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► **To cite this version:**

Duboux Thibaut, Offret Yoann. Maximum Entropy Random Walks on Infinite Graphs: the case of Spider Networks and their Scaling Limits. 2023. hal-03582680v3

HAL Id: hal-03582680

<https://hal.science/hal-03582680v3>

Preprint submitted on 13 Nov 2023 (v3), last revised 4 Dec 2024 (v4)

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Maximum Entropy Random Walks on Infinite Graphs: the case of Spider Networks and their Scaling Limits

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November 8, 2023

Abstract. In this article, we lay solid foundations for the study of Maximal Entropy Random Walks (MERWs) on infinite graphs. We propose an appropriate definition that extends the original one from finite to infinite setting. Unlike conventional simple random walks, which maximize entropy locally, MERWs maximize entropy globally along their paths, representing a significant paradigm shift. They were introduced by physicists and computer scientists in [1] and are characterized by strong localization properties in heterogeneous environments. In the infinite setting, however, MERWs are not always unique and might not even exist. Despite this, they continue to maximize the entropy rate, albeit in a less direct way. Our investigation focuses on weighted spider networks, where we explicitly describe the MERWs and highlight a phase transition phenomenon related to the model's parameters. In studying the scaling limits of these random walks, we obtain some interesting Walsh diffusions and the three-dimensional Bessel process. A unified proof is presented based on submartingale problems. Notably, when applying our findings to particles satisfying to the Pauli exclusion principle, we can deduce the electrostatic force by solely relying on the maximal entropy principle. We also touch upon how to achieve these results for continuous time stochastic processes, utilizing the Kullback–Leibler divergence.

Key words. Random walks, Maximum entropy principle, Functional scaling limits, Reflected diffusions, Submartingale problem.

Mathematics Subject Classification (2000). 60G50, 60F17, 60G42, 60G44, 60J10, 60J60, 60K99, 60C05, 82B41, 82B26, 05C38, 05A15, 94A17.

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1 Introduction

The most popular way to randomly explore a locally finite graph G without any further information is to assume that the walker sitting at a node jumps onto any neighboring node chosen with a uniform probability and does this independently at every time. This Markov process is known as a Generic Random Walk (GRW). This choice among all possible random walks can be justified by its property of maximizing the entropy production at each step.

Entropy rate. The concept of entropy, introduced by Ludwig Boltzmann in the 1870s, is fundamental in the fields of Statistical Physics and Thermodynamics. Similarly, the field of Information Theory, developed by Claude Shannon in the 1940s, also recognizes the importance of this quantity. We refer to their groundbreaking papers [2, 3]. Here, all we need to know is that the entropy of a distribution μ on a countable set V is defined by

$$H(\mu) = - \sum_{x \in V} \mu(x) \ln(\mu(x)) \in [0, \infty]. \quad (1.1)$$

When X is a random variable on V , the quantity $H(X)$ represents the entropy of the distribution of X . Furthermore, if $\text{card}(V) = N$ is finite, the maximum value of $H(\mu)$ is attained when μ is the uniform probability measure on V , and it equals to $\ln(N)$. Concerning Markov chain theory on a countable state space V , a quantity of significant interest is the entropy rate

$$h = \lim_{n \rightarrow \infty} \frac{H((X_0, \dots, X_n))}{n}. \quad (1.2)$$

When $(X_n)_{n \geq 0}$ is irreducible and positive recurrent, h is independent of the initial distribution and only depends on the invariant probability measure π and the transition kernel P as

$$h = - \sum_{x, y \in V} \pi(x) P(x, y) \ln(P(x, y)). \quad (1.3)$$

For more details, one can refer to [4, 5]. As an example, the entropy rate of a GRW on a finite lattice with vertex set V is given by

$$h_{\text{GRW}} = \frac{\sum_{x \in V} d(x) \ln(d(x))}{\sum_{x \in V} d(x)}, \quad (1.4)$$

where $d(x)$ denotes the out-degree of the vertex x .

