\theta$  for all orientation of the surface. b, the relation between the angle of the incoming and outgoing waves can be imposed at the interface by considering the condition that the difference between the propagation phase shift of two light rays of the incident wavefront ( $\Delta\phi_i$ ), impinging on the surface at points separated by a distance  $(\Delta u, \Delta v)$ , and the propagation phase shift after the interface ( $\Delta\phi_t$ ) is exactly compensated by the phase shift introduced at the interface at those points. However, local phase retardations are not the only physical quantities to take into consideration. Because the orientation of the surface specifically, the orientation of the unit vector field  $\mathbf{N}$  normal to  $\mathbf{S}$  changes as one moves across it, the induced dipole moments at the surface are excited and radiate differently depending on the location. To account for the aforementioned changes in phase, amplitude and polarization, susceptibility tensors are calculated from Eq. 37 and 38-41. c and d respectively show a conventional and an anomalous grating. Light incident on a metasurface-grating (d) can blaze the diffraction towards a single order (black curves in e and f) but it can also refract light at any other user-specified angle (red curves in e and f). Solid (dashed) curves represent the imaginary (real) part of the susceptibility tensors. g shows the calculation of the real part of the electric fields for free-standing metasurfaces in air, that is, when the refractive indices on either side of the boundary are set to be equal.

### 3.1 Steering interface

In our first example, we investigate a unidirectional steering interface -also called "carpet cloak", a subject inspired by the works of Estakhri and Alù [20]- to achieve a unidirectional mirror reflection. Let  $\mathcal{S} = n^{-1}(0)$ , where  $n(x, y, z) = z - \cos(x) - \cos(y)$ , for  $x, y$  such that  $\cos(x) + \cos(y) > 0$ . If we define  $f(u, v) = \cos(u) + \cos(v)$ , one can see that  $\mathcal{S}$  is the graph of  $f$  [see figure 4]. Since there will be no transmission, for any field  $\mathbf{V} = \mathbf{E}, \mathbf{H}$ , we have  $\mathbf{V}^+ = 0$ , and  $\mathbf{V}^- = \mathbf{V}^{inc} + \mathbf{V}^{ref}$ . Let  $\theta \in (0, \frac{\pi}{2})$  be the angle between incident (and reflected) plane wave and the  $z$ -axis on the  $x, z$  plane. In order for the metasurface to be independent of the polarisation of the incident fields, we need to compute the susceptibilities using two orthogonally polarised plane wave for a given  $\theta$ .

Polarisation 1:

$$\begin{aligned} \mathbf{E}^{inc} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} A_{TE} \xi^{inc}, \quad \mathbf{E}^{ref} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} A_{TE} \xi^{ref}, \\ \mathbf{H}^{inc} &= \begin{pmatrix} -\cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix} \frac{A_{TE}}{\eta} \xi^{inc}, \quad \mathbf{H}^{ref} = \begin{pmatrix} \cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix} \frac{A_{TE}}{\eta} \xi^{ref}, \end{aligned} \tag{49}$$

Polarisation 2:

$$\begin{aligned} \mathbf{E}^{inc} &= \begin{pmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{pmatrix} A_{TM} \xi^{inc}, \quad \mathbf{E}^{ref} = \begin{pmatrix} -\cos(\theta) \\ 0 \\ -\sin(\theta) \end{pmatrix} A_{TM} \xi^{ref}, \\ \mathbf{H}^{inc} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{A_{TM}}{\eta} \xi^{inc}, \quad \mathbf{H}^{ref} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{A_{TM}}{\eta} \xi^{ref}, \end{aligned} \tag{50}$$

where  $\xi^{inc} = e^{i(\omega t - k_0(x \sin(\theta) + z \cos(\theta)))}$ ,  $\xi^{ref} = e^{i(\omega t - k_0(x \sin(\theta) - z \cos(\theta)))}$ ,  $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}}$ , and  $A_m, A_e$  are constants.

