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Febuary 22, 2022

Abstract

We use the Landau-de Gennes energy to describe a particle immersed into nematic liquid crystals with a constant applied magnetic field. We derive a limit energy in a regime where both line and point defects are present, showing quantitatively that the close-to-minimal energy is asymptotically concentrated on lines and surfaces nearby or on the particle. We also discuss regularity of minimizers and optimality conditions for the limit energy.
Keywords: Nematic liquid crystal colloids, Landau-de Gennes model, $\Gamma-$convergence, flat chains, singular limits
MSC2020: 49J45, 49S05, 76A15

Contents

1 Introduction

The history of interaction between variational problems and geometry has been long and of great mutual influence [28], starting from the geometrically motivated problem of the brachistochrone curve [7, 57], Fermat's principle in optics [10], material science [5] to general relativity [30, 47].

One particularly important problem arises when the size of geometrical objects themselves is to be minimized leading to so called *minimal surfaces* [41]. A classical example is the two dimensional soap film spanning between predefined (fixed) boundary curves, called Plateau's problem [23,60,67]. Some solutions can be constructed explicitly [21, 37] or studied through means of harmonic and complex analysis [18, 36, 56], but a general theory was not available until the development of geometric measure theory and its language of currents, flat chains, mass and varifolds to describe the objects and how to measure them [3, 24, 25, 55, 68].

Another question giving rise to problems involving minimal surfaces is given by the classical $\Gamma-$convergence result of Modica and Mortola [52] (see also [50]) of a weighted Dirichlet energy and a penalizing double-well potential to the perimeter functional. A constraint such as a prescribed volume ensures the problem to be non trivial. The energy typically is concentrated in regions where none of the favourable states of the potential are attained. For the limsup inequality, one constructs a one dimensional profile that minimizes the transition between the favoured states.

Another variational problem in which geometry appears is given by the Ginzburg-Landau model. In the famous work [9], the (logarithmically diverging) leading order term and (after a rescaling) a limit problem have been derived. The limit energy is stated geometrically as finding an optimal distribution of points in the plane subject to constraints coming from the topological degree of the initial problem. This approach stimulated research which lead to a large literature [2, 11, 16, 32, 42, 49, 62], in particular for problems in micromagnetics [33, 40], superconductors [38, 64, 66] and liquid crystals [6, 34, 43].

Our work combines many of the before mentioned ideas to describe the different contributions and effects that take place in our problem. For example, we use the generalized three dimensional analogue of estimations in [9] as considered in [14, 16, 17, 35, 61] to make appear a length minimization problem for curves. Coupled with this optimization problem, we show using a Modica-Mortola type argument that the remaining part of the energy concentrates on hypersurfaces which end either on the boundary of the domain or on the described line.

This article is the continuation of the work started in [4], our main theorem (Theorem 3.1) is a generalisation of Theorem 3.1 in [4] (see Remark 3.3). In particular, our new theorem holds for an arbitrary manifold $\mathcal{M}$ of class $C^{1,1}$ instead of a sphere and we remove the hypothesis of rotational equivariance of Q.

We place ourself in the context of the Landau-de Gennes model for nematic liquid crystals, although the applied ideas could be used to carry out a similar analysis for a larger class of energy functionals.

2 Preliminaries

Before we can state our results, we give a short introduction to the Landau-de Gennes model that we use here and the concept of flat chains, stating some results that will be used later in the proof section.

2.1 Landau-de Gennes model for nematic liquid crystals

Our article has been motivated by the study of liquid crystal colloids with external magnetic field. The Landau-de Gennes energy with additional magnetic field term [59, Ch. 6, Secs. 3-4 and Ch. 10, Sec. 2.3] (see also [22, Ch. 3, Secs. 1-2]) can be stated in dimensionless form as

$$\mathcal{E}_{\eta,\xi}(Q) \;=\; \int_\Omega \frac{1}{2}|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) + \frac{1}{\eta^2} g(Q) + C_0 \,\mathrm{d}x\,, \tag{1}$$

where the energy density f is given by

$$f(Q) \;=\; C - \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}(Q^2)^2\,. \tag{2}$$

We consider the case when the parameters η and ξ converge to zero in a regime where $\eta|\ln(\xi)| \to \beta \in (0,\infty)$. The constant $C_0 = C_0(\eta,\xi)$ (resp. C) is chosen such that the energy density (resp. f) becomes non-negative.

The following properties of f are going to be used in the sequel:

1. The function f is non-negative and $\mathcal{N} := f^{-1}(0)$ is a smooth, closed, compact, connected manifold, diffeomorphic to the real projective plane $\mathbb{R}P^2$. Note that $\mathcal{N}$ is given by

$$\mathcal{N} = \left\{ s_* \left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathrm{Id} \right) \,:\, \mathbf{n}\in\mathbb{S}^2 \right\},$$

   for $s_* = \frac{1}{4c}(b + \sqrt{b^2+24ac})$ (cf. [46]).

2. We need f to behave uniformly quadratic close to its minima, i.e. we assume that there exist constants $\delta_0, \gamma_1 > 0$ such that for all $Q \in \mathrm{Sym}_0$ with $\mathrm{dist}(Q,\mathcal{N}) \le \delta_0$ it holds

$$f(Q) \;\ge\; \gamma_1 \mathrm{dist}^2(Q,\mathcal{N})\,.$$

3. Lastly, we need to quantify the growth of f. More precisely, we assume that there exist constants $C_1, C_2 > 0$, such that for all $Q \in \mathrm{Sym}_0$

$$f(Q) \ge C_1 \left(|Q|^2 - \frac{2}{3}s_*^2 \right)^2, \qquad Df(Q) : Q \ge C_1\,|Q|^4 - C_2\,.$$

It can be checked that f given in (2) satisfies these assumptions [4, 12, 14, 46]. The exponent 4 in the last assumption is not crucial but only needs to outweigh the growth of g.

We also recall the following facts about the geometry of Sym_0:

1. Elements $Q \in \mathrm{Sym}_0$ admit the following decomposition: There exist $s \in [0,\infty)$ and $r \in [0,1]$ such that

$$Q = s\left(\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathrm{Id} \right) + r\left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}\mathrm{Id} \right) \right), \tag{3}$$

   where $\mathbf{n}, \mathbf{m}$ are normalized, orthogonal eigenvectors of Q. The values s and r are continuous functions of Q.

2. The set where decomposition (3) is not unique, is given by $\mathcal{C} := \{Q \in \mathrm{Sym}_0 \setminus \{0\} \,:\, r(Q) = 1\} \cup \{0\}$. Another characterization of $\mathcal{C}$ is $\mathcal{C} = \{Q \in \mathrm{Sym}_0 \,:\, \lambda_1(Q) = \lambda_2(Q)\}$, where the two leading eigenvalues of Q are denoted by λ_1, λ_2. Moreover, $\mathcal{C}$ has the structure of a cone over $\mathbb{R}P^2$ and $\mathcal{C}\setminus\{0\} \cong \mathbb{R}P^2 \times \mathbb{R}$.

3. There exists a continuous retraction $\mathcal{R} : \mathrm{Sym}_0 \setminus \mathcal{C} \to \mathcal{N}$ such that $\mathcal{R}(Q) = Q$ for all $Q \in \mathcal{N}$. One can choose $\mathcal{R}$ to be the nearest point projection onto $\mathcal{N}$. In this case, $\mathcal{R}(Q) = s_*(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathrm{Id})$ for $Q \in \mathrm{Sym}_0 \setminus \mathcal{C}$ decomposed as in (3).

The energy density g in (1) incorporates an external magnetic field into the model. This motivates the following assumption:

1. The function g favours Q having an eigenvector equal to the direction of the external field, in our case chosen to be along $\mathbf{e}_3$. More precisely, assume g is invariant by rotations around the $\mathbf{e}_3-$axis and the function $O(3) \ni R \mapsto g(R^\top Q R)$ is minimal if $\mathbf{e}_3$ is eigenvector to the maximal eigenvalue of $R^\top Q R$. Decomposing Q as in (3) with $\mathbf{n} = \mathbf{e}_3$ and keeping s and $\mathbf{m}$ fixed, then $g(Q)$ is minimal for $r = 0$. For a uniaxial $Q \in \mathcal{N}$, i.e. $Q = s_*(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathrm{Id})$ for $s_* \geq 0$ and $\mathbf{n} \in \mathbb{S}^2$ we have

$$g(Q) = c_*^2(1 - \mathbf{n}_3^2)\,. \tag{4}$$

The precise formula for g in (4) is not important to our analysis, but for simplicity we assume this particular form. It would be enough to assume that $g|_{\mathcal{N}}$ has a strict minimum in $Q = s_*(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{2}\mathrm{Id})$, see Remark 4.18 in [4]. Besides this physical assumption, our analysis requires g to satisfy the following hypothesis:

2. The function $g : \mathrm{Sym}_0 \to \mathbb{R}$ is of class C^2 away from $Q = 0$ and in particular satisfies the Lipschitz condition close to $\mathcal{N}$: There exist constants $\delta_1, C > 0$ such that if $Q \in \mathrm{Sym}_0$ with $\mathrm{dist}(Q, \mathcal{N}) < \delta$ for $0 < \delta < \delta_1$, then

$$|g(Q) - g(\mathcal{R}(Q))| \leq C\,\mathrm{dist}(Q, \mathcal{N})\,. \tag{5}$$

3. The growth of g is slower than f, namely

$$|g(Q)| \;\leq\; C\,(1 + |Q|^4)\,, \tag{6}$$

$$|Dg(Q)| \;\leq\; C\,(1 + |Q|^3)\,, \tag{7}$$

for all $Q \in \mathrm{Sym}_0$ and a constant $C > 0$.

A physically motivated example that satisfies those assumptions [4, Prop. A.1] is for example given by

$$g(Q) \;=\; \frac{2}{3}s_* - Q_{33}\,. \tag{8}$$

Under these assumptions on f and g, it has been shown in [4, Prop. 2.4 and Prop. 2.6] that g acts on f as a perturbation in the following sense:

Proposition 2.1. *For $\xi, \eta > 0$ with $\xi \ll \eta$, there exists a smooth manifold $\mathcal{N}_{\eta,\xi} \subset \mathrm{Sym}_0$, diffeomorphic to $\mathcal{N}$ such that*

$$f(Q) + \frac{\xi^2}{\eta^2}g(Q) + \xi^2 C_0(\xi, \eta) \geq \gamma_2\,\mathrm{dist}^2(Q, \mathcal{N}_{\eta,\xi}) \tag{9}$$

for a constant $\gamma_2 > 0$. In addition, there exists a constant $C > 0$ such that

$$\sup_{Q \in \mathcal{N}_{\eta,\xi}} \mathrm{dist}(Q, \mathcal{N}) \leq C\frac{\xi^2}{\eta^2}\,. \tag{10}$$

Furthermore, there exists a unique $Q_{\infty,\xi,\eta} \in \mathcal{N}_{\eta,\xi}$ such that

$$Q_{\infty,\xi,\eta} = \operatorname*{argmin}_{Q \in \mathrm{Sym}_0} \frac{1}{\xi^2}f(Q) + \frac{1}{\eta^2}g(Q)\,,$$

given by $Q_{\infty,\xi,\eta} = s_{*,\xi^2/\eta^2}(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}\mathrm{Id})$, where $|s_{*,t} - s_*| \leq Ct$.

This shows that the constant C_0 in (1) can be chosen to be $C_0(\xi, \eta) = -\frac{1}{\xi^2}f(Q_{\infty,\xi,\eta}) - \frac{1}{\eta^2}g(Q_{\infty,\xi,\eta}) \geq 0$ and it also holds true that $C_0(\xi, \eta) \leq C\xi^2/\eta^4$.

Since $s_{*,\xi^2/\eta^2} \to s_{*,0} = s_*$ for $\xi, \eta \to 0$ in our regime, it is convenient to also introduce $Q_\infty := s_*(\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3}\mathrm{Id})$ which minimizes $\xi^{-2}f(Q) + \eta^{-2}g(Q)$ among $Q \in \mathcal{N}$.

2.2 Flat chains

In the statement of our main theorem, we will use the language of flat chains. We therefore give a quick overview of the most important results. For a detailed and complete presentation of flat chains and geometric measure theory, we refer to [24–26, 55, 65].

Polyhedral flat chains. Let G be an abelian group (written additively) with neutral element 0 and $|\cdot| : G \to [0,\infty)$ a function satisfying $|g| = 0$ if and only if $g = 0$, $|-g| = |g|$ and $|g+h| \leq |g| + |h|$ for all $g, h \in G$. In this paper, we are only concerned with the easiest case of $G = \mathbb{Z}_2$ and $|\cdot|$ the normal absolute value. For $n, k \in \mathbb{N}$, $k \leq n$, we denote by $\mathcal{P}^k$ the group of polyhedral chains of dimension k with coefficients in G i.e. the set of formal sums of compact, convex, oriented polyhedra of dimension k in $\mathbb{R}^n$ with coefficients in G together with the obvious addition. We identify a polyhedron that results from glueing along a shared face (and compatible orientation) with the sum of the individual polyhedra. Also, we identify a polyhedron of opposite orientation with the negative of the original polyhedron. An element $P \in \mathcal{P}^k$ can thus be written as

$$P \;=\; \sum_{i=1}^{p} g_i \sigma_i \,, \tag{11}$$

where $g_i \in G$ and σ_i are compact, convex, oriented polyhedra that can be chosen to be non-overlapping. Note that in our case of $G = \mathbb{Z}_2$, the non trivial coefficients g_i all equal 1 and that the orientational aspect of the above definition becomes trivial. The *boundary* $\partial\sigma$ of a polyhedron σ is the formal sum of the $(k-1)-$dimensional polyhedral faces of σ with the induced orientation and coefficient 1 under the above mentioned identifications. Note that $\partial(\partial\sigma) = 0$. We can linearly extend this operator to a boundary operator $\partial : \mathcal{P}^k \to \mathcal{P}^{k-1}$.

Mass and flat norm. For a polyhedral chain $P \in \mathcal{P}^k$ written as in (11), we define the *mass* $\mathbb{M}(P) = \sum_{i=1}^{p} |g_i| \mathcal{H}^k(\sigma_i)$ and the *flat norm* $\mathbb{F}(P)$ by

$$\mathbb{F}(P) \;=\; \inf\{\mathbb{M}(Q) + \mathbb{M}(R) \;:\; P = \partial Q + R \,, Q \in \mathcal{P}^{k+1} \,, R \in \mathcal{P}^k\} \,.$$

Obviously it holds $\mathbb{F}(P) \leq \mathbb{M}(P)$ and $\mathbb{F}(\partial P) \leq \mathbb{F}(P)$. One can show that $\mathbb{F}$ defines a norm on $\mathcal{P}^k$ [26, Ch. 2].

Flat chains and associated measures. We define the space of *flat chains* $\mathcal{F}^k$ to be the $\mathbb{F}-$completion of $\mathcal{P}^k$. The boundary operator ∂ extends to a continuous operator $\partial : \mathcal{F}^k \to \mathcal{F}^{k-1}$ and we still denote by $\mathbb{M}$ the largest lower semicontinuous extension of the mass which was defined on $\mathcal{P}^k$. Furthermore, one can show [26, Thm 3.1] that for all $A \in \mathcal{F}^k$

$$\mathbb{F}(A) \;=\; \inf\{\mathbb{M}(Q) + \mathbb{M}(R) \;:\; P = \partial Q + R \,, Q \in \mathcal{F}^{k+1} \,, R \in \mathcal{F}^k\} \,.$$

For a measurable set $X \subset \mathbb{R}^n$, we can define the *restriction* $A \llcorner X$ via an approximation by polyhedral chains [26, Ch. 4]. To each flat chain $A \in \mathcal{F}^k$, there exists an associated measure μ_A (see [26, Ch. 4]) such that for each μ_A-measurable set X, $A \llcorner X$ is a flat chain and $\mu_A(X) = \mathbb{M}(A \llcorner X)$. The *support* of A is denoted supp (A) and given (if it exists) by the smallest closed set X such that for every open set $U \supset X$ there exists a sequence of polyhedral chains $(P_j)_j$ approximating A and such that all cells of all P_j lie inside U. If A is of finite mass, then supp (A) = supp (μ_A) (see [26, Thm. 4.3]).

Cartesian products and induced mappings. In the case of finite mass flat chains A, B (or one of the two chains having finite mass *and* finite boundary mass), it is possible to define the product $A \times B$ (by polyhedral approximation), see e.g. [26, Sec. 6]. In particular, it is always possible to define $[0,1] \times B$. For $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ open sets and a Lipschitz function $f : U \to V$, one can define an induced mapping $f_\#$ on the level of flat chains, i.e. for a flat chain A supported in U, $f_\# A$ is a flat chain supported in V (see [26, Sec. 5] and [25, Sec. 2 and 3]).

Generic properties and Thom transversality theorem. A property of an object (such as a function or a set) that can be achieved by an arbitrarily small perturbation of the object is called generic. In this work we encounter two such properties: Two dimensional planes have the generic property of not containing a fixed single point (can be achieved by shifting normal to the plane). The second one is that smooth maps intersect a submanifold transversely. The latter will be used to apply Thom's transversality theorem [70] in the form given in [31, Thm. 2.7].

Deformations. In certain situations it is beneficial to approximate a flat chain A by a polyhedral chain P. The usual way to construct P is through pushing A onto cells of a grid. In this paper, a (cubic) grid of size h is understood to be a cell complex in $\mathbb{R}^3$ which consists of cubes of side length h. Then, every $A \in \mathcal{F}^k$ can be written as $A = P + B + \partial C$, where $P \in \mathcal{F}^k$ is a polyhedral chain, $B \in \mathcal{F}^k$ and $C \in \mathcal{F}^{k+1}$ satisfy the estimates $\mathbb{M}(P) \lesssim \mathbb{M}(A) + h\mathbb{M}(\partial A)$, $\mathbb{M}(\partial P) \lesssim \mathbb{M}(\partial A)$, $\mathbb{M}(B) \lesssim h\mathbb{M}(\partial A)$ and $\mathbb{M}(C) \lesssim h\mathbb{M}(A)$, see [26, Thm. 7.3].

Compactness. One point of importance from the perspective of calculus of variations are the compactness properties of flat chains whose mass and the mass of their boundary is bounded. We will use the result from [26, Cor. 7.5] which holds for coefficient groups G such that for all $M > 0$ the set $\{g \in G : |g| \leq M\}$ is compact. This is trivially true in our case where $G = \mathbb{Z}_2$. Let $K \subset \mathbb{R}^n$ be compact and $C_1 > 0$. Then the corollary states that

$$\{A \in \mathcal{F}^k \,:\, \operatorname{supp}(A) \subset K \text{ and } \mathbb{M}(A) + \mathbb{M}(\partial A) \leq C_1\}$$

is compact.

Rectifiability. Another aspect of flat chains concerns their regularity and if one can define objects originating in smooth differential geometry such as tangent spaces. It turns out that this can be achieved a.e. provided the flat chain is rectifiable. By definition, rectifiability of a flat chain $A \in \mathcal{F}^k$ means that there exists a countable union of $k-$dimensional C^1-submanifolds N of $\mathbb{R}^n$ such that $A = A \llcorner N$ [72, Sec. 1.2]. For finite groups G, finite mass $\mathbb{M}(A) < \infty$ implies rectifiability of A, see [26, Thm 10.1].

3 Statement of result

Our main result concerns the asymptotic behaviour of the energy $\mathcal{E}_{\eta,\xi}$ for $\eta, \xi \to 0$. Physically speaking, we consider the regime of large particles and weak magnetic fields, see [4, 27] for more discussion of the physical interpretation of our limit.

The liquid crystal occupies a region Ω outside a solid particle E, i.e. $\Omega = \mathbb{R}^3 \setminus E$. We assume the boundary of the particle $\mathcal{M} := \partial E$ to be sufficiently smooth for our analysis, that is we require $\mathcal{M}$ to be a closed, compact and oriented manifold of class at least $C^{1,1}$. The regularity will be needed to ensure that the outward unit normal field $\nu \in W^{1,\infty}$ or in other words $\mathcal{M}$ has bounded curvature. Furthermore, we assume that

$$\Gamma := \{\omega \in \mathcal{M} \,:\, \nu_3(\omega) = 0\}$$

is a C^2-curve (or a union thereof) in $\mathcal{M}$, see also Remark 3.2.

In order to make the minimization of the energy $\mathcal{E}_{\eta,\xi}$ non trivial, we impose the following boundary condition on $\mathcal{M}$:

$$Q \;=\; s_*\left(\nu \otimes \nu - \frac{1}{3}\mathrm{Id}\right) \quad \text{ on } \mathcal{M}\,. \tag{12}$$

Indeed, without (12) the minimizer of $\mathcal{E}_{\eta,\xi}$ would be the constant function $Q_{\eta,\xi,\infty}$. We define the class of admissible functions $\mathcal{A} := \{Q \in H^1(\Omega, \mathrm{Sym}_0) + Q_{\eta,\xi,\infty} \,:\, Q \text{ satisfies (12)}\}$. It is convenient

to define the energy $\mathcal{E}^{\mathcal{A}}_{\eta,\xi}$ for $Q \in H^1(\Omega, \mathbb{R}^{3\times 3}) + Q_{\eta,\xi,\infty}$ by

$$\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q) := \begin{cases} \mathcal{E}_{\eta,\xi}(Q) & \text{if } Q \in \mathcal{A}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

We also use the notation $\mathcal{E}_{\eta,\xi}(Q,U)$ (resp. $\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q,U)$) for the energy $\mathcal{E}_{\eta,\xi}$ (resp. $\mathcal{E}^{\mathcal{A}}_{\eta,\xi}$) of the function Q on the set U.