A brief history of MERWs. Maximum Entropy Random Walks represent a paradigm shift from a local to a global perspective. In essence, these are RWs that maximize entropy along their paths or, equivalently, the entropy rate. This approach has been recently introduced in [1, 6, 7]. Among their findings, the authors emphasize the strong localization phenomenon of MERWs in slightly disordered environments. This is especially relevant in Quantum Mechanics, especially in the context of the Anderson localization phenomenon (refer to [8] for a mathematical survey). More broadly, MERWs appear to hold significant implications for statistical physics. The concept of MERW is closely related to that of Parry measures for sub-shifts of finite type, as defined in [9] and recently explored in [10]. This idea is also subtly present in [11, 12]. Furthermore, MERWs could be instrumental in studying and modeling complex networks, as suggested in [13–15]. Lastly, the MERW concept has found applications in diverse scientific areas, such as community detection [16, 17], link prediction [18], and even quasispecies evolution [19].

The finite setting. While significant progress has been made in the mathematical framework of MERWs, further inquiry is still required. Current advancements mostly pertain to finite graphs, which present advantageous properties. Specifically, when dealing with an irreducible finite graph, the Perron-Frobenius theorem guarantees the existence and uniqueness of a MERW. As illustrated in [6], its Markov kernel and invariant probability measure, for all vertices x, y , are given by

$$P(x, y) = A(x, y) \frac{\psi(y)}{\rho \psi(x)} \quad \text{and} \quad \pi(x) = \varphi(x) \psi(x), \quad (1.5)$$

where A represents the adjacency matrix of graph G , ρ is its spectral radius, and ψ (respectively, φ) is the associated positive right (left) eigenfunction of A . The latter is normalized so that π defines a probability measure. The corresponding entropy rate is $h_{\text{MERW}} = \ln(\rho)$. Intriguingly, all trajectories of length n between vertices x and y have the same probability, given by

$$\mathbb{P}(X_0 = x, \dots, X_n = y) = \frac{\psi(y)}{\rho^n \psi(x)}. \quad (1.6)$$

While the trajectory distribution is not uniform, it becomes uniform when conditioned on trajectory length and endpoints. This property suggests the rich combinatorial features inherent in MERWs. Equation (1.5) evokes the well-known Doob h -transform, commonly encountered when conditioning stochastic processes to remain within a specific domain. For relevant references we refer to [20] and [21], as well as Remark 3.2. To broadening the scope, one can substitute the adjacency matrix A with a weighted variant (strictly positive across edges) and require the MERW to maximize

$$h(Q) = - \sum_{x, y \in G} \mu(x) Q(x, y) \ln \left(\frac{Q(x, y)}{A(x, y)} \right), \quad (1.7)$$

over the positive-recurrent Markov kernels Q on G . Here, μ is the invariant distribution of Q . When the entries $A(x, y)$ are non-negative integers, this formulation is interpreted as a MERW on a multi-edge graph. Additional constraints, like energy conditions, can be introduced as discussed in [22] and [6]. The positive eigenfunction ψ is prominent when assessing node influence in complex networks, forming the crux of the eigenvector centrality method [23]. For physicists, ψ can be interpreted as a wave function, specifically the ground state of a discrete Schrödinger equation

$$- \Delta \psi(x) + H(x) \psi(x) = -\rho \psi(x), \quad (1.8)$$

where Δ is the graph Laplacian and H is the potential defined by

$$\Delta f(x) = \sum_{y \in G} A(x, y)(f(y) - f(x)) \quad \text{and} \quad H(x) = - \sum_{y \in G} A(x, y). \quad (1.9)$$

For symmetric matrices A , we have $\pi(x) = \psi(x)^2$: the stationary probability distribution is the square of the wave function. There are only a limited number of solvable models where the spectral radius and the associated wave function are explicitly known; determining these in general is a challenging task. For specific examples, such as (truncated) Cayley trees and ladder graphs, we refer to [24]. For small graphs, it is feasible to compute these values numerically and carry out computer simulations of the MERW.

Beyond the finite situation. Importantly, and central to this paper's focus, there is a lack of consistent results for infinite networks. Infinite periodic lattices are discussed in [7], where diffusion coefficients are computed, but many questions remain unanswered:

- How should the MERW be properly defined on an weighted infinite graph ?
- Does a unique MERW exist ?
- How do the scaling limits of MERWs compare with those of GRWs?

