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# ON QUOTIENTS OF BOUNDED HOMOGENEOUS DOMAINS BY UNIPOTENT DISCRETE GROUPS

CHRISTIAN MIEBACH

ABSTRACT. We show that the quotient of any bounded homogeneous domain by a unipotent discrete group of automorphisms is holomorphically separable. Then we give a necessary condition for the quotient to be Stein and prove that in some cases this condition is also sufficient.

## 1. INTRODUCTION

Given a Lie group of holomorphic transformations of a Stein space  $X$ , one would like to have a complex quotient space whose holomorphic functions are the invariant holomorphic functions on  $X$  and which is again Stein. In the case that the Lie group is *compact*, it is possible to average holomorphic functions over this group and to construct in this way an invariant-theoretic Stein quotient space, see [Hei91]. On the other hand, when an infinite discrete group acts properly by holomorphic transformations on a Stein space, then the orbit space is again a complex space. However, in general there is no averaging method in order to construct holomorphic functions on this quotient space. In fact, it is not hard to find examples where the quotient space is compact, hence where all invariant holomorphic functions are constant.

There are numerous results in the literature where certain quotients of Stein manifolds by proper actions of infinite discrete groups are shown to be holomorphically separable or Stein. In [dFI01] the authors showed that quotients of the unit ball  $\mathbb{B}_n \subset \mathbb{C}^n$  by proper  $\mathbb{Z}$ -actions are Stein, which was then generalized to simply-connected bounded domains of holomorphy in  $\mathbb{C}^2$  ([MO09]), to arbitrary bounded homogeneous domains in  $\mathbb{C}^n$  ([Mie10a]), and to Akhiezer-Gindikin domains ([Vit14]). Quotients of the unit ball and Akhiezer-Gindikin domains by discrete groups that act cocompactly on a real form of these domains are shown to be Stein in [BS76, Proposition 6.4] and [BHH03, Corollary 7], respectively. An analogous result is shown in [Loe85] for quotients of complex solvable Lie groups by discrete subgroups that act cocompactly on a real form having purely imaginary spectrum. Quotients of complex Olshanski semigroups by certain discrete groups were studied in [ABK04] and [Mie10b]. In [Che13] actions of discrete groups on Kähler-Hadamard manifolds and their quotients are investigated. In [MO18] Schottky group actions on the unit ball having Stein quotients are constructed. Most of these results strongly rely on Lie theory.

In this paper, we are concerned with the action of a *unipotent* discrete group  $\Gamma$  of holomorphic automorphisms of a bounded homogeneous domain. To state the main results, let  $D \subset \mathbb{C}^n$  be a bounded domain and recall that its automorphism group  $\text{Aut}(D)$  is a real Lie group that acts properly on  $D$ , see [Car35]. Let  $G$  be the connected component of  $\text{Aut}(D)$  that contains the identity. The domain  $D$  is *homogeneous* if  $\text{Aut}(D)$  and hence  $G$  act transitively on it. Let us fix a base point  $p_0 \in D$ . Its isotropy group  $K := G_{p_0}$  is a maximal compact subgroup of  $G$  and there exists a decomposition  $G = KR$  where  $R$  is a simply-connected split

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solvable Lie group, see [Kan67] and [Vin61]. Consequently,  $R$  is isomorphic to a semi-direct product  $A \ltimes N$  where  $N$  is the nilradical of  $R$  and  $A \cong (\mathbb{R}^{>0})^r$ .

Let  $D \subset \mathbb{C}^n$  be a bounded homogeneous domain and let  $G = KAN$  be the decomposition of  $G = \text{Aut}^0(D)$  introduced above. A discrete subgroup of  $G$  will be called *unipotent* if it is conjugate to a subgroup of  $N$ . The research presented in this paper was motivated by the following result.

**Proposition 1.1.** *Let  $D \subset \mathbb{C}^n$  be a bounded homogeneous domain and let  $\Gamma$  be a unipotent discrete group of automorphisms of  $D$ . Then the complex manifold  $D/\Gamma$  is holomorphically separable.*

It is therefore natural to ask under which conditions the quotient manifold  $D/\Gamma$  is Stein for unipotent discrete  $\Gamma$ . Suppose from now on that  $\Gamma$  is a discrete subgroup of  $N$ . It is well known that the simply-connected nilpotent group  $N$  admits a unique structure as a real-algebraic group such that its Zariski closed subgroups are precisely its connected Lie subgroups, see [Vin94, Chapter 2.4.2]. Hence, we can define the Zariski closure  $N_\Gamma$  of  $\Gamma$  in  $N$ . Then  $N_\Gamma$  is again a simply-connected nilpotent Lie group such that  $N_\Gamma/\Gamma$  is compact.

We have the following necessary condition for  $D/\Gamma$  to be Stein.

**Proposition 1.2.** *Let  $\Gamma \in N$  be a discrete subgroup with Zariski closure  $N_\Gamma$ . If  $D/\Gamma$  is Stein, then all  $N_\Gamma$ -orbits in  $D$  are totally real.*

Since  $R \cong A \ltimes N$  acts simply transitively on  $D$ , we can identify its Lie algebra  $\mathfrak{r} = \mathfrak{a} \oplus \mathfrak{n}$  with  $T_{p_0}D = \mathbb{C}^n$  and thus obtain a complex structure  $j \in \text{End}(\mathfrak{r})$ . Let  $N_\Gamma$  be the Zariski closure of a discrete subgroup  $\Gamma \subset N$  and let  $\mathfrak{n}_\Gamma$  be its Lie algebra. The orbit  $N_\Gamma \cdot p_0$  is totally real if and only if  $\mathfrak{n}_\Gamma \cap j(\mathfrak{n}_\Gamma) = \{0\}$ , in which case we call  $\mathfrak{n}_\Gamma$  a *totally real subalgebra* of  $\mathfrak{n}$ . We have the following sufficient criterion for  $D/\Gamma$  to be Stein.

**Proposition 1.3.** *Let  $\Gamma \in N$  be a discrete subgroup with Zariski closure  $N_\Gamma$ . If  $\mathfrak{n}_\Gamma$  is contained in a maximal totally real subalgebra of  $\mathfrak{n}$ , then  $D/\Gamma$  is Stein.*

The methods and results described above allow us to prove the main result of this paper, which in particular answers the question raised in [Che13, Remark 7.6] in the negative.

**Theorem 1.4.** *Let  $D$  be the unit ball or the Lie ball and let  $\Gamma$  be a unipotent discrete group of automorphisms of  $D$  having Zariski closure  $N_\Gamma \subset N$ . Then  $D/\Gamma$  is Stein if and only if  $\mathfrak{n}_\Gamma$  is totally real.*

Let us outline the structure of this paper. In Section 2 we review some parts of the structure theory of bounded homogeneous domains and prove Propositions 1.1, 1.2, and 1.3. Sections 3 and 4 contain the proves of Theorem 1.4 for the unit ball and the Lie ball, respectively. In the last section 5 we are going to present an example that shows that Theorem 1.4 does *not* hold for arbitrary bounded homogeneous domains.

## 2. RESULTS FOR ARBITRARY BOUNDED HOMOGENEOUS DOMAINS

We begin by reviewing the structure of the automorphism group of a bounded homogeneous domain, its unbounded realization as a Siegel domain, and its associated normal  $j$ -algebra. As main references we refer the reader to [Kan67] and [PS69]. We then show that the quotient of a bounded homogeneous domain by a unipotent discrete group  $\Gamma$  of automorphisms is always holomorphically separable, and prove the necessary and sufficient conditions for  $D/\Gamma$  to be Stein.

**2.1. Bounded homogeneous domains and normal  $j$ -algebras.** Let  $D \subset \mathbb{C}^n$  be a bounded homogeneous domain with base point  $p_0 \in D$  and consider the decomposition  $G = KR$  where  $G = \text{Aut}^0(D)$  and  $K = G_{p_0}$ . The elements of the Lie algebra  $\mathfrak{g}$  will be viewed as complete holomorphic vector fields on  $D$ . Since the group  $R$  is split solvable, we have  $R \cong A \ltimes N$  where  $N$  is the nilradical of  $R$  and the adjoint representation of the Abelian group  $A$  on  $\mathfrak{n}$  is diagonalizable over  $\mathbb{R}$ , compare [Kan67, Proposition 2.8].

It is well known that every bounded homogeneous domain  $D \subset \mathbb{C}^n$  is biholomorphic to a Siegel domain of the second kind  $\widehat{D} \subset \mathbb{C}^n$ , see [VGP63].

*Example.* The unit ball  $\mathbb{B}_n \subset \mathbb{C}^n$  is biholomorphically equivalent to

$$\widehat{\mathbb{B}}_n = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}; \text{Im}(z) - \|w\|^2 > 0\},$$

the Lie ball  $\mathbb{L}_n \subset \mathbb{C}^n$  is biholomorphic to

$$\widehat{\mathbb{L}}_n = \{z \in \mathbb{C}^n; \text{Im}(z_n)^2 - (\text{Im}(z_1)^2 + \cdots + \text{Im}(z_{n-1})^2) > 0\},$$

and the Siegel disk  $\mathbb{S}_n \subset \text{Sym}(n, \mathbb{C})$  may be realized as Siegel's upper half plane

$$\widehat{\mathbb{S}}_n = \{Z \in \text{Sym}(n, \mathbb{C}); \text{Im}(Z) \text{ is positive definite}\}.$$

The split-solvable group  $R \cong A \ltimes N$  acts by affine transformations on  $\widehat{D}$ . In particular, this  $R$ -action extends to the whole of  $\mathbb{C}^n$ . It follows from the explicit description of its affine automorphism group, see e.g. [Kan67, Chapters 2 and 3], that  $N$  and hence  $N^{\mathbb{C}}$  act *algebraically* by affine transformations on  $\mathbb{C}^n$  and that every transformation in  $N^{\mathbb{C}}$  has Jacobi determinant equal to 1.