Since our equations are still underdetermined, we may further assume that for  $a = e, m$ ,  $\chi_a^{uz} = \chi_a^{vz} = 0$ . Note that this does not imply that  $\chi_a^{un} = 0$  nor  $\chi_a^{vn} = 0$ . Given the above assumptions,

the relation between the virtual susceptibilities and the fields are given by

$$\begin{aligned}
\chi_e^{ux} &= \frac{i}{\omega\epsilon_0} \left( \frac{E_y^{\bar{a}v}(H'_y|_{-}^{\pm} + \frac{\partial f}{\partial v}H'_z|_{-}^{\pm}) - E_y^{\bar{a}v}(H_y|_{-}^{\pm} + \frac{\partial f}{\partial v}H_z|_{-}^{\pm})}{E_x^{\bar{a}v}E_y^{\bar{a}v} - E_x^{av}E_y^{\bar{a}v}} \right), \\
\chi_e^{uy} &= \frac{i}{\omega\epsilon_0} \left( \frac{E_x^{\bar{a}v}(H_y|_{-}^{\pm} + \frac{\partial f}{\partial v}H_z|_{-}^{\pm}) - E_x^{\bar{a}v}(H'_y|_{-}^{\pm} + \frac{\partial f}{\partial v}H'_z|_{-}^{\pm})}{E_x^{\bar{a}v}E_y^{\bar{a}v} - E_x^{av}E_y^{\bar{a}v}} \right), \\
\chi_e^{vx} &= \frac{i}{\omega\epsilon_0} \left( \frac{E_y^{\bar{a}v}(H_x|_{-}^{\pm} + \frac{\partial f}{\partial u}H_z|_{-}^{\pm}) - E_y^{\bar{a}v}(H'_x|_{-}^{\pm} + \frac{\partial f}{\partial u}H'_z|_{-}^{\pm})}{E_x^{\bar{a}v}E_y^{\bar{a}v} - E_x^{av}E_y^{\bar{a}v}} \right), \\
\chi_e^{vy} &= \frac{i}{\omega\epsilon_0} \left( \frac{E_x^{\bar{a}v}(H'_x|_{-}^{\pm} + \frac{\partial f}{\partial u}H'_z|_{-}^{\pm}) - E_x^{\bar{a}v}(H_x|_{-}^{\pm} + \frac{\partial f}{\partial u}H_z|_{-}^{\pm})}{E_x^{\bar{a}v}E_y^{\bar{a}v} - E_x^{av}E_y^{\bar{a}v}} \right), \\
\chi_m^{ux} &= \frac{i}{\omega\mu_0} \left( \frac{H_x^{\bar{a}v}(E_y|_{-}^{\pm} + \frac{\partial f}{\partial v}E_z|_{-}^{\pm}) - H_x^{\bar{a}v}(E'_y|_{-}^{\pm} + \frac{\partial f}{\partial v}E'_z|_{-}^{\pm})}{H_x^{\bar{a}v}H_y^{\bar{a}v} - H_x^{\bar{a}v}H_y^{\bar{a}v}} \right), \\
\chi_m^{uy} &= \frac{i}{\omega\mu_0} \left( \frac{H_x^{\bar{a}v}(E'_y|_{-}^{\pm} + \frac{\partial f}{\partial v}E'_z|_{-}^{\pm}) - H_x^{\bar{a}v}(E_y|_{-}^{\pm} + \frac{\partial f}{\partial v}E_z|_{-}^{\pm})}{H_x^{\bar{a}v}H_y^{\bar{a}v} - H_x^{\bar{a}v}H_y^{\bar{a}v}} \right), \\
\chi_m^{vx} &= \frac{i}{\omega\mu_0} \left( \frac{H_y^{\bar{a}v}(E'_x|_{-}^{\pm} + \frac{\partial f}{\partial u}E'_z|_{-}^{\pm}) - H_y^{\bar{a}v}(E_x|_{-}^{\pm} + \frac{\partial f}{\partial u}E_z|_{-}^{\pm})}{H_x^{\bar{a}v}H_y^{\bar{a}v} - H_x^{\bar{a}v}H_y^{\bar{a}v}} \right), \\
\chi_m^{vy} &= \frac{i}{\omega\mu_0} \left( \frac{H_x^{\bar{a}v}(E_x|_{-}^{\pm} + \frac{\partial f}{\partial u}E_z|_{-}^{\pm}) - H_x^{\bar{a}v}(E'_x|_{-}^{\pm} + \frac{\partial f}{\partial u}E'_z|_{-}^{\pm})}{H_x^{\bar{a}v}H_y^{\bar{a}v} - H_x^{\bar{a}v}H_y^{\bar{a}v}} \right),
\end{aligned} \tag{51}$$

which for a flat surface ( $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$ ), are exactly those derived in [??]. Therefore using (47), we obtain the actual susceptibilities,