Regarding the last question, a quintessential example is Donsker's celebrated result [25]. This has led to a rich body of literature. Under certain conditions, Donsker's result indicates that the scaling limits of GRWs are Brownian motions. A key objective of ours is to showcase that many usual continuous-time stochastic processes can be interpreted as scaling limits of MERWs.

Outline of the article. In Section 2, we first provide a generalized definition of MERWs on (possibly infinite) weighted irreducible graphs that does not involve entropy. Subsequently, we present the necessary and sufficient conditions for their existence and uniqueness. Examples where either existence or uniqueness is absent are discussed. We further motivate our definition and we elucidate how these RWs optimize the entropy rate and establish a connection with the entropy of some dynamical systems. An open question about finite approximations of these MERWs is posed, accompanied by an intriguing example. We also describe how to leverage potential symmetries to compute MERWs. In Section 3, we delve into MERWs on weighted spider networks. A phase transition phenomenon is spotlighted: depending on the parameters, they can be positive-recurrent, null-recurrent, or transient. The spectral radius and its associated eigenfunctions are determined explicitly. In Section 4, we state the functional scaling limits we have obtained. In the recurrent situations, these involve Walsh diffusions, specifically the standard Walsh Brownian motion and, in a manner of speaking, attractive Ornstein-Uhlenbeck Walsh diffusions. In the transient case, scaling limits are related to the renowned three-dimensional Bessel process. Additionally, in the more intricate context of continuous-time stochastic processes, we explain how the latter diffusions can be directly interpreted as those minimizing some Kullback-Leibler divergence. Finally, the proofs of the functional scaling limits are furnished in Section 5.

2 General Framework

In what follows, let G represent a countable irreducible weighted graph. We use A to denote the weighted adjacency matrix and \mathcal{E} for the set of edges. For simplicity, we will refer to G as

the set of vertices. Additionally, we shall assume that

$$\sup_{x \in G} \sum_{y \in G} A(x, y) < \infty. \quad (2.1)$$

2.1 An Expanded Definition of MERWs

For the primary results on infinite positive matrices that we make use of, we refer to [26].

Definition 2.1. For any arbitrary $x, y \in G$, the combinatorial spectral radius, denoted by ρ , is defined as the inverse of the radius of convergence for $\sum_{n=0}^{\infty} A^n(x, y)z^n$. Notably, it is independent of the choice of x and y .

In essence, the leading asymptotic behavior of the number of n -step trajectories from x to y is on the order of ρ^n .

Definition 2.2. A random walk on G is termed a MERW if, for all vertices $x, y \in G$, its Markov kernel is defined as

$$P(x, y) = A(x, y) \frac{\psi(y)}{\rho \psi(x)}, \quad (2.2)$$

where ψ represents a positive eigenfunction of A associated with the spectral radius ρ .

Analogously to (1.5), if φ is a positive left eigenfunction of A associated with the eigenvalue ρ then $\pi(x) = \varphi(x)\psi(x)$ is an invariant measure of the MERW. Without loss of generality we can assume that $\psi(o) = 1$ for a chosen base point $o \in G$. Moreover the set of positive solutions to $A\psi = \rho\psi$ with $\psi(o) = 1$ is a convex compact set for the pointwise topology. By the Krein–Milman and Choquet theorems such a solution can be expressed as

$$\psi(x) = \int \psi_e(x) \mu(de), \quad (2.3)$$

where $\{\psi_e : e \in K\}$ denotes the set of extremal solutions and μ is a probability measure on K .

2.2 Existence and Uniqueness

To ensure the existence and uniqueness of such MERW, we require further assumptions related to the recurrence and transience of Markov processes.

Definition 2.3. Let $x, y \in G$ be arbitrary. The matrix A is termed \mathcal{R} -recurrent (resp. \mathcal{R} -transient) if

$$\sum_{n \geq 0} \frac{A^n(x, y)}{\rho^n} = \infty \quad \left(\text{resp. } \sum_{n \geq 0} \frac{A^n(x, y)}{\rho^n} < \infty \right). \quad (2.4)$$

If A is \mathcal{R} -recurrent, it is termed \mathcal{R} -null (resp. \mathcal{R} -positive) if $A^n(x, y)\rho^{-n}$ tends to zero (resp. does not tend to zero). Notably, these definitions are independent of the choice of x and y .