Since  $R$  acts simply transitively on  $D$ , we may identify its Lie algebra  $\mathfrak{r} = \mathfrak{a} \oplus \mathfrak{n}$  with  $T_{p_0}D = \mathbb{C}^n$  and thus obtain an integrable complex structure  $j \in \text{End}(\mathfrak{r})$ .

In order to show that  $(\mathfrak{r}, j)$  is a normal  $j$ -algebra we need to prove existence of a linear form  $\lambda \in \mathfrak{r}^*$  such that for all  $x, y \in \mathfrak{r}$  we have

$$\lambda[j(x), j(y)] = \lambda[x, y] \quad \text{and} \quad \lambda[j(x), x] > 0 \text{ if } x \neq 0,$$

see part (III) of the definition in [PS69, p. 51].

In [Kan67] and [PS69] existence of this linear form  $\lambda \in \mathfrak{r}^*$  is deduced from [Kos55]. Let us explain here how  $\lambda$  can be constructed from the Bergman metric of  $D$  via a moment map. It is shown in [Ish06] that there exists an  $R$ -equivariant holomorphic embedding of  $D$  into Siegel's upper half plane  $\widehat{\mathbb{S}}_n$ . Since the domain  $\mathbb{S}_n$  is symmetric, the group  $\text{Aut}^0(\mathbb{S}_n)$  is semisimple and thus there exists an equivariant moment map for the action of  $\text{Aut}^0(\mathbb{S}_n)$  on  $\mathbb{S}_n$ . Pulling back this moment map to  $D$  we obtain an  $R$ -equivariant moment map  $\mu: D \rightarrow \mathfrak{r}^*$  with respect to the Bergman metric  $\omega$  of  $D$ . Note that  $\mu$  is a diffeomorphism onto its image, an open coadjoint orbit. Moreover,  $\mu$  is a symplectomorphism with respect to the Kostant-Kirillov form on this coadjoint orbit, i.e., we have

$$(2.1) \quad \omega_{p_0}(x(p_0), y(p_0)) = \mu(p_0)[x, y]$$

for all  $x, y \in \mathfrak{r}$ , see [GS84, Equations (26.2) and (26.8)]. In other words, we can take  $\lambda := \mu(p_0) \in \mathfrak{r}^*$  in the definition of the normal  $j$ -algebra associated with  $D$ . For the structure theory of normal  $j$ -algebras we refer the reader to [PS69, Chapter 2].

*Remark.* For a connected closed subgroup  $R'$  of  $R$  the orbit  $R' \cdot p_0$  is totally real if and only if  $\mathfrak{r}' \cap j(\mathfrak{r}') = \{0\}$ . In this case, we call  $\mathfrak{r}'$  a totally real subalgebra of  $\mathfrak{r}$ . Note that every isotropic orbit (with respect to the Bergman metric) is totally real. Moreover, every orbit of an Abelian subgroup of  $R$  is totally real since  $\mu(p_0)[x, j(x)] > 0$  holds for  $x \neq 0$ . This last statement is also a consequence of the fact that  $D$  is Kobayashi-hyperbolic, since an Abelian

group of automorphisms that is not totally real would yield a non-constant holomorphic map  $\mathbb{C} \rightarrow D$ .

Conversely, every abstract normal  $j$ -algebra is the normal  $j$ -algebra associated with some bounded homogeneous domain, see [PS69, Appendix].

**2.2. Quotients by unipotent discrete groups are holomorphically separable.** Let  $D \subset \mathbb{C}^n$  be a bounded homogeneous domain and let  $\Gamma$  be a unipotent subgroup of  $G = \text{Aut}^0(D)$ .

*Proof of Proposition 1.1.* Any discrete subgroup  $\Gamma \subset G$  acts on  $D \times \mathbb{C}$  by

$$\gamma \cdot (z, t) := (\gamma(z), t \det d\gamma(z)^{-1}),$$

hence we have the quotient manifold  $L := D \times_{\Gamma} \mathbb{C} := (D \times \mathbb{C})/\Gamma$ . Moreover, the projection onto the first factor  $D \times \mathbb{C} \rightarrow D$  is  $\Gamma$ -equivariant and the induced map  $L = D \times_{\Gamma} \mathbb{C} \rightarrow D/\Gamma$  defines a holomorphic line bundle on  $D/\Gamma$ . Holomorphic sections in the  $k$ -fold tensor product  $L^{\otimes k}$  can be constructed via Poincaré series for  $k \geq 2$  and separate the points of  $D/\Gamma$  in the following sense. For every pair of elements  $p, q \in D/\Gamma$  with  $p \neq q$  there exist  $k \geq 2$  (depending on  $p$  and  $q$ ) and a holomorphic section  $s$  in  $L^{\otimes k}$  such that  $s(p) = 0$  and  $s(q) \neq 0$ , see [PS69, Lemma 3.4.1]. We will finish the proof by showing that for unipotent  $\Gamma$  the line bundle  $L$ , and thus all of its powers, are holomorphically trivial.

To do so, let  $\varphi: D \rightarrow \widehat{D}$  be a  $\Gamma$ -equivariant biholomorphic map to a Siegel domain of the second kind on which  $\Gamma$  acts by affine transformations such that every  $\gamma \in \Gamma$  verifies  $\det d\gamma(z) = 1$  for all  $z \in \widehat{D}$ . We define  $s: D \rightarrow \mathbb{C}^*$  by  $s(z) := \det d\varphi(z)$ . Then the chain rule shows that  $s$  induces a holomorphic section of the line bundle  $L \rightarrow D/\Gamma$  without zeros, hence that  $L$  is indeed holomorphically trivial.  $\square$

**2.3. A necessary condition for Steinness.** In this subsection we prove Proposition 1.2.

An important ingredient of this proof is the following result of Loeb. Let  $\Gamma \subset G$  be a discrete group of unipotent automorphisms of  $D$ . After conjugation we may suppose that  $\Gamma$  is contained in  $N$ . Let  $N_{\Gamma}$  be the real Zariski closure of  $\Gamma$  in  $N$  and let  $N_{\Gamma}^{\mathbb{C}}$  be its universal complexification. Then the complex homogeneous space  $N_{\Gamma}^{\mathbb{C}}/\Gamma$  is Stein due to [Loe85, Théorème 1].

Recall that the complexification  $N^{\mathbb{C}}$  is a unipotent complex algebraic group and its action on the Siegel domain  $\widehat{D}$  extends to an algebraic action on  $\mathbb{C}^n$ . In particular, the orbits of any algebraic subgroup of  $N^{\mathbb{C}}$  are closed in  $\mathbb{C}^n$ , see [Bor91, Proposition 4.10].

**Lemma 2.1.** *The orbit  $N_{\Gamma} \cdot z$  is totally real in  $\widehat{D}$  if and only if the isotropy group  $(N_{\Gamma}^{\mathbb{C}})_z$  is trivial.*

*Proof.* For this, let  $x + iy \in \mathfrak{n}_{\Gamma} \oplus i\mathfrak{n}_{\Gamma} = \mathfrak{n}_{\Gamma}^{\mathbb{C}}$  be a holomorphic vector field on  $\widehat{D}$  and consider  $x(z) + j_q y(z) \in T_z \widehat{D}$  where  $j_z$  is the complex structure of  $T_z \widehat{D}$  for some  $z \in \widehat{D}$ . Suppose that  $N_{\Gamma} \cdot z$  is totally real. Then  $x(z) + j_q y(z) = 0$  implies  $x(z) = y(z) = 0$ , and since  $N_{\Gamma}$  acts freely on  $\widehat{D}$ , we obtain  $x = y = 0$ . Consequently, the isotropy  $(N_{\Gamma}^{\mathbb{C}})_z$  is discrete. Since  $N_{\Gamma}^{\mathbb{C}}$  acts algebraically and has no finite subgroups, we conclude that  $(N_{\Gamma}^{\mathbb{C}})_z$  is trivial.

If  $N_{\Gamma} \cdot z$  is not totally real for some  $z \in \widehat{D}$ , then there are  $x, y \in \mathfrak{n}_{\Gamma}$  such that  $x(z) = j_z y(z)$ . Thus the vector field  $x - iy \in \mathfrak{n}_{\Gamma}^{\mathbb{C}}$  vanishes at  $z$ , i.e.,  $N_{\Gamma}^{\mathbb{C}}$  does not act freely.  $\square$

We are now in position to prove Proposition 1.2.

*Proof of Proposition 1.2.* Suppose that  $D/\Gamma$  is a Stein manifold. In the first step we are going to show that  $N_{\Gamma}$  has at least one totally real orbit in  $D$ .

Since  $D/\Gamma$  is Stein, the domain  $D$  admits a  $\Gamma$ -invariant smooth strictly plurisubharmonic function  $\rho$  that is exhaustive modulo  $\Gamma$ . We may assume that 0 is a global minimum of  $\rho$ .

Since  $N_\Gamma/\Gamma$  is compact, we can suppose without loss of generality that  $\rho$  is invariant under  $N_\Gamma$  and an exhaustion modulo  $N_\Gamma$ , see [Loe85, Lemme 2.1]. Let  $N_\Gamma \cdot z$  be an orbit lying in the minimal set of  $\rho$ . Then  $N_\Gamma \cdot z$  is totally real due to [HW73].