$$\begin{aligned}
\chi_a^{uu} &= \frac{1}{\gamma}(\chi_a^{ux}(\frac{\partial n}{\partial z} - \frac{\partial f}{\partial v}\frac{\partial n}{\partial y}) + \chi_a^{uy}(\frac{\partial f}{\partial v}\frac{\partial n}{\partial x})), \\
\chi_a^{uv} &= \frac{1}{\gamma}(\chi_a^{ux}(\frac{\partial n}{\partial y}\frac{\partial f}{\partial u}) + \chi_a^{uy}(\frac{\partial n}{\partial z} - \frac{\partial f}{\partial u}\frac{\partial n}{\partial x})), \\
\chi_a^{un} &= \frac{-1}{\gamma}(\chi_a^{ux}(|\nabla n|\frac{\partial f}{\partial u}) + \chi_a^{uy}(|\nabla n|\frac{\partial f}{\partial v})), \\
\chi_a^{vu} &= \frac{1}{\gamma}(\chi_a^{vx}(\frac{\partial n}{\partial z} - \frac{\partial f}{\partial v}\frac{\partial n}{\partial y}) + \chi_a^{vy}(\frac{\partial f}{\partial v}\frac{\partial n}{\partial x})), \\
\chi_a^{vv} &= \frac{1}{\gamma}(\chi_a^{vx}(\frac{\partial n}{\partial y}\frac{\partial f}{\partial u}) + \chi_a^{vy}(\frac{\partial n}{\partial z} - \frac{\partial f}{\partial u}\frac{\partial n}{\partial x})), \\
\chi_a^{vn} &= \frac{-1}{\gamma}(\chi_a^{vx}(|\nabla n|\frac{\partial f}{\partial u}) + \chi_a^{vy}(|\nabla n|\frac{\partial f}{\partial v})),
\end{aligned} \tag{52}$$

for  $a = e, m$ . Substituting the values given above, we obtain

$$\begin{aligned}
\chi_e^{ux} = \chi_m^{ux} &= \frac{2ic}{\omega \cos(\theta)} \left( \frac{\xi^{inc} + \xi^{ref}}{\xi^{ref} - \xi^{inc}} \right), \quad \chi_e^{uy} = \chi_m^{uy} = \frac{2ic}{\omega} \sin(\theta) \sin(v), \quad \chi_e^{vx} = \chi_m^{vx} = 0, \\
\chi_e^{vy} = \chi_m^{vy} &= \frac{2ic}{\omega} \left( \frac{\cos(\theta)(\xi^{ref} - \xi^{inc}) - \sin(u) \sin(\theta)(\xi^{inc} + \xi^{ref})}{\xi^{inc} + \xi^{ref}} \right),
\end{aligned} \tag{53}$$

which give us

$$\begin{aligned}
\chi_a^{uu} &= \frac{2ic}{\omega\gamma} \left( \frac{1 + \sin^2(v)}{\cos(\theta)} \left( \frac{\xi^{inc} + \xi^{ref}}{\xi^{ref} - \xi^{inc}} \right) - \sin(\theta) \sin(u) \sin^2(v) \right), \\
\chi_a^{uv} &= \frac{2ic}{\omega\gamma} \sin(v) \left( \sin(\theta)(1 + \sin^2(u)) - \frac{\sin(u)}{\cos(\theta)} \left( \frac{\xi^{inc} + \xi^{ref}}{\xi^{ref} - \xi^{inc}} \right) \right), \\
\chi_a^{un} &= \frac{2ic}{|\nabla n|^2 \omega} \left( \frac{\sin(u)}{\cos(\theta)} \left( \frac{\xi^{inc} + \xi^{ref}}{\xi^{ref} - \xi^{inc}} \right) + \sin(\theta) \sin^2(v) \right), \\
\chi_a^{vu} &= \frac{2ic}{\omega\gamma} \sin(v) \sin(u) \left( \sin(u) \sin(\theta) - \cos(\theta) \left( \frac{\xi^{ref} - \xi^{inc}}{\xi^{inc} + \xi^{ref}} \right) \right), \\
\chi_a^{vv} &= \frac{2ic}{\gamma\omega} (1 + \sin^2(u)) \left( \cos(\theta) \left( \frac{\xi^{ref} - \xi^{inc}}{\xi^{inc} + \xi^{ref}} \right) - \sin(u) \sin(\theta) \right), \\
\chi_a^{vn} &= \frac{2ic}{|\nabla n|^2 \omega} \left( \cos(\theta) \left( \frac{\xi^{ref} - \xi^{inc}}{\xi^{inc} + \xi^{ref}} \right) - \sin(u) \sin(\theta) \right),
\end{aligned} \tag{54}$$