Proposition 2.1. Suppose that A is \mathcal{R} -recurrent. Then, ρ is an eigenvalue of A , and there exist unique left and right eigenfunctions associated with ρ (up to a multiplicative constant). Consequently, there is a unique MERW, which is recurrent. Moreover, this MERW is positive recurrent if and only if A is \mathcal{R} -positive.

A toy example. In the \mathcal{R} -transient case, neither existence nor uniqueness of the MERW is assured. Consider the example where $G = \mathbb{N}_0 = \{0, 1, \dots\}$. Define the adjacency matrix A such that $A(n, n-1) = 1$, $A(0, n) = \alpha_n$ for all $n \geq 1$, and $A(x, y) = 0$ elsewhere. It can be observed that

$$\sum_{n \geq 1} A^n(0, 0) z^n = \frac{1}{1 - \sum_{n \geq 1} \alpha_n z^{n+1}}. \quad (2.5)$$

To go further, if $\sum_{n \geq 1} \alpha_n z^{n+1}$ has a radius of convergence equal to 1 and $\sum_{n \geq 1} \alpha_n < 1$, then $\rho = 1$ and A is \mathcal{R} -transient. Furthermore, one can see that no solution exists for the system $\psi_0 = 1$, $\psi_n = \psi_{n-1}$ for all $n \geq 1$, and $\psi_0 = \sum_{n \geq 1} \alpha_n \psi_n$. Consequently, a MERW does not exist for this setup.

As a matter of facts, it is possible to establish a necessary and sufficient condition for the existence of a MERW. This condition is closely tied to taboo-like probabilities and draws upon the foundational work by Harris and Veech on the existence of an invariant measure for a transient Markov chain. A detailed presentation of this result can be found in [27].

Theorem 2.1. *The equation $A\psi = \lambda\psi$ with $\lambda > 0$ has a positive solution ψ if and only if one of the following conditions is satisfied:*

(i) $\lambda = \rho$ and A is \mathcal{R} -recurrent;

(ii) (a) $\lambda = \rho$ and A is \mathcal{R} -transient, or (b) $\lambda > \rho$, and in both cases, there exists an infinite subset $K \subset G$ and an exhaustive nested sequence $(G_j)_{j \geq 0}$ of G with $\text{card}(G_j) = j$ such that

$$\lim_{j \rightarrow \infty, k \rightarrow \infty, k \in K} \frac{\sum_{y \notin G_j} A(x, y) {}_x F_{y,k}(\lambda^{-1})}{{}_x F_{x,k}(\lambda^{-1})} = 0, \quad (2.6)$$

where the power series ${}_x F_{y,k}(z) = \sum_{n=0}^{\infty} {}_x f_{y,k}^{(n)} z^n$ are recursively defined by

$${}_x f_{y,k}^{(n+1)} = \sum_{w \neq x} A(y, w) {}_x f_{w,k}^{(n)} \quad \text{and} \quad {}_x f_{y,k}^{(0)} = \delta_{y,k}(1 - \delta_{x,y}). \quad (2.7)$$

Here $k \rightarrow \infty$ in G in the sense of the Alexandroff extension, and $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Remark 2.1. *If G is locally finite then the Harris-Veech condition (2.6) is met. Consequently, there exists at least one MERW in that case.*

2.3 Entropy Rate characterizations

It should be noted that in case (ii)(b) of Theorem 2.1, one can define a Markov kernel by replacing ρ in (2.2) with the corresponding $\lambda > \rho$. Conditionally on their length and their extremities, the probability of any trajectory remains proportional to its weight. One might question the reason for not replacing ρ in Definition 2.2 with an arbitrary $\lambda \geq \rho$ when feasible. The primary motivation is that we want MERWs to genuinely maximize the entropy production along the paths, in a manner yet to be defined.