Due to Lemma 2.1, it is enough to show that  $N_\Gamma^{\mathbb{C}}$  acts freely on  $N_\Gamma^{\mathbb{C}} \cdot \widehat{D} \subset \mathbb{C}^n$ . To do this, let  $z \in D$  and consider the algebraic subgroup  $H := (N_\Gamma^{\mathbb{C}})_z$  of  $N_\Gamma^{\mathbb{C}}$ . Since  $N_\Gamma^{\mathbb{C}} \cdot z$  is closed in  $\mathbb{C}^n$ , the intersection  $\Omega := D \cap (N_\Gamma^{\mathbb{C}} \cdot z)$  is an  $N_\Gamma$ -invariant Stein open neighborhood of  $N_\Gamma \cdot eH \cong N_\Gamma$  in  $N_\Gamma^{\mathbb{C}}/H \cong \mathbb{C}^k$  such that  $\Omega/\Gamma$  is Stein. Applying [HW73] and [Loe85] again, we see that  $N_\Gamma \cdot eH$  must be totally real in  $N_\Gamma^{\mathbb{C}}/H$  and hence that  $H = \{e\}$ , as was to be shown.  $\square$

*Remark.* Every orbit lying in the minimal set of  $\rho$  is isotropic with respect to the Kähler form  $i\partial\bar{\partial}\rho$ . In general, it is not isotropic with respect to the Bergman metric of  $D$ .

**2.4. A sufficient condition for  $D/\Gamma$  to be Stein.** In this subsection we are going to prove Proposition 1.3.

As above, let  $\Gamma \subset N$  be a discrete group of unipotent automorphisms of a bounded homogeneous domain  $D$  and let  $N_\Gamma$  be its Zariski closure in  $N$ . We realize  $D$  as a Siegel domain of the second kind  $\widehat{D} \subset \mathbb{C}^n$  such that  $N$  acts by affine transformations on  $\widehat{D}$ .

*Proof of Proposition 1.3.* By assumption, there exists a maximal totally real subalgebra  $\widehat{\mathfrak{n}}_\Gamma \subset \mathfrak{n}$  which contains  $\mathfrak{n}_\Gamma$ . On the group level we thus find a connected subgroup  $\widehat{N}_\Gamma \subset N$  containing  $N_\Gamma$  such that  $\widehat{N}_\Gamma \cdot p_0$  is maximally totally real.

Consider the complexifications  $N_\Gamma^{\mathbb{C}} \subset \widehat{N}_\Gamma^{\mathbb{C}} \subset N^{\mathbb{C}}$  as well as their algebraic actions by affine transformations on  $\mathbb{C}^n$ . Since  $\widehat{N}_\Gamma \cdot p_0$  is maximally totally real, the orbit  $\widehat{N}_\Gamma^{\mathbb{C}} \cdot p_0$  is open in  $\mathbb{C}^n$  and the  $\widehat{N}_\Gamma^{\mathbb{C}}$ -isotropy at  $p_0$  is trivial, see Lemma 2.1. Since  $\widehat{N}_\Gamma^{\mathbb{C}} \cdot p_0$  is also closed in  $\mathbb{C}^n$ , we obtain  $N_\Gamma^{\mathbb{C}} \cdot p_0 = \mathbb{C}^n$ . In other words,  $\widehat{N}_\Gamma^{\mathbb{C}}$  acts freely and transitively on  $\mathbb{C}^n$ .

It follows that  $N_\Gamma^{\mathbb{C}}$  acts freely and properly on  $\mathbb{C}^n \cong \widehat{N}_\Gamma^{\mathbb{C}}$  and that  $\mathbb{C}^n/N_\Gamma^{\mathbb{C}} \cong \mathbb{C}^m$ . Therefore the corresponding holomorphic principal bundle is holomorphically trivial which implies together with [Loe85, Théorème 1] that  $\mathbb{C}^n/\Gamma \cong (N_\Gamma^{\mathbb{C}}/\Gamma) \times \mathbb{C}^m$  is a Stein manifold. Since  $\widehat{D}/\Gamma$  is a locally Stein domain in  $\mathbb{C}^n/\Gamma$ , the quotient  $\widehat{D}/\Gamma$  is likewise Stein by [DG60].  $\square$

*Remark.* The proof of Proposition 1.3 shows that, if  $\mathfrak{n}_\Gamma$  is contained in a maximal totally real subalgebra of  $\mathfrak{n}$ , then all  $N_\Gamma$ -orbits are totally real in  $D$ .

*Remark.* Under the hypotheses of Proposition 1.3,  $\widehat{N}_\Gamma^{\mathbb{C}} \cong \widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{D} = \mathbb{C}^n$  is the universal globalization of the induced local  $\widehat{N}_\Gamma^{\mathbb{C}}$ -action on  $\widehat{D}$  in the sense of [Pal57].

### 3. THE CASE OF THE UNIT BALL

In this section we consider the unit ball  $\mathbb{B}_n \subset \mathbb{C}^n$ . First we illustrate Theorem 1.4 by two examples that can be analyzed by ad hoc methods. Then we review the structure of the normal  $j$ -algebra  $\mathfrak{b}_n$  of  $\mathbb{B}_n$ , which will be used to show that every totally real subalgebra of  $\mathfrak{b}_n$  is contained in a maximal totally real one.

**3.1. Two examples.** We identify the unit ball  $\mathbb{B}_n$  with its unbounded realization

$$\widehat{\mathbb{B}}_n = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}; \operatorname{Im}(z) - \|w\|^2 > 0\}.$$

For an explicit description of the vector fields belonging to its normal  $j$ -algebra  $\mathfrak{b}_n$  as well as of the corresponding one-parameter groups we refer the reader to [Mie10a, Table 1, p. 341].

First, let us present an example of a nonabelian discrete group  $\Gamma$  such that  $\mathbb{B}_3/\Gamma$  is Stein, thus answering the question raised in [Che13, Remark 7.6] in the negative. Note that 3 is the smallest dimension in which such an example can exist.

*Example.* Let us consider the complete holomorphic vector fields  $x_1 := 2iw_1 \frac{\partial}{\partial z} + \frac{\partial}{\partial w_1}$ ,  $x_2 := 2(w_1 + w_2) \frac{\partial}{\partial z} + i \frac{\partial}{\partial w_1} + i \frac{\partial}{\partial w_2}$ , and  $x_3 := \frac{\partial}{\partial z}$  on  $\widehat{\mathbb{B}}_3$ . Since their only non-vanishing Lie bracket is  $[x_1, x_2] = 4x_3$ , they generate a three-dimensional subalgebra in the nilradical of  $\mathfrak{b}_3$ , isomorphic to the three-dimensional Heisenberg algebra. Moreover, this algebra is defined over  $\mathbb{Q}$  and therefore the corresponding connected subgroup admits a cocompact discrete subgroup  $\Gamma$ , which justifies the notation  $N_\Gamma = \exp(\mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \mathbb{R}x_3)$ . One verifies directly that every  $N_\Gamma$ -orbit in  $\widehat{\mathbb{B}}_3$  is maximally totally real. This implies that the universal globalization of the local  $N_\Gamma^{\mathbb{C}}$ -action on  $\widehat{\mathbb{B}}_3$  is isomorphic to  $N_\Gamma^{\mathbb{C}}$  with left multiplication. Since  $N_\Gamma^{\mathbb{C}}/\Gamma$  is a Stein manifold which contains  $\widehat{\mathbb{B}}_3/\Gamma$  as a domain, we see that  $\mathbb{B}_3/\Gamma$  is Stein while  $\Gamma$  is not Abelian.

Secondly, we present an explicit example of a unipotent discrete group  $\Gamma$  where  $\mathbb{B}_2/\Gamma$  is non-Stein but admits a (singular) Stein envelope.

*Example.* Let  $\Gamma$  be the discrete group consisting of the automorphisms

$$(z, w) \mapsto (z + 2(n + im)w + i(m^2 + n^2) + 2k, w + m + in) =: \varphi_{2k, m+in}(z, w),$$

where  $m, n, k \in \mathbb{Z}$  and  $(z, w) \in \widehat{\mathbb{B}}_2$ . Although we do not need this fact, let us remark that the Lie algebra  $\mathfrak{n}_\Gamma$  of the Zariski closure of  $\Gamma$  coincides with the nilradical of  $\mathfrak{b}_2$ .

In order to determine the quotient  $\widehat{\mathbb{B}}_2/\Gamma$ , we consider first the action of the normal subgroup  $\Gamma_0 := \{\varphi_{2k, 0}; k \in \mathbb{Z}\} \triangleleft \Gamma$ . The map  $p: \widehat{\mathbb{B}}_2 \rightarrow \mathbb{C}^* \times \mathbb{C}$  given by  $p(z, w) = (e^{i\pi z}, w)$  is  $\Gamma_0$ -invariant and yields

$$\widehat{\mathbb{B}}_2/\Gamma_0 \cong p(\widehat{\mathbb{B}}_2) = \{(z, w) \in \mathbb{C}^* \times \mathbb{C}; |z| < e^{-\pi|w|^2}\}.$$

The induced action of  $\mathbb{Z} \oplus i\mathbb{Z} \cong \Gamma/\Gamma_0$  is

$$(m + in) \cdot (z, w) = (e^{i\pi(2(n+im)w + i(m^2+n^2))} z, w + m + in).$$

It follows that the action of  $\mathbb{Z} \oplus i\mathbb{Z}$  on  $\widehat{\mathbb{B}}_2$  extends to a proper action on the whole of  $\mathbb{C}^2$  and that the equivariant map  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ ,  $(z, w) \mapsto w$ , induces a holomorphic line bundle  $L := \mathbb{C}^2/\Gamma \rightarrow E = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ . We see that  $\widehat{\mathbb{B}}_2/\Gamma$  embeds into  $L$  as a Kobayashi-hyperbolic open neighborhood of the zero section minus this zero section. Since  $\partial\widehat{\mathbb{B}}_2$  is strictly pseudoconvex,  $L$  is negative in the sense of Grauert and hence the zero section can be blown down to yield a Stein space  $Y$  with an isolated singularity containing  $\widehat{\mathbb{B}}_2/\Gamma$  as a neighborhood of this singularity minus the singularity. Consequently,  $\mathbb{B}_2/\Gamma$  is holomorphically separable but not Stein.