for  $a = e, m$ , where  $\gamma = (1 + \sin^2(u) + \sin^2(v))^{\frac{3}{2}}$ ,  $|\nabla n|^2 = 1 + \sin^2(u) + \sin^2(v)$ , and we may factorize out the expressions  $e^{i\omega t}$ , and take  $\xi^{inc} = e^{-ik_0 \cos(\theta)(\cos(u) + \cos(v))}$  and  $\xi^{ref} = e^{ik_0 \cos(\theta)(\cos(u) + \cos(v))}$ . Observe that the susceptibilities depended on the angle  $\theta$ , meaning that with all the assumptions above, we are only able to implement cloaking for a single direction  $\theta$ .

Although a mantle cloak designed with this technique hides an object by mimicking the reflection from the ground, our technique can in principle be used to create true optical illusions, in which the image of an object is replaced by the virtual holographic image of another [19]. Consider a detector with coordinates  $(\xi, \eta)$ , and let  $g(\xi, \eta)$  be the graph of the detector. Let  $E_{k'}^{obs}(\xi, \eta)$  and  $E_{k'}^{ref}(u, v)$ , for  $k' = x, y, z$ , be the electric field of the virtual image at the detector plane and the electric fields reflected from the surface  $\mathcal{S}$  respectively. The relationship between the two electric fields is given by the first Rayleigh-Sommerfield integral as

$$E_{k'}^{obs}(\xi, \eta) = -\frac{1}{2\pi} \int_{\mathcal{S}} E_{k'}^{ref}(u, v) \left( \nabla \left( \frac{e^{ikR}}{R} \right) \cdot \frac{\nabla n}{|\nabla n|} \right) dudv, \tag{55}$$

where  $R = ((u - \xi)^2 + (v - \eta)^2 + (f(u, v) - g(\xi, \eta))^2)^{\frac{1}{2}}$ . Using the approximation  $\nabla \left( \frac{e^{ikR}}{R} \right) \approx \frac{ik e^{ikR}}{R^2}$ , the above expression can be simplified to become

$$\begin{aligned}
E_{k'}^{obs}(\xi, \eta) &= \frac{1}{i\lambda} \int_{\mathcal{S}} E_{k'}^{ref}(u, v) \frac{e^{ikR}}{|\nabla n|R^2} \left( \nabla n \cdot \begin{pmatrix} u - \xi \\ v - \eta \\ f - g \end{pmatrix} \right) dudv \\
&= \frac{1}{i\lambda} \int_{\mathcal{S}} E_{k'}^{ref}(u, v) \frac{e^{ikR}}{|\nabla n|R^2} \left( \frac{\partial n}{\partial x}(u - \xi) + \frac{\partial n}{\partial y}(v - \eta) + \frac{\partial n}{\partial z}(f(u, v) - g(\xi, \eta)) \right) dudv
\end{aligned} \tag{56}$$

After obtaining  $E_{k'}^{ref}$  analytically, or by numerical methods in general, we are able to calculate  $H_{k'}^{ref}$  using Maxwell's equations. Therefore given any incident fields  $E_{k'}^{inc}$  and  $H_{k'}^{inc}$ , using the recipe prescribed above, we are able to compute the susceptibilities needed to create the virtual image at the detector plane.

### 3.2 Curved transmission metasurface

In our second example, we design a reflection-less optical interface [48] defined by a periodic structure of period  $\Gamma$  greater than the wavelength of the light, ( $\Gamma > \Lambda_0$ ). Considering light incident at an angle  $\theta_i$  from a lossless dielectric of refractive index  $n_i$ , as in Fig. 4b, conventional diffraction in Fig 4.c should create plane waves at angles  $\theta_t$  given by the grating formula:

$$m\lambda_0 = \Gamma(n_i \sin \theta_i - \sin \theta_t) \quad (57)$$

However, by depositing a metasurface conformable to the periodically undulating interface, one can tailor the diffracted light. As shown in Fig. 4b, the interface has to delay the incoming fields such that the transmitted light from each point along the interface constructively interferes along the user preferred direction, creating plane waves that travel along the angle  $\theta_t$ . We consider a sinusoidal surface such that it diffracts normally incident fields at a specified angle. Let  $\mathcal{S} = n^{-1}(0)$ , where  $n(x, y, z) = z - \sin(\frac{2\pi x}{\Gamma})$  for some  $\Gamma > \lambda$ . Defining  $f(u, v) = \sin(\frac{2\pi u}{\Gamma})$ , then  $\mathcal{S}$  is the graph of  $f$ . Now given a incoming polarised plane wave

$$\mathbf{E}^- = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} A\xi^-, \quad \mathbf{H}^- = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \frac{A}{\eta}\xi^-, \quad (58)$$

where  $\xi^- = e^{i(\omega t - k_0 z)}$  and  $A$  is a constant, assuming no reflection, we want it such that the fields leaving  $\mathcal{S}$  are given by (see figure ??)

$$\mathbf{E}^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} A\xi^+, \quad \mathbf{H}^+ = \begin{pmatrix} -\cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix} \frac{A}{\eta}\xi^+, \quad (59)$$

where  $\xi^+ = e^{i(\omega t - k_0(x \sin(\theta) + z \cos(\theta)))}$ ,  $\theta \in (0, \frac{\pi}{2})$  is the angle between the exiting plane wave and the  $z$ -axis, on the  $x, z$  plane. Using Eq. (48), we have

$$\begin{aligned} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \end{pmatrix} \begin{pmatrix} 0 \\ A(\xi^+ - \xi^-) \\ 0 \end{pmatrix} &= i\omega\mu_0 \begin{pmatrix} \chi_m^{ux} & \chi_m^{uy} & \chi_m^{uz} \\ \chi_m^{vx} & \chi_m^{vy} & \chi_m^{vz} \end{pmatrix} \begin{pmatrix} -\frac{A}{2\eta}(\xi^- + \xi^+ \cos(\theta)) \\ 0 \\ \frac{A}{2\eta}\xi^+ \sin(\theta) \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \end{pmatrix} \begin{pmatrix} \frac{A}{\eta}(\xi^- - \xi^+ \cos(\theta)) \\ 0 \\ \frac{A}{\eta}\xi^+ \sin(\theta) \end{pmatrix} &= -i\omega\epsilon_0 \begin{pmatrix} \chi_e^{ux} & \chi_e^{uy} & \chi_e^{uz} \\ \chi_e^{vx} & \chi_e^{vy} & \chi_e^{vz} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{A}{2}(\xi^- + \xi^+) \\ 0 \end{pmatrix}. \end{aligned} \quad (60)$$

From the above, we may assume  $\chi_m^{uy} = \chi_m^{vy} = \chi_e^{ux} = \chi_e^{uy} = \chi_e^{uz} = \chi_e^{vx} = \chi_e^{vz} = 0$ . Further, we may assume  $\chi_m^{vz} = 1$  and  $\chi_m^{uz} = 0$  to simplify the equations. Then we end up with

$$\begin{aligned} \chi_m^{ux} &= \frac{2c}{i\omega} \left( \frac{\xi^- - \xi^+}{\xi^- + \xi^+ \cos(\theta)} \right), \quad \chi_m^{vx} = \frac{\xi^+ \sin \theta}{\xi^- + \xi^+ \cos \theta}, \\ \chi_e^{vy} &= \frac{2c}{i\omega} \left( \frac{\xi^- + \xi^+ (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \sin(\theta) - \cos(\theta))}{\xi^+ + \xi^-} \right), \end{aligned} \quad (61)$$

where we may take  $\xi^- = e^{-ik_0 \sin(u)}$  and  $\xi^+ = e^{-ik_0(u \sin(\theta) + \sin(u) \cos(\theta))}$ . Thus using (47), we have

$$\begin{aligned} \chi_m^{uu} &= \frac{2c}{i\omega(1 + (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}))^2)^{3/2}} \left( \frac{\xi^- - \xi^+}{\xi^- + \xi^+ \cos(\theta)} \right), \\ \chi_m^{un} &= \frac{2c \cos(u)}{i\omega(1 + (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}))^2)} \left( \frac{\xi^+ - \xi^-}{\xi^- + \xi^+ \cos(\theta)} \right), \\ \chi_m^{vn} &= \frac{1}{1 + (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}))^2} \left( \frac{\xi^+ \sin \theta}{(1 + (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}))^2)^{1/2} (\xi^- + \xi^+ \cos \theta)} - 1 \right), \\ \chi_e^{vv} &= \frac{2c}{i\omega \sqrt{1 + (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}))^2}} \left( \frac{\xi^- + \xi^+ (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \sin(\theta) - \cos(\theta))}{\xi^+ + \xi^-} \right), \end{aligned} \quad (62)$$

and the rest of the components of the susceptibilities are zero.