Theorem 3.1. *Suppose that*

$$\eta|\ln(\xi)| \to \beta \in (0,\infty) \quad \text{as } \eta \to 0\,. \tag{13}$$

Then $\eta\,\mathcal{E}^{\mathcal{A}}_{\eta,\xi} \to \mathcal{E}_0$ in a variational sense, where the limiting energy $\mathcal{E}_0$ is given by

$$\mathcal{E}_0(T,S) = 2s_*c_*E_0(\mathcal{M},\mathbf{e}_3) + 4s_*c_*\int_{\mathcal{M}} |\cos(\theta)|\,\mathrm{d}\mu_{T\llcorner\Omega} + \frac{\pi}{2}s_*^2\beta\mathbb{M}(S) + 4s_*c_*\mathbb{M}(T\llcorner\Omega) \tag{14}$$

for $(T,S) \in \mathcal{A}_0 := \{(T,S) \in \mathcal{F}^2 \times \mathcal{F}^1 \,:\, \partial T = S + \Gamma\}$ and where

$$E_0(\mathcal{M},\mathbf{e}_3) := \int_{\{\nu_3>0\}} (1-\cos(\theta))\,\mathrm{d}\omega + \int_{\{\nu_3\le 0\}} (1+\cos(\theta))\,\mathrm{d}\omega\,.$$

The variational convergence is to be understood in the following sense: Along any sequence $\eta_k, xi_k \to 0$ with $\eta_k|\ln(\xi_k)| \to \beta$ (not labelled in the following statements):

1. *Compactness and $\Gamma-$liminf: For any sequence $Q_{\eta,\xi} \in H^1(\Omega,\mathbb{R}^{3\times 3}) + Q_{\eta,\xi,\infty}$ such that there exists a constant $C > 0$ with*

$$\eta\,\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q_{\eta,\xi}) \le C\,, \tag{15}$$

there exists $(T,S) \in \mathcal{A}_0$, $\widetilde{Q_{\eta,\xi}} \in C^\infty(\Omega, \mathrm{Sym}_0)$ with $\lim_{\eta,\xi\to 0}\|Q_{\eta,\xi} - \widetilde{Q_{\eta,\xi}}\|_{H^1} = 0$, $\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(\widetilde{Q_{\eta,\xi}}, B_R) \le \mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q_{\eta,\xi}, B_R) + C_R$ and $Y_{\eta,\xi} \in Sym$ with $\|Y_{\eta,\xi}\| \to 0$ such that $T_{\eta,\xi} = (\widetilde{Q_{\eta,\xi}} - Y_{\eta,\xi})^{-1}(\mathcal{T})$, $S_{\eta,\xi} = (\widetilde{Q_{\eta,\xi}} - Y_{\eta,\xi})^{-1}(\mathcal{C})$ are smooth flat chains with

$$\partial T_{\eta,\xi} = S_{\eta,\xi} \;+\; \Gamma_{\eta,\xi}\,, \tag{16}$$

and, up to a subsequence, for any bounded measurable set $B \subset \Omega$

$$\lim_{\eta,\xi\to 0} \mathbb{F}(T_{\eta,\xi} - T, B) = 0\,, \quad \lim_{\eta,\xi\to 0} \mathbb{F}(S_{\eta,\xi} - S, B) = 0\,. \tag{17}$$

Here, $\Gamma_{\eta,\xi}$ is a smooth approximation of Γ which converges to Γ in $C^{1,1}$. Furthermore, we have

$$\liminf_{\eta\to 0} \eta\,\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q_{\eta,\xi}) \ge \mathcal{E}_0(T,S)\,. \tag{18}$$

2. *$\Gamma-$limsup: For any $(T,S) \in \mathcal{A}_0$, there exists a sequence $Q_{\eta,\xi} \in \mathcal{A}$ with $\|Q_{\eta,\xi}\|_{L^\infty} \le \sqrt{\frac{2}{3}}s_{\eta,\xi,*}$ such that (16),(17) hold and*

$$\limsup_{\eta\to 0} \eta\,\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q_{\eta,\xi}) \le \mathcal{E}_0(T,S)\,. \tag{19}$$

Remark 3.2. *1. We note that due to our assumptions $\beta \in (0,\infty)$, the global energy bound (15) can be reformulated as*

$$\mathcal{E}^{\mathcal{A}}_{\eta,\xi}(Q_{\eta,\xi}) \;\le\; \tilde{C}\,|\ln(\xi)|\,.$$

This reflects the classical behaviour of a logarithmic divergence of the energy close to singularities as already observed in earlier works e.g. in [9].

2. *If $Q_{\eta,\xi}$ is smooth enough (for example C^2) and verifies a Lipschitz estimate as in (23) for $n \sim \xi^{-1}$, we can choose $\widetilde{Q_{\eta,\xi}} = Q_{\eta,\xi}$ in the above Theorem. This is particularly true if $Q_{\eta,\xi}$ is a minimizer of (1). Indeed, from the Euler-Lagrange equations, one can deduce the regularity and the required estimate on the gradient [8, Lemma A.2].*

3. *The compactness claim holds for almost every $Y \in \mathrm{Sym}_0$ with $\|Y\|$ small enough. The norm converging to zero is needed to recover the condition $\partial T = S + \Gamma$, the stated energy densities on F, F^c, and the coefficient in front of $\mathbb{M}(T \llcorner \Omega)$.*

4. *Another possibility of introducing the set $S_{\eta,\xi}$ is by using the operator $\mathbf{S}$ defined in [15, 16]. It holds that $\mathbb{F}(\mathbf{S}_Y(Q_{\eta,\xi}) - S) \to 0$ as $\xi, \eta \to 0$.*

5. *The assumption of $\{\omega \in \mathcal{M} \,:\, \nu(\omega) \cdot \mathbf{e}_3 = 0\}$ being a C^2-curve is not very restrictive. In fact, this can already be achieved by a slight deformation of $\mathcal{M}$ which changes the energies $\mathcal{E}_{\eta,\xi}$ and $\mathcal{E}_0$ in a continuous way.*

Remark 3.3. 1. *We can express the energy (14) in a slightly different way by writing $\mu_{T \llcorner \mathcal{M}} = \chi_G \mathcal{H}^2 \llcorner \mathcal{M}$ for a mesurable set $G \subset \mathcal{M}$ and defining*

$$F = \{\omega \in \mathcal{M} \setminus G \,:\, \nu(\omega) \cdot \mathbf{e}_3 > 0\} \cup \{\omega \in \mathcal{M} \cap G \,:\, \nu(\omega) \cdot \mathbf{e}_3 \leq 0\}\,. \tag{20}$$

Then, (14) reads

$$\begin{aligned} \mathcal{E}_0(T,S) \;&=\; 2s_*c_* \int_F (1-\cos(\theta))\,\mathrm{d}\omega + 2s_*c_* \int_{\mathcal{M}\setminus F} (1+\cos(\theta))\,\mathrm{d}\omega \\ &+\; \frac{\pi}{2} s_*^2 \beta \mathbb{M}(S) + 4s_*c_* \mathbb{M}(T \llcorner \Omega)\,. \end{aligned} \tag{21}$$

2. *For convex particles E, there exists a orthogonal projection $\Pi : \Omega \to \mathcal{M}$. By convexity of E, we find that $\mathcal{E}_0(\Pi_\# T, \Pi_\# S) \leq \mathcal{E}_0(T,S)$, so that we can restrict ourselves to the case $T \llcorner \Omega = 0 = S \llcorner \Omega$. Using (20), we find that $\partial F = \Pi_\# S$ and (21) becomes*

$$\mathcal{E}_0(\Pi_\# T, \Pi_\# S) \;=\; 2s_*c_* \int_F (1-\cos(\theta))\,\mathrm{d}\omega + 2s_*c_* \int_{\mathcal{M}\setminus F} (1-\cos(\theta))\,\mathrm{d}\omega + \frac{\pi}{2} s_*^2 \beta \mathbb{M}(\partial F)\,.$$

In particular, (14) is a generalization of the limit energy $\mathcal{E}_0$ defined in [4].

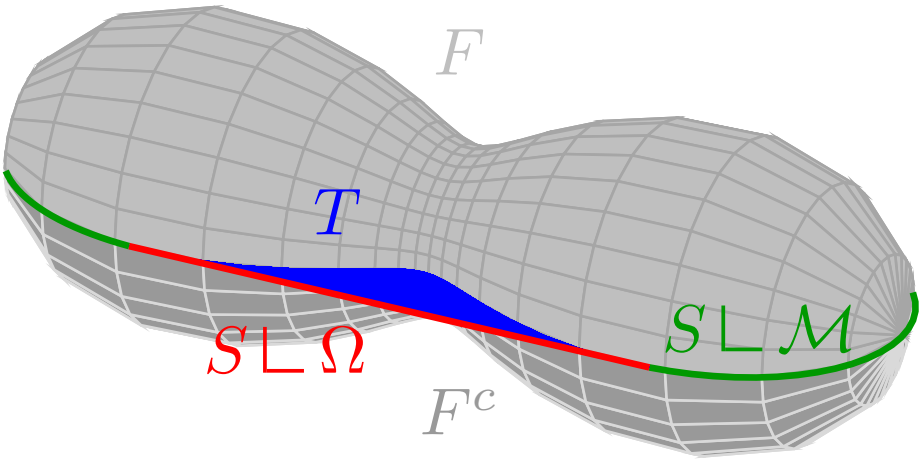

Figure 1: Illustration of flat chains T, S and the sets F, F^c appearing in the limit energy $\mathcal{E}_0$.

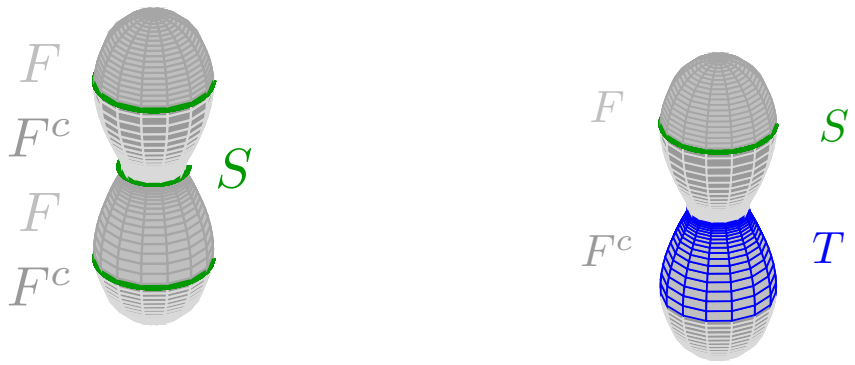

Figure 2: Expected minimizers of $\mathcal{E}_0$ for $\beta \ll 1$ (left) and intermediate β (right). For small β the line S has the tendency to stick to $\mathcal{M}$ and optimize F, thus no T appears. For larger β one may find a configuration as depicted on the right, i.e. the energy is decreased by joining two parts of S by a surface T glued to $\mathcal{M}$.

4 Compactness

The structure of this section is as follows. We regularize the sequence $Q_{\eta,\xi}$ in the first subsection. For this new sequence $Q_{\eta,\xi,n}$, we define a 2−chains $T_{\eta,\xi,n} \in \mathcal{F}^2$ and 1−chains $S_{\eta,\xi,n} \in \mathcal{F}^1$ such that $\partial T_{\eta,\xi,n} = S_{\eta,\xi,n}$ and we have bounds on the masses to get the existence of limit objects T and S with $\partial T = S$. This construction is carried out in steps in the subsections two, three and four, where we distinguish the case of $Q_{\eta,\xi,n}$ being almost uniaxial and $Q_{\eta,\xi,n}$ being biaxial, e.g. close to the boundary S. The passage to the limit is to happen in the last subsection.

4.1 Approximating sequence

This section is devoted to the definition of a sequence of smooth functions $Q_{\eta,\xi,n}$, replacing $Q_{\eta,\xi}$ in our analysis and proving the properties required for the estimates in the following chapters. More precisely, we need that

- the sequence $Q_{\eta,\xi,n}$ approximates $Q_{\eta,\xi}$ in H^1,
- verifies the same energy bound $\eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}) \leq \tilde{C}$ and
- the estimate $\mathrm{Lip}(Q_{\eta,\xi,n}) \leq C\, n$ holds.

For technical reasons, we are going to extend $Q_{\eta,\xi}$ into a small neighbourhood into the interior of E. Since $\mathcal{M}$ is compact and of class $C^{1,1}$, we can fix a small radius $\mathbf{r}_0 > 0$ such that $\mathcal{M}$ satisfies the inner ball condition for all radii $r \leq 2\mathbf{r}_0$. In particular, $\mathbf{r}_0$ is smaller than the minimal curvature radius of $\mathcal{M}$. For $x \in E$ such that $\mathrm{dist}(x, \mathcal{M}) < 2r_0$, define

$$Q_{\eta,\xi}(x) \;=\; s_* \left(\overline{\nu}(x) \otimes \overline{\nu}(x) - \frac{1}{3}\mathrm{Id} \right),$$

where $\overline{\nu}(x) = \nu(\Pi_{\mathcal{M}}(x))$, $\Pi_{\mathcal{M}}$ the orthogonal projection onto $\mathcal{M}$, is the obvious extension of the outward normal unit vector field ν in E.

Let $\Pi_R : \mathrm{Sym}_0 \to B_R(0) \subset \mathrm{Sym}_0$ be the orthogonal projection with $\sqrt{\frac{2}{3}} s_* \leq R < \infty$ to be fixed later. Furthermore, let $\varrho \in C_c^\infty(\mathbb{R}^3)$ be a convolution kernel with $0 \leq \varrho \leq 1$, $\varrho(x) = 0$ if

$|x| > 1$, $\int_{\mathcal{R}^3} \varrho(x)\,\mathrm{d}x = 1$ and $\|\nabla\varrho\|_\infty \le 1$. We set $\varrho_n(x) = n^3\varrho(nx)$. Then, for $n \ge r_0^{-1}$ we define $Q_{\eta,\xi,n}(x)$ for $x \in \overline{\Omega}$ as the convolution

$$Q_{\eta,\xi,n}(x) \;:=\; \left((\Pi_R Q_{\eta,\xi}) * \varrho_n\right)(x)\,. \tag{22}$$

Remark 4.1. *1. This definition also extends $Q_{\eta,\xi,n}$ into the interior of E up to distance r_0 to $\mathcal{M}$.*

2. Through the convolution, we change the boundary values of $Q_{\eta,\xi}$, i.e. $Q_{\eta,\xi,n}$ does not necessarily satisfy (12). We will see in Proposition 4.2 that $Q_{\eta,\xi,n} \to Q_{\eta,\xi}$ in H^1 for $n \to \infty$. This implies convergence of the traces in $H^{\frac{1}{2}}$ and thus pointwise a.e. on $\mathcal{M}$ for a subsequence. This is sufficient for the compactness and lower bound.

3. The convolution also changes that the approximations of T that we are about to construct will not end on Γ, but on a set Γ_n (which is generically again a line) in the neighbourhood of Γ and Γ_n converges uniformly to Γ as a consequence of the next proposition.

The following proposition shows that this sequence has indeed the desired properties.

Proposition 4.2. *The sequence $Q_{\eta,\xi,n}$ defined in (22) verifies:*

1. The functions $Q_{\eta,\xi,n}$ are smooth and there exists a constant $C > 0$ such that

$$\|\nabla Q_{\eta,\xi,n}\|_{L^\infty} \;\le\; C\,n\,. \tag{23}$$

2. We have convergence $Q_{\eta,\xi,n} \to Q_{\eta,\xi}$ in H^1 for $n \to \infty$, for all $\xi, \eta > 0$ fixed. Moreover, $Q_{\eta,\xi,n}|_{\mathcal{M}} \rightrightarrows Q_{\eta,\xi}|_{\mathcal{M}}$ for $n \to \infty$ uniformly on $\mathcal{M}$.

3. There exists $R \ge \sqrt{\frac{2}{3}}s_*$ and constants $C_1, C_2 > 0$ such that for all measurable sets $\Omega' \subset \Omega$ with $|\Omega'| < \infty$ the energy of $Q_{\eta,\xi,n}$ can be bounded as

$$\begin{aligned}\eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n},\Omega') \;\le\; C_1\left(1 + \frac{1}{\xi^2 n^2} + \frac{1+R^2}{n^2}\right)\eta\,\mathcal{E}_{\eta,\xi}\left(Q_{\eta,\xi}, B_{\frac{1}{n}}(\Omega')\cap\Omega\right) \\ + C_2\frac{\eta}{n} \;+\; C_R\left(\frac{|\Omega'|}{n^2}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi},\Omega')\right)^{\frac{1}{2}}\,,\end{aligned} \tag{24}$$

where $B_r(\Omega')$ denotes the $r-$neighbourhood around Ω'.

Proof. The smoothness of the functions $Q_{\eta,\xi,n}$ is clear by standard convolution arguments, since ϱ is smooth. The bound on the gradient follows from the computation

$$|\nabla Q_{\eta,\xi,n}(x)| \;\le\; \|\nabla\varrho_n\|_{L^\infty}\int_{B_1(x)} |\Pi Q_{\eta,\xi}(y)|\,\mathrm{d}y \;\le\; \frac{4}{3}\pi R\,n\,.$$

The H^1-convergence is again a well known result. The uniform convergence follows since $\mathcal{M}$ is compact and $Q_{\eta,\xi}|_{\mathcal{M}}$ is continuous since it verifies the boundary condition (12). It remains only to prove the energy bound. For this, we first note that

$$\begin{aligned}\int_{\Omega'} |\nabla Q_{\eta,\xi,n}|^2\,\mathrm{d}x \;&=\; \int_{\Omega'} |(\nabla Q_{\eta,\xi}) * \varrho_n|^2\,\mathrm{d}x \;\le\; \int_{\Omega'}\left|\int_{B_{\frac{1}{n}}(0)} (\nabla_x Q_{\eta,\xi})(x-y)n^3\varrho(ny)\,\mathrm{d}y\right|^2\,\mathrm{d}x \\ &\le\; |B_1|\int_{\Omega'}\int_{B_1(0)} |(\nabla_x Q_{\eta,\xi})(x-\frac{z}{n})|^2|\varrho(z)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &=\; |B_1|\int_{B_1(0)}|\rho(z)|^2\int_{\Omega'}|(\nabla_x Q_{\eta,\xi})(x-\frac{z}{n})|^2\,\mathrm{d}x\,\mathrm{d}z \\ &\le\; |B_1|\int_{B_{\frac{1}{n}}(\Omega')}|\nabla Q_{\eta,\xi}|^2\,\mathrm{d}x\,.\end{aligned}$$

Writing $B_{\frac{1}{n}}(\Omega') = (B_{\frac{1}{n}}(\Omega') \cap \Omega) \cup (B_{\frac{1}{n}}(\Omega') \cap E)$ and using that $|\nabla \nu(\Pi_{\mathcal{M}}(x))|$ is bounded, the integral can be further estimated by

$$\int_{B_{\frac{1}{n}}(\Omega')} \frac{1}{2}|\nabla Q_{\eta,\xi,n}|^2 \,\mathrm{d}x \;\leq\; \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, B_{\frac{1}{n}}(\Omega') \cap \Omega) + \frac{C}{n}\,. \tag{25}$$

Next, we compare the bulk energy of $Q_{\eta,\xi,n}$ and $Q_{\eta,\xi}$. To this goal, we use the triangle inequality to get

$$\int_{\Omega'} F(Q_{\eta,\xi,n}) - F(Q_{\eta,\xi}) \,\mathrm{d}x \;\leq\; \int_{\Omega'} |F(Q_{\eta,\xi,n}) - F(\Pi Q_{\eta,\xi})| \,\mathrm{d}x + \int_{\Omega'} F(\Pi Q_{\eta,\xi}) - F(Q_{\eta,\xi}) \,\mathrm{d}x\,, \tag{26}$$

where we used the notation $F(Q) = f(Q) + \frac{\xi^2}{\eta^2} g(Q) + \xi^2 C_0$. As in [4, Prop. 4.1] we fix R such that $F(Q) \geq F(\Pi Q)$ for all $Q \in \mathrm{Sym}_0$. Hence $\int_\Omega f(\Pi Q_{\eta,\xi}) - f(Q_{\eta,\xi}) \,\mathrm{d}x \leq 0$. It remains to estimate the first integral of the RHS of (26). We calculate

$$\begin{aligned}\int_{\Omega'} |Q_{\eta,\xi,n} - \Pi Q_{\eta,\xi}|^2 \,\mathrm{d}x \;&=\; \int_{\Omega'} |(\Pi Q_{\eta,\xi}) * \varrho_n - \Pi Q_{\eta,\xi}|^2 \,\mathrm{d}x \;\leq\; \frac{1}{n^2}\int_{\Omega'} |\nabla(\Pi Q_{\eta,\xi})|^2 \,\mathrm{d}x \\ &\leq\; \frac{1}{n^2}\int_{\Omega'} |\nabla Q_{\eta,\xi}|^2 \,\mathrm{d}x \;\leq\; \frac{C}{n^2}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, \Omega')\,.\end{aligned} \tag{27}$$

Combined with the $L^\infty-$bounds $|Q_{\eta,\xi,n}|, |\Pi Q_{\eta,\xi}| \leq R$ that gives

$$\int_{\Omega'} |f(Q_{\eta,\xi,n}) - f(\Pi Q_{\eta,\xi})| \,\mathrm{d}x \leq \frac{C_f}{n^2}(1 + R + R^2)\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, \Omega')\,. \tag{28}$$

It remains the estimate of $g(Q_{\eta,\xi,n}) - g(\Pi Q_{\eta,\xi})$. It is enough to consider the set $\Omega'' := \Omega' \cap \{x \in \Omega \,:\, |Q_{\eta,\xi,n}(x)| \geq \frac{1}{2}\sqrt{\frac{2}{3}} s_*\}$, on $\Omega' \setminus \Omega''$ we use Proposition 4.2 in [4]. By smoothness of g on $\{Q \in \mathrm{Sym}_0 \,:\, |Q| \in [\sqrt{\frac{2}{3}} s_*, R]\}$, we find

$$\begin{aligned}\int_{\Omega'} |g(Q_{\eta,\xi,n}) - g(\Pi Q_{\eta,\xi})| \,\mathrm{d}x \;&\leq\; C\frac{\xi^2}{\eta}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, \Omega') + C\|\nabla g\|_{L^\infty(\Omega'')} \int_{\Omega''} |Q_{\eta,\xi,n} - \Pi Q_{\eta,\xi}| \,\mathrm{d}x \\ &\leq\; C\frac{\xi^2}{\eta}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, \Omega') + C\|\nabla g\|_{L^\infty(\Omega'')} \left(|\Omega'| \frac{1}{n^2}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, \Omega')\right)^{\frac{1}{2}}\,.\end{aligned}$$

Combining this with (25) and (28), we substract $C\frac{\xi^2}{\eta}\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, \Omega')$ from both sides and divide by $1 - C\frac{\xi^2}{\eta^2}$ to get the estimate (24). □

Having established these properties of $Q_{\eta,\xi,n}$, we are able to identify the size and structure of the set where $Q_{\eta,\xi,n}$ is close to being uniaxial as stated in the next Lemma.

Lemma 4.3. *There exists a constant $C > 0$ such that for all $\delta > 0$, there exists a finite set $I \subset \Omega$ which satisfies*

1. *the following inclusions*

$$U_\delta \;\subset\; \bigcup_{x \in I} B_{\frac{\delta}{2n}}(x) \;\subset\; U_{\delta/2}\,, \tag{29}$$

   where $U_\delta := \{x \in \Omega \,:\, \mathrm{dist}(Q_{\eta,\xi,n}(x), \mathcal{N}) > \delta\}$,

2. *and*

$$\#I \;\leq\; C\,\frac{n^3}{\eta\, f^\delta_{\min}\, \delta^3}\left(\xi^2 + \frac{1}{n^2}\right)\,, \tag{30}$$

   where $f^\delta_{\min} = \min\{f(Q) \,:\, \mathrm{dist}(Q, \mathcal{N}) \geq \delta/2\}$.