**3.2. The normal  $j$ -algebra of the unit ball.** The automorphism group of the unit ball  $\mathbb{B}_n$  is  $G = \text{Aut}(\mathbb{B}_n) \cong \text{PSU}(n, 1)$ . Let  $G = KAN$  be an Iwasawa decomposition with maximal compact subgroup  $K = G_{p_0} \cong \text{U}(n)$  for  $p_0 = 0$ . It is well known that  $G$  is of real rank 1, i.e., that  $\dim A = 1$ . Moreover, we can rewrite the moment map condition (2.1) as

$$(3.1) \quad [x, y](p_0) = \omega_{p_0}(x(p_0), y(p_0))\zeta(p_0)$$

for all  $x, y \in \mathfrak{n}$ , compare [PS69, p. 52]. This means that  $\mathfrak{n}$  is the Heisenberg algebra of dimension  $2n - 1$  with center  $\mathbb{R}\zeta$  which is associated with the symplectic form  $\omega_{p_0}$  induced by the Bergman metric of  $\mathbb{B}_n$ . As we shall see in the following subsection, it is this close relation between the geometry of  $(\mathbb{B}_n, \omega)$  and the structure of  $N$  that enables us to prove Theorem 1.4 for the unit ball.

Recall from [Mie10a, Section 4.2] that the normal  $j$ -algebra  $\mathfrak{b}_n = \mathfrak{a} \oplus \mathfrak{n}$  can be written as

$$\mathfrak{a} = \mathbb{R}\alpha \quad \text{and} \quad \mathfrak{n} = \bigoplus_{k=1}^{n-1} \mathbb{R}\zeta_k \oplus \bigoplus_{k=1}^{n-1} \mathbb{R}\zeta_k' \oplus \mathbb{R}\zeta,$$

where the only non-zero Lie brackets are

$$[\xi_k, \xi'_k] = \zeta, [\alpha, \xi_k] = -\xi_k, [\alpha, \xi'_k] = -\xi'_k, \text{ and } [\alpha, \zeta] = -2\zeta$$

for all  $1 \leq k \leq n-1$ , and where the complex structure  $j: \mathfrak{b}_n \rightarrow \mathfrak{b}_n$  is given by

$$j(\zeta) = \alpha \text{ and } j(\xi_k) = \xi'_k$$

for all  $1 \leq k \leq n-1$ .

**3.3. Proof of Theorem 1.4 for the unit ball.** Let  $\mathfrak{b}_n = \mathfrak{a} \oplus \mathfrak{n}$  be the normal  $j$ -algebra of  $\mathbb{B}_n$ . In this subsection we are going to show that every totally real subalgebra of  $\mathfrak{n}$  is contained in a maximal totally real subalgebra, which generalizes [Mie10a, Lemma 4.1]. Consequently, we can apply Proposition 1.3 in order to prove Theorem 1.4 for  $\mathbb{B}_n$ .

**Proposition 3.1.** *Every totally real subalgebra  $\mathfrak{n}'$  of  $\mathfrak{n}$  is contained in a maximal totally real subalgebra  $\widehat{\mathfrak{n}}'$  of  $\mathfrak{n}$ . Moreover, if  $\mathfrak{n}'$  is Abelian, then  $\widehat{\mathfrak{n}}'$  can also be chosen to be Abelian.*

*Proof.* Let  $\Phi: \mathfrak{n} \rightarrow \mathfrak{n} \cdot p_0$  be the linear isomorphism given by  $\Phi(x) := x(p_0)$ . Let  $V \subset \mathfrak{n} \cdot p_0$  be a real vector subspace. The following is a direct consequence of equation (3.1).

- (a) The preimage  $\Phi^{-1}(V)$  is an Abelian subalgebra of  $\mathfrak{n}$  if and only if  $V$  is isotropic with respect to  $\omega_{p_0}$ .
- (b) The preimage  $\Phi^{-1}(V)$  is a nonabelian subalgebra of  $\mathfrak{n}$  if and only if  $V$  is not isotropic and contains  $\zeta(p_0)$ .

Now suppose first that  $\mathfrak{n}'$  is Abelian. Then  $\mathfrak{n}' \cdot p_0$  is not only totally real but isotropic, and by basic symplectic linear algebra there exists a Lagrangian subspace  $V$  of  $\mathfrak{n} \cdot p_0$  that contains  $\mathfrak{n}' \cdot p_0$ . As we have noted above, the preimage  $\widehat{\mathfrak{n}}' := \Phi^{-1}(V)$  is an Abelian subalgebra of maximal dimension of  $\mathfrak{n}$  that contains  $\mathfrak{n}'$ .

In the case that  $\mathfrak{n}'$  is not Abelian, we choose any maximally totally real subspace  $V$  of  $\mathfrak{n} \cdot p_0$  that contains  $\mathfrak{n}' \cdot p_0$ . Since  $\mathfrak{n}' \cdot p_0$  contains  $\zeta(p_0)$ , the same holds for  $V$ . Hence, it follows that  $\widehat{\mathfrak{n}}' := \Phi^{-1}(V)$  has the required properties.  $\square$

#### 4. THE CASE OF THE LIE BALL

In this section we describe first the structure of the normal  $j$ -algebra  $\mathfrak{l}_n$  of the Lie ball  $\mathbb{L}_n$  in order to establish in particular the existence of a holomorphic submersion from  $\mathbb{L}_n$  onto the unit disk  $\mathbb{B}_1$  whose fibers are biholomorphic to  $\mathbb{B}_{n-1}$ . Since Proposition 3.1 does *not* hold for  $\mathfrak{l}_n$ , this submersion is crucial in the proof of Theorem 1.4 for  $\mathbb{L}_n$ .

**4.1. The normal  $j$ -algebra of the Lie ball.** The  $n$ -dimensional Lie ball is biholomorphically equivalent to the tube domain over the symmetric cone

$$\Omega := \{p \in \mathbb{R}^n; p_n^2 - (p_1^2 + \cdots + p_{n-1}^2) > 0, p_n > 0\}.$$

For  $n = 1$  this is just the upper half plane  $\mathbb{H}^+$ , while for  $n = 2$  we get  $\mathbb{H}^+ \times \mathbb{H}^+$ . Therefore, we will concentrate on the case  $n \geq 3$ .

In the notation of [Hel01, Table V, Chapter X] the Lie ball  $\mathbb{L}_n$  corresponds to the item  $BD I$  with  $p = 2$  and  $q = n$ . Its connected automorphism group is isomorphic to  $G = \text{SO}^0(2, n)$  and the subgroup of affine automorphisms is  $G(\Omega) \times \mathbb{R}^n$  with  $G(\Omega) = \mathbb{R}^{>0}\text{SO}^0(1, n-1)$ , see [FaKo94, Chapter X.5].

In order to describe the normal  $j$ -algebra  $\mathfrak{l}_n = \mathfrak{a} \oplus \mathfrak{n}$  of the Lie ball, it is sufficient to find a maximal triangular subalgebra of  $\mathfrak{so}(1, n-1)$  and to apply [Kan67, Proposition 2.8]. By



doing so, we can choose the basis of  $\mathfrak{n}$  given by the vector fields

$$\begin{aligned}\zeta_k &= \frac{\partial}{\partial z_k} \quad (1 \leq k \leq n-2), \\ \zeta'_k &= (z_n - z_{n-1}) \frac{\partial}{\partial z_k} + z_k \frac{\partial}{\partial z_{n-1}} + z_k \frac{\partial}{\partial z_n} \quad (1 \leq k \leq n-2), \\ \zeta &= \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_n}, \\ \eta &= \frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_{n-1}}.\end{aligned}$$

The only non-vanishing Lie brackets are

$$[\zeta_k, \zeta'_k] = \zeta \quad \text{and} \quad [\eta, \zeta'_k] = 2\zeta_k$$

for all  $1 \leq k \leq n-2$ . In particular, we see that  $\mathbb{R}\zeta$  is the center of  $\mathfrak{n}$  and that the Abelian Lie algebra  $\mathfrak{n}' = \bigoplus_{k=1}^{n-2} \mathbb{R}\zeta_k \oplus \mathbb{R}\zeta$  is its derived algebra.

A basis of  $\mathfrak{a}$  is given by

$$\begin{aligned}\delta &:= z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n} \quad (\text{the Euler field}), \text{ and} \\ \alpha &:= z_n \frac{\partial}{\partial z_{n-1}} + z_{n-1} \frac{\partial}{\partial z_n}.\end{aligned}$$

The action of  $\mathfrak{a}$  on  $\mathfrak{n}$  is determined by

$$[\delta, \zeta_k] = -\zeta_k, \quad [\delta, \zeta] = -\zeta, \quad [\delta, \eta] = -\eta$$

for all  $1 \leq k \leq n-2$  and by

$$[\alpha, \zeta'_k] = -\zeta'_k, \quad [\alpha, \zeta] = -\zeta, \quad [\alpha, \eta] = \eta$$

for all  $1 \leq k \leq n-2$ .

To obtain the complex structure  $j: \mathfrak{l}_n \rightarrow \mathfrak{l}_n$ , let us fix the base point  $p_0 = ie_n$  in  $\widehat{\mathbb{L}}_n = \mathbb{R}^n + i\Omega$ . This yields

$$j(\zeta_k) = \zeta'_k, \quad j(\zeta) = \alpha + \delta =: \alpha_1, \quad j(\eta) = \delta - \alpha =: \alpha_2$$

for all  $1 \leq k \leq n-2$ .