Next, we show that given the surface susceptibilities in (61), using the same incoming fields  $\mathbf{E}^-$  and  $\mathbf{H}^-$ , we are indeed able to obtain the transmitted fields  $\mathbf{E}^+$  and  $\mathbf{H}^+$ . Let  $A = 1$ , and we will work in the  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  frame, but use  $(u, v)$  for the coordinates since all the points lies on the



surface  $\mathcal{S}$ . Because we are assuming a plane wave, the transmitted fields can be written as

$$\mathbf{E}^+ = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \xi'^+, \quad \mathbf{H}^+ = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} k_y E_z - E_y k_z \\ k_z E_x - k_x E_z \\ k_x E_y - k_y E_x \end{pmatrix} \frac{\xi'^+}{\omega \mu_0}, \quad (63)$$

where we may take  $\xi'^+ = e^{i(k_x u + k_y v + k_z \sin(u))}$ . We need to find the above 6 unknowns using

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \end{pmatrix} \begin{pmatrix} E_x \xi'^+ \\ E_y \xi'^+ - \xi^- \\ E_z \xi'^+ \end{pmatrix} = i\omega \mu_0 \begin{pmatrix} \chi_m^{ux} & 0 & 0 \\ \chi_m^{vx} & 0 & \chi_m^{vz} \end{pmatrix} \begin{pmatrix} \frac{1}{2}(H_x - \frac{\xi^-}{\eta}) \\ \frac{H_y}{2} \\ \frac{H_z}{2} \end{pmatrix}, \quad (64)$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \end{pmatrix} \begin{pmatrix} H_x + \frac{\xi^-}{\eta} \\ H_y \\ H_z \end{pmatrix} = -i\omega \epsilon_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \chi_e^{vy} & 0 \end{pmatrix} \begin{pmatrix} \frac{E_x \xi'^+}{2} \\ \frac{E_y \xi'^+ + \xi^-}{2} \\ \frac{E_z \xi'^2}{2} \end{pmatrix}.$$

From the above we obtain  $H_y = 0$ , implying that  $k_x E_z = k_z E_x$ . Since  $k_z \neq 0$ , then  $E_z = \frac{k_z}{k_x} E_x$ .

After some simple manipulation, we obtain

$$-E_x \xi'^+ \left(1 + \frac{k_z}{k_x} \frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma})\right) = \frac{\chi_m^{vx}}{\chi_m^{ux}} (E_y \xi'^+ - \xi^-) + \frac{i\xi'^+}{2} (k_x E_y - k_y E_x), \quad (65)$$

which gives us

$$E_x \left(\frac{ik_y}{2} - 1\right) = \frac{\chi_m^{vx}}{\xi'^+ \chi_m^{ux}} (E_y \xi'^+ - \xi^-) - E_x \frac{k_z}{k_x} \frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) + \frac{ik_x}{2} E_y. \quad (66)$$

Notice that the left hand side is a constant, but the right hand side is a function of  $u$ . And since  $k_y$  must be a real number, necessarily  $E_x = 0$ , thus  $E_z = 0$ . Now since we are assuming no lost or gain in amplitude, then  $|\mathbf{E}^+| = 1$ , giving us  $E_y = 1$ . Similarly  $\frac{1}{\eta} = |\mathbf{H}^+| = \frac{\sqrt{k_z^2 + k_x^2}}{\omega \mu_0}$  implies that  $k_y = 0$ .