Proof. Let $\delta > 0$ and $x_0 \in U_\delta$. By Lipschitz continuity of $Q_{\eta,\xi,n}$ (Proposition 4.2), we can deduce that for any $x \in B_{\frac{\delta}{2n}}(x_0)$ it holds

$$\operatorname{dist}(Q_{\eta,\xi,n}(x),\mathcal{N}) \;\geq\; \operatorname{dist}(Q_{\eta,\xi,n}(x_0),\mathcal{N}) - \|\nabla Q_{\eta,\xi,n}\|_\infty \frac{\delta}{2n} \;\geq\; \frac{\delta}{2}\,,$$

so that $x \in U_{\delta/2}$. From this, we get that

$$U_\delta \;\subset\; \bigcup_{x\in U_\delta} B_{\frac{\delta}{2n}}(x) \;\subset\; U_{\delta/2}\,.$$

By Vitali covering theorem, we find a subset $I \subset U_\delta$ with the same property and $B_{\frac{1}{3}\frac{\delta}{2n}}(x_i) \cap B_{\frac{1}{3}\frac{\delta}{2n}}(x_j) = \emptyset$ for $i \neq j$ and $x_i, x_j \in I$. Furthermore, using Proposition 4.2

$$\begin{aligned}
\frac{C\,\xi^2}{\eta} \;&\geq\; \int_\Omega f(Q_{\eta,\xi})\,\mathrm{d}x \;\geq\; \int_\Omega f(Q_{\eta,\xi,n})\,\mathrm{d}x - \frac{C}{\eta}\left(\xi^2 + \frac{1}{n^2}\right) \\
&\geq\; \int_{U_{\delta/2}} f(Q_{\eta,\xi,n})\,\mathrm{d}x - \frac{C}{\eta}\left(\xi^2 + \frac{1}{n^2}\right) \;\geq\; C\#I|B_{\frac{\delta}{2n}}|f_{\min} - \frac{C}{\eta}\left(\xi^2 + \frac{1}{n^2}\right) \\
&\geq\; C\#I\frac{\delta^3 f_{\min}}{n^3} - \frac{C}{\eta}\left(\xi^2 + \frac{1}{n^2}\right)\,,
\end{aligned}$$

where we used that $f \geq f_{\min} > 0$ on $U_{\delta/2}$. From this it follows that

$$\#I \;\leq\; C\frac{n^3}{\eta\, f_{\min}\delta^3}\left(\xi^2 + \frac{1}{n^2}\right)\,.$$

□

In [4] a similar result was obtained using a regularization related to the energy and using the Euler-Lagrange equation to derive the Lipschitz continuity. This approach would also work in the new setting and one could obtain Lemma 4.3 with $n = \xi^{-1}$. However, our new approach has two major advantages: The first one is that the proofs are shorter and more elegant. The second (and main) reason is that we now have control over the gradient of the approximation as well, contrary to the approach in [4].

From (30) it follows that the volume of the union of balls in (29) converges to zero for $\eta, \xi \to 0$ and $n \sim \xi^{-1}$. The same holds true for the union of the surfaces of those balls. Note however that the sum of the diameters is not bounded and diverges like η^{-1}. With the tool developed in [9] and used in [4,14] it would be possible to derive a bound, namely the sum of diameters can be shown to be bounded.

4.2 Definition of the line singularity

The goal of this section is to define a $1-$chain $S_{\eta,\xi,n}$ of finite length which satisfies the compactness properties announced in 3.1. The necessary analysis has already been carried out in [15, 16] but for the reader's convenience we recall the important steps and results.

Following Section 3 in [15], we note that there exists a smooth retraction $\mathcal{R} : \mathrm{Sym}_0 \setminus \mathcal{C} \to \mathcal{N}$, where $\mathcal{C}$ is the cone of biaxial $Q-$tensors seen as a smooth simplicial complex of codimension 2 in Sym_0. Evoking Thom's transversality theorem, one can assume that $Q_{\eta,\xi,n} - Y$ is transverse to all cells of $\mathcal{C}$ for almost every $Y \in \mathrm{Sym}_0$. Subdividing the preimages of the cells under the map $Q_{\eta,\xi,n} - Y$ if necessary, $(Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{C})$ defines a smooth, simplicial, finite complex of codimension 2 which we call $S_{\eta,\xi,n}$. Note that $S_{\eta,\xi,n}$ depends on the choice of Y.

The relevant estimates on $S_{\eta,\xi,n}$ are formulated in Theorem C and Section 4 in [16]:

Theorem 4.4. *There exists a finite mass chain S such that one can choose a subsequence $S_{\eta,\xi,n}$ (not relabelled) and $\alpha > 0$ with*

$$\mathbb{F}(S_{\eta,\xi,n} - S) \;\to\; 0 \qquad \textit{for almost every } Y \in B_\alpha(0)\,.$$

Furthermore, for any open subset $U \subset \mathbb{R}^3$ it holds

$$\liminf_{\xi,\eta\to 0} \eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, U\cap\Omega) \;\geq\; \beta\, \mathbb{M}(S \llcorner U)\,.$$

In our situation, by construction of $Q_{\eta,\xi,n}$ and for $Y \in B_\alpha(0)$ (α small enough) it holds that

$$(Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{C}) \;\subset\; U_\delta \;\subset\; \bigcup_{x\in I} B_{\frac{\delta}{2n}}(x)\,.$$

Hence $\operatorname{supp}(S_{\eta,\xi,n}) \subset \bigcup_{x\in I} B_{\frac{\delta}{2n}}(x)$ and in view of the lower bound in Theorem 4.4 we deduce that the energy coming from $S_{\eta,\xi,n}$ in U is already contained in $U \cap \bigcup_{x\in I} B_{\frac{\delta}{2n}}(x)$.

4.3 Construction of T and estimates for Q close to uniaxial

In this subsection we carry out the first steps to define the $2-$chain T. We start by defining

$$\mathcal{T} \;:=\; \{Q \in \mathrm{Sym}_0 \;:\; s > 0\,, 0 \leq r < 1\,, n_3 = 0\}\,, \tag{31}$$

where $r, s, \mathbf{n}$ are defined as in (3). From this we want to define $T_{\eta,\xi,n}$ close to $Q^{-1}_{\eta,\xi,n}(\mathcal{T})$. As carried out in [15] and described in Subsection 4.2, for almost every Y the set $(Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{T})$ is in fact a smooth finite complex. In Lemma 4.6, we show that in addition for a.e. $Y \in \mathrm{Sym}_0$, the definition

$$T_{\eta,\xi,n} \;:=\; (Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{T})$$

allows to control the area in regions where $Q_{\eta,\xi,n}$ is close to being uniaxial. Since both the constructions of $S_{\eta,\xi,n}$ and $T_{\eta,\xi,n}$ are valid for a.e. Y, we can choose the same Y and hence $\partial T_{\eta,\xi,n} \cap \Omega = S_{\eta,\xi,n}$. In parts of Ω where $Q_{\eta,\xi,n}$ is far from $\mathcal{N}$, e.g. close to $S_{\eta,\xi,n}$, we need to modify $T_{\eta,\xi,n}$. This will be the subject of the next subsection.

At first, we recall the (intuitively obvious) result that $\mathcal{T}$ is well behaved close to $\mathcal{N}$ in the sense that the level sets $\{n_3 = s\}$ for s small have a similar $\mathcal{H}^4-$volume as $\mathcal{T}$. This can be interpreted as control on the curvature of $\mathcal{T} \cap \mathcal{N}$.

Lemma 4.5. *There exists $\alpha_0, \alpha_1, C > 0$ such that for $Q \in \mathrm{Sym}_0$, $\operatorname{dist}(Q, \mathcal{N}) \leq \alpha_0$ and $\alpha \in (0, \alpha_1)$ it holds that*

$$\lim_{s\to 0} \mathcal{H}^4(\{Y \in B_\alpha(0) \;:\; n_3(Q - Y) = s\}) \;=\; \mathcal{H}^4(B_\alpha(Q) \cap \mathcal{T})\,.$$

In the smooth case this lemma follows as in [51, Lemma 3], however we give a proof here for completeness.

Proof. The parameter α_0 needs to be small enough to avoid problems far from $\mathcal{N}$ due to the non-smoothness of $\mathcal{T}$ at the singularity $0 \in \mathrm{Sym}_0$. So we choose $0 < \alpha_0 < \frac{1}{8}\operatorname{dist}(0, \mathcal{N})$. To avoid dealing with the topology of the sets involved, we pick $0 < \alpha_1 < \frac{1}{8}\operatorname{diam}(\mathcal{N})$. Hence, $B_\alpha(Q) \cap T$ is diffeomorphic to a $4-$dimensional ball.

We define $\phi(Y) := n_3(Y)$ for $Y \in B_\alpha(Q)$ and note that $B_\alpha(Q) \cap \mathcal{T} = \phi^{-1}(0)$. One can calculate $D\phi(Q) = D_Y n_3(Q)$ and by the calculations in the proof of Lemma A.4, $D\phi(Q)$ is parallel to the normal vector N_Q. Hence, for α_0, α_1 small enough $\operatorname{rank}(D\phi(Q)) = 1$. By the implicit function theorem, there exists a function ψ such that $\phi(Q + y + \psi(y)N_Q) = s$ for $y \in B_\alpha(0)$ with $y \perp N_Q$. Furthermore, $D\psi(Q + y) = (Dn_3(Q + y + \psi(y)) : N_Q)^{-1}(Dn_3(Q + y + \psi(y)))$. Since Dn_3 is parallel to N_Q in first order, for each $\epsilon > 0$ and $s_\epsilon > 0$ small enough it holds that $1 - \epsilon \leq (\det(D\psi^\top D\psi))^{\frac{1}{2}} \leq 1 + \epsilon$ on $\{\phi \leq s_\epsilon\}$. With a change of variables this becomes for $s \leq s_\epsilon$

$$(1-\epsilon)\int_{\{\phi=s\}} \psi(y)\, \mathrm{d}y \;\leq\; \mathcal{H}^4(\phi^{-1}(0)) \;\leq\; (1+\epsilon)\int_{\{\phi=s\}} \psi(y)\, \mathrm{d}y\,.$$

In the limit $s \to 0$ we obtain $(1+\epsilon)\int_{\{\phi=0\}} \psi(y)\, \mathrm{d}y \leq (1+\epsilon)^2\mathcal{H}^4(\phi^{-1}(0))$. Analogously $(1-\epsilon)\int_{\{\phi=0\}} \psi(y)\, \mathrm{d}y \geq (1-\epsilon)^2\mathcal{H}^4(\phi^{-1}(0))$. Since $\epsilon > 0$ was arbitrary, the claim follows. □

For $\delta > 0$, we introduce the set $A_\delta \subset \Omega$ in which $Q_{\eta,\xi,n}$ is close to being uniaxial as

$$A_\delta \;:=\; \{x \in \Omega \,:\, \mathrm{dist}(Q_{\eta,\xi,n}(x), \mathcal{N}) < \delta\}\,. \tag{32}$$

The next lemma shows that (in average) the $\mathcal{H}^2-$measure of $(Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{T})$ that lies in A_δ is controlled by the energy.

Lemma 4.6. *There exists $\alpha_0, \delta_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$, $\delta \in (0, \delta_0)$ one can find a constant $C > 0$ such that*

$$\int_{B_\alpha(0)} \mathcal{H}^2(A_\delta \cap (Q_{\eta,\xi,n} - Y)^{-1}(\mathcal{T})\}) \,\mathrm{d}Y \;\le\; C\,\eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, A_\delta)\,. \tag{33}$$

Proof. Let $\alpha, \delta > 0$ small enough such that for $Y \in B_\alpha(0)$, the map $Q \mapsto n_3(Q - Y)$ is smooth on $\{Q \in \mathrm{Sym}_0 \,:\, \mathrm{dist}(Q, \mathcal{N}) < \delta\}$. Let A_δ be defined as in (32). In order for the map $x \mapsto n_3(Q_{\eta,\xi,n}(x) - Y)$ to be well defined, we need to restrict ourselves to a simply connected subset of A_δ. For this, take $x_0 \in A_\delta$ and $r > 0$ such that $A_\delta \cap B_r(x_0)$ is simply connected. We carry out the analysis on $A_\delta \cap B_r(x_0)$, noting that we can cover A_δ by such balls to find the estimate (33). With $x_0 \in A_\delta$ and $r > 0$ fixed as described, we can calculate

$$\begin{aligned}
&\int_{B_\alpha(0)} \mathcal{H}^2(B_r(x_0) \cap A_\delta \cap (Q_{\eta,\xi,n}(x) - Y)^{-1}(\mathcal{T})\}) \,\mathrm{d}Y \\
&\quad= \int_{B_\alpha(0)} |D\chi_{\{x\in\Omega | n_3(Q_{\eta,\xi,n}(x)-Y)>0\}}|(B_r(x_0) \cap A_\delta) \,\mathrm{d}Y \\
&\quad\le \liminf_{\epsilon\to 0} \int_{B_\alpha(0)} \int_{B_r(x_0)\cap A_\delta} |\nabla_x (h_\epsilon \circ n_3 \circ (Q_{\eta,\xi,n} - Y))(x)| \,\mathrm{d}x \,\mathrm{d}Y \\
&\quad= \liminf_{\epsilon\to 0} \int_{B_\alpha(0)} \int_{B_r(x_0)\cap A_\delta} |h'_\epsilon(n_3(Q_{\eta,\xi,n}(x) - Y))\nabla_Q n_3(Q_{\eta,\xi,n}(x) - Y) : \nabla_x Q(x)| \,\mathrm{d}x \,\mathrm{d}Y\,,
\end{aligned}$$

where $h_\epsilon \in C^1(\mathbb{R}, [0,1])$ is an approximation of the Heaviside function, i.e. $h_\epsilon(x) = 0$ for $x \le 0$, $h_\epsilon(x) = 1$ for $x \ge \epsilon$ and $h'_\epsilon > 0$ on $(0, \epsilon)$. The above inequality is then just the lower semi continuity of the total variation. With the identity $h'_\epsilon(n_3(Q_{\eta,\xi,n}(x) - Y))\nabla_Q n_3(Q_{\eta,\xi,n}(x) - Y) = -\nabla_Y(h_\epsilon \circ n_3 \circ (Q_{\eta,\xi,n}(x) - Y))$ and the Fubini theorem we can rewrite

$$\begin{aligned}
&\int_{B_\alpha(0)} \int_{B_r(x_0)\cap A_\delta} |h'_\epsilon(n_3(Q_{\eta,\xi,n}(x) - Y))\nabla_Q n_3(Q_{\eta,\xi,n}(x) - Y) : \nabla_x Q_{\eta,\xi,n}(x)| \,\mathrm{d}x \,\mathrm{d}Y \\
&\quad\le \int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \int_{B_\alpha(0)} |\nabla_Y (h_\epsilon \circ n_3 \circ (Q_{\eta,\xi,n} - Y))| \,\mathrm{d}Y \,\mathrm{d}x \\
&\quad= \int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \int_0^1 \mathcal{H}^4(\{Y \in B_\alpha(0) \,:\, h_\epsilon(n_3(Q_{\eta,\xi,n}(x) - Y)) = s\}) \,\mathrm{d}s \,\mathrm{d}x \\
&\quad= \int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \int_0^1 \mathcal{H}^4(\{Y \in B_\alpha(0) \,:\, n_3(Q_{\eta,\xi,n}(x) - Y) = h_\epsilon^{-1}(s)\}) \,\mathrm{d}s \,\mathrm{d}x\,,
\end{aligned}$$

where we also used the coarea formula. By Lemma 4.5 in the liminf $\epsilon \to 0$ this equals

$$\begin{aligned}
&\liminf_{\epsilon\to 0} \int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \int_0^1 \mathcal{H}^4(\{Y \in B_\alpha(0) \,:\, n_3(Q_{\eta,\xi,n}(x) - Y) = h_\epsilon^{-1}(s)\}) \,\mathrm{d}s \,\mathrm{d}x \\
&\qquad= \int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \,\mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T}) \,\mathrm{d}x
\end{aligned}$$

by translation invariance of $\mathcal{H}^4$. Applying the elementary inequality $2ab \le a^2 + b^2$ we get

$$\begin{aligned}
&\int_{B_r(x_0)\cap A_\delta} |\nabla Q_{\eta,\xi,n}| \,\mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T}) \,\mathrm{d}x \\
&\qquad\le \int_{B_r(x_0)\cap A_\delta} \frac{\eta}{2}|\nabla Q_{\eta,\xi,n}|^2 \;+\; \frac{1}{2\eta}\mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T})^2 \,\mathrm{d}x\,.
\end{aligned}$$

The Dirichlet term appears in the energy, so it remains to estimate $\mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T})^2$ in terms of $g(Q_{\eta,\xi,n})$. We first note that $\mathcal{T} \cap B_\alpha(Q_{\eta,\xi,n}(x)) = \emptyset$ if $\operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{T}) > \alpha$ and since $\operatorname{dist}(Q_{\eta,\xi,n}, \mathcal{N}) < \delta$ we have $\mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T}) \leq C_\delta \alpha^4$ by Proposition A.5. Hence, we get

$$\int_{B_r(x_0)\cap A_\delta} \mathcal{H}^4(B_\alpha(Q_{\eta,\xi,n}) \cap \mathcal{T})^2 dx \;\leq\; (C_\delta \alpha^4)^2 |B_r(x_0) \cap A_\delta \cap \{x \in \Omega \,:\, \operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{T}) < \alpha\}| \,.$$

For $x \in A_\delta \cap \{x \in \Omega \,:\, \operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{T}) < \alpha\}$ we can estimate

$$\begin{aligned} g(Q_{\eta,\xi,n}(x)) \;&\geq\; g(\mathcal{R}(Q_{\eta,\xi,n}(x))) - C_g \operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{N}) \;\geq\; \sqrt{\frac{3}{2}}(1 - n_3^2(Q_{\eta,\xi,n}(x))) - C_g\delta \\ &\geq\; \sqrt{\frac{3}{2}}(1 - C_{\mathcal{T}}\alpha) - C_g\delta \;\geq\; G > 0 \end{aligned}$$

for $\alpha, \delta \ll 1$ small enough. Hence,

$$G|B_r(x_0) \cap A_\delta \cap \{x \in \Omega \,:\, \operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{T}) < \alpha\}| \;\leq\; \int_{B_r(x_0)\cap A_\delta} g(Q_{\eta,\xi,n}) \, \mathrm{d}x \,.$$

□

We remark that although Lemma 4.6 control the size for a.e. *fixed* $Y \in B_\alpha(0)$, but degenerates with α. Hence it does not provide a uniform bound in α allowing to pass to the limit $Y \to 0$. A bound independent of α will be derived in the section on the lower bound.

4.4 Estimates near singularities

At points $x \in \Omega$ where $\operatorname{dist}(Q_{\eta,\xi,n}(x), \mathcal{N}) > \delta$, the estimates we derived in the previous subsection are no longer available and we need new tools to bound the mass of $T_{\eta,\xi,n}$. We are concerned with two different cases: The first case is the one of $x \in T_{\eta,\xi,n}$ far from the boundary $S_{\eta,\xi,n}$. We can simply "cut out" those pieces and replace them by parts of surfaces of spheres which are controlled in mass. This will be made precise using Lemma 4.3. The second case is more challenging. We will modify $T_{\eta,\xi,n}$ close to the boundary $S_{\eta,\xi,n}$ by using a construction similar to the one used in the deformation theorem (see Lemma 4.8). This will allow us to express the mass of the modified $2-$chain in terms of the surface of cubes and Lemma 4.3 permits us to control the number of such cubes.

Lemma 4.7 (Deformation in the interior)**.** *Let $I^{\mathrm{int}} \subset I$ be the subset of points $x_0 \in I$ such that $\operatorname{dist}(x_0, S_{\eta,\xi,n}) > \frac{\delta}{2n}$ and $\operatorname{dist}(x_0, T_{\eta,\xi,n}) < \frac{\delta}{2n}$. Then, there exists a flat $2-$chain $\widetilde{T}$ with values in $\pi_1(\mathcal{N})$ and support in $B^{\mathrm{int}} := \bigcup_{x\in I^{\mathrm{int}}} \overline{B_{\frac{\delta}{2n}}(x)}$ such that*

1. $\partial\widetilde{T} = \partial(T_{\eta,\xi,n} \llcorner B^{\mathrm{int}})$,
2. *and* $\mathbb{M}(\widetilde{T}) \lesssim \frac{n}{\eta}\left(\xi^2 + \frac{1}{n^2}\right)$.

Proof. Since $B^{\mathrm{int}} \cap \operatorname{supp}(T_{\eta,\xi,n}) \neq \emptyset$ and $B^{\mathrm{int}} \cap \operatorname{supp}(S_{\eta,\xi,n}) = \emptyset$ we know that $\emptyset \neq \partial(T_{\eta,\xi,n} \llcorner B^{\mathrm{int}}) \subset \partial B^{\mathrm{int}}$. Furthermore, since $\partial^2 = 0$ it follows that $\partial(T_{\eta,\xi,n} \llcorner B^{\mathrm{int}})$ consists of closed curves and divides $\partial B^{\mathrm{int}}$ into domains. Let D be the set of these domains. Now pick any subset $D' \subset D$ such that $\partial\left(\bigcup_{U\in D'} U\right) = \partial(T_{\eta,\xi,n} \llcorner B)$. We define $\widetilde{T} := \sum_{U\in D'} [U]$. Then, by definition $T_{\eta,\xi,n} \llcorner B^{\mathrm{int}}$ and $\widetilde{T}$ have the same boundary and since $\widetilde{T} \subset \partial B^{\mathrm{int}}$ we also have

$$\mathbb{M}(\widetilde{T}) \;\leq\; \mathbb{M}(\partial B^{\mathrm{int}}) \;\leq\; \sum_{x\in I^{\mathrm{int}}} \mathbb{M}(\partial B_{\frac{\delta}{2n}}) \;\leq\; \#I \, 4\pi \, \frac{\delta^2}{n^2} \;\lesssim\; \frac{n}{\eta}\left(\xi^2 + \frac{1}{n^2}\right) .$$

□

At the boundary we cannot remove a disk without the risk of creating new boundary which might not be controlled, so another method has to be used. The idea is the following: Take a cube K of size $\frac{\delta}{n}$ which contains a part of the singular line $S_{\eta,\xi,n}$ and intersects with $T_{\eta,\xi,n}$. We then modify (deform) the "surface" connecting $T_{\eta,\xi,n} \cap \partial K$ and $S_{\eta,\xi,n} \cap K$ by pushing it onto a part of ∂K (see also Figure 3). The result is a modified $T_{\eta,\xi,n}$ with the same boundary as before and the surface inside the cube is controlled by the surface area of K and the length of the singular line. We point out that this procedure and its proof is closely related to the deformation theorem (for flat chains) (see [71], Chapter 5 in [25], Theorem 7.3 in [26] and Chapter 4.2 in [24]) but differs in some details so that we give a full proof here.

Lemma 4.8 (Deformation close to the boundary)**.** *Let $I^{\mathrm{bdry}} \subset I$ be the subset of points $x_0 \in I$ such that $\operatorname{dist}(x_0, S_{\eta,\xi,n}) < \frac{\delta}{2n}$ and $\operatorname{dist}(x_0, T_{\eta,\xi,n}) < \frac{\delta}{2n}$. Then there exists a flat $2-$chain $\widetilde{T}$ with values in $\pi_1(\mathcal{N})$ and support in B^{bdry} verifying $\bigcup_{x \in I^{\mathrm{bdry}}} \overline{B_{\frac{\delta}{2n}}(x)} \subset B^{\mathrm{bdry}}$ such that*

1. $\partial \widetilde{T_{\eta,\xi,n}} = \partial (T_{\eta,\xi,n} \llcorner B^{\mathrm{bdry}})$,
2. *and*

$$\mathbb{M}(\widetilde{T_{\eta,\xi,n}}) \;\lesssim\; \frac{n}{\eta}\left(\xi^2 + \frac{1}{n^2}\right). \tag{34}$$

Proof. For the sake of readability we drop the dependences on ξ, η, n in the notation of this proof. Covering S with a cubic grid of size $h = \frac{\delta}{n}$ such that S is in a general position, we can assume that the center x_K of all cubes K that contain parts of S does not intersects S or T, i.e. $x_K \notin \operatorname{supp}(T), \operatorname{supp}(S)$. Indeed, this is possible S intersects only a finite number of cubes according to Lemma 4.3. Let G be the set of those cubes and X the set of its centers.