**4.2. The equivariant fibration of  $\mathbb{L}_n$ .** It follows directly from the bracket relations described above that the subalgebra

$$\mathfrak{b}_{n-1} := \mathbb{R}\alpha_1 \oplus \bigoplus_{k=1}^{n-2} \mathbb{R}\zeta_k \oplus \bigoplus_{k=1}^{n-2} \mathbb{R}\zeta'_k \oplus \mathbb{R}\zeta$$

is a  $j$ -invariant ideal in  $\mathfrak{l}_n$  isomorphic to the normal  $j$ -algebra of the  $(n-1)$ -dimensional unit ball. Moreover, the quotient  $\mathfrak{b}_1 \cong \mathfrak{l}_n / \mathfrak{b}_{n-1}$  is isomorphic to the  $j$ -subalgebra  $\mathbb{R}\alpha_2 \oplus \mathbb{R}\eta$  of  $\mathfrak{l}_n$ . In other words,  $\mathfrak{l}_n$  decomposes into the semi-direct product  $\mathfrak{l}_n = \mathfrak{b}_{n-1} \oplus \mathfrak{b}_1$ . Let

$$L_n \cong B_{n-1} \ltimes B_1$$

be the corresponding decomposition on the group level.

Geometrically, this corresponds to the  $L_n$ -equivariant holomorphic submersion  $\pi: \widehat{\mathbb{L}}_n \rightarrow \mathbb{C}$  given by  $\pi(z) = z_n - z_{n-1}$ . In fact, an elementary argument shows that for  $z \in \widehat{\mathbb{L}}_n$

we have  $z_n - z_{n-1} \in \mathbb{H}^+$  and  $z_n + z_{n-1} \in \mathbb{H}^+$ . Moreover, for  $a := z_n - z_{n-1} \in \mathbb{H}^+$  and  $w := z_n + z_{n-1} \in \mathbb{H}^+$ , we obtain

$$\pi^{-1}(a) \cong \left\{ (z_1, \dots, z_{n-2}, w) \in \mathbb{C}^{n-1} \mid \operatorname{Im}(w) > \frac{\operatorname{Im}(z_1)^2 + \dots + \operatorname{Im}(z_{n-2})^2}{\operatorname{Im}(a)} \right\},$$

which is a realization of the  $(n-1)$ -dimensional unit ball as an unbounded tube domain. Hence, all  $\pi$ -fibers are isomorphic to  $\mathbb{B}_{n-1}$ .

*Remark.* Since the Lie ball  $\mathbb{L}_n$  is irreducible for  $n \geq 3$ , the holomorphic submersion  $\pi$  is *not* holomorphically locally trivial unless  $n = 2$ .

**4.3. Proof of Theorem 1.4 for the Lie ball.** Let  $\Gamma$  be a discrete subgroup of  $\operatorname{Aut}^0(\mathbb{L}_n)$  such that  $\Gamma \subset N$ . Let  $N_\Gamma$  be the Zariski closure of  $\Gamma$  in  $N$  and suppose that  $\mathfrak{n}_\Gamma$  is totally real, i.e., that  $\mathfrak{n}_\Gamma \cap j(\mathfrak{n}_\Gamma) = \{0\}$ .

We start by giving an example that shows that Proposition 3.1 does no longer hold for the Lie ball.

*Example.* Let us consider the normal  $j$ -algebra  $\mathfrak{l}_3 = \mathfrak{a} \oplus \mathfrak{n}$  of the three-dimensional Lie ball. We claim that the totally real subalgebra  $\mathbb{R}\zeta'_1 \subset \mathfrak{n}$  is not contained in a maximal totally real subalgebra of  $\mathfrak{n}$ .

Since  $\mathfrak{b}_3$  is 3-dimensional and not totally real, a maximal totally real subalgebra of  $\mathfrak{n}$  must contain an element  $x = a\zeta_1 + b\zeta'_1 + c\zeta + d\eta$  with  $a, b, c, d \in \mathbb{R}$  and  $d \neq 0$ . Thus it also must contain

$$[x, \zeta'_1] = a\zeta + 2d\zeta_1 \quad \text{and} \quad [[x, \zeta'_1], \zeta'_1] = 2d\zeta,$$

and therefore  $\zeta$  and  $\zeta_1 = -j(\zeta'_1)$ , too, which is impossible.

The following proposition is weaker than Proposition 3.1 but will still allow us to prove Theorem 1.4 for the Lie ball.

**Proposition 4.1.** *Let  $\mathfrak{n}'$  be a totally real subalgebra of the nilradical  $\mathfrak{n}$  of  $\mathfrak{l}_n$ . If  $\mathfrak{n}'$  is not contained in  $\mathfrak{b}_{n-1}$ , then there exists a maximal totally real subalgebra  $\widehat{\mathfrak{n}}' \subset \mathfrak{n}$  which contains  $\mathfrak{n}'$ .*

*Proof.* Since by assumption  $\mathfrak{n}'$  is not contained in  $\mathfrak{b}_{n-1}$ , there exists an element

$$x_0 = \sum a_k \zeta_k + \sum b_l \zeta'_l + c\zeta + \eta \in \mathfrak{n}'.$$

Moreover, we have  $\dim(\mathfrak{n}' \cap \mathfrak{b}_{n-1}) = \dim \mathfrak{n}' - 1$ .

We claim that  $\mathfrak{n}'$  is totally real if and only if  $\mathfrak{n}' \cap \mathfrak{b}_{n-1}$  is so. To see this, suppose that  $\mathfrak{n}' \cap \mathfrak{b}_{n-1}$  is totally real, the other implication being trivial. Let  $x \in \mathfrak{n}' \cap j(\mathfrak{n}')$  and write  $x = x_1 + tx_0$  with  $x_1 \in \mathfrak{n}' \cap \mathfrak{b}_{n-1}$ . From  $j(x) = j(x_1) + tj(x_0) \in \mathfrak{n}'$  we deduce  $t = 0$ , hence  $x = x_1 \in \mathfrak{n}' \cap \mathfrak{b}_{n-1}$ . Since the latter is assumed to be totally real we obtain  $x_1 = 0$ , as was to be shown.

In the next step we observe that if  $\zeta \notin \mathfrak{n}'$ , then  $\mathfrak{n}' \oplus \mathbb{R}\zeta$  is again a totally real subalgebra, for the following reason. If  $\zeta \notin \mathfrak{n}'$ , then  $\mathfrak{n}' \cap \mathfrak{b}_{n-1}$  is Abelian. Therefore,  $(\mathfrak{n}' \cap \mathfrak{b}_{n-1}) \oplus \mathbb{R}\zeta$  is also Abelian, hence totally real, which due to the previous step implies that  $\mathfrak{n}' \oplus \mathbb{R}\zeta$  is totally real.

Consequently, we may assume that  $\zeta \in \mathfrak{n}'$ . Let us consider the projection  $\pi: \mathfrak{b}_{n-1} \rightarrow \mathfrak{b}_{n-1}$  onto  $\bigoplus_k \mathbb{R}\zeta_k \oplus \mathbb{R}\zeta$  with kernel  $\bigoplus_l \mathbb{R}\zeta'_l$ . If the restriction

$$\pi|_{\mathfrak{n}' \cap \mathfrak{b}_{n-1}}: \mathfrak{n}' \cap \mathfrak{b}_{n-1} \rightarrow \bigoplus_k \mathbb{R}\zeta_k \oplus \mathbb{R}\zeta$$

is surjective, then  $\dim(\mathfrak{n}' \cap \mathfrak{b}_{n-1}) \geq n-1$ , hence  $\dim \mathfrak{n}' = n$ , and we are done.

Therefore suppose that  $\zeta_{k_0} \notin \pi(\mathfrak{n}' \cap \mathfrak{b}_{n-1})$  for some  $k_0$ . Note that in particular  $\zeta_{k_0} \notin \mathfrak{n}'$ . From  $\zeta \in \mathfrak{n}'$  it follows that  $(\mathfrak{n}' \cap \mathfrak{b}_{n-1}) \oplus \mathbb{R}\zeta_{k_0}$  is a subalgebra of  $\mathfrak{n}' \oplus \mathfrak{b}_{n-1}$ . Since moreover,

$\eta$  and  $\zeta_{k_0}$  commute, we see that  $\mathfrak{n}' \oplus \mathbb{R}\zeta_{k_0}$  is a subalgebra of  $\mathfrak{n}$ . We claim that  $\mathfrak{n}' \oplus \mathbb{R}\zeta_{k_0}$  is totally real. For this it is sufficient to show that  $(\mathfrak{n}' \cap \mathfrak{b}_{n-1}) \oplus \mathbb{R}\zeta_{k_0}$  is totally real. This follows from the fact that  $[x_0, j(\zeta_{k_0})]$  has a non-zero contribution from  $\zeta_{k_0}$ .

Iterating this procedure we eventually construct a maximal totally real subalgebra of  $\mathfrak{n}$  that contains  $\mathfrak{n}'$ , as was to be shown.  $\square$

*Proof of Theorem 1.4 for the Lie ball.* We shall prove that  $\mathbb{L}_n/\Gamma$  is Stein where  $\Gamma$  satisfies the conditions stated at the beginning of this subsection.

Let us assume first that  $\mathfrak{n}_\Gamma$  is not contained in the  $j$ -ideal  $\mathfrak{b}_{n-1}$ . Due to Proposition 4.1 there exists a maximal totally real subalgebra  $\widehat{\mathfrak{n}}_\Gamma \subset \mathfrak{n}$  that contains  $\mathfrak{n}_\Gamma$ . Then the result follows from Proposition 1.3.