A toy example. For illustration, consider the standard lattice $G = \mathbb{Z}$. The set of extremal positive solutions with $\psi(0) = 1$ exists for any $\lambda = 2 \cosh(\alpha)$, $\alpha \geq 0$, and is given by $\psi_{\pm}(x) = e^{\pm \alpha x}$. The corresponding RWs are the usual biased RWs defined by the transition probabilities

$$P(x, x+1) = 1 - P(x, x-1) = \frac{e^{\pm \alpha}}{e^{\alpha} + e^{-\alpha}}. \quad (2.8)$$

Starting from any point, it is straightforward to confirm that the rate of entropy, as defined in (1.2), is maximized for $\alpha = 0$, corresponding to $\lambda = \rho = 2$.

To delve deeper, remember that $h(Q)$ is defined in (1.7). Let us introduce

$$h^*(G) = \sup\{h(Q) : Q \text{ is a positive-recurrent kernel on } G\}, \quad (2.9)$$

It is important to note that $h^*(G) < \infty$ because it is bounded above by the logarithm of (2.1). For a finite graph G , the supremum of $h(Q)$ is attained at a unique positive recurrent kernel represented by the MERW (1.5), and we have $h^*(G) = \ln(\rho(G))$ where $\rho(G)$ denotes the spectral radius of G . Moving forward, when considering a subgraph $H \subset G$, it is naturally endowed with the weight-structure of G through the restriction of the weighted adjacency matrix. Broadly speaking, the following result can be inferred from sources such as [28–37]. We provide some detailed of the results below.

Theorem 2.2. *It holds that $h^*(G) = \ln(\rho)$. Additionally, the supremum in (2.9) is actually a maximum if and only if G is \mathcal{R} -positive. When this condition is met, the maximum is attained by the unique MERW transition kernel. Moreover, one can express*

$$h^*(G) = \sup\{h^*(H) : H \subset G \text{ is finite and irreducible}\}. \quad (2.10)$$

To find the sequence that maximizes this, one can select any exhaustive and increasing sequence of finite, irreducible subgraphs, denoted as (H_n) . Let P_n represent the unique MERW transition kernel on each H_n .

1. *If G is \mathcal{R} -recurrent, then the sequence (P_n) converges pointwise to the unique MERW transition kernel.*
2. *If G is locally finite and \mathcal{R} -transient, then the sequence (P_n) is tight, and any of its limit points is a MERW transition kernel.*

To be more specific, for unweighted graphs, $h^*(G)$ is the Gurevich entropy of the associated topological Markov chain. It has been established that an equilibrium measure exists if and only if G is \mathcal{R} -positive, and in such cases, it is unique. We refer to [31, 32, 35] for a review of these results. Those can be extended to weighted graphs and further, using the concepts of topological pressure and potential (see, for instance, [33, 36]). Regarding the convergence of a maximizing sequence, references [28–30] and [37] provide insights, with the latter highlighting a connection to the theory of large deviations. The reason we assume G is locally finite for transient graphs is due to the need to exchange the limit in

$$\rho_n \psi_n(x) = \sum_{y \in H_n} A(x, y) \psi_n(y), \quad x, y \in H_n, \quad (2.11)$$

where ψ_n represents the eigenfunction associated with the spectral radius of H_n , and $\psi_n(o) = 1$ for a base point o that is present in all the H_n .

An open question and an intriguing example. In light of the previous Theorem, an interesting question arises: can all MERWs be obtained from finite approximations when the graph is \mathcal{R} -transient? To gauge the breadth of the problem, let us contemplate a rudimentary example. Suppose $G = \mathbb{Z}$ carries the standard weight-structure, except that $A(0, \pm 1) = \gamma$ for a certain $\gamma > 0$. In this context, this represents the symmetric spider lattice with two legs, having $\gamma_3 = 1$,

$\gamma_2^{(1)} = \gamma_2^{(2)} = \gamma$, and $\gamma_1 = 0$. We find that G is \mathcal{R} -transient if and only if $\gamma < 1$. Under these conditions, $\rho = 2$, and the two extremal eigenfunctions are

$$\psi^{(\pm)}(n) = \begin{cases} 1 + \Lambda n & \text{if } \pm n \geq 0, \\ 1 & \text{if } \pm n \leq 0, \end{cases} \quad \text{with } \Lambda = \frac{2(1-\gamma)}{\gamma}. \quad (2.12)$$