Step 1 (Construction and properties of the retraction map). Let $K \in G$ be a cube and let $x_K \in X$ be its center. Let P be the central projection onto ∂K originating in x_K. We define a homotopy $\Phi : [0,1] \times (K \setminus \{x_K\}) \to K$ between the identity on K and P by simply taking $\Phi(t,x) = (1-t)x + tPx$. Note that by definition this homotopy is relative to ∂K, i.e. $\Phi(t,x) = x$ for all $t \in [0,1]$ and $x \in \partial K$. Furthermore, for all $x \in K \setminus \{x_k\}$ and $t \in [0,1]$ it holds

$$\operatorname{dist}(\Phi(t,x), x_K) \;\geq\; \operatorname{dist}(x, x_K). \tag{35}$$

Since $|\partial_t \Phi(t,x)| = |-x + Px| \leq \sqrt{3}h$ and by (35) we deduce that Φ is locally Lipschitz continuous and $\operatorname{Lip}(\Phi(t,x)) \leq C\, h \operatorname{dist}(x, x_K)^{-1}$. Since Φ is relative to ∂K we can glue together all those functions defined on the cubes $K \in G$ with the identity on cubes $K \notin G$ to get a function Φ defined everywhere in $\mathbb{R}^3 \setminus X$.

Step 2 (Intermediate estimate). In this step we want to show that if we allow for a small translation of the chain S, then the mass of $\Phi_\#([0,1] \times S)$ can be bounded by $\mathbb{M}(S)$ times the size of the cube h, up to a constant.

Applying Corollary 2.10.11 in [24] (or Section 2.7 in [25]) we get as in [71, Lemma 2.1]

$$\begin{aligned}\mathbb{M}(\Phi_\#([0,1] \times S)) \;&\leq\; \|\mathrm{Id} - P\|_\infty \int_{\mathbb{R}^3} \sup_{t \in [0,1]} \operatorname{Lip}(\Phi(t,x))\, \mathrm{d}\mu_S(x) \\ &\leq\; C\, h^2 \int_{\mathbb{R}^3} \operatorname{dist}(x, X)^{-1}\, \mathrm{d}\mu_S(x).\end{aligned}$$

Taking the mean over translation by a vector $y \in [0,1]^3$, we arrive at

$$\begin{aligned}\int_{[0,1]^3} \mathbb{M}(\Phi_\#([0,1]\times(\tau_{hy}S))\,\mathrm{d}y &= C\,h^2 \int_{[0,1]^3}\int_{\mathbb{R}^3} \mathrm{dist}(x,X)^{-1}\,\mathrm{d}\mu_{\tau_{hy}S}(x)\,\mathrm{d}y\\ &= C\,h^2 \int_{[0,1]^3}\int_{\mathbb{R}^3} \mathrm{dist}(x+hy,X)^{-1}\,\mathrm{d}\mu_S(x)\,\mathrm{d}y\\ &= C\,h^2 \int_{\mathbb{R}^3}\int_{[0,1]^3} \mathrm{dist}(x+hy,X)^{-1}\,\mathrm{d}y\,\mathrm{d}\mu_S(x)\\ &\le C\,h \int_{\mathbb{R}^3} \mathrm{d}\mu_S(x)\\ &= C\,h\,\mathbb{M}(S)\,.\end{aligned}$$

Hence, we can assume that S is in a position such that

$$\mathbb{M}(\Phi_\#([0,1]\times S)) \;\le\; C\,h\,\mathbb{M}(S)\,. \tag{36}$$

Step 3 (Definition of $\tilde{T}$). We define

$$\tilde{T} \;:=\; \partial(\Phi_\#([0,1]\times T)) - T\,.$$

Considering a cube $K \in G$, one can think of this construction as the boundary of the three dimensional object created by filling the space between T and its projection onto ∂K according to Step 1 and then removing the original part T. Another but equivalent point of view is to take $\tilde{T}$ as all the points along the path created by projecting $T \llcorner \partial K$, S together with the projection $P_\#(T)$, see also Figure 3. Indeed, one can calculate for $K \in G$

$$\begin{aligned}\partial(\Phi_\#([0,1]\times(T\llcorner K))) &= \Phi_\#((\partial[0,1])\times(T\llcorner K)) + \Phi_\#([0,1]\times(\partial T)\llcorner K) + \Phi_\#([0,1]\times T\llcorner(\partial K))\\ &= P_\#(T\llcorner K) - (\mathrm{Id}_K)_\#(T) + \Phi_\#([0,1]\times(S\llcorner K)) + \Phi_\#([0,1]\times T\llcorner(\partial K))\,.\end{aligned}$$

Thus, we have the formula

$$\tilde{T}\llcorner K \;=\; P_\#(T\llcorner K) \;+\; \Phi_\#([0,1]\times(S\llcorner K)) \;+\; \Phi_\#([0,1]\times T\llcorner(\partial K))\,.$$

Since $P_\#(T\llcorner K) + \Phi_\#([0,1]\times T\llcorner(\partial K)) \subset \partial K$ from which we derive the bound on the mass of $\tilde{T}$

$$\mathbb{M}(\tilde{T}\llcorner K) \;\le\; \mathbb{M}(\partial K) \;+\; \mathbb{M}(\Phi_\#([0,1]\times(S\llcorner K))) \;\le\; 6\,h^2 \;+\; C\,h\,\mathbb{M}(S\llcorner K)\,, \tag{37}$$

where we also used the estimate (36) on K of Step 2. On all cubes $K \notin G$, $\tilde{T}\llcorner K = 0$, so that we find $\mathrm{supp}\,(\tilde{T}) \subset \bigcup_{K\in G} \overline{K}$. Adding cubes if necessary, we can achieve the inclusion $\bigcup_{x\in I^{\mathrm{bdry}}} \overline{B_{\frac{\delta}{2n}}(x)} \subset B^{\mathrm{bdry}}$. Since $\partial\circ\partial = 0$, the boundary of $\tilde{T}$ coincides with ∂T. Since all calculations in Step 3 were local and Φ is relative to the boundaries of the cubes, (34) follows from summing up (37) over all cubes $K \in G$. □

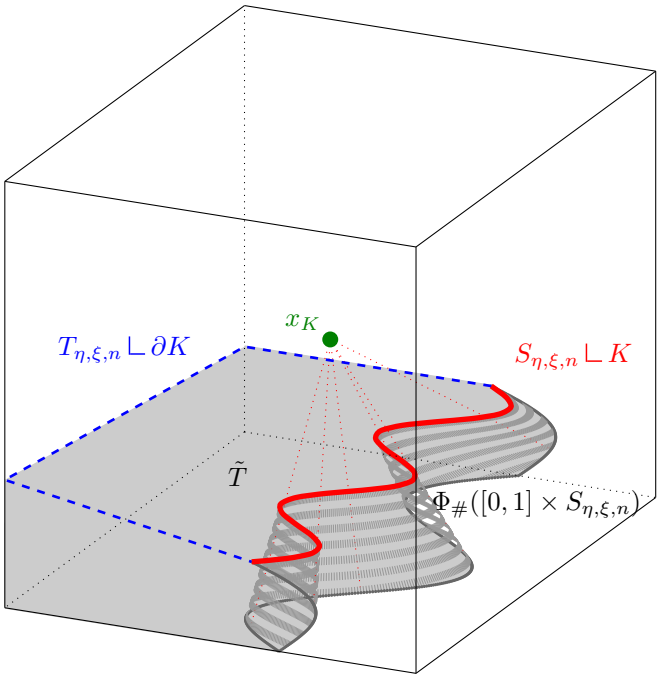

Figure 3: Construction near the boundary inside a cube: The newly created area (grey) is controlled by the surface of the cube and the length of the line singularity (red).

As a direct consequence of Lemma 4.6, Lemma 4.7 and Lemma 4.8 we have the following corollary:

Corollary 4.9. *There exists a flat $2-$chain $\widetilde{T_{\eta,\xi,n}}$ with values in $\pi_1(\mathcal{N})$ such that*

1. $\partial\widetilde{T_{\eta,\xi,n}} = S_{\eta,\xi,n}$,
2. *and for all $x_0 \in \Omega$ and $R > 0$*

$$\mathbb{M}(\widetilde{T_{\eta,\xi,n}} \llcorner B_R(x_0)) \;\lesssim\; \eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, B_R(x_0)) + \frac{n}{\eta}\left(\xi^2 + \frac{1}{n^2}\right). \tag{38}$$

4.5 Proof of compactness for fixed Y

Let $B \subset \Omega$ open, bounded and choose $n := \xi^{-1}$. Then, by Lemma 4.6 and Corollary A.3, we deduce that for all $\alpha > 0$ and almost every $Y \in B_\alpha(0) \subset \mathrm{Sym}_0$, our construction yields a flat chain $T_{\eta,\xi,n} \in \mathcal{F}^2$ such that $\partial T_{\eta,\xi,n} = S_{\eta,\xi,n} + \Gamma_n$ and

$$\mathbb{M}(T_{\eta,\xi,n} \llcorner B) \;\le\; C\left(\eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, B_R(x_0)) + \frac{\xi}{\eta}\right) \;\le\; C\left(\eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, B_R(x_0)) + \frac{\xi}{\eta}\right),$$

where we also used (24) of Proposition 4.2. In particular the energy bound (15) implies that $\mathbb{M}(T_{\eta,\xi,n} \llcorner B)$ is bounded. Applying a compactness theorem for flat chains as stated in the preliminary part ([26, Cor. 7.5]), there exists a subsequence (which we do not relabel) and a flat chain $T \in \mathcal{F}^2$ with support in $\overline{\Omega}$ such that

$$\mathbb{F}(T_{\eta,\xi,n} - T) \to 0 \qquad \text{for almost every } Y \in B_\alpha(0).$$

Since the boundary operator ∂ is continuous we conclude with Theorem 4.4 that $\partial T = S + \Gamma$. The finite mass of T and S immediately implies rectifiability [26, Thm 10.1]. Expressing the measure μ_T restricted to $\mathcal{M}$ as $\mu_{T \llcorner \mathcal{M}} = \chi_G \mathcal{H}^2 \llcorner \mathcal{M}$ for a measurable set $G \subset \mathrm{supp}\,(T \llcorner \mathcal{M}) \subset \mathcal{M}$ we can define

$$F := \{\omega \in \mathcal{M} \setminus G \,:\, \nu(\omega) \cdot \mathbf{e}_3 > 0\} \cup \{\omega \in \mathcal{M} \cap G \,:\, \nu(\omega) \cdot \mathbf{e}_3 \le 0\}.$$

Consequently,

$$F^c := \{\omega \in \mathcal{M} \setminus G \,:\, \nu(\omega)\cdot \mathbf{e}_3 < 0\} \cup \{\omega \in \mathcal{M} \cap G \,:\, \nu(\omega)\cdot \mathbf{e}_3 \geq 0\}\,.$$

5 Lower bound

This section is devoted to the $\Gamma-$liminf inequality of Theorem 3.1. The proof necessary to deduce the line energy has already been given in [16], so that we will only state the result for completeness (Proposition 5.1). The energy contributions of T far from $\mathcal{M}$ are to be derived in Subsection 5.1. In the remaining, we are concerned with the energy of T and F close resp. on $\mathcal{M}$.

The precise cost of a singular line in our setting has been derived first in [13] based on ideas in [35, 61]. In our case, the result reads as follows.

Proposition 5.1. *Let $B \subset \overline{\Omega}$ be a bounded open set and $U_\eta := \{x \in \Omega \,:\, \operatorname{dist}(x, S_{\eta,\xi,n}) \leq \sqrt{\eta}\}$. Then*

$$\liminf_{\eta,\xi\to 0} \eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, U_\eta \cap B) \;\geq\; \frac{\pi}{2} s_*^2 \beta\, \mathbb{M}(S \llcorner B)\,. \tag{39}$$

Proof. See Theorem C and Proposition 4.1 in [16] for a proof of the version we used here. □

To derive the exact minimal energy for the lower bound related to T, we introduce the following auxiliary problem:

$$I(r_1, r_2, a, b) \;:=\; \inf_{\substack{n_3 \in H^1([r_1,r_2],[-1,1]) \\ n_3(r_1)=a,\, n_3(r_2)=b}} \int_{r_1}^{r_2} \frac{s_*^2 |n_3'|^2}{1-n_3^2} + c_*^2(1-n_3^2)\,\mathrm{d}r \tag{40}$$

for $0 \leq r_1 \leq r_2 \leq \infty$, $a, b \in [-1, 1]$. It is one dimensional and only takes into account the derivative along the integration path. Problem (40) is equivalent to minimizing $\int \left(\frac{1}{2}|\partial_r Q|^2 + g(Q)\right)\,\mathrm{d}r$ subject to a $\mathcal{N}-$valued function Q and fitting boundary conditions. This reflects that by Lemma 4.3, the regions where $Q_{\eta,\xi,n}$ is far from $\mathcal{N}$ are small. The functional in (40) has been previously studied in [1] and [4, Lemma 4.17] from which we need the following lemma:

Lemma 5.2. *Let $0 \leq r_1 \leq r_2 \leq \infty$. Then,*

1. *$I(r_1, r_2, -1, 1) \geq 4 s_* c_*$.*
2. *Let $\theta \in [0, \pi]$. Then the minimizer $\mathtt{n}_3$ of $I(0, \infty, \cos(\theta), 1)$ is explicitly given by*

$$\mathtt{n}_3(r, \theta) = \frac{A(\theta) - \exp(-2c_*/s_* r)}{A(\theta) + \exp(-2c_*/s_* r)}\,, \quad A(\theta) = \frac{1+\cos(\theta)}{1-\cos(\theta)} \tag{41}$$

and

$$I(0, \infty, \cos(\theta), \pm 1) = 2 s_* c_* (1 \mp \cos(\theta))\,. \tag{42}$$

In this lemma, we use that g reduces to $c_*^2(1-n_3^2)$ for Q in $\mathcal{N}$, as demanded in (4). However, as pointed out in Remark 4.18 in [4], this is not necessary.

During the blow up procedure in the next subsection, we want to quantify the energy necessary for a close to uniaxial $Q_{\eta,\xi,n}$ to pass from $n_3(Q_{\eta,\xi,n}) \approx \pm 1$ to $n_3(Q_{\eta,\xi,n} - Y) = 0$, i.e. to intersect $T_{\eta,\xi,n}$. Since problem (40) does not take into account the perturbation made by subtracting $Y \in B_\alpha(0)$ from $Q_{\eta,\xi,n}$, we also introduce for $\alpha > 0$ small enough

$$I_\alpha(r_1, r_2, a, b) \;:=\; \inf\{I(r_1, r_2, a, n_3(Q)) \,:\, Q \in \mathrm{Sym}_0,\, n_3(Q-Y) = b,\, Y \in B_\alpha(0)\}\,. \tag{43}$$

We will only be concerned with the case $b = 0$. Note that $I_\alpha(r_1, r_2, a, b) \to I(r_1, r_2, a, b)$ for $\alpha \to 0$.

The knowledge about the optimal profile in (42) is also used in the construction of the upper bound, in particular the fact that $|\mathfrak{n}_3| - 1$ and all derivatives of $\mathfrak{n}_3$ decay fast enough (here exponentially) as $r \to \infty$. The result that for minimizers of (40), n_3^2 approaches 1 exponentially fast is complemented by the next lemma. It states that for a bounded energy configuration on a line, n_3^2 cannot always stay far from 1.

Lemma 5.3. *There exist constants $\mathfrak{C} > 0$ and $\delta_0 > 0$ such that for a line ℓ and $Q \in H^1(\ell, \mathrm{Sym}_0)$ and $K > 0$ a constant such that $\eta\, \mathcal{E}_{\eta,\xi}(Q, \ell) \leq K < \infty$ it holds: For $\delta \in (0, \delta_0)$, there exist a set $I_\delta \subset \ell$ and $C_\delta > 0$ such that*

$$|\ell \setminus I_\delta| \;\leq\; \frac{K+1}{C_\delta}\eta \quad \textit{and} \quad |n_3(Q)| \geq 1 - \mathfrak{C}\sqrt{\delta} \textit{ on } I_\delta\,.$$

Proof. Let

$$g_{\min} \;:=\; \min\{g(Q) \,:\, Q \in \mathrm{Sym}_0,\, \mathrm{dist}(Q, \mathcal{N}) \leq \delta,\, |Q - Q_\infty| \geq \mathfrak{a}\sqrt{\delta}\}\,,$$

where $\mathfrak{a} > 0$ is chosen as in [4] and for $\delta > 0$ small enough. Proposition 2.5 in [4] then implies that $g_{\min} > 0$. Then, we can estimate

$$K \;\geq\; \eta\, \mathcal{E}_{\eta,\xi}(Q, \ell) \;\geq\; \frac{1}{\eta} g_{\min} |\{x \in \ell \,:\, \mathrm{dist}(Q(x), \mathcal{N}) \leq \delta\} \cap \{x \in \ell \,:\, |Q - Q_\infty| \geq \mathfrak{a}\sqrt{\delta}\}|\,.$$

In view of Proposition 2.1, it holds that $|\ell \setminus \{\mathrm{dist}(Q, \mathcal{N}) \geq \delta/2\}| \leq C\xi^2/\eta^2$. Furthermore, a straightforward calculation shows that if $|n_3(Q)| \leq 1 - 2\frac{5}{4\sqrt{2}s_*}\mathfrak{a}\sqrt{\delta}$, then $|Q - Q_\infty| \geq \mathfrak{a}\sqrt{\delta}$. Hence,

$$K \;\geq\; \frac{1}{\eta} g_{\min} \left| \left\{ x \in \ell \,:\, |n_3(Q(x))| \leq 1 - 2\frac{5}{4\sqrt{2}s_*}\mathfrak{a}\sqrt{\delta} \right\} \right| - C\frac{\xi^2}{\eta^2} - C\frac{K}{\gamma_2 \delta}\xi^2\,,$$

from which the claims follow for $\mathfrak{C} := 2\frac{5}{4\sqrt{2}s_*}\mathfrak{a}$ and $I_\delta := \{x \in \ell \,:\, |n_3(Q(x))| \geq 1 - \mathfrak{C}\sqrt{\delta}\}$. □

In the following two sections, we detail how Lemma 5.2 combined with Lemma 5.3 can be applied in the case of $T \llcorner \Omega$ and on the surface $\mathcal{M}$.

5.1 Blow up

Let $x_0 \in \Omega$ be a point of rectifiability of T. Let $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega$, $B_{r_0}(x_0) \cap \mathrm{supp}(S) = \emptyset$ and let $(r_k)_k$ be a sequence of real numbers converging to 0. Defining the measures $\mu_{\eta,\xi}(U) := \eta \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, U)$, we have the following lemma:

Lemma 5.4. *There exists a constant $C > 0$ independent of x_0 such that*

$$\frac{\mu_{\eta,\xi}(B_{r_k}(x_0))}{\mu_T(B_{r_k}(x_0))} \;\geq\; 2\, I_\alpha(0, \infty, 1, 0) - C\left(\delta r_k + \frac{\eta}{\delta r_k^3}\right) - o(1)\,. \tag{44}$$

Proof. We start by introducing some notation. We set $T_{r_k} := T \llcorner B_{r_k}(x_0)$ and $Q_k(y) := Q_{\eta,\xi,n}(x_0 + r_k y)$. The latter allows us to express

$$\eta\, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, B_{r_k}(x_0)) = \int_{B_1(0)} \frac{\eta r_k}{2}|\nabla Q_k|^2 + \frac{r_k^3}{\eta} g(Q_k) + \frac{\eta r_k^3}{\xi^2} f(Q_k) + \eta r_k^3 C_0 \,\mathrm{d}y\,. \tag{45}$$

Furthermore, we note that by rectifiability of T in x_0 that there exists a unit vector $\nu \in \mathbb{S}^2$ such that

$$\mathbb{F}\left(\frac{T_{r_k} - x_0}{r_k^2} \llcorner B_1 - P_\nu \llcorner B_1\right) \to 0 \quad \text{for } k \to \infty\,, \tag{46}$$

where $P_\nu = \{\nu\}^\perp$ is the two dimensional plane perpendicular to ν passing through 0. Indeed, by Theorem 10.2 in [45] we know that $(T_{r_k} - x_0)/r_k^2$ approaches P_ν in a weak sense and by Theorem

31.2 in [65] we get the equivalence between the weak convergence and convergence in the $\mathbb{F}-$norm in our case of T having integer coefficients and $T, \partial T$ being of bounded mass.

From (46) we infer that there exist flat chains $A_{2,k} \in \mathcal{F}^2$ and $A_{3,k} \in \mathcal{F}^3$ with $\mathbb{M}(A_{2,k}), \mathbb{M}(\partial A_{3,k}) \to 0$ (for $k \to \infty$) such that

$$\frac{T_{r_k} - x_0}{r_k^2} \llcorner B_1 = P_\nu \llcorner B_1 + A_{2,k} + \partial A_{3,k} \,. \tag{47}$$

Since P_ν is a plane, we can write $P_\nu = \partial H_\nu$, where H_ν is the half-space $H_\nu = \{p + t\nu : p \in P_\nu, t > 0\}$. Hence, (47) can be rewritten to

$$\frac{T_{r_k} - x_0}{r_k^2} \llcorner B_1 - A_{2,k} = \partial \left(H_\nu \llcorner B_1 + A_{3,k} \right) \,. \tag{48}$$

We are interested in the energy coming from lines in the ball B_1 parallel to ν. In order to relate this energy to the mass of T, we need to know how much energy each line carries and that a substantial portion of those lines intersect T. So let's first consider a line $\ell \subset B_1$ parallel to ν that intersects T. Because of the mirror symmetry or our problem w.r.t. P_ν, it is in fact enough to consider lines $\ell \subset H_\nu \cap B_1$ parallel to ν and to show that ℓ contributes half of the energy demanded, i.e. $I_\alpha(0, \infty, 1, 0) - C\eta$. In view of (44) that we want to prove, we can restrict ourselves even further to the case when

$$\int_\ell \frac{\eta r_k}{2} |\nabla Q_k|^2 + \frac{r_k^3}{\eta} g(Q_k) + \frac{\eta r_k^3}{\xi^2} f(Q_k) + \eta r_k^3 C_0 \, \mathrm{d}y \;<\; I_\alpha(0, \infty, 1, 0) \,, \tag{49}$$

otherwise there is nothing to prove. But then (49) implies with Lemma 5.3 that whenever $|\ell| > \frac{I_\alpha(0,\infty,1,0)+1}{C_\delta r_k^3} \eta$, there exists a point $p \in \ell$ such that $|n_3(Q_k(p))| > 1 - \mathfrak{C}\sqrt{\delta}$. We can apply Lemma 5.2 to get

$$\int_\ell \frac{\eta r_k}{2} |\nabla Q_k|^2 + \frac{r_k^3}{\eta} g(Q_k) + \frac{\eta r_k^3}{\xi^2} f(Q_k) + \eta r_k^3 C_0 \, \mathrm{d}y \;\geq\; I_\alpha(0, \infty, 1, 0) r_k^2 - C \, \|\mathrm{dist}(Q_k, \mathcal{N})\|_{L^\infty(\ell)} \, r_k \,.$$

It remains to estimate the measure of the lines which do not intersect T or are shorter than $\frac{I_\alpha(0,\infty,1,0)+1}{C_\delta r_k^3} \eta$. For the former lines, we can use (48) and (46) to show that this number is negligible in the limit $k \to \infty$. For the latter, a simple computation shows that the measure of those lines is bounded by a constant times $(\frac{I_\alpha(0,\infty,1,0)+1}{C_\delta r_k^3} \eta)^2$ □

Lemma 5.5. *Let $B \subset \Omega$ be a open and bounded set. Then,*

$$\liminf_{\eta, \xi \to 0} \eta \, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, B) \;\geq\; 2 \, I_\alpha(0, \infty, 1, 0) \, \mathbb{M}(T \llcorner B) - C|B|\delta \,. \tag{50}$$

Proof. The global energy bound (15) implies that the sequence $\mu_{\eta,\xi}$ is bounded and hence there exists s subsequence (not relabelled) and a non-negative measure μ_0 such that $\mu_{\eta,\xi} \rightharpoonup \mu_0$. This implies in particular that $\mu_{\eta,\xi}(B) \to \mu_0(B)$ for all bounded measurable sets B.