If  $\mathfrak{n}_\Gamma$  is contained in  $\mathfrak{b}_{n-1}$ , we find a maximal totally real subalgebra  $\widehat{\mathfrak{n}}_\Gamma$  of  $\mathfrak{b}_{n-1}$  that contains  $\mathfrak{n}_\Gamma$ , see Proposition 3.1. The holomorphic submersion  $\pi: \widehat{\mathbb{L}}_n \rightarrow \mathbb{H}^+$  is  $\widehat{N}_\Gamma$ -invariant and extends to a  $\widehat{N}_\Gamma^{\mathbb{C}}$ -invariant holomorphic submersion  $\widehat{\pi}: \widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n \rightarrow \mathbb{H}^+$ . It follows that the fibers of  $\widehat{\pi}$  are precisely the  $\widehat{N}_\Gamma^{\mathbb{C}}$ -orbits, isomorphic to  $\mathbb{C}^{n-1}$ . In other words,  $\widehat{\pi}$  defines an  $\widehat{N}_\Gamma^{\mathbb{C}}$ -principal bundle, which then must be holomorphically trivial. This implies that

$$(\widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n)/\Gamma \cong (\widehat{N}_\Gamma^{\mathbb{C}}/\Gamma) \times \mathbb{H}^+$$

and hence  $\widehat{\mathbb{L}}_n/\Gamma$  are Stein.  $\square$

## 5. THE CASE OF THE SIEGEL DISK

In this section we present an example that shows that Theorem 1.4 and Proposition 4.1 do not hold for arbitrary bounded homogeneous domains. This example will be constructed on a 5-dimensional bounded homogeneous subdomain of the Siegel disk  $S_3$ . Therefore we describe in a first step the normal  $j$ -algebra  $\mathfrak{s}_n$  of  $S_n$ .

**5.1. The normal  $j$ -algebra of Siegel's upper half plane.** Recall that the Siegel disk  $S_n$  can be realized as Siegel's upper half plane which is the symmetric tube domain associated with the cone of positive definite real symmetric matrices. The automorphism group of  $S_n$  is isomorphic to the symplectic group  $\mathrm{Sp}(n, \mathbb{R})$  and  $S_n$  corresponds to the hermitian symmetric space of type  $CI$  in [Hel01]. For  $n = 1$  Siegel's upper half plane is the usual upper half plane, while for  $n = 2$  it is isomorphic to the 3-dimensional Lie ball.

The linear automorphism group of the symmetric cone of positive definite real symmetric matrices is  $\mathrm{GL}(n, \mathbb{R})$  acting by  $g \cdot A := gAg^t$ , see [FaKo94, p. 213]. Hence, the normal  $j$ -algebra  $\mathfrak{s}_n$  of  $S_n$  can be determined as before using [FaKo94, Proposition X.5.4] and [Kan67, Proposition 2.8]. Our description of  $\mathfrak{s}_n$  follows [Ish06].

Let us denote by  $\mathfrak{u}_n$  the solvable Lie algebra of lower triangular real  $n \times n$  matrices. We have the decomposition  $\mathfrak{u}_n = \mathfrak{a}_n \oplus \mathfrak{u}'_n$  where  $\mathfrak{a}_n$  is the Abelian Lie algebra of diagonal matrices in  $\mathbb{R}^{n \times n}$ . The normal  $j$ -algebra of Siegel's upper half plane can be realized as

$$\mathfrak{s}_n := \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \mid A \in \mathfrak{u}_n, B \in \mathrm{Sym}(n, \mathbb{R}) \right\}.$$

Note that  $\varphi: \mathfrak{u}_n \rightarrow \mathrm{Sym}(n, \mathbb{R})$ ,  $\varphi(A) = A + A^t$ , is a linear isomorphism. The complex structure  $j: \mathfrak{s}_n \rightarrow \mathfrak{s}_n$  is given by

$$j \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} = \begin{pmatrix} \varphi^{-1}(B) & -\varphi(A) \\ 0 & -\varphi^{-1}(B)^t \end{pmatrix}.$$

The linear form  $\lambda \in \mathfrak{s}_n^*$  corresponding to the Bergman metric is given by

$$\lambda \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} = \text{Tr}(B).$$

One verifies directly that the subspace of  $\mathfrak{s}_n$  consisting of all matrices  $A = (a_{kl})$  and  $B = (b_{kl})$  such that  $a_{kl} = b_{kl} = 0$  for all  $1 \leq k, l \leq n-1$  is a  $j$ -invariant ideal isomorphic to  $\mathfrak{b}_n$ , while the subspace consisting of all matrices where  $a_{nl} = b_{nl} = 0$  for all  $l$  is a  $j$ -invariant complementary subalgebra isomorphic to  $\mathfrak{s}_{n-1}$ . Consequently,  $\mathfrak{s}_n$  decomposes as a semi-direct sum  $\mathfrak{s}_{n-1} \oplus \mathfrak{b}_n$ . Geometrically, this decomposition corresponds to the equivariant fibration  $\pi: \widehat{\mathbb{S}}_n \rightarrow \mathbb{H}^+$ ,  $\pi(Z) = z_{11}$ .

In the following we will concentrate on the case  $n = 3$ . Here we have  $\mathfrak{s}_3 = \mathfrak{b}_3 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_1$  where  $\mathfrak{b}_3$  is a  $j$ -ideal in  $\mathfrak{s}_3$  and  $\mathfrak{b}_2$  is a  $j$ -ideal in  $\mathfrak{s}_2 = \mathfrak{b}_2 \oplus \mathfrak{b}_1$ . For the rest of this section let  $D := \pi^{-1}(i)$  be the 5-dimensional bounded homogeneous (non-symmetric) domain corresponding to the normal  $j$ -algebra  $\mathfrak{b}_3 \oplus \mathfrak{b}_2$ . Let us consider the bases  $(\alpha_3, \zeta_{31}, \zeta_{32}, \zeta'_{31}, \zeta'_{32}, \zeta_3)$  of  $\mathfrak{b}_3$  and  $(\alpha_2, \zeta_{21}, \zeta'_{21}, \zeta_2)$  of  $\mathfrak{b}_2$ . Let us realize these elements explicitly as matrices as well as vector fields in the coordinates of the entries of  $Z = (z_{kl}) \in \text{Sym}(3, \mathbb{C})$ :

$$\begin{aligned} \alpha_3 &= \begin{pmatrix} E_{33} & 0 \\ 0 & -E_{33} \end{pmatrix} \mapsto 2z_{33} \frac{\partial}{\partial z_{33}}, & \zeta_3 &= -2 \begin{pmatrix} 0 & E_{33} \\ 0 & 0 \end{pmatrix} \mapsto -2 \frac{\partial}{\partial z_{33}}, \\ \zeta_{31} &= \begin{pmatrix} 0 & E_{13} + E_{31} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{13}}, & \zeta'_{31} &= \begin{pmatrix} E_{31} & 0 \\ 0 & -E_{13} \end{pmatrix} \mapsto z_{11} \frac{\partial}{\partial z_{13}} + z_{12} \frac{\partial}{\partial z_{23}} + 2z_{13} \frac{\partial}{\partial z_{33}}, \\ \zeta_{32} &= \begin{pmatrix} 0 & E_{23} + E_{32} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{23}}, & \zeta'_{32} &= \begin{pmatrix} E_{32} & 0 \\ 0 & -E_{23} \end{pmatrix} \mapsto z_{12} \frac{\partial}{\partial z_{13}} + z_{22} \frac{\partial}{\partial z_{23}} + 2z_{23} \frac{\partial}{\partial z_{33}}, \\ \alpha_2 &= \begin{pmatrix} E_{22} & 0 \\ 0 & -E_{22} \end{pmatrix} \mapsto 2z_{22} \frac{\partial}{\partial z_{22}}, & \zeta_2 &= -2 \begin{pmatrix} 0 & E_{22} \\ 0 & 0 \end{pmatrix} \mapsto -2 \frac{\partial}{\partial z_{22}}, \\ \zeta_{21} &= \begin{pmatrix} 0 & E_{12} + E_{21} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{12}}, & \zeta'_{21} &= \begin{pmatrix} E_{21} & 0 \\ 0 & -E_{12} \end{pmatrix} \mapsto z_{11} \frac{\partial}{\partial z_{12}} + 2z_{12} \frac{\partial}{\partial z_{22}} + z_{13} \frac{\partial}{\partial z_{23}}. \end{aligned}$$

The representation of  $\mathfrak{b}_2$  on  $\mathfrak{b}_3$  is defined by

$$\begin{aligned} [\alpha_2, \zeta_{32}] &= \zeta_{32} & [\alpha_2, \zeta'_{32}] &= -\zeta'_{32} \\ [\zeta_{21}, \zeta'_{31}] &= -\zeta_{32} & [\zeta_{21}, \zeta'_{32}] &= -\zeta_{31} \\ [\zeta'_{21}, \zeta_{31}] &= \zeta_{32} & [\zeta'_{21}, \zeta'_{32}] &= -\zeta'_{31} \\ [\zeta_2, \zeta'_{32}] &= 2\zeta_{32}, \end{aligned}$$

all other brackets being zero.

**5.2. A counterexample.** We present an example that shows that Proposition 4.1 and the implication Theorem 1.4 do not hold for  $D$ .

Let  $\mathfrak{n}$  denote the nilradical of  $\mathfrak{b}_3 \oplus \mathfrak{b}_2$  and consider the elements  $x_1, x_2, x_3 \in \mathfrak{n}$  given by

$$x_1 := \zeta'_{31} + \zeta_3 + \zeta_{21}, \quad x_2 := -\zeta_{31} + \zeta'_{21}, \quad x_3 := \zeta_3 + \zeta_2.$$

Since  $[x_1, x_2] = x_3$ , they generate a Lie subalgebra  $\mathfrak{n}_\Gamma := \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \mathbb{R}x_3$  that is isomorphic to the 3-dimensional Heisenberg algebra and projects surjectively onto the nilradical of  $\mathfrak{b}_2$ . Moreover, the corresponding group  $N_\Gamma$  of automorphisms of  $D$  admits a cocompact discrete subgroup, which justifies the notation  $\mathfrak{n}_\Gamma$ .