Next using the remaining information from (64), substituting what we have from above, we have the following equations

$$\xi^- - \xi'^+ = \left(\frac{\xi^- - \xi^+}{\xi^- + \xi^+ \cos \theta}\right) \left(\xi^- + \frac{k_z \xi'^+}{k_0}\right), \quad (67)$$

$$\frac{-\frac{\xi'^+ k_z}{k_0} + \xi^- + \frac{\xi'^+ k_x}{k_0} \frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma})}{\xi'^+ + \xi^-} = \frac{\xi^- + \xi^+ (\frac{2\pi}{\Gamma} \cos(\frac{2\pi u}{\Gamma}) \sin \theta - \cos \theta)}{\xi^- + \xi^+},$$

giving us 2 equations

$$\begin{aligned} \frac{\xi^- - \xi'^+}{\xi^- + \frac{k_z \xi'^+}{k_0}} &= \frac{\xi^- - \xi^+}{\xi^- + \xi^+ \cos \theta}, \\ \frac{\xi^- + \xi'^+ \left( \frac{k_x}{k_0} \frac{2\pi}{\Gamma} \cos\left(\frac{2\pi u}{\Gamma}\right) - \frac{k_z}{k_0} \right)}{\xi'^+ + \xi^-} &= \frac{\xi^- + \xi^+ \left( \frac{2\pi}{\Gamma} \cos\left(\frac{2\pi u}{\Gamma}\right) \sin \theta - \cos \theta \right)}{\xi^+ + \xi^-}, \end{aligned} \quad (68)$$

with 2 unknowns  $k_z$  and  $k_x$ . Therefore the unique solution given by  $\frac{k_x}{k_0} = \sin \theta$  and  $\frac{k_z}{k_0} = \cos \theta$  satisfy the equations. With that, we indeed recover the given transmitted fields given above.

## 4 Discussions

Our use of coordinate transformations to control abrupt changes of light at interfaces is complementary to and distinct from – the technique of transformation optics, which uses coordinate transformations to control the propagation of light in bulk media. Whereas transformation optics uses a coordinate system that conforms to the direction of light propagation, the concept of conformal boundary optics uses a coordinate system that conforms to the geometry of the interface. With the advent of transformation optics and metamaterials, many creative ways to manipulate light have been proposed that involve negative refraction, hyperlensing, cloaking and backward Cherenkov emission [45-49]. Transformation optics has also been used to redirect the propagation of surface electromagnetic waves [25-28], and to suppress the surface wave scattering losses [50]. New and interesting optical effects are expected with the added control of fields at interfaces of metamaterials with arbitrary geometries (Fig.5).

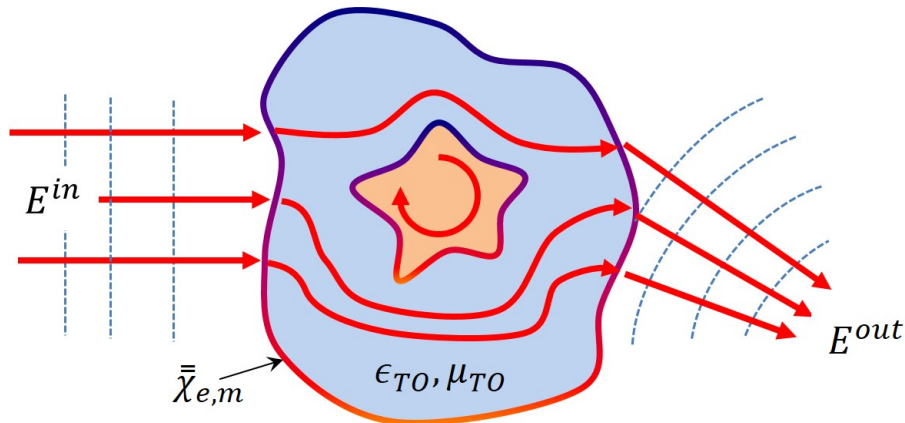


Figure 5: Combining the use of conformal boundary optics and transformation optics can be useful to achieve complete, geometry-independent control over the behavior of light at interfaces and in bulk media.

The concept of conformal boundary optics has important implications even in the realm of con-

ventional optics, where the surface susceptibility tensors are zero. In this case, the metasurface becomes a simple interfaces and  $\chi_m = \chi_e = 0$ . Considering the Equ.42-45 and dropping the assumption of having no free surface charge and current, then we recover the usual electromagnetic boundary conditions for nonplanar surfaces:

$$[ij]\Lambda_j^{k'} E_{k'}|_{\pm}^{\pm} \mathbf{e}_i = 0, \quad (69)$$

$$[ij]\Lambda_j^{k'} H_{k'}|_{\pm}^{\pm} \mathbf{e}_i = J_S^i \mathbf{e}_i, \quad (70)$$

$$\Lambda_n^{k'} D_{k'}|_{\pm}^{\pm} = \rho_S, \quad (71)$$

$$\Lambda_n^{k'} B_{k'}|_{\pm}^{\pm} = 0, \quad (72)$$

where  $\mathbf{J}_S$  is the surface free current, and  $\rho_S$  is the surface charge density.