Choosing a sequence $r_\eta \searrow 0$ depending on η, such that $\eta r_\eta^{-3} \to 0$ for $\eta \to 0$, we can apply Lemma 5.4 for this sequence and get that

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mu_T}(x) \;=\; \lim_{\eta, \xi \to 0} \frac{\mu_{\eta,\xi}(B_{r_\eta}(x))}{\mu_T(B_{r_\eta}(x))} \;\geq\; 2 \, I_\alpha(0, \infty, 1, 0) \,.$$

This implies

$$\begin{aligned} \eta \, \mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n}, B) \;&=\; \int_B 1 \, \mathrm{d}\mu_{\eta,\xi} \;=\; \int_B 1 \, \mathrm{d}\mu_0 + |\mu_{\eta,\xi}(B) - \mu_0(B)| \;=\; \int_B \frac{\mathrm{d}\mu_0}{\mathrm{d}\mu_T} \, \mathrm{d}\mu_T + o(1) \\ &\geq\; 2 \, I_\alpha(0, \infty, 1, 0) \, \mathbb{M}(T \llcorner B) + o(1) \,. \end{aligned}$$

□

5.2 Surface energy

In this section we do the necessary calculations to find the announced energy contribution of F and $\mathcal{M}\setminus F$. For this, we estimate the energy close to $\mathcal{M}$. More precisely, we define $\mathcal{M}_{2\sqrt{\eta}} := \{x \in \Omega \,:\, \mathrm{dist}(x,\mathcal{M}) \leq 2\sqrt{\eta}\}$.

The goal is then to apply Lemma 5.2 to the rays perpendicular to $\mathcal{M}$ on which $Q_{\eta,\xi,n}$ is taking values close to $\mathcal{N}$. For $\omega \in \mathcal{M}$ and $r>0$ we define

$$L_{\omega,r} := \{\omega + t\nu(\Omega) \,:\, t \in [0,r]\} \tag{51}$$

and recall that for $\delta>0$ we denote $U_\delta = \{x \in \Omega \,:\, \mathrm{dist}(Q_{\eta,\xi,n}(x),\mathcal{N}) > \delta\}$. We assume η small enough such that $2\sqrt{\eta} < \mathbf{r}_0$. We recall that $\mathbf{r}_0$ was fixed in the beginning of Section 4 such that $\mathbf{r}_0$ is smaller than the minimal curvature radius of $\mathcal{M}$.

For the estimation, we set proxies for F

$$\begin{aligned} F_{\eta,\delta} := {} & \{\omega \in \mathcal{M} \,:\, L_{\omega,\sqrt{\eta}} \cap U_\delta = \emptyset,\ \nu_3(\omega) > 0 \text{ and } L_{\omega,\sqrt{\eta}} \cap T_{\eta,\xi,n} = \emptyset\} \\ & \cup \{\omega \in \mathcal{M} \,:\, L_{\omega,2\sqrt{\eta}} \cap U_\delta = \emptyset,\ \nu_3(\omega) < 0 \text{ and } L_{\omega,\sqrt{\eta}} \cap T_{\eta,\xi,n} \neq \emptyset\} \end{aligned} \tag{52}$$

and analogously for $\mathcal{M}\setminus F$

$$\begin{aligned} \widetilde{F}_{\eta,\delta} := {} & \{\omega \in \mathcal{M} \,:\, L_{\omega,\sqrt{\eta}} \cap U_\delta = \emptyset,\ \nu_3(\omega) < 0 \text{ and } L_{\omega,\sqrt{\eta}} \cap T_{\eta,\xi,n} = \emptyset\} \\ & \cup \{\omega \in \mathcal{M} \,:\, L_{\omega,2\sqrt{\eta}} \cap U_\delta = \emptyset,\ \nu_3(\omega) > 0 \text{ and } L_{\omega,\sqrt{\eta}} \cap T_{\eta,\xi,n} \neq \emptyset\}\,. \end{aligned}$$

We now rewrite the energy so that the line integrals over $L_{\omega,\sqrt{\eta}}$ appear. We note that for $0<\eta\ll 1$ the map $\mathcal{M}\times[0,2\sqrt{\eta}] \to \Omega$ given by $(\omega,r)\mapsto \omega + r\nu(\omega)$ is injective. The differential of this map is given by $\mathrm{Id}_{T_\omega\mathcal{M}} + r\,\mathrm{d}_\omega\nu + \nu$. Using the normalized eigenvectors $\nu, \overline{\omega_1}, \overline{\omega_2}$ corresponding to the eigenvalues $1,\kappa_1,\kappa_2$ with κ_i being the principal curvatures of $\mathcal{M}$ at ω, i.e. the eigenvalues of the Gauss map $\mathrm{d}_\omega\nu$. Then

$$\det(\mathrm{Id} + r\,\mathrm{d}_\omega\nu(\omega)) = (1+r\kappa_1)(1+r\kappa_2)$$

and the gradient transforms as

$$|\nabla Q_{\eta,\xi,n}|^2 = |\partial_r Q_{\eta,\xi,n}|^2 + \frac{1}{|1+r|\kappa_1||^2}|\partial_{\overline{\omega_1}} Q_{\eta,\xi,n}|^2 + \frac{1}{|1+r|\kappa_2||^2}|\partial_{\overline{\omega_2}} Q_{\eta,\xi,n}|^2\,.$$

In order to shorten our formulas, we still use the notation $\nabla Q_{\eta,\xi,n}$. The energy can then be rewritten as

$$\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n},\mathcal{M}_{2\sqrt{\eta}}) = \int_{\mathcal{M}}\int_0^{2\sqrt{\eta}} \left(\frac{1}{2}|\nabla Q_{\eta,\xi,n}|^2 + \frac{1}{\xi^2}f(Q_{\eta,\xi,n}) + \frac{1}{\eta^2}g(Q_{\eta,\xi,n})\right)\prod_{i=1}^{2}(1+r\kappa_i)\,\mathrm{d}r\,\mathrm{d}\omega\,.$$

Consider a point $\omega \in F_{\eta,\delta}$. We can assume that $\eta\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n},L_{\omega,\sqrt{\eta}}) < 2I_\alpha(0,\infty,1,0)$, otherwise the result is trivial. If $\omega\in F_{\eta,\delta}$ such that $L_{\omega,\sqrt{\eta}}\cap T = \emptyset$, then Lemma 5.3 implies that for $\delta>0$ and $\eta>0$ small enough, there exists $t_\omega \in [0,\sqrt{\eta}]$ such that $|n_3(Q_{\eta,\xi,n})(\omega + t_\omega\nu(\omega))| \geq 1 - \mathfrak{C}\sqrt{\delta}$. Since furthermore $L_{\omega,\sqrt{\eta}}\cap U_\delta = \emptyset$, and for $\eta \ll 1$ we can apply Lemma 5.2 to get

$$\begin{aligned} \eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n},L_{\omega,2\sqrt{\eta}}) &\geq \int_0^{2\sqrt{\eta}} \left(\frac{\eta}{2}|\partial_r Q_{\eta,\xi,n}|^2 + \frac{\eta}{\xi^2}f(Q_{\eta,\xi,n}) + \frac{1}{\eta}g(Q_{\eta,\xi,n}) + \eta C_0\right)\prod_{i=1}^{2}(1+r\kappa_i)\,\mathrm{d}r \\ &\geq I(0,t_\omega,\cos(\theta_\eta),1) - C\sqrt{\delta} + o(1)\,. \end{aligned}$$

Note that since $Q_{\eta,\xi,n}$ does not verify the boundary condition (12), the lower bound contains an angle θ_η instead of θ. But due to Proposition 4.2, θ_η converges to θ uniformly for $\eta\to 0$. If $L_{\omega,\sqrt{\eta}}\cap T \neq \emptyset$, this implies that there exist $t'_\omega \in (0,\sqrt{\eta})$ and $t_\omega \in (\sqrt{\eta},2\sqrt{\eta})$ such that $|n_3(Q_{\eta,\xi,n} - Y)(\omega + t'_\omega\nu(\omega))| \leq \sqrt{\delta}$ and $|n_3(Q_{\eta,\xi,n})(\omega+t_\omega\nu(\omega))| \geq 1-\mathfrak{C}\sqrt{\delta}$ and hence

$$\eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi,n},L_{\omega,2\sqrt{\eta}}) \geq I_\alpha(0,t'_\omega,\cos(\theta_\eta),0) + I_\alpha(t'_\omega,t_\omega,1,0) - C\sqrt{\delta} + o(1)\,.$$

Mutatis mutandis, we get the analogous estimates for $\widetilde{F}_{\eta,\delta}$ and we can pass to the limits $\eta \to 0$ and $\delta \to 0$.

It remains the passage $Y \to 0$. Since $I_\alpha \to I$ for $\alpha \to 0$, the bound (50) of Lemma 5.5 is uniform in Y (and hence α) and since $\mathcal{M}$ is compact, we can pass to the limit $\alpha \to 0$. Using a diagonal sequence, we can also choose $Y_{\eta,\xi} \to 0$ as claimed in the Theorem.

6 Upper bound

This section is devoted to the construction of the recovery sequence of Theorem 3.1. Essentially, there are three steps in this construction:

1. We approximate T by a sequence T_n, solution to a minimization problem. The advantage of replacing T by T_n is the gain of regularity. Indeed, as we will see in Subsection 6.1, T and its boundary inside Ω will be of class $C^{1,1}$. Furthermore, by a comparison argument, we can show that $\partial(T_n \llcorner \mathcal{M})$ is a line of finite length.
2. We introduce local coordinate systems in which we can define $Q_{\eta,\xi,n}$ and estimate its energy.
3. Choosing a diagonal sequence $n(\xi,\eta)$ we find the recovery sequence.

6.1 A first regularity result for (almost) minimizers

In this subsection, we rewrite the limit energy $\mathcal{E}_0$ in a way that it only depends on T:

$$\mathcal{E}_0(T) \;=\; E_0(\mathcal{M}, \mathbf{e}_3) + 4s_*c_* \int_{\mathcal{M}} |\cos(\theta)| \, \mathrm{d}\mu_{T\llcorner\mathcal{M}} + 4s_*c_* \mathbb{M}(T \llcorner \Omega) + \frac{\pi}{2} s_*^2 \beta \mathbb{M}(\partial T - \Gamma)\,, \qquad (53)$$

where $\Gamma \in \mathcal{F}^1$ is given by the curve $\{\nu_3 = 0\} \subset \mathcal{M}$. For the approximation of a flat chain $T \in \mathcal{F}^2$ we are going to study the following minimization problem:

$$\min_{\tilde{T} \in \mathcal{F}^2} \mathcal{E}_0(\tilde{T}) + n\, \mathbb{F}(\tilde{T} - T)\,, \qquad (54)$$

for $n \in \mathbb{N}$. The existence of a minimum of (54) is imminent since by assumption T verifies $\mathcal{E}_0(T) + n\mathbb{F}(T - T) = \mathcal{E}_0(T) < \infty$, the energy is non-negative and lower semi-continuous with respect to convergence in the flat norm. We have the following result:

Proposition 6.1. *Let $T \in \mathcal{F}^2$ with $\mathcal{E}_0(T) < \infty$. For all $n \in \mathbb{N}$, the problem (54) has a solution $T_n \in \mathcal{F}^2$. The minimizer T_n verifies*

1. *$T_n \to T$ for $n \to \infty$ in the flat norm.*
2. *$T_n \llcorner \Omega$ is of class C^1 up to the boundary $\partial(T_n \llcorner \Omega)$.*
3. *$(\partial T_n) \llcorner \Omega$ is of class $C^{1,1}$.*

We note that the above Proposition also holds true for $n = 0$, i.e. minimizers of (53) and hence of our limit problem are of class C^1 up to the boundary in Ω which is of class $C^{1,1}$. As we will see later, the minimizers of $\mathcal{E}_0$ are in fact smooth (see Proposition 7.1). In order to make this subsection more readable and simplify notation, we divide (53) by $4s_*c_*$ and redefine the parameter β to replace the constant $\frac{1}{8}\frac{s_*}{c_*}\beta$. Also, we will simply write n instead of $\frac{n}{4c_*s_*}$. Since in this subsection we are only concerned with the regularity of minimizers, this change of notation does not impact our results.

The proof of Proposition 6.1 makes use of a series of lemmas which we are going to state and prove first. The main ideas for the regularity of T_n and ∂T_n have already been developed in earlier papers [19, 20, 53, 68], so it remains to check that we can apply them in our case. For the sake of simple notation, we drop the subscript n for the rest of this section and define $S := \partial T - \Gamma$.

Lemma 6.2. *Let* $\mathrm{rect}(S)$ *be the rectifiability set of* S*. Then* $\mathrm{supp}\,(S) = \overline{\mathrm{rect}(S)}$ *and* $\mathcal{H}^1(\mathrm{supp}\,(S) \setminus \mathrm{rect}(S)) = 0$.

Proof. Let's show first that S is supported by a closed 1-dimensional set.

For this, we prove that S cannot contain subcycles of arbitrary small length. Assume that S_1 is a subcycle of S, i.e. $\mathbb{M}(S) = \mathbb{M}(S_1) + \mathbb{M}(S - S_1)$ and $\partial S_1 = 0$, and that S_1 is supported in $B_r(x_0)$ for $r \in (0, \frac{1}{2}\mathbf{r}_0)$. By (7.6) in [26], there exists a constant $b > 0$ and $T_1 \in \mathcal{F}^2$ such that $S_1 = \partial T_1$ and $\mathbb{M}(T_1) \leq b\mathbb{M}(S_1)^2$. By projecting T_1 onto $B_r(x_0) \cap \overline{\Omega}$, we can furthermore assume that T_1 is supported in $B_r(x_0)$ and lies within $\overline{\Omega}$. Projecting onto $B_r(x_0)$ does not affect the previous estimate since it decreases the mass. Projecting $T_1 \llcorner E$ onto $\mathcal{M}$ has Lipschitz constant less than $1 + 4\frac{r}{\mathbf{r}_0}$ and hence, the estimate stays true with an additional factor of $1 + 4\frac{r}{\mathbf{r}_0}$. We estimate by minimality of T

$$\begin{aligned}
\mathcal{E}_0(T) + n\mathbb{F}(T - T_0) &\leq \mathcal{E}_0(T + T_1) + n\mathbb{F}(T + T_1 - T_0) \\
&\leq \mathcal{E}_0(T) + \mathbb{M}(T_1) - \beta\mathbb{M}(S_1) + n\mathbb{F}(T - T_0) + n\mathbb{M}(T_1) \\
&\leq \mathcal{E}_0(T) - \beta\mathbb{M}(S_1) + n\mathbb{F}(T - T_0) + (n+1)(1 + 4\frac{r}{\mathbf{r}_0})b\mathbb{M}(S_1)^2\,,
\end{aligned}$$

and thus $\beta\mathbb{M}(S_1) - b(n+1)(1 + 4\frac{r}{\mathbf{r}_0})\mathbb{M}(S_1)^2 \leq 0$. We hence find that either $\mathbb{M}(S_1) = 0$ or that $\mathbb{M}(S_1) \geq \beta/(3b(n+1))$.

Now, let x_0 be a point of rectifiability of S and $r \leq \beta/(6b(n+1))$. Assume that $\mu_S(B_r(x_0)) < 2r$. Then, since

$$\int_0^r \mu_S(\partial B_s(x_0))\,\mathrm{d}s \leq \mu_S(B_r(x_0)) < 2r\,,$$

we can evoke Theorem 5.7 of [26] to deduce that there exists a set of positive measure $I \subset [0, r]$ such that $\mu_S(\partial B_s(x_0)) < 2$ for all $s \in I$. Thus, we can find radii $s < r$ such that $\mathbb{M}(\partial(S \cap B_s(x_0))) \leq 1$. But a bounded 1-chain cannot have just one end, so that for $S_1 := S \cap B_s(x_0)$ we conclude that $\partial S_1 = 0$. In addition $\mathbb{M}(S_1) < 2r$ by assumption. Hence, we have $\mathbb{M}(S_1) < \beta/(3b(n+1))$ and the above calculation shows that necessarily $\mathbb{M}(S_1) = 0$. In particular, x_0 is not in the support of S which is a contradiction.

Let us conclude now that S is indeed a closed set. Let $\mathrm{rect}(S)$ be the rectifiability set of S. Since S has coefficients in a finite group, it is rectifiable [72] with $\mu_S = \mathcal{H}^1 \llcorner \mathrm{rect}(S)$. Now, take a sequence $x_k \in \mathrm{rect}(S)$ and assume $x_k \to x$. By the above reasoning it holds $\mu_S(B_r(x_k)) \geq 2r$ for all $r \leq \beta/(6b(n+1))$ and in the limit $k \to \infty$ also $\mu_S(B_r(x)) \geq 2r$. Hence $x \in \mathrm{rect}(S)$. This allows us to conclude that $\mathcal{H}^1(\mathrm{supp}\,(S) \setminus \mathrm{rect}(S)) = 0$. □

After having established this basic property of S, we can state a first regularity result:

Lemma 6.3. *The flat chain* S *is supported on a finite union of closed* $C^{1,\frac{1}{2}}-$*curves.*

Proof. Our goal is to prove that $S \llcorner \Omega$ is an almost minimizer of the length functional $\mathbb{M}$ and apply Theorem 3.8 in [53] to deduce $C^{1,\frac{1}{2}}-$regularity.

Let $x_0 \in \Omega$ and $\overline{r} \in (0, \frac{1}{2}\mathbf{r}_0)$ such that $B_{\overline{r}}(x_0) \subset \Omega$. Consider $T' \in \mathcal{F}^2$ with $\mathrm{supp}\,(T - T') \subset B_{\overline{r}}(x_0) \subset \Omega$. Then, for almost every $r \in (0, \overline{r})$, it holds that $S_r := S \cap B_r(x_0)$ is a flat chain with boundary $\partial S_r = \partial S \cap B_r(x_0)$. In this case, $S'_r := \partial T' \cap B_r(x_0)$ has the same boundary. Hence, the flat chain $A := S_r + S'_r = \partial T + \partial T'$ is a cycle, i.e. verifies $\partial A = 0$ and is supported inside $B_r(x_0)$. We can construct the cone C' with vertex x_0 over A. Then, $\partial C' = A$ and $\mathbb{M}(C') \leq cr\mathbb{M}(A)$. Now, we distinguish two cases: It holds either $\mathbb{M}(S_r) \leq \mathbb{M}(S'_r)$ (which is enough for our conclusion as we will see below) or $\mathbb{M}(S_r) \geq \mathbb{M}(S'_r)$ and hence $\mathbb{M}(A) \leq 2\mathbb{M}(S_r)$. Comparing T to $T + C'$ and by minimality of T we get that

$$\beta\mathbb{M}(S_r) \;\leq\; \beta\mathbb{M}(S'_r) + (n+1)\mathbb{M}(C') \;\leq\; \beta\mathbb{M}(S'_r) + 2c(n+1)r\mathbb{M}(S_r)\,.$$

For r small enough we conclude that

$$\mathbb{M}(S_r) \;\leq\; \left(1+\frac{4c(n+1)}{\beta}\,r\right)\mathbb{M}(S_r')\,. \tag{55}$$

In case T' is not entirely contained in Ω, we need to project those parts of T' and of the boundary S_r' onto $\mathcal{M}$. Since we assumed $r \leq \frac{1}{2}\mathbf{r}_0$, the Lipschitz constant of the projection can be estimated by $1+4\frac{r}{\mathbf{r}_0}$, i.e. our analysis and in particular (55) holds true if we replace $\mathbb{M}(S_r')$ by $(1+4\frac{r}{\mathbf{r}_0})\mathbb{M}(S_r')$. Together with Lemma 6.2, (55) allows us to apply Theorem 3.8 in [53] which gives the $C^{1,1/2}-$regularity and the decomposition of supp (S) into a finite union of curves, possibly meeting at triple points. Finally, since our flat chains take values only in $\pi_1(\mathcal{N}) = \{0,1\}$, we can exclude the existence of triple points since they would create boundary. Hence, S is a union of curves. □

The regularity of S in Lemma 6.3 is not optimal. The following Lemma provides us with the smoothness we announced in Proposition 6.1:

Lemma 6.4. *The flat chain S is supported on a finite union of closed $C^{1,1}-$curves. In particular, the curvature of S is bounded.*

Proof. Let $x_0 \in$ supp (S) and take $r > 0$ such that $B_r(x_0) \subset \Omega$ and $\mu_S(\partial B_r(x_0)) = 2$. Let $\{x_1, x_2\} := \text{supp}\,(S) \cap \partial B_r(x_0)$ and define $S_r := S \llcorner B_r(x_0)$. We compare S_r (and $T \llcorner B_r(x_0)$) to two competitors.

The first one is the geodesic segment S_g joining x_1 and x_2 in $\partial B_r(x_0)$. For the corresponding T_g we use a piece of $\partial B_r(x_0)$ between $T \llcorner (\partial B_r(x_0))$ and S_g. By minimality of S_r we find

$$\beta\mathbb{M}(S_r) \;\leq\; 2\pi r\,(\beta + 4(n+1)r)\;. \tag{56}$$

Our second competitor is the flat chain supported on the straight line segment joining x_1 to x_2 which we call S'. Then $S' + S_r$ is supported in $B_r(x_0)$ and is closed, i.e. $\partial(S'+S_r) = 0$. By the construction (7.6) in [26], we get the existence of a flat chain $T' \in \mathcal{F}^2$ supported in Ω and a constant $b > 0$ (depending only on the dimensions of the flat chains and the ambient space) such that $\partial T' = S' + S_r$ and $\mathbb{M}(T') \leq b(\mathbb{M}(S') + \mathbb{M}(S_r))^2$. Since $x_0 \in$ supp (S) it also holds that $\mathbb{M}(S_r) \geq 2r$. This, together with the minimality of S_r and (56) implies that

$$\begin{aligned} 2\beta r \;\leq\; \beta\mathbb{M}(S_r) \;&\leq\; \beta\mathbb{M}(S') + b(n+1)(\mathbb{M}(S') + \mathbb{M}(S_r))^2 \\ &\leq\; \beta\mathbb{M}(S') + b(n+1)\left(\mathbb{M}(S') + 2\pi r\left(1+\frac{4(n+1)}{\beta}r\right)\right)^2 \\ &\leq\; \beta\mathbb{M}(S') + C_1 r^2\,, \end{aligned} \tag{57}$$

for $C_1 = 2(2+2\pi)^2 b(n+1)$ and r small enough. Hence, (57) implies $(2 - (C_1/\beta)r)r \leq \mathbb{M}(S')$. If we now choose r even smaller to assure $r \leq r_1 := (C_1)^{-1}\beta$, one gets even $\mathbb{M}(S') \geq r$, i.e. the points x_1 and x_2 must not be too close.