We shall see that all  $N_\Gamma$ -orbits in  $D$  are totally real, while  $\mathfrak{n}_\Gamma$  is not contained in any maximal totally real subalgebra of  $\mathfrak{n}$ . This shows that Proposition 4.1 does not hold for the domain  $D$ , even under the stronger assumption that *all*  $N_\Gamma$ -orbits are totally real in  $D$ .

*Remark.* A subalgebra of codimension 1 in a nilpotent Lie algebra is automatically normal. We apply this result in the following way: If  $y \in \mathfrak{n}$  is an element such that  $\mathfrak{n}_\Gamma \oplus \mathbb{R}y$  is a subalgebra, then  $y$  normalizes  $\mathfrak{n}_\Gamma$ .

Let us prove first that  $\mathfrak{n}_\Gamma$  is not contained in a maximal totally real subalgebra of  $\mathfrak{n}$ . For this, we write down the basis elements of  $\mathfrak{n}_\Gamma$  as vector fields:

$$\begin{aligned} x_1 &= \frac{\partial}{\partial z_{12}} + z_{11} \frac{\partial}{\partial z_{13}} + z_{12} \frac{\partial}{\partial z_{23}} + 2(z_{13} - 1) \frac{\partial}{\partial z_{33}}, \\ x_2 &= z_{11} \frac{\partial}{\partial z_{12}} - \frac{\partial}{\partial z_{13}} + 2z_{12} \frac{\partial}{\partial z_{22}} + z_{13} \frac{\partial}{\partial z_{23}}, \\ x_3 &= -2 \frac{\partial}{\partial z_{22}} - 2 \frac{\partial}{\partial z_{33}}. \end{aligned}$$

A direct calculation shows that the normalizer of  $\mathfrak{n}_\Gamma$  in  $\mathfrak{n}$  is given by

$$\mathcal{N}_{\mathfrak{n}}(\mathfrak{n}_\Gamma) = \mathfrak{n}_\Gamma \oplus \mathbb{R}\zeta_{32} \oplus \mathbb{R}\zeta_3 = \mathbb{R}(\zeta'_{31} + \zeta_{21}) \oplus \mathbb{R}(-\zeta_{31} + \zeta'_{21}) \oplus \mathbb{R}\zeta_{32} \oplus \mathbb{R}\zeta_3 \oplus \mathbb{R}\zeta_2.$$

A maximal totally real subalgebra of  $\mathcal{N}_{\mathfrak{n}}(\mathfrak{n}_\Gamma)$  containing  $\mathfrak{n}_\Gamma$  is given by  $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$  with

$$y_\tau = \zeta_{32} + \tau\zeta_3 = \frac{\partial}{\partial z_{23}} - 2\tau \frac{\partial}{\partial z_{33}} \quad (\tau \in \mathbb{R}^*).$$

Observing that  $y_\tau$  centralizes  $\mathfrak{n}_\Gamma$ , one obtains

$$\mathcal{N}_{\mathfrak{n}}(\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau) = \mathcal{N}_{\mathfrak{n}}(\mathfrak{n}_\Gamma),$$

which is *not* totally real. Hence,  $\mathfrak{n}_\Gamma$  is not contained in a maximal totally real subalgebra of  $\mathfrak{n}$ , i.e., Proposition 4.1 does not hold for the domain  $D$ .

Let  $\widehat{N}_\Gamma$  be the connected Lie group having Lie algebra  $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$ . Since  $y_\tau$  centralizes  $\mathfrak{n}_\Gamma$ , the group  $\widehat{N}_\Gamma$  admits cocompact discrete subgroups. We will show that the quotient of  $D$  with respect to any cocompact discrete subgroup of  $\widehat{N}_\Gamma$  is *not* Stein, hence that Theorem 1.4 does not hold for  $D$ . For this it is enough to find an  $\widehat{N}_\Gamma$ -orbit in  $D$  which is not totally real. Note that this implies again that  $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$  cannot be contained in a maximal totally real subalgebra. Indeed, for every  $\alpha > (1 + \tau^2)^{-2}$  the matrix

$$Z_0 = \begin{pmatrix} i & \frac{\tau+i}{1+\tau^2} & 0 \\ \frac{\tau+i}{1+\tau^2} & i\alpha & 0 \\ 0 & 0 & i \end{pmatrix}$$

belongs to  $D$ , and one sees without difficulty that  $\widehat{N}_\Gamma \cdot Z_0$  is not totally real.

Nevertheless, we shall prove in the rest of this subsection that  $D/\Gamma$  is a Stein manifold where  $\Gamma$  is any cocompact discrete subgroup of  $N_\Gamma$ . Using the description of the elements of  $\mathfrak{s}_\mathfrak{n}$  as matrices, we can realize  $N_\Gamma^{\mathbb{C}}$  as the matrix group

$$(5.1) \quad N_\Gamma^{\mathbb{C}} = \left\{ \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & a & -b \\ b & 1 & 0 & a & -2c & -\frac{a^2+b^2}{2} \\ a & 0 & 1 & -b & \frac{a^2+b^2}{2} & -2(a+c) \\ 0 & 0 & 0 & 1 & -b & -a \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid a, b, c \in \mathbb{C} \right\}.$$

The action of  $N_\Gamma^{\mathbb{C}}$  on  $\text{Sym}(3, \mathbb{C})$  is given by

$$Z = (z_{kl}) \mapsto \begin{pmatrix} z_{11} & z_{12} + bz_{11} + a & z_{13} + az_{11} - b \\ z_{12} + bz_{11} + a & z_{22} + 2bz_{12} + b^2z_{11} + ab - 2c & z_{23} + az_{12} + bz_{13} + abz_{11} + \frac{a^2 - b^2}{2} \\ z_{13} + az_{11} - b & z_{23} + az_{12} + bz_{13} + abz_{11} + \frac{a^2 - b^2}{2} & z_{33} + 2az_{13} + a^2z_{11} - ab - 2(a + c) \end{pmatrix}.$$

It is not difficult to verify that  $N_\Gamma^{\mathbb{C}}$  acts freely on  $\text{Sym}(3, \mathbb{C})$ , which in particular implies that  $N_\Gamma$  has only totally real orbits in  $D$ , see Lemma 2.1.

In the following we will show that  $N_\Gamma^{\mathbb{C}}$  acts properly on  $\text{Sym}(3, \mathbb{C})$  with quotient manifold  $\text{Sym}(3, \mathbb{C})/N_\Gamma^{\mathbb{C}} \cong \mathbb{C}^3$ . It then follows that the principal bundle  $\text{Sym}(3, \mathbb{C}) \rightarrow \text{Sym}(3, \mathbb{C})/\mathbb{C}^3$  is holomorphically trivial. Consequently,  $\text{Sym}(3, \mathbb{C})/\Gamma$  and  $D/\Gamma$  are Stein.

In order to prove that the  $N_\Gamma^{\mathbb{C}}$ -action is proper we shall use the following lemma.

**Lemma 5.1.** *Let  $G$  be a Lie group acting smoothly and freely on a manifold  $M$  and let  $H$  be a closed normal subgroup of  $G$ . Then  $G$  acts properly on  $M$  if and only if  $H$  acts properly on  $M$  and  $G/H$  acts properly on  $M/H$ .*

*Proof.* Suppose first that  $G$  acts properly on  $M$ . Since  $H$  is closed in  $G$ , the  $H$ -action on  $M$  is proper and we get a smooth action of  $G/H$  on the quotient manifold  $M/H$ . Properness of this latter action was shown in [Pal61, Proposition 1.3.2].

Conversely, suppose that the actions of  $H$  on  $M$  and of  $G/H$  on  $M/H$  are proper. Let  $(g_n)$  and  $(p_n)$  be sequences in  $G$  and  $M$ , respectively, such that  $(g_n \cdot p_n, p_n)$  converges to  $(q_0, p_0)$ . We must show that  $(g_n)$  has a convergent subsequence.

For this, note that  $(g_n H \cdot [p_n], [p_n])$  converges to  $([q_0], [p_0])$  in  $M/H \times M/H$  where  $[p] \in M/H$  denotes the class of  $p$  modulo  $H$ . Thus  $(g_n H)$  has a convergent subsequence. To simplify the notation we shall assume that  $g_n H \rightarrow g_0 H$ . Hence, there is a sequence  $(h_n)$  in  $H$  such that  $h_n^{-1} g_n \rightarrow g_0$  in  $G$ , from which we conclude that

$$(h_n \cdot (h_n^{-1} g_n \cdot p_n), h_n^{-1} g_n \cdot p_n) \rightarrow (q_0, g_0 \cdot p_0).$$

Since  $H$  acts properly on  $M$ , it follows that  $(h_n)$  and hence  $(g_n)$  have convergent subsequences.  $\square$

We apply Lemma 5.1 first to the action of the center of  $N_\Gamma^{\mathbb{C}}$  on  $\text{Sym}(3, \mathbb{C})$ . The center of  $N_\Gamma^{\mathbb{C}}$  is given by all matrices of the form (5.1) having  $a = b = 0$  and acts on  $\text{Sym}(3, \mathbb{C})$  by

$$c \cdot \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} - 2c & z_{23} \\ z_{13} & z_{23} & z_{33} - 2c \end{pmatrix}.$$