In many optics textbooks, the boundary conditions are derived by considering an Amperian loop and a Gaussian pillbox at an interface, often graphically represented – for aesthetical reasons – as a nonplanar surface. This, however, can be misleading because these conditions are originally derived in the coordinate system that conforms to the interface whereas the electromagnetic fields are expressed in the ambient – typically Cartesian, spherical or cylindrical – coordinate system. Because the surface normal of the nonplanar interface is changing, the boundary conditions vary depending on the position along the interface. This observation shows that our approach is already significant for conventional interfaces that involve only regular dielectric materials. Prior to this work, boundary conditions existed only for cases where the interface – which the coordinate system conforms to – has a basic geometry e.g., Cartesian, cylindrical, spherical. We believe that the approach discussed in this paper constitutes a first step towards the development of a general theory for waves in systems containing interfaces with complex geometries. Surface susceptibility tensors have to be considered as intermediate tool to predict the electromagnetic behavior at a given optical interface. Experimentally, however, susceptibilities are realized by averaging the overall contribution of subwavelength unit cells or scatterers, each individually designed to have a specific optical response. It is therefore important to discuss the feasibility of designing and fabricating nanostructures with scattering properties that, in practice, can realize the synthesized susceptibilities. In practice, the susceptibility of arrays of identical scatterers can be computed numerically or obtained experimentally. In the numerical approach, this is done by discretizing the interface according to a two dimensional array of unit cells [8] for which the electromagnetic response, including phase, amplitude and polarization, are essentially obtained after performing a series of full electromagnetic wave simulations. In some cases, the design and the physical realization of the scattering particles might be difficult and even impossible to realize experimentally [45].

However, as has been pointed out in [45], obtaining the surface susceptibility tensor is generally an under-determined problem due to the large number of degrees of freedom involved. This gives one the flexibility to choose susceptibility tensors that are experimentally realizable. We note that to date, there exists no theoretical tool that directly connects shape and materials properties of nanostructures with macroscopic values of metasurfaces, and this remains an interesting topic of future research. The connection between the near-field local polarization response of individual scatterer with the overall surface susceptibility tensor might be achieved by considering multipole-moments decomposition and near-field to far field transformation[55,56]. A recently published paper shows that it is in principle possible to experimentally realize a carpet cloak metasurface to hide an arbitrarily shaped object [57]. Given the height of a 3D object with respect to a reference plane, the authors designed the interface phase response point-by-point to address the phase of the reflected wavefront at the interface. The concept of conformal boundary optics presented in our work goes far beyond a method that considers only phase retardation of reflected fields. With this method, one can design the optical response at any object to produce any output beam both for the reflection and/or transmission. The susceptibilities obtained from our conformal boundary method are complex in general. Complex surface impedances, admittances and susceptibilities have been reported previously [35, 42-46] and this may indicate that the interface has some sort of spatially varying loss and/or gain mechanism to achieve the desired optical effect. However, the actual presence of gain and/or loss at interfaces can only be confirmed after taking all terms of the susceptibilities into account, as compensation can produce a real response [45]. It is worth pointing out that conformal boundary optics applies only to smooth interface, i.e. for which the surface is a level set meaning that it can be parametrized by an analytical expression.

We have introduced the concept of conformal boundary optics, a versatile design theory that enables metasurfaces of arbitrary geometries to find even greater applications in the emerging fields of stretchable, camouflage photonics and augmented reality. This strategy will certainly help to design macroscopic electromagnetic parameters of arbitrary interfaces. However, further fundamental works are needed to connect macroscopic quantities with the microscopic optical response of nanoscale objects. Controlling light at curved geometries will find widespread industrial applications, for example, in the design of the next generation of contact lenses and flexible, stretchable optical surfaces. One can also realise unusually-shaped optical cavities that support designer resonant modes. By allowing us to transcend the limitations imposed by physical geometries, this concept has fundamental implications for the way in which we understand and structure light in systems of arbitrary geometries.

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