Our goal is now to show that S_r is in fact close to S' and that S' is almost a diameter of $B_r(x_0)$, in the sense that S_r lies in a small neighbourhood of S' and the distance between x_0 and S' is of order r^2. Let's denote $\ell := \mathbb{M}(S') = |x_2 - x_1|$. Suppose $\mathbb{M}(S_r) \leq \ell + \alpha$ for a $\alpha > 0$ and let $\rho > 0$ be the smallest positive number such that S_r lies within a $\rho-$neighbourhood of S'. Then, $\mathbb{M}(S_r) \geq \sqrt{\ell^2 + 4\rho^2}$ and hence $\ell^2 + 4\rho^2 \leq \mathbb{M}(S_r) \leq (\ell+\alpha)^2$ which yields the bound

$$\rho \;\leq\; \sqrt{\frac{\ell\alpha}{2} + \frac{\alpha^2}{4}} \;\leq\; \sqrt{2r\alpha}\,, \tag{58}$$

provided $\alpha \leq 4r$ and $\ell \leq 2r$. Applying this result to our case where $\alpha = \beta^{-1}C_1 r^2$, we get S_r is contained in a neighbourhood of S' of radius $\rho \leq \sqrt{2\beta^{-1}C_1 r^3}$.

In addition, if S_r is supported in a $\rho-$cylinder around S', there exists a $T_\rho \in \mathcal{F}^2$ and a constant c (depending only on the space dimension) such that $\mathbb{M}(T_\rho) \leq c\rho\mathbb{M}(S_r)$ and $\partial T_\rho = S' + S_r$. This

implies that $\mathbb{M}(S_r) \leq \ell + \beta^{-1}(n+1)c\rho\mathbb{M}(S_r)$. Previously, we have also shown that $\mathbb{M}(S_r) \leq \ell + \beta^{-1}C_1 r^2 \leq 3r$, leading to

$$\mathbb{M}(S_r) \;\leq\; \ell + C_2\rho r\,, \quad \text{where } C_2 = 3c\frac{n+1}{\beta}\,. \tag{59}$$

Now, we want to iterate this procedure. Let $\alpha_0 := \beta^{-1}C_1 r^2$ as start of our induction.

1. Knowing that $\mathbb{M}(S_r) \leq \ell + \alpha_k$ (either by (57) or by induction hypothesis) and by (58) we can deduce that S_r lies in a ρ_k-neighbourhood of S', for $\rho_k = \sqrt{2r\alpha_k}$.
2. Since S_r lies in a ρ_k-neighbourhood of S', one can use (59) with $\rho = \rho_k$ to obtain $\mathbb{M}(S_r) \leq \ell + \alpha_{k+1}$, where $\alpha_{k+1} := C_2 r\rho_k$.

Throughout this iteration, α_k and ρ_k verify $\rho_{k+1} = \sqrt{2r\alpha_{k+1}} = \sqrt{2C_2\rho_k}\,r$. Thus, ρ_k converges to $2C_2r^2$ in the limit $k \to \infty$. We can conclude that the distance between a point in S_r and S' is at most $2C_2r^2$. In particular, since $x_0 \in \operatorname{supp}(S_r)$, it holds that $\operatorname{dist}(x_0, \operatorname{supp}(S'))$ is of order r^2 which shows that the line S' is close to being a diameter.

Let us turn now to the assertion of the lemma. For $x_0 \in \operatorname{supp}(S)$ and $r > 0$ chosen small enough, we denote $\tau_r(x_0)$ the vector $\frac{x_2-x_1}{\|x_2-x_1\|}$, where x_1, x_2 are constructed as before. By our previous calculations, we know that the corresponding points for $\frac{r}{2}$ are at most at distance $2C_2r^2$ from the line connecting x_1 and x_2 which gives $\|\tau_r(x_0) - \tau_{\frac{r}{2}}(x_0)\| \leq C_3 r$. This shows that the limit $\tau(x_0) = \lim_{r\to 0}\tau_r(x_0)$ exists and that $\|\tau_r(x_0) - \tau(x_0)\| \leq 2C_3 r$. The triangle inequality then yields the existence of another constant $C_4 > 0$, depending on β and n, such that for $x, y \in \operatorname{supp}(S)$ with $|x-y| =: r$ small enough we have $\|\tau(x) - \tau(y)\| \leq C_4 r$.

□

Having reached the optimal regularity for S, we now turn to the properties of T.

Lemma 6.5. *The flat chain $T \llcorner \Omega$ is supported on a hypersurface of class C^1 up to the boundary.*

Proof. We first discuss the regularity in the interior of $T \llcorner \Omega$. Let $x_0 \in \Omega$, $r > 0$ such that $B_r(x_0) \cap \operatorname{supp}(T \llcorner \Omega) \neq \emptyset$ and consider a variation T' of T in $B_r(x_0)$. Then, by minimality of T we find

$$\mathbb{M}(T) \;\leq\; \mathbb{M}(T') \;+\; n\mathbb{F}(T - T') \;\leq\; \mathbb{M}(T') \;+\; \frac{4}{3}\pi n r^3\,.$$

We can then apply the result of Taylor [68], or more general Theorem 1.15 in [19] to obtain $C^{1,\alpha}-$regularity in Ω, for some $\alpha > 0$.

For the regularity up to the boundary we want to apply Theorem 31.1 in [20]. In order to do this we need to show that on a certain scale, the boundary S is close to a straight line and T is almost flat.

Take a point of rectifiability $x_0 \in S$. We define a blow-up sequence $r_k \searrow 0$. Since S is supported by $C^{1,1}-$curves, a blow up of S converges to a straight line. We claim that a blow up of T converges to a limit T_0 which is a half plane. For this, we use the minimality of T to deduce for $r > 0$ small enough that

$$\begin{aligned}
\mathbb{M}(T \llcorner B_r(x_0)) + 2\beta r \;&\leq\; \mathbb{M}(T \llcorner B_r(x_0)) + \beta\mathbb{M}((\partial T) \llcorner B_r(x_0)) \\
&\leq\; \mathbb{M}(\operatorname{cone}(T \llcorner \partial B_r(x_0))) + \beta\mathbb{M}(\operatorname{cone}((\partial T) \llcorner \partial B_r(x_0))) \\
&\leq\; \frac{r}{2}\mathbb{M}(T \llcorner \partial B_r(x_0)) + \beta r\mathbb{M}((\partial T) \llcorner \partial B_r(x_0)) \\
&=\; \frac{r}{2}\mathbb{M}(T \llcorner \partial B_r(x_0)) + 2\beta r\,.
\end{aligned}$$

This implies that $\mathbb{M}(T \llcorner B_r(x_0))/r^2$ is monotonically increasing and thus admits a unique limit d. We define $T_{r_k} = (T - x_0)/r_k$ and by monotonicity we get for $s_1 < s_2$ that $\mathbb{M}(T_{r_k} \llcorner B_{s_1})/s_1^2 \leq$

$\mathbb{M}(T_{r_k} \llcorner B_{s_2})/s_2^2$. For $r_k \to 0$ both sides converge to the same limit πd. But this means that $\mathbb{M}(T_0 \llcorner B_{s_1})/s_1^2 = \mathbb{M}(T_0 \llcorner B_{s_2})/s_2^2$, i.e. T_0 is a cone and hence a half-plane. Since a half plane has density $\frac{1}{2}$, we find $d = \frac{1}{2}$. In particular, we have for k large enough

$$\mathbb{M}\left(\frac{T_{r_k} - x_0}{r_k} \llcorner B_1\right) \;=\; \frac{\pi}{2} + o(1)\,,$$

from which it follows that condition (31.6) in [20] holds and thus we can apply Theorem 31.1 on a length scale $R \leq r_k$. We remark that by convergence in the flat norm, following [48], we also verify the condition (31.4) of Theorem 31.1 in [20]. By compactness of the boundary $(\partial T) \llcorner \Omega$, we find a finite cover with balls of uniformly positive radius to which we can apply Theorem 31.1. This allows us to conclude. □

Proof of Proposition 6.1. We have already established the existence of a minimizer of (54). The convergence $\mathbb{F}(T_n - T_0) \to 0$ for $n \to \infty$ is obvious since $n\,\mathbb{F}(T_n - T_0) \leq \mathcal{E}_0(T_0) < \infty$ for all $n \in \mathbb{N}$.

The regularity of T_n follows from Lemma 6.4 and Lemma 6.5. □

6.2 Construction of the recovery sequence

In this section we will use the approximation of T given by the minimizer of (54) to construct our recovery sequence. First we establish the following Proposition which yields additional control over $\partial(T \llcorner \mathcal{M}) \setminus \partial T$ and its boundary which will be necessary for the final construction in Proposition 6.9.

Proposition 6.6. *Let $T \subset \overline{\Omega}$ be a flat $2-$chain of finite mass and $S \subset \overline{\Omega}$ be a flat $1-$chain of finite mass such that $\partial S = 0$ and $\partial T = S + \Gamma$. Then, there exist finite mass flat chains $T_n \in \mathcal{F}^2$ of class* Lip *up to the boundary and $S_n \in \mathcal{F}^1$ of class $C^{1,1}$ such that*

1. *$\partial S_n = 0$ and $\partial T_n = S_n + \Gamma$,*
2. *$\mathbb{F}(T_n - T) \to 0$ and $\mathcal{E}_0(T_n) \to \mathcal{E}_0(T)$ as $n \to \infty$,*
3. *and there exists a constant $C_n > 0$ such that $\mathbb{M}(\partial(T_n \llcorner \mathcal{M}) \setminus \partial T_n) \leq C_n$ and $\mathbb{M}(\partial(\partial(T_n \llcorner \mathcal{M}) \setminus \partial T_n)) \leq C_n$.*

Essentially, the first two parts of Proposition are proved by Proposition 6.1, saying that the minimizer T_n of (54) fulfils our claims. It remains to prove the last assertion i.e. that we can modify T_n to control the length of the set where the T_n attaches to $\mathcal{M}$. For this, we need the following average argument stating that we can find radii r such that the corresponding sets $T_n \llcorner \mathcal{M}_r$, for $\mathcal{M}_r := \{x \in \Omega : \operatorname{dist}(x, \mathcal{M}) = r\}$, are of finite length.

Lemma 6.7. *Let T_n be as constructed in the previous subsection. There exist a constant $c > 0$ and a radius $r \in (0, \frac{1}{2}\mathbf{r}_0)$ such that*

$$\mathbb{M}(T_n \llcorner \mathcal{M}_r) \;\leq\; \frac{4c\mathbb{M}(T_n)}{\mathbf{r}_0}\,. \tag{60}$$

Proof. Assume that $\mathbb{M}(T_n \llcorner \mathcal{M}_r) > \frac{4c\mathbb{M}(T_n)}{\mathbf{r}_0}$ for all $r \in (0, \frac{1}{2}\mathbf{r}_0)$ and some $c > 0$. This implies

$$\int_0^{\mathbf{r}_0/2} \mu_{T_n}(\mathcal{M}_r)\,\mathrm{d}r \;>\; 2c\mathbb{M}(T_n)\,.$$

Now, there exists a constant $c > 0$ such that $\int_0^{\mathbf{r}_0/2} \mu_{T_n}(\mathcal{M}_r)\,\mathrm{d}r \leq c\mathbb{M}(T_n)$ (see (5.7) in [26]). Hence, the lemma is proved. □

Now, we can modify T_n by replacing a small part close to $\mathcal{M}$ by a projection to control the boundary of $T_n \llcorner \mathcal{M}$ which is not included in S.

Proof of Proposition 6.6. We construct T_n as in Proposition 6.1. To ensure the additional estimate, we choose a radius r and a slice $\mathcal{M}_r$ as in Lemma 6.7. With the same argument as in Lemma 6.7 for S_n one can choose $r \in (0, \frac{1}{2}\mathbf{r}_0)$ for which additionally $\mathbb{M}(S_n \llcorner \mathcal{M}_r)$ is finite. Let $\Pi : \Omega_{\mathbf{r}_0} \to \mathcal{M}$ be the projection onto $\mathcal{M}$. We define $\Phi : \mathcal{M}_r \times [0,1] \to \overline{\Omega}$ by $\Phi(x,t) := (1-t)x + t\Pi x$. Then, by [25, Sec. 2.7], [24, Cor. 2.10.11], $\mathbb{M}(\Phi_\#(T_n \llcorner \mathcal{M}_r \times [0,1])) \leq Cr\mathbb{M}(T_n \llcorner \mathcal{M}_r)$ and also $\mathbb{M}(\Pi_\#(T_n \llcorner \mathcal{M}_r)) \leq C\mathbb{M}(T_n \llcorner \mathcal{M}_r)$. Again by the same argument, we get $\mathbb{M}(\partial \Pi_\#(T_n \llcorner \mathcal{M}_r)) \leq C\mathbb{M}(\partial(T_n \llcorner \mathcal{M}_r))$. This procedure can be applied to almost every $r \in (0, \frac{1}{2}\mathbf{r}_0)$, in particular, we can choose a sequence $r_n \to 0$ as $n \to \infty$. Replacing T_n close to $\mathcal{M}$ with these projections, we get the desired estimates.

The convergence of the energy $\mathcal{E}_0(T_n)$ to $\mathcal{E}_0(T)$ is a consequence of the convergence statements in Proposition 6.1 and the fact that $T_n \llcorner \mathcal{M}$ approaches $T \llcorner \mathcal{M}$. □

The recovery sequence $Q_{\eta,\xi}$ for our problem will be constructed around the regularized sequence of T. The gained regularity permits us to define $Q_{\eta,\xi}$ directly and without the need to write T as a complex and 'glue' together the parts of $Q_{\eta,\xi}$ on each simplex.

Before starting, we need one final ingredient as stated in the following lemma stating that the normal field on $\mathcal{M}$ can be extended to Ω. It will be used to fix choices of orientation consistently across Ω. The only crucial point is that the gradient of the extension must be bounded in order for our estimates to work out.

Lemma 6.8. *Let $\mathcal{M}$ be a closed compact manifold of class $C^{1,1}$. Then, there exists a Lipschitz continuation v of the outward normal field ν on $\mathcal{M}$ to Ω with the same Lipschitz constant.*

Proof. Since $\mathcal{M} \in C^{1,1}$, the outward normal ν is Lipschitz continuous. Then, the existence of a Lipschitz extension with the same Lipschitz constant is a direct consequence from Kirszbraum's theorem [39] (see also Theorem 7.2 in [45] or in full generality Theorem 1.31 in [63]). □

Proposition 6.9. *There exists a recovery sequence $Q_{\eta,\xi}$ for the problem* (19).

The construction relies on the approximation and regularisation made in the previous subsection. We will construct $Q_{\eta,\xi}$ step by step: The straightforward parts are the profile on F and $\mathcal{M} \setminus F$ away from ∂F, as well as the transition across T. In order to be compatible with the latter, we have to adjust the construction made in [4] for the singular line S. The profile of the part of S that approaches the surface $\mathcal{M}$ can be chosen as in [4]. Last, we need to connect $\partial F \setminus S$ to the profile of T already constructed. This last part is a bit subtle since the $\partial F \setminus S$ does not appear in the energy. The control we obtained in Proposition 6.6 is artificial and indeed we do not control the length of $\partial F \setminus S$. Hence, we will choose n depending on η, ξ in a way to allow this length to slowly grow to infinity while the energy contribution of $Q_{\eta,\xi}$ in this region vanishes.

Proof. From now on we choose n depending on ξ, η such that $\mathbb{M}(\partial(T_n \llcorner \mathcal{M})), \mathbb{M}(\partial(\partial(T_n \llcorner \mathcal{M}) \setminus \partial T_n)) \leq C|\ln(\eta)|$ and that the curvature of S_n is bounded by $C/\sqrt{\eta}$. Indeed, if the constant C_n in Proposition 6.6 is bounded in n, then for η small enough this condition is trivial. If $C_n \to \infty$ for $n \to \infty$ (choosing a subsequence we can furthermore assume that $C_n \nearrow \infty$), we can define $n(\eta) := \sup\{m \in \mathbb{N} : C_m \leq -\ln(\eta)\}$. Since $-\ln(\eta) \to \infty$ for $\eta \to 0$, it also holds that $n(\eta) \to \infty$ and the bound $\mathbb{M}(\partial(T_{n(\eta)} \llcorner \mathcal{M})), \mathbb{M}(\partial(\partial(T_{n(\eta)} \llcorner \mathcal{M}) \setminus \partial T_{n(\eta)})) \leq C|\ln(\eta)|$ holds.

Furthermore, whenever this doesn't lead to confusion, we drop the subscript parameters η, ξ and n in order to make the construction more readable.

Step 1: Adaptation of the optimal profile. The goal of this step is to construct a one dimensional profile close to the optimal one in Lemma 5.2, but where the transition takes place on a finite length and which gives the correct energy density (42) for $\eta \to 0$. To this goal, we introduce the "artificial" length scale η^γ for $\gamma \in (\frac{1}{2}, 1)$ and define

$$\Phi^\pm_\eta(t,\theta,v) := \begin{cases} s_*(\mathbf{n}^\pm(t/\eta,\theta) \otimes \mathbf{n}^\pm(t/\eta,\theta) - \frac{1}{3}\mathrm{Id}) & t \in [0,\eta^\gamma], \\ s_*(\mathbf{n}^\pm(\eta^\gamma/\eta,\theta) \otimes \mathbf{n}^\pm(\eta^\gamma/\eta,\theta) - \frac{1}{3}\mathrm{Id})(2\eta^\gamma - t) + (t-\eta^\gamma)Q_\infty & t \in (\eta^\gamma, 2\eta^\gamma], \\ (3\eta^\gamma - t)Q_\infty + (t-2\eta^\gamma)Q_{\eta,\xi,\infty} & t \in (2\eta^\gamma, 3\eta^\gamma], \end{cases} \tag{61}$$

with $\mathbf{n}^\pm = (\sqrt{1-\mathtt{n}_3^2}(v_1,v_2), \pm\mathtt{n}_3)$, where $\mathtt{n}_3(t,\theta)$ is the optimal profile from (41) and $(v_1,v_2)\in\mathbb{S}^1$. The choice of η^γ permits us to conclude that $\mathbf{n}^\pm(\eta^\gamma/\eta)\to\pm\mathbf{e}_3$ for $\eta\to\infty$. On the other hand, η^γ is large enough to verify $(\eta^\gamma)^2/\eta\to 0$ which ensures that undesired energy contributions vanish, as we will see in the following steps.

Step 2: Construction on F and F^c. Let $\omega\in F_{3\eta^\gamma} := \{\omega\in F : \operatorname{dist}(\omega,\partial F)\geq 3\eta^\gamma\}\subset\mathcal{M}$ and let $0\leq r<3\eta^\gamma\leq\frac{1}{2}\mathbf{r}_0$. We define

$$Q_{\eta,\xi}(\omega+r\nu(\omega)) \;:=\; \Phi^+(r,\theta,v(x)) \quad \text{where } \theta=\arccos(\nu(\omega)\cdot\mathbf{e}_3)\,. \tag{62}$$

Since $|\nabla v|$ is bounded, one can then easily calculate

$$\begin{aligned}
\eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi},F_{3\eta^\gamma,3\eta^\gamma}) \;&=\; \int_{F_{3\eta^\gamma}}\int_0^{3\eta^\gamma}\left(\frac{\eta}{2}|\nabla Q_{\eta,\xi}|^2+\frac{\eta}{\xi^2}f(Q_{\eta,\xi})+\frac{1}{\eta}g(Q_{\eta,\xi})+\eta C_0\right)\prod_{i=1}^2(1+r\kappa_i)\,\mathrm{d}r\,\mathrm{d}\omega\\
&\leq\; \int_{F_{3\eta^\gamma}}\int_0^{\eta^\gamma}\left(\frac{\eta}{2}|\partial_r Q_{\eta,\xi}|^2+\frac{1}{\eta}g(Q_{\eta,\xi})\right)\prod_{i=1}^2(1+r\kappa_i)\,\mathrm{d}r\,\mathrm{d}\omega+C\eta\\
&\leq\; \int_{F_{3\eta^\gamma}} I\left(0,\frac{\eta^\gamma}{\eta},\cos(\theta),+1\right)\,\mathrm{d}\omega+o(1)\,,
\end{aligned}$$

where $F_{3\eta^\gamma,R} := \{x\in\Omega : x=\omega+r\nu(\omega),\,\omega\in F_{3\eta^\gamma},\,r\in[0,R]\}$ for $R>0$. Note that $\frac{\eta^\gamma}{\eta}\to\infty$ for $\eta\to 0$ since we chose $\gamma\in(\frac{1}{2},1)$. Analogously, we can define $Q_{\eta,\xi}$ on F^c away from ∂F by using Φ^- and estimate its energy. Note that this construction may already create the part of T that attaches to the surface $\mathcal{M}$ in the limit $\eta,\xi\to 0$. Indeed, if a point ω is contained in F although the energy density corresponding to F^c would be lower, the profile constructed passes trough $n_3=0$ within a distance η^γ from $\mathcal{M}$ and hence is included in the limiting T.