Let us introduce $H_{p,q} = \{-q+1, -q+2, \dots, p-2, p-1\} \subset G$ for $p, q \geq 1$. Observe that $\rho(H_{p,q}) \uparrow 2$ as p, q grow to infinity. Let $\psi_{p,q}$ be the eigenfunction associated with the spectral radius and satisfying $\psi_{p,q}(0) = 1$. We can write $\rho(H_{p,q}) = 2 \cos(\theta_{p,q})$ where $\theta_{p,q} \in (0, \pi/2)$ and thereafter

$$\psi_{p,q}(n) = \cos(\theta_{p,q}n) + (b_- \mathbf{1}_{n < 0} + b_+ \mathbf{1}_{n > 0}) \sin(\theta_{p,q}n). \quad (2.13)$$

Analyzing the boundary conditions at points $0, p$, and $-q$, we derive

$$b_+ - b_- = \Lambda \cotan(\theta_{p,q}), \quad b_+ = -\cotan(p\theta_{p,q}) \quad \text{and} \quad b_- = \cotan(q\theta_{p,q}). \quad (2.14)$$

One can further express

$$\psi_{p,q}(n) = \frac{\sin((p-n)\theta_{p,q})}{\sin(p\theta_{p,q})} \mathbf{1}_{n \geq 0} + \frac{\sin((q+n)\theta_{p,q})}{\sin(q\theta_{p,q})} \mathbf{1}_{n < 0}. \quad (2.15)$$

As $\theta_{p,q} \rightarrow 0$ and since $\psi_{p,q}$ is positive, to maximize $2 \cos(\theta_{p,q})$ we need

$$\theta_{p,q} \sim \frac{\pi}{\max(p, q)}. \quad (2.16)$$

Assume first that $q \sim \alpha p$ for some $0 < \alpha < 1$. Then, $b_- \sim \cot(\alpha\pi)$ and

$$b_+ \sim \frac{1}{\pi - p\theta_{p,q}} \sim \frac{\Lambda p}{\pi}. \quad (2.17)$$

From this, we infer that $\psi_{p,q} \rightarrow \psi^+$ pointwise, and subsequently, $\psi_{q,p} \rightarrow \psi^-$. As a result, we can identify the two extremal MERWs. Next, let us assume that $q \sim p$ with $p - q = \delta$ for some $\delta \in \mathbb{N}_0$. Under this assumption,

$$\pi - q\theta_{p,q} \sim \left(1 + \delta \frac{\theta_{p,q}}{\pi - p\theta_{p,q}}\right) (\pi - p\theta_{p,q}). \quad (2.18)$$

We find that $b_{\pm} \sim \mu_{\pm} \Lambda$ where $\mu_- = 1 - \mu_+$ and $\mu_+ \left(1 + \frac{1}{1 + \delta \mu_+ \Lambda}\right) = 1$. Interestingly, the non-extremal MERWs, obtainable through finite approximations, are quantified and represented by the eigenfunctions $\mu_{\delta} \psi^{(+)} + (1 - \mu_{\delta}) \psi^-$ for $\delta \in \mathbb{Z}$, with

$$\mu_{\delta} = \frac{\sqrt{\Lambda^2 \delta^2 + 4} + \Lambda \delta - 2}{2\delta \Lambda} \in (0, 1). \quad (2.19)$$

This formula is extended by continuity at $\delta = 0$ by defining $\mu_0 = 1/2$.

2.4 Automorphism and Reduced Models

In general, computing the combinatorial spectral radius and the associated eigenfunctions can be quite challenging. In this section, we provide tools to explore simpler models when the graph exhibits symmetries. For a deeper understanding of graph automorphisms and amenable groups, we refer the reader to [38]. Let us define \mathcal{T} as a subgroup of

$$\text{Aut}(G) = \{\tau \in \mathfrak{S}(G) : \forall x, y \in G, A(\tau x, \tau y) = A(x, y)\}, \quad (2.20)$$

where $\text{Aut}(G)$ denotes the group of graph automorphisms. The orbit of an element $x \in G$ under the action of \mathcal{T} is represented by \bar{x} , and the entire orbit space is denoted by \overline{G} .

Definition 2.4. The set \overline{G} is canonically endowed with a weighted graph structure inherited from that