Hence, the corresponding  $\mathbb{C}$ -principal bundle is holomorphically trivial and given by

$$\pi: \text{Sym}(3, \mathbb{C}) \rightarrow \mathbb{C}^5, \quad \pi(z_{kl}) = (z_{11}, z_{12}, z_{13}, z_{22} - z_{33}, z_{23}).$$

The induced action of  $\mathbb{C}^2 \cong N_\Gamma^{\mathbb{C}}/Z(N_\Gamma^{\mathbb{C}})$  on  $\mathbb{C}^5$  is given by

$$(a, b) \cdot z = \begin{pmatrix} z_1 \\ z_2 + bz_1 + a \\ z_3 + az_1 - b \\ z_4 + 2bz_2 - 2az_3 + (b^2 - a^2)z_1 + 2ab + 2a \\ z_5 + az_2 + bz_3 + abz_1 + (a^2 - b^2)/2 \end{pmatrix}.$$

In the next step we consider only the action of the one parameter group  $\mathbb{C}_a$  of elements  $(a, 0)$ . A direct calculation shows that the  $\mathbb{C}_a$ -action on  $\mathbb{C}^5$  becomes a translation in the first coordinate after conjugation by the biregular map  $\Phi: \mathbb{C}^5 \rightarrow \mathbb{C} \times \mathbb{C}^4$ ,

$$z \mapsto (z_2, z_1, z_3 - z_1 z_2, z_4 + 2z_2 z_3 - 2z_1 z_5 - 2z_2, z_2^2 - 2z_5).$$

It follows that  $\mathbb{C}_a$  acts properly on  $\mathbb{C}^5$  and that the corresponding  $\mathbb{C}_a$ -principal bundle  $\mathbb{C}^5 \rightarrow \mathbb{C}^5/\mathbb{C}_a \cong \mathbb{C}^4$  is holomorphically trivial. Moreover, the induced  $\mathbb{C}_b$ -action on  $\mathbb{C}^4$  is of the form

$$b \cdot w = \begin{pmatrix} w_1 \\ w_2 - b(w_1^2 + 1) \\ w_3 - 2bw_1 \\ w_4 - 2bw_2 + b^2(w_1^2 + 1) \end{pmatrix}.$$

Note that the projection onto the first three coordinates is equivariant; therefore we obtain a free algebraic  $\mathbb{C}$ -action on  $\mathbb{C}^3$  of the form  $t \cdot w = (w_1, w_2 + tf(w_1), w_3 + tg(w_1))$  where  $f(w_1) = -(w_1^2 + 1)$  and  $g(w_1) = -2w_1$ . Due to a result of Kaliman, see [Kal04], every free algebraic  $\mathbb{C}$ -action on  $\mathbb{C}^3$  is conjugate under a biregular map to a translation. Hence, it follows that  $\mathbb{C}_b$  acts properly on  $\mathbb{C}^3$  and thus on  $\mathbb{C}^4$  as well as that  $\mathbb{C}^4/\mathbb{C}_b \cong \mathbb{C}^3$ . Consequently, Lemma 5.1 applies to show that  $N_\Gamma^{\mathbb{C}}$  acts properly on  $\text{Sym}(3, \mathbb{C})$  with quotient  $\mathbb{C}^3$ .

*Remark.* In our case it is possible to give an elementary proof of Kaliman's result which applies in a slightly more general setup. For this, let  $f, g \in \mathcal{O}(\mathbb{C})$  be two entire functions and consider the holomorphic  $\mathbb{C}$ -action on  $\mathbb{C}^3$  given by  $t \cdot z = (z_1, z_2 + tf(z_1), z_3 + tg(z_1))$ . This action is free if and only if  $f$  and  $g$  have no common zeros. In this case, there exist entire functions  $\varphi, \psi \in \mathcal{O}(\mathbb{C})$  such that  $\varphi f + \psi g = 1$ , see [Hel40]. Then the biholomorphic map  $\Phi: \mathbb{C}^3 \rightarrow \mathbb{C}^3$  given by

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(z_1) & g(z_1) \\ 0 & -\psi(z_1) & \varphi(z_1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

transforms the  $\mathbb{C}$ -action into a translation.

**5.3. Open problems and concluding remarks.** Let  $\Gamma \subset N$  be a discrete subgroup with Zariski closure  $N_\Gamma$ . As we have seen, in general the condition that only one  $N_\Gamma$ -orbit is totally real in  $D$  does not imply that  $D/\Gamma$  is Stein. It is natural to ask, however, whether  $D/\Gamma$  is Stein if all  $N_\Gamma$ -orbits in  $D$  are totally real. In this case, we obtain a free algebraic action of  $N_\Gamma^{\mathbb{C}}$  on the domain  $N_\Gamma^{\mathbb{C}} \cdot \widehat{D} \subset \mathbb{C}^n$ .

Since it seems impossible to apply Proposition 1.3 for general  $D$ , one could try a direct approach like the one in the previous subsection, which poses two major problems. First of all, one has to show that  $N_\Gamma^{\mathbb{C}}$  acts properly on  $N_\Gamma^{\mathbb{C}} \cdot \widehat{D} \subset \mathbb{C}^n$ . Secondly, if  $N_\Gamma^{\mathbb{C}}$  does indeed act properly, one must prove that  $(N_\Gamma^{\mathbb{C}} \cdot \widehat{D})/N_\Gamma^{\mathbb{C}}$  is a Stein manifold. These questions are far from being trivial, bearing in mind that there is an example of a proper algebraic  $\mathbb{C}^2$ -action on  $\mathbb{C}^6$  by affine-linear transformations such that  $\mathbb{C}^6/\mathbb{C}^2$  is quasi-affine but not Stein, see [Win90].

## REFERENCES

- [ABK04] D. Achab, F. Betten, and B. Krötz, *Discrete group actions on Stein domains in complex Lie groups*, Forum Math. **16** (2004), no. 1, 37–68.
- [Bor91] A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
- [BHH03] D. Burns, S. Halverscheid, and R. Hind, *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke Math. J. **118** (2003), no. 3, 465–491.

- [BS76] D. Burns, Jr. and S. Shnider, *Spherical hypersurfaces in complex manifolds*, Invent. Math. **33** (1976), no. 3, 223–246.
- [Car35] H. Cartan, *Sur les groupes de transformations analytiques*, (Exposés mathématiques IX.) Actual. scient. et industr., vol. 198, Hermann, Paris, 1935.
- [Che13] B.-Y. Chen, *Discrete groups and holomorphic functions*, Math. Ann. **355** (2013), no. 3, 1025–1047.
- [DG60] F. Docquier and H. Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.
- [dFI01] C. de Fabritiis and A. Iannuzzi, *Quotients of the unit ball of  $\mathbb{C}^n$  for a free action of  $\mathbb{Z}$* , J. Anal. Math. **85** (2001), 213–224.
- [FaKo94] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994.
- [GS84] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.
- [HW73] F. Reese Harvey and R. O. Wells, Jr., *Zero sets of non-negative strictly plurisubharmonic functions*, Math. Ann. **201** (1973), 165–170.
- [Hei91] P. Heinzner, *Geometric invariant theory on Stein spaces*, Math. Ann. **289** (1991), no. 4, 631–662.
- [Hel01] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, corrected reprint of the 1978 original, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI, 2001.
- [Hel40] O. Helmer, *Divisibility properties of integral functions*, Duke Math. J. **6** (1940), 345–356.
- [Ish06] H. Ishi, *On symplectic representations of normal  $\mathfrak{j}$ -algebras and their application to Xu’s realizations of Siegel domains*, Differential Geom. Appl. **24** (2006), no. 6, 588–612.
- [Kal04] S. Kaliman, *Free  $\mathbb{C}_+$ -actions on  $\mathbb{C}^3$  are translations*, Invent. Math. **156** (2004), no. 1, 163–173.
- [Kan67] S. Kaneyuki, *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo Sect. I **14** (1967), 89–130 (1967).
- [Kos55] J. L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canadian J. Math. **7** (1955), 562–576.
- [Loe85] J.-J. Loeb, *Action d’une forme réelle d’un groupe de Lie complexe sur les fonctions plurisousharmoniques*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 59–97.
- [Mie10a] C. Miebach, *Quotients of bounded homogeneous domains by cyclic groups*, Osaka J. Math. **47** (2010), no. 2, 331–352.
- [Mie10b] ———, *Sur les quotients discrets de semi-groupes complexes*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 2, 269–276.
- [MO09] C. Miebach and K. Oeljeklaus, *On proper  $\mathbb{R}$ -actions on hyperbolic Stein surfaces*, Doc. Math. **14** (2009), 673–689.
- [MO18] ———, *Schottky group actions in complex geometry*, Geometric complex analysis, Springer Proc. Math. Stat., vol. 246, Springer, Singapore, 2018, pp. 257–268.
- [Pal57] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. **22** (1957), p. iii+123.
- [Pal61] ———, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323.
- [PS69] I. I. Pyateskii-Shapiro, *Automorphic functions and the geometry of classical domains*, Translated from the Russian. Mathematics and Its Applications, Vol. 8, Gordon and Breach Science Publishers, New York, 1969.
- [Vin61] È. B. Vinberg, *The Morozov-Borel theorem for real Lie groups*, Dokl. Akad. Nauk SSSR **141** (1961), 270–273.
- [Vin94] È. B. Vinberg (ed.), *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994.
- [VGP63] È. B. Vinberg, S. G. Gindikin, and I. I. Pyateskii-Shapiro, *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moskov. Mat. Obšč. **12** (1963), 359–388.
- [Vit14] S. Vitali, *Quotients of the crown domain by a proper action of a cyclic group*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3227–3239.
- [Win90] J. Winkelmann, *On free holomorphic  $\mathbb{C}$ -actions on  $\mathbb{C}^n$  and homogeneous Stein manifolds*, Math. Ann. **286** (1990), no. 1-3, 593–612.

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