Step 3: Construction on T. Let $x\in T_\eta := \{x\in\operatorname{supp}(T) : \operatorname{dist}(x,\mathcal{M})>3\eta^\gamma \text{ and } \operatorname{dist}(x,S)>3\eta^\gamma\}$. For each connected component of T (and thus of T_η) we can associate a sign depending on the sign of the degree of the singularity line S (if the component of T has such). This must be compatible with the part of T that reaches $\mathcal{M}$ and already has been constructed in Step 2. The compatibility corresponds to the choice of the signs of $\phi_\eta^\pm$ and of the distance function, viewing T_η as a boundary, locally. Assuming that in Step 2 we chose Φ_η^+ whenever $\operatorname{dist}(\cdot,T_\eta)>0$ and Φ_η^- for $\operatorname{dist}(\cdot,T_\eta)<0$, we define

$$Q_{\eta,\xi,n}(x) \;:=\; \Phi_\eta^+(\operatorname{dist}(x,T),\frac{\pi}{2},v(x))\,.$$

Since $|\nabla v|$ is bounded, and writing $T_{\eta,t} := \{x\in\Omega : \operatorname{dist}(x,T_\eta)=\operatorname{dist}(x,T) \text{ and } \operatorname{dist}(x,T_\eta)\leq t\}$ for $t\geq 0$ we can estimate by Lemma (5.2) and the coarea formula

$$\begin{aligned}
&\int_{T_{\eta,3\eta^\gamma}}\frac{\eta}{2}|\nabla Q_{\eta,\xi,n}|^2+\frac{\eta}{\xi^2}f(Q_{\eta,\xi,n})+\frac{1}{\eta}g(Q_{\eta,\xi,n})+\eta C_0\,\mathrm{d}x\\
&\quad\leq\; \int_{T_{\eta,\eta^\gamma}}\frac{\eta}{2}|\nabla Q_{\eta,\xi,n}|^2+\frac{1}{\eta}g(Q_{\eta,\xi,n})\,\mathrm{d}x+C\eta\,\mathbb{M}(T)\\
&\quad=\; 2s_*c_*\int_{T_{\eta,\eta^\gamma}}|\mathtt{n}_3'(\operatorname{dist}(x,T_n)/\eta)|\,\mathrm{d}x+C\eta\,\mathbb{M}(T)\\
&\quad=\; 2s_*c_*\int_0^{\eta^\gamma}\int_{T_{\eta,\eta^\gamma}\cap\{\operatorname{dist}(\cdot,T)=s\}}|\mathtt{n}_3'(s/\eta)|\,\mathrm{d}s+o(1)\\
&\quad=\; 2s_*c_*\int_0^{\eta^\gamma}\mathcal{H}^2(T_{\eta,\eta^\gamma}\cap\{\operatorname{dist}(\cdot,T)=s\})|\mathtt{n}_3'(s/\eta)|\,\mathrm{d}s+o(1)\\
&\quad\leq\; 2s_*c_*\big(2\mathbb{M}(T)+o(1)\big)\int_0^{\eta^\gamma}|\mathtt{n}_3'(s/\eta)|\,\mathrm{d}s+o(1)\\
&\quad=\; 4s_*c_*|\mathtt{n}_3(\eta^\gamma/\eta)|\,\mathbb{M}(T)+o(1)\,,
\end{aligned}$$

where we also used that $\mathcal{H}^2(T_{\eta,\eta^\gamma} \cap \{\mathrm{dist}(\cdot,T) = s\}) \to 2\mathbb{M}(T)$ for $s \to 0$. Note that $|\mathfrak{n}_3(t)| \to 1$ as $t \to \infty$. Hence, for $\eta, \xi \to 0$ we end up with

$$\begin{aligned}\limsup_{\eta,\xi\to 0} \int_{T_{\eta,3\eta^\gamma}} \frac{\eta}{2}|\nabla Q_{\eta,\xi,n}|^2 + \frac{\eta}{\xi^2} f(Q_{\eta,\xi,n}) + \frac{1}{\eta} g(Q_{\eta,\xi,n}) + \eta C_0 \,\mathrm{d}x \\ \leq\ 4 s_* c_* \mathbb{M}(T_n)\,.\end{aligned}$$

Step 4: Construction on $S \llcorner \Omega$. Following the notation we used in Step 2 and 3, we introduce the region

$$S_{3\eta^\gamma} \ :=\ \{x \in \Omega\ :\ \exists y \in T \text{ with } \mathrm{dist}(y,S) \leq 3\eta^\gamma \text{ and } \mathrm{dist}(x,T) = \|x-y\| \leq 3\eta^\gamma\} \tag{63}$$

around the singular line S (see also Figure 5). We will construct $Q_{\eta,\xi,n}$ as follows: Depending on the sign attributed to the connected component of T in Step 3 or the change between F and F^c in Step 2, we place a singularity of degree $-\frac{1}{2}$ (resp. $\frac{1}{2}$) as in Lemma 5.2 in [4] in the center of $S_{3\eta^\gamma}$. We do so by setting $Q = 0$ in a disk of radius ξ (perpendicular to S) and Q uniaxial with director field $(\sin(\phi/2), 0, \cos(\phi/2))$ on the annulus between the radii 2ξ and η, interpolating linearly in radial direction between these two regions. From the circle of radius η to the boundary of the region (63), we use the profile $\Phi^\pm_\eta$ to make a transition to Q_∞ along $\nabla \mathrm{dist}(\cdot,T)$. By doing so, we get the compatibility between the construction made for T and S.

More precisely, we define as in [4](Lemma 5.2, Step 3, Equation (55))

$$Q_B(r,\phi) \ :=\ \begin{cases} 0 & r \in [0,\xi)\,, \\ \left(\frac{r}{\xi} - 1\right) Q(\phi) & r \in [\xi, 2\xi)\,, \\ Q(\phi) & r \in [2\xi, \eta)\,, \end{cases}$$

where $r \in [0,\eta)$, $\phi \in [0, 2\pi)$ and

$$Q(\phi) \ =\ s_* \left(\mathbf{n}(\phi) \otimes \mathbf{n}(\phi) - \frac{1}{3}\mathrm{Id}\right) \quad \text{with} \quad \mathbf{n}(\phi) \ =\ \begin{pmatrix} \sin(\phi/2) \\ 0 \\ \cos(\phi/2) \end{pmatrix}.$$

We use this to define $Q_{\eta,\xi}$ on a small $\eta-$neighbourhood of S as follows. For η small enough, we can assume that the $\eta-$neighbourhood is parametrized by the projection onto S, the radius $\mathrm{dist}(\cdot,S)$ and an angle ϕ.

Modifying T close to S if necessary, we can furthermore assume that on each (small) plane disk perpendicular to S, the restriction of T to this disk is given by a straight line segment. Indeed, as in Lemma 6.7 we can select a radius $r \in (\eta, 2\eta)$ and a slice T_r of T at $\mathrm{dist}(\cdot,S) = r$ such that T_r is of finite length. One can then replace T by a $\tilde{T}$ inside the tubular neighbourhood $\{\mathrm{dist}(x,S) \leq r\}$ where $\tilde{T}$ is defined by the straight lines connecting S to T_r on each disk perpendicular to S.

Consider $x_0 \in S$. By applying rotations if necessary, we can assume that a normal vector of T_n in x_0 is $\nu_T = \mathbf{e}_3$ and the outward normal vector of S seen as boundary of T verifies $\nu_S = \mathbf{e}_1$. We then set

$$Q_{\eta,\xi,n}(x) \ :=\ Q_B(\mathrm{dist}(x,S), \phi(x))\,,$$

where

$$\phi(x) = \begin{cases} \arccos\left(\nu_S \cdot \frac{x-x_0}{\|x-x_0\|}\right) & \text{if } \nu_T \cdot \frac{x-x_0}{\|x-x_0\|} \geq 0\,, \\ 2\pi - \arccos\left(\nu_S \cdot \frac{x-x_0}{\|x-x_0\|}\right) & \text{otherwise.} \end{cases}$$

Note that if $\phi(x) \in (\frac{\pi}{2}, \frac{3\pi}{2})$, then $\mathrm{dist}(x,T) < \mathrm{dist}(x,S)$ and thus we can also write

$$\phi(x) \ =\ \arccos\left(\frac{\mathrm{dist}(x,T)}{\mathrm{dist}(x,S)}\right) + \frac{\pi}{2}\,.$$

It remains the transition from the set $\{\mathrm{dist}(\cdot,S)=\eta\}$ to the boundary of (63). Let Π be the projection along $\nabla\mathrm{dist}(\cdot,T)$ onto $\{\mathrm{dist}(\cdot,S)=\eta\}\cup(T\cap\{\mathrm{dist}(\cdot,S)\geq\eta\})$. The function $Q_{\eta,\xi}$ is already defined on the first set in the union, for the second we simply pose $Q_{\eta,\xi}(x)=s_*((v(x),0)\otimes(v(x),0)-\frac{1}{3}\mathrm{Id})$ in order to be compatible with Step 3. For $x\in S_{3\eta^\gamma}\setminus(\{\mathrm{dist}(\cdot,S)\leq\eta\}\cup(T\cap\{\mathrm{dist}(\cdot,S)\geq\eta\}))$ we then define

$$Q_{\eta,\xi}(x) \;:=\; \Phi^+_\eta(\|x-\Pi x\|,\theta(x),v(x))\,,$$

where $\theta(x)$ is the angle between $\mathbf{e}_3$ and the director field that we have already constructed in Πx, i.e. $\theta(x)=\arccos(\mathbf{n}(\phi(x))\cdot\mathbf{e}_3)$ or $\theta(x)=\arccos(v(\Pi x)\cdot\mathbf{e}_3)$ depending on which set contains Πx.

It is easy to see that since f,g and C_0 are uniformly bounded and the curvature of S is bounded by Lemma 6.4, we get

$$\begin{aligned}
&\int\frac{\eta}{2}|\nabla Q_{\eta,\xi,n}|^2+\frac{\eta}{\xi^2}f(Q_{\eta,\xi,n})+\frac{1}{\eta}g(Q_{\eta,\xi,n})+\eta C_0\,\mathrm{d}x\\
&\leq\;\frac{\eta}{2}\int_{2\xi}^{\eta}\int_{\{\mathrm{dist}(\cdot,S)=r\}}|\nabla Q(\phi(x))|^2\,\mathrm{d}r\;+\;C\frac{\eta}{\xi^2}\mathcal{H}^3(\{\mathrm{dist}(\cdot,S)\leq 2\xi\})\;+\;C\frac{\eta}{\eta^2}\mathcal{H}^3(\{\mathrm{dist}(\cdot,S)\leq\eta\})\\
&\leq\;\frac{\eta}{2}\int_{2\xi}^{\eta}\int_{\{\mathrm{dist}(\cdot,S)=r\}}|\nabla Q(\phi(x))|^2\,\mathrm{d}r\;+\;C\eta\,(1+K_S)\,\mathbb{M}(S)\,.
\end{aligned}$$

Estimating the gradient at distance $r:=\mathrm{dist}(\cdot,S)\in[2\xi,\eta)$ yields

$$\begin{aligned}
\frac{1}{2}|\nabla Q(\phi(x))|^2 \;&=\; s_*^2|\nabla\mathbf{n}(\phi(x))|^2\;\leq\;\frac{s_*^2}{4}\left|\frac{1}{r}\partial_\phi\mathbf{n}(\phi(x))\nabla\phi(x)\right|^2+\frac{s_*^2}{4}|\partial_r\mathbf{n}(x)|^2+C(|\nabla\tau_S|^2+|\nabla\nu_S|^2)\\
&\leq\;\frac{s_*^2}{4r^2}\,(1+Cr)+C\,.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{\eta}{2}\int_{2\xi}^{\eta}\int_{\{\mathrm{dist}(\cdot,S)=r\}}|\nabla Q(\phi(x))|^2\,\mathrm{d}r\;&\leq\;\frac{s_*^2\pi}{2}\mathbb{M}(S)\int_{2\xi}^{\eta}\frac{1}{r}\,\mathrm{d}r+o(1)\\
&\leq\;\frac{\pi}{2}s_*^2\eta|\ln(\xi)|\mathbb{M}(S)+o(1)\,.
\end{aligned}$$

For the remaining part of the domain defined in (63) we get that

$$\eta\int\frac{1}{2}|\nabla Q_{\eta,\xi,n}|^2+\frac{1}{\xi^2}f(Q_{\eta,\xi,n})+\frac{1}{\eta^2}g(Q_{\eta,\xi,n})+C_0\,\mathrm{d}x\;\leq\;C\eta\frac{(\eta^\gamma)^2}{\eta^2}\mathbb{M}(S)+o(1)\,.$$

Since we chose $\gamma>\frac{1}{2}$, we get that $\eta^{2\gamma-1}\to 0$ as $\eta\to 0$ and the energy contribution vanishes in the limit.

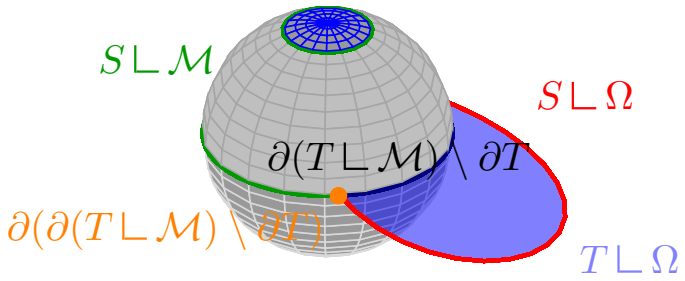

Figure 4: Schematic view of the different parts of T and S that are constructed in Step 2 to 6

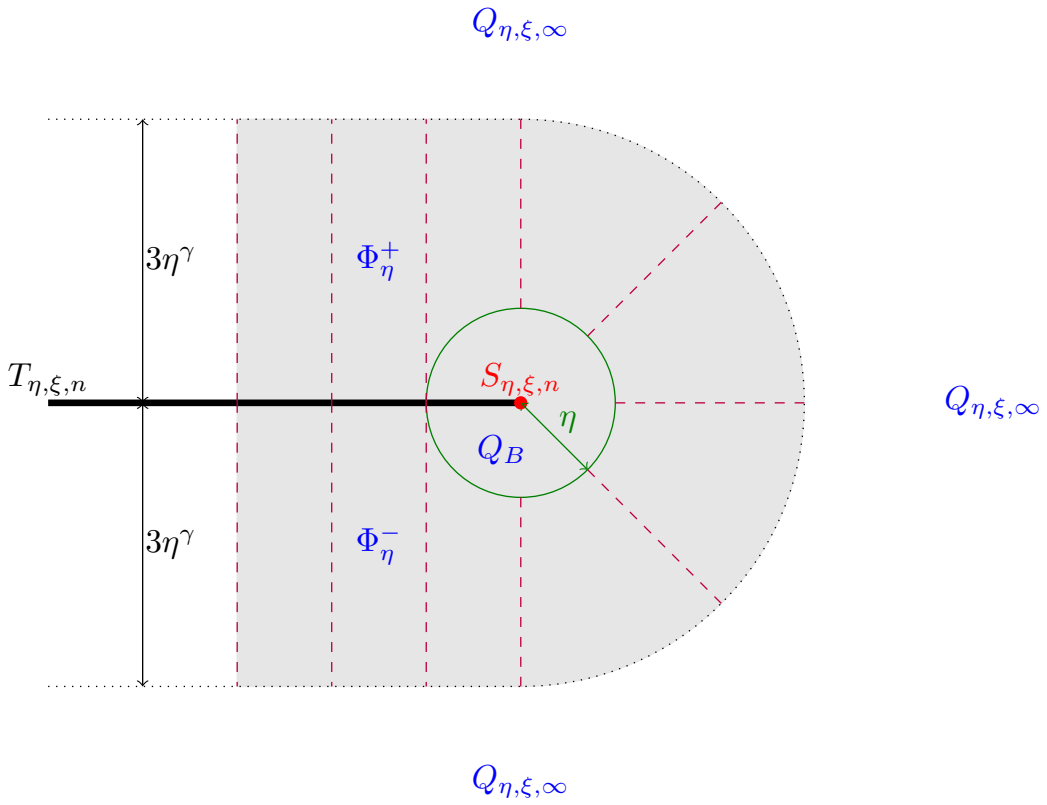

Figure 5: Sketch of the construction for $Q_{\eta,\xi,n}$ in Step 5 in the region $S_{3\eta^\gamma}$ defined in (63) (grey shaded area). Dashed lines indicate the direction of the projection Π.

Step 5: Construction on $S \llcorner \mathcal{M}$. The domain

$$S_{0,3\eta^\gamma} := \{x \in \Omega : \operatorname{dist}(x,S) \le 3\eta^\gamma, \operatorname{dist}(x,M) \le 3\eta^\gamma \text{ and } \operatorname{dist}(x, \partial(\partial(T \llcorner \mathcal{M}) \setminus \partial T)) \ge 3\eta^\gamma\}$$

can essentially be treated in the same manner as in Step 4 or as in [4, p.1444, Step 3]. To give some more details, we can reuse the profile Q_B from the previous step (assuming a $+\frac{1}{2}-$singularity) for defining $Q_{\eta,\xi}$ in a ball of radius η centered in x_S in the middle of $S_{0,3\eta^\gamma}$, seen as family of plane sets perpendicular to S. Note that $Q_{\eta,\xi}$ has already been defined on the boundary on each of those plane sets. Thus, a simple two dimensional interpolation of the phase angle along $\nabla \operatorname{dist}(\cdot, x_S)$ as in [4, eq (56-64)] shows that the energy contribution is

$$\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, S_{0,3\eta^\gamma}) \le (1 + C\eta)\frac{\pi}{2} s_*^2 |\ln(\xi)| \mathbb{M}(S \llcorner \mathcal{M}) + C\frac{(\eta^\gamma)^2}{\eta^2}.$$

Step 6: Construction on $\partial F \setminus S$. It remains to fill the "gaps" left by the Steps 2 to 5. The important part is the transition between the part of T that approaches $\mathcal{M}$ (and which was constructed in Step 2) and the part that stays bounded away, including the region where S detaches from $\mathcal{M}$. At distance larger than $3\eta^\gamma$ from $\mathcal{M}$, we set $Q_{\eta,\xi} = Q_{\eta,\xi,\infty}$ for all points where we haven't defined $Q_{\eta,\xi}$ so far. Note that this is compatible with the previous constructions.

Let's consider the set $\Upsilon_{3\eta^\gamma} := \{x \in \Omega : \operatorname{dist}(x, \partial(T \llcorner \mathcal{M}) \setminus \partial T) \le 3\eta^\gamma \text{ and } \operatorname{dist}(x, \partial(\partial(T \llcorner \mathcal{M}) \setminus \partial T)) \ge 3\eta^\gamma\}$. Considering the slices of $\Upsilon_{3\eta^\gamma}$ orthogonal to and parametrized by $\partial(T \llcorner \mathcal{M})$, we note that Steps 2 to 5 ensure that $Q_{\eta,\xi}$ takes values in $\mathcal{N}$ whenever meeting the boundary of the slice and $Q_{\eta,\xi}$ having trivial homotopy class. For an arbitrary $Q \in N$, we can define $Q_{\eta,\xi} := Q$ on a disk of size η in the middle of the slice and again by linear interpolation of the phase towards the boundary of the disk. We thus get a function $Q_{\eta,\xi} \in H^1(\Upsilon_{0,3\eta^\gamma}, \mathcal{N})$ respecting the previous constructions and $Q = Q_b$ on $\mathcal{M}$. Furthermore, the interpolation allows us to estimate $|\nabla Q|^2 \le C((\eta^\gamma)^{-2} + \eta^{-2})$ and since g is bounded, $f(Q) = 0$ (because Q takes values in $\mathcal{N}$) the energy contribution can be estimated

$$\begin{aligned} \eta \mathcal{E}_{\eta,\xi}(Q, \Upsilon_{0,3\eta^\gamma}) &\le C\, \eta\, |\Upsilon_{0,3\eta^\gamma}| \left(\frac{1}{(\eta^\gamma)^2} + \frac{1}{\eta^2} + 1\right) \\ &\le C\, \mathbb{M}(\partial(T \llcorner \mathcal{M}) \setminus \partial T) \left(\eta + \frac{(\eta^\gamma)^2}{\eta} + \eta(\eta^\gamma)^2\right), \end{aligned}$$

which vanishes in the limit $\eta, \xi \to 0$ due to our hypothesis about the size of $\partial(T \llcorner \mathcal{M}) \setminus \partial T$.

It remains the region where S detaches from $\mathcal{M}$ or in other words $\Upsilon_{1,3\eta^\gamma} := \{x \in \Omega : \operatorname{dist}(x, \partial(\partial(T \llcorner \mathcal{M}) \setminus S)) \le 3\eta^\gamma\}$. We can also use interpolation to construct $Q_{\eta,\xi}$ and estimate its energy but we need to be a bit more careful since this time $f(Q_{\eta,\xi})$ cannot be chosen to be zero. This is due to the isotropic core of our construction around S. So we connect the 'core' parts from Step 4 and 5 where we defined S in Ω and close to $\mathcal{M}$ that is the profile Q_B which has been used in both steps. Around this tube, we can again apply the previous idea of linear interpolation of the phase, this time on slices perpendicular to the tube. We end up with

$$\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}, \Upsilon_{1,3\eta^\gamma}) \le C\eta^\gamma \mathbb{M}(\partial(\partial(T \llcorner \mathcal{M}) \setminus \partial T)),$$

which vanishes in the limit $\eta \to 0$ in view of the bound $\mathbb{M}(\partial(\partial(T \llcorner \mathcal{M}) \setminus \partial T)) \le C|\ln(\eta)|$. □

7 Regularity and optimality conditions for the limit problem

Let us first state an improved regularity results for minimizers of the energy $\mathcal{E}_0$:

Proposition 7.1. *Let T be a minimizer of (14) and $S = \partial T - \Gamma$. Then each component of $T \llcorner \Omega$ is an embedded manifold-with-boundary of class C^∞.*

Proof. The main work has been already carried out in the proof of Proposition 6.1 for $n = 0$. The higher regularity can be obtained by Schauder theory. For details we refer to Theorem 2.1 in [54]. □

Next, we give a characterization of minimizers of the limit energy. Because of the regularity given by Proposition 7.1, we can take variations of $T \llcorner \Omega$ and $S \llcorner \Omega$ in the classical sense to derive the optimality conditions. Furthermore, we can obtain a version of Young's law [58,69]

Proposition 7.2. *Let T be a minimizer of (14) and $S = \partial T - \Gamma$. Then $T \llcorner \Omega$ has zero mean curvature and $S \llcorner \Omega$ is of constant curvature $\frac{8}{\pi}\frac{c_*}{s_*}\beta^{-1}$. Furthermore, Young's law holds*

$$\nu_{\partial(T\cap\Omega)} \cdot \nu_{\partial F^+} \;=\; \nu_{\mathcal{M}} \cdot \mathbf{e}_3 \quad \text{on } \partial(T \llcorner \Omega) \setminus S\,,$$

i.e. T meets $\mathcal{M}$ in an angle $\theta = \arccos(\nu_{\mathcal{M}} \cdot \mathbf{e}_3)$.

Proof. The first claim is a well known fact since the variation of $\mathbb{M}(T \llcorner \Omega)$ along a smooth vector field Ξ in Ω yields [44, Proposition 2.1.3]

$$(\mathbb{M}(T \llcorner \Omega))'(\Xi) \;=\; \int_{T\cap\Omega} H_T(\Xi \cdot \nu_T)\,\mathrm{d}x \;+\; \int_{\partial(T\cap\Omega)} (\Xi \cdot \nu_{\partial T})\,\mathrm{d}x\,, \tag{64}$$

where H_T is the mean curvature of T, ν_T is a normal vector of T and $\nu_{\partial T}$ is the inward normal vector of $\partial(T \llcorner \Omega)$ in the tangent space of T. With the same argument and since $\partial S = 0$, we get that

$$(\mathbb{M}(S))'(\Xi) \;=\; \int_S K_S(\Xi \cdot \nu_S)\,\mathrm{d}x\,, \tag{65}$$

where K_S is the curvature of S and ν_S is the normal vector of S in $\mathbb{R}^3$, such that the plane for the circle of maximal curvature is spanned by ν_S and a tangent vector to S. This yields for the boundary that

$$0 \;=\; \int_S \Xi \cdot \Big(4 s_* c_* \nu_{\partial T} + \frac{\pi}{2} s_*^2 \beta K_S \nu_S\Big)\,\mathrm{d}x\,,$$

from which we deduce $\nu_{\partial T} = \pm\nu_S$ and $K_S = \pm\frac{8}{\pi}\frac{c_*}{s_*}\beta^{-1}$. In particular, the circle of maximal curvature for S lies in the plane spanned by the tangent space of T. Finally, taking variations on $\mathcal{M}$ we get

$$\Big(\int_{F^\pm} 1 \mp \cos(\theta)\,\mathrm{d}\omega\Big)'(\Xi) \;=\; \int_{\partial F^\pm} (1 \mp \cos(\theta))\,(\Xi \cdot \nu_{\partial F^\pm})\,\mathrm{d}\omega\,.$$

Since $\nu_{\partial F^-} = -\nu_{\partial F^-}$, we hence get

$$\Big(\int_{F^+} 1 - \cos(\theta)\,\mathrm{d}\omega + \int_{F^-} 1 + \cos(\theta)\,\mathrm{d}\omega\Big)'(\Xi) \;=\; -\int_{\partial F^+} 2\cos(\theta)(\Xi \cdot \nu_{\partial F^+})\,\mathrm{d}\omega\,. \tag{66}$$

As in the proof of Theorem 19.8 in [45], (64) and (66) combine to

$$0 \;=\; \int_{\partial F^+} \Xi \cdot (4 s_* c_* \nu_T|_{\mathcal{M}} - 4 s_* c_* \cos(\theta)\nu_{\partial F^+})\;\mathrm{d}\,.$$

If we take a variation with $\Xi \cdot \nu_{\mathcal{M}} = 0$ and write

$$\Xi \cdot \nu_T|_{\mathcal{M}} \;=\; \Xi \cdot ((\nu_{\partial T} \cdot \nu_{\partial F^+})\nu_{\partial F^+}) \;+\; \Xi \cdot ((\nu_T|_{\mathcal{M}} \cdot \tau)\tau)$$

where τ is a unit tangent vector to $\mathcal{M}$, perpendicular to $\nu_{\partial F^+}$, we get

$$\Xi \cdot ((\nu_{\partial T} \cdot \tau)\tau) \;=\; 0 \qquad \text{and} \qquad \nu_{\partial T} \cdot \nu_{\partial F^+} \;=\; \cos(\theta)\,.$$

The first equality is automatically true since $\nu_{\partial T} \cdot \tau = 0$ ($\nu_{\partial T}$ can only have a component in direction $\nu_{\partial F^+}$ and one in direction $\nu_{\mathcal{M}}$) and the second one implies that

$$\nu_{\partial T} \cdot \nu_{\partial F^+} \;=\; \nu_{\mathcal{M}} \cdot \mathbf{e}_3\,.$$

□

A The complex $\mathcal{T}$

In this section, we collect and prove all results in connection to the structure of $\mathcal{T}$ as defined in Section 4.3. Recall that

$$\mathcal{T} \coloneqq \{Q \in \mathrm{Sym}_0 \,:\, s > 0\,, 0 \leq r < 1\,, n_3 = 0\}\,.$$

Our first result is a characterization of $\mathcal{T}$ that provides us with a more accessible parametrization.

Proposition A.1. *Every matrix $Q \in \mathcal{T}$ can be written as*

$$Q = \lambda(\mathbf{n} \otimes \mathbf{n} - R_{\mathbf{n}}^{\top} M R_{\mathbf{n}})\,,$$

where $\lambda > 0$, $\mathbf{n} = (n_1, n_2, 0) \in \mathbb{S}^2$, $R_{\mathbf{n}}$ is the rotation around $\mathbf{n} \wedge \mathbf{e}_3$, such that $R_{\mathbf{n}}\mathbf{n} = \mathbf{e}_3$ and

$$M \;=\; \begin{pmatrix} & & 0 \\ & M' & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $M' \in \mathbb{R}^{2\times 2}$ symmetric, $\mathrm{tr}(M') = 1$ and $\langle M'v, v\rangle > -1$ for all $v \in \mathbb{S}^1$. The matrix Q is uniaxial if and only if $M' = \frac{1}{2}\mathrm{Id}$.

Proof. A matrix Q of the above form $Q = \lambda(\mathbf{n} \otimes \mathbf{n} - R_{\mathbf{n}}^{\top} M R_{\mathbf{n}})$ has $\mathbf{n}$ as an eigenvector to the eigenvalue λ and $\mathbf{n}_3 = 0$ by definition. Furthermore, since $\min_{v\in\mathbb{S}^1}\langle M'v, v\rangle > -1$ the eigenvalue λ is strictly bigger than the other eigenvalues, thus $r < 1$ and $Q \in \mathcal{T}$. Conversly, we can write every $Q \in \mathrm{Sym}_0$ as

$$Q \;=\; \lambda_1 \mathbf{n} \otimes \mathbf{n} + \lambda_2 \mathbf{m} \otimes \mathbf{m} + \lambda_3 \mathbf{p} \otimes \mathbf{p}\,,$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\mathbf{n}, \mathbf{m}, \mathbf{p} \in \mathbb{S}^2$ pairwise orthogonal eigenvectors of Q to $\lambda_1, \lambda_2, \lambda_3$. By definition of $\mathcal{T}$, $n_3 = 0$ as required for our parametrization and clearly we can identify $\lambda = \lambda_1$. Setting $M = -R_{\mathbf{n}}(\frac{\lambda_2}{\lambda_1}\mathbf{m} \otimes \mathbf{m} + \frac{\lambda_3}{\lambda_1}\mathbf{p} \otimes \mathbf{p})R_{\mathbf{n}}^{\top}$, it is obvious that M is of the above form and that $Q \in \mathcal{T}$ can be written as claimed.

If $M' = \frac{1}{2}\mathrm{Id}$ then

$$Q \;=\; \lambda(\mathbf{n} \otimes \mathbf{n} - R_{\mathbf{n}}^{\top} M R_{\mathbf{n}}) \;=\; \frac{3}{2}\lambda(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathrm{Id})\,,$$

i.e. Q is uniaxial. The reversed implication follows similarly, since the matrices $R_{\mathbf{n}}^{\top}, R_{\mathbf{n}}$ are invertible.

□

Remark A.2. *Given a vector $u \in \mathbb{R}^3$ as axis of rotation and an angle θ, then this rotation is described by the matrix R with*

$$R = \begin{pmatrix} \cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) - u_3\sin\theta & u_1u_3(1-\cos\theta) + u_2\sin\theta \\ u_1u_2(1-\cos\theta) + u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) - u_1\sin\theta \\ u_1u_3(1-\cos\theta) - u_2\sin\theta & u_2u_3(1-\cos\theta) + u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta) \end{pmatrix}.$$

Corollary A.3. *$\mathcal{T}$ is a four dimensional smooth complex and $\partial\mathcal{T} = \mathcal{S}$.*

Proof. From the characterization in Proposition A.1, it is clear that one can use the map $Q \mapsto (\lambda, \mathbf{n}, m_{11}, m_{12})$ to make $\mathcal{T}$ a four dimensional manifold with a conical singularity in $Q = 0$. In particular, $\mathcal{T}$ is a smooth complex.

Proposition A.1 furthermore implies that the boundary of $\mathcal{T}$ consists of matrices of the form $\lambda = 0$ (from which follows directly $Q = 0$) or M' has the eigenvalue -1 (which corresponds to $r = 1$). In particular, the matrices with $r = 0$ are not included in $\partial\mathcal{T}$ as one may think from

the definition in (31). This implies the inclusion $\partial\mathcal{T} \subset \mathcal{S}$. For the inverse inclusion, take $Q \in \mathcal{S}$ with orthogonal eigenvectors $\mathbf{m}, \mathbf{p} \in \mathbb{S}^2$ associated to the biggest eigenvalue $\lambda_1 = \lambda_2$. So in fact we have a two dimensional subspace of eigenvectors to this eigenvalue spanned by $\mathbf{m}$ and $\mathbf{p}$. Since the hyperplane defined through $\{n_3 = 0\}$ is of codimension one, there exists a unit vector $\mathbf{n} \in \{n_3 = 0\} \cap \operatorname{span}\{\mathbf{m}, \mathbf{p}\}$ which we were looking for. The unit eigenvector orthogonal to $\mathbf{n}$ in the plane $\operatorname{span}\{\mathbf{m}, \mathbf{p}\}$ requires M' to have the eigenvalue -1 or in other words $\min_{v \in \mathbb{S}^1} \langle M'v, v\rangle = -1$, so that $Q \in \partial\mathcal{T}$. □

Lemma A.4. *Let $Q \in \mathcal{T} \cap \mathcal{N}$. Then, the normal vector N_Q on $\mathcal{T}$ at Q is given by*

$$N_Q = \frac{3}{2}\lambda \begin{pmatrix} 0 & 0 & n_1 \\ 0 & 0 & n_2 \\ n_1 & n_2 & 0 \end{pmatrix},$$

where $\mathbf{n} = (n_1, n_2, 0) \in \mathbb{S}^2$ is the eigenvector associated to the biggest eigenvalue λ_1.

Proof. We are going to prove a slightly more general result by first considering $Q \in \mathcal{T}$ and calculating the tangent vectors to $\mathcal{T}$ in Q. We use the representation from Proposition A.1 and vary $\lambda, \mathbf{n}, m_{11}, m_{12}$ one after another.

- First, we can easily take the derivative with respect to λ and obtain $\mathbf{T}_1 = (\mathbf{n} \otimes \mathbf{n} - R_{\mathbf{n}}^\top M R_{\mathbf{n}})$.
- Second, we vary the parameter $\mathbf{n}$. So, let's consider $\mathbf{n} = (n_1, n_2, 0) \in \mathbb{S}^2$. Without loss of generality we assume that $n_2 \neq 0$ and write $\mathbf{n}(t) = (n_1 + t, n_2 - \frac{n_1}{n_2}t)$. Then $|\mathbf{n}(t)|^2 = 1 + O(t^2)$ and

$$\mathbf{n}(t) \otimes \mathbf{n}(t) \;=\; \mathbf{n} \otimes \mathbf{n} + tD_{\mathbf{n}\otimes\mathbf{n}} + O(t^2), \qquad D_{\mathbf{n}\otimes\mathbf{n}} = \begin{pmatrix} 2n_1 & n_2 - \frac{n_1^2}{n_2} & 0 \\ n_2 - \frac{n_1^2}{n_2} & -2n_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  The derivative of the second term $R_{\mathbf{n}(t)}^\top M R_{\mathbf{n}(t)}$ can be calculated using Remark A.2 with the axis $\mathbf{n}^\perp(t) := \mathbf{n}(t) \wedge \mathbf{e}_3$. Since $\mathbf{n}(t) \perp \mathbf{e}_3$ we can write

$$R_{\mathbf{n}(t)} \;=\; R_{\mathbf{n}} + tD_{R_{\mathbf{n}}} + O(t^2), \qquad D_{R_{\mathbf{n}}} = \frac{1}{n_2}\begin{pmatrix} -2n_1n_2 & -n_2^2 + n_1^2 & -n_2 \\ -n_2^2 + n_1^2 & 2n_1n_2 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}.$$

  The second tangent vector $\mathbf{T}_2$ is thus given by $\mathbf{T}_2 = \lambda(D_{\mathbf{n}\otimes\mathbf{n}} - D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}} - R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}})$.
- Third, we can take the derivative with respect to m_{11}. This is straightforward and we obtain

$$\mathbf{T}_3 = \lambda R_{\mathbf{n}}^\top \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix} R_{\mathbf{n}}.$$

- Last, varying m_{12} we easily calculate

$$\mathbf{T}_4 = \lambda R_{\mathbf{n}}^\top \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} R_{\mathbf{n}}.$$

Before proceeding, we want to calculate a fifth vector by varying $\mathbf{n}_3$. As it will turn out later, this is indeed the normal vector.

- Writing once again $\mathbf{n} = (n_1, n_2, 0)$ and defining $\mathbf{n}(t) := (n_1\sqrt{1-t^2}, n_2\sqrt{1-t^2}, t)$ we can express

$$\mathbf{n}(t) \otimes \mathbf{n}(t) \;=\; \mathbf{n} \otimes \mathbf{n} + t(\mathbf{n} \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{n}) + O(t^2).$$

As for the second tangent vector, we use Remark A.2 and the rotation around $\mathbf{n}^\perp(t) = \mathbf{n}(t) \wedge \mathbf{e}_3$. Unlike previously, $\mathbf{n}(t)$ is no longer orthogonal to $\mathbf{e}_3$ for $t \neq 0$, namely $\theta = \arccos(\langle \mathbf{n}(t), \mathbf{e}_3 \rangle) = t$. Substituting this our expression of the rotation matrix we get

$$R_{\mathbf{n}(t)} \;=\; R_{\mathbf{n}} + tD_3 + O(t^2)\,, \qquad D_3 = \begin{pmatrix} 1-n_2^2 & n_1 n_2 & 0 \\ n_1 n_2 & 1-n_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Adding the two partial results, we get

$$N \;\coloneqq\; \lambda(\mathbf{n}\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{n} - D_3^\top M R_{\mathbf{n}} - R_{\mathbf{n}}^\top M D_3)\,.$$

It remains to show that $\{\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, N\}$ are pairwise orthogonal if Q is uniaxial. Indeed, then it follows that N is a normal vector, since it is orthogonal to $T_Q\mathcal{T}$.

It is easy to see that since the trace is invariant by change of basis and since $R_{\mathbf{n}}^\top = R_{\mathbf{n}}^{-1}$

$$\langle \mathbf{T}_3, \mathbf{T}_4 \rangle \;=\; \lambda^2 \mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \;=\; \lambda^2 \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \;=\; 0\,.$$

Noting that $\mathbf{n}\otimes\mathbf{n} R_{\mathbf{n}}^\top M R_{\mathbf{n}} = 0$ for $M \in \mathrm{Sym}_0$ with $m_{ij} = 0$ if $i = 3$ or $j = 3$, we get

$$\begin{aligned} \langle \mathbf{T}_1, \mathbf{T}_3 \rangle \;&=\; \lambda \mathrm{tr}\Big((\mathbf{n}\otimes\mathbf{n} - R_{\mathbf{n}}^\top M R_{\mathbf{n}})(R_{\mathbf{n}}^\top \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix} R_{\mathbf{n}}) \Big) \\ &=\; \lambda \mathrm{tr}(M \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}) \;=\; \lambda \mathrm{tr}(\begin{pmatrix} m_{11} & -m_{12} & \\ m_{12} & -m_{22} & \\ & & 0 \end{pmatrix}) \;=\; \lambda(2m_{11} - 1)\,. \end{aligned}$$

With the same argument we also find

$$\begin{aligned} \langle \mathbf{T}_1, \mathbf{T}_4 \rangle \;&=\; \lambda \mathrm{tr}\Big((\mathbf{n}\otimes\mathbf{n} - R_{\mathbf{n}}^\top M R_{\mathbf{n}})(R_{\mathbf{n}}^\top \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} R_{\mathbf{n}}) \Big) \\ &=\; \lambda \mathrm{tr}(M \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}) \;=\; \lambda \mathrm{tr}(\begin{pmatrix} m_{12} & m_{11} & \\ m_{22} & m_{12} & \\ & & 0 \end{pmatrix}) \;=\; 2\lambda m_{12}\,. \end{aligned}$$

Furthermore, we claim that

$$\langle \mathbf{T}_1, \mathbf{T}_2 \rangle \;=\; \lambda\, \mathrm{tr}((\mathbf{n}\otimes\mathbf{n} - R_{\mathbf{n}}^\top M R_{\mathbf{n}})(D_{\mathbf{n}\otimes\mathbf{n}} - D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}} - R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}})) \;=\; 0\,.$$

Indeed, one can check that

$$\begin{aligned} \mathrm{tr}(\mathbf{n}\otimes\mathbf{n} D_{\mathbf{n}\otimes\mathbf{n}}) \;&=\; 0 \;=\; \mathrm{tr}(\mathbf{n}\otimes\mathbf{n} D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}})\,, \\ \mathrm{tr}(\mathbf{n}\otimes\mathbf{n} R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}}) \;&=\; 0 \;=\; \mathrm{tr}(R_{\mathbf{n}}^\top M R_{\mathbf{n}} D_{\mathbf{n}\otimes\mathbf{n}})\,, \\ \mathrm{tr}(R_{\mathbf{n}}^\top M R_{\mathbf{n}} D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}}) \;&=\; 0 \;=\; \mathrm{tr}(R_{\mathbf{n}}^\top M R_{\mathbf{n}} R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}})\,. \end{aligned}$$

This implies that

$$\langle N, \mathbf{T}_3 \rangle \;=\; \lambda^2 \mathrm{tr}\Big((\mathbf{n}\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{n} - D_3^\top M R_{\mathbf{n}} - R_{\mathbf{n}}^\top M D_3)(R_{\mathbf{n}}^\top \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix} R_{\mathbf{n}}) \Big) \;=\; 0\,,$$

since again the traces of all cross terms vanish. Similarly,

$$\langle N, \mathbf{T}_4 \rangle \;=\; 0\,.$$

Next, we have the equality

$$\langle \mathbf{T}_2, \mathbf{T}_3 \rangle \;=\; -4\lambda^2 \frac{m_{12}}{n_2}\,.$$

This follows since $\operatorname{tr}(D_{\mathbf{n}\otimes\mathbf{n}}\mathbf{T}_3) = 0$ and $\operatorname{tr}(D_3 M R_{\mathbf{n}}\mathbf{T}_3) = \frac{2m_{12}}{n_2}$. The latter fact is evident if one calculates $M\begin{pmatrix}1 & 0 & \\ 0 & -1 & \\ & & 0\end{pmatrix} = \begin{pmatrix}m_{11} & -m_{12} & \\ m_{12} & -m_{22} & \\ & & 0\end{pmatrix}$ and $R_{\mathbf{n}}D_3^\top = \begin{pmatrix}0 & -1/n_2 & 1\\ 1/n_2 & 0 & -n_1/n_2\\ -1 & n_1/n_2 & 0\end{pmatrix}$.
This also permits us to derive

$$\langle \mathbf{T}_2, \mathbf{T}_4\rangle \;=\; 2\lambda^2\frac{2m_{11}-1}{n_2}\,.$$

Again, we simply calculate the traces of all cross terms. For example

$$\begin{aligned}
\operatorname{tr}(\mathbf{n}\otimes\mathbf{e}_3 D_{\mathbf{n}\otimes\mathbf{n}}) &= 0\,,\\
\operatorname{tr}(\mathbf{n}\otimes\mathbf{e}_3 R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}}) &= 0\,,\\
\operatorname{tr}(\mathbf{n}\otimes\mathbf{e}_3 D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}}) &= \frac{m_{12}}{n_2}(n_1^2-n_2^2) - n_1(2m_{11}-1)\,,\\
\operatorname{tr}(D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}} D_{\mathbf{n}\otimes\mathbf{n}}) &= 2\frac{m_{11}n_1}{n_2} + \frac{1}{n_2^2}(n_1^2(2m_{11}-1)+m_{11})\,,\\
\operatorname{tr}(D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}} R_{\mathbf{n}}^\top M D_{R_{\mathbf{n}}}) &= -2\frac{n_1 m_{12}}{n_2} + \frac{1}{n_2^2}\left(3(m_{11}^2+m_{12}^2) - (1+n_1^2)(2m_{11}-1)\right)\,,\\
\operatorname{tr}(D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}} D_{R_{\mathbf{n}}}^\top M R_{\mathbf{n}}) &= 2\frac{m_{11}m_{22}+m_{12}^2}{n_2^2}\,,
\end{aligned}$$

We end up with

$$\langle N, \mathbf{T}_2\rangle \;=\; \frac{6\lambda^2 m_{12}(n_1^2-n_2^2)}{n_2} - 6\lambda^2 n_1(2m_{11}-1)\,.$$

Another straightforward calculation shows that

$$\begin{aligned}
\langle N, \mathbf{T}_1\rangle \;=\;& \lambda 2 n_1 m_{12}(n_1^5 m_{12} - 2n_1^4 n_2 m_{11} - 2n_1^3 m_{12} - 2n_1^2 n_2^3 m_{11} + 3n_1^2 n_2 m_{11}\\
&- 2n_1^2 n_2 - n_1 n_2^4 m_{12} + n_1 m_{12} + n_2^3 m_{11} - n_2 m_{11} - 2n_2^3 + 2n_2)\,.
\end{aligned}$$

After these calculations, it is apparent that for $Q\in\mathcal{N}$, i.e. $M' = \frac{1}{2}\mathrm{Id}$ all inner products vanish. In order to form a basis, we must prove that the vectors themselves never vanish. We find

$$\begin{aligned}
\|\mathbf{T}_1\|^2 &= 2(m_{11}^2 - m_{11} + m_{12} + 1)\,,\\
\|\mathbf{T}_2\|^2 &= \frac{2}{n_2^2}(6n_1^2(1-2m_{11}) - 6m_{12}n_1n_2 + 5m_{11}^2 - 2m_{11} + 5m_{12} + 2)\,,\\
\|\mathbf{T}_3\|^2 &= 2\lambda^2\,,\\
\|\mathbf{T}_4\|^2 &= 2\lambda^2\,,\\
\|N\|^2 &= \lambda^2(12m_{11}n_1^2 - 6n_1^2 + 12m_{12}n_1n_2 + 2m_{11}^2 - 8m_{11} + 2m_{12}^2 + 8)\,,
\end{aligned}$$

and thus for $M' = \frac{1}{2}\mathrm{Id}$ it holds that $\|\mathbf{T}_1\|^2 = \frac{6}{4}$, $\|\mathbf{T}_2\|^2 = \frac{9}{2}\lambda^2 n_2^{-2}$ and $\|N\|^2 = \frac{9}{2}\lambda^2$.

This concludes the proof that $\{\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4\}$ form indeed a basis of $T_Q\mathcal{T}$, and since N is orthogonal to $T_Q\mathcal{T}$, the result follows. □

Proposition A.5. *There exists $C, \alpha_0 > 0$ such that for all $\alpha\in(0,\alpha_0)$ and $Q\in\mathcal{N}$ it holds*

$$\mathcal{H}^4(B_\alpha(Q)\cap\mathcal{T}) \;\le\; C\alpha^4\,.$$

Proof. As seen before, $\mathcal{T}$ has the structure of a smooth manifold around $\mathcal{N}$. By invariance of $\mathcal{N}$ under rotations, it is enough to show that the claim holds around one $Q\in\mathcal{N}$. The Ricci curvature κ of $\mathcal{N}$ is bounded so that we can choose $\alpha_0 > 0$ small enough such that $B_\alpha(Q)\cap\mathcal{T}$ is contained in the geodesic ball in $\mathcal{T}$ of size 2α around Q for all $\alpha\in(0,\alpha_0)$. Furthermore, if needed, we can choose $\alpha_0 > 0$ even smaller such that $1 - \frac{\kappa}{36\alpha_0^2} \le 2$. Theorem 3.1 in [29] then implies that

$$\mathcal{H}^4(B_\alpha(Q)\cap\mathcal{T}) \;\le\; \mathrm{vol}_{\mathcal{T}}(B_{2\alpha}(Q)) \;\le\; 16\pi^2\alpha^4\,.$$

□

Acknowledgment. The authors thank Antoine Lemenant and Guy David for the useful discussions and comments on the regularity for almost minimizers of length and surface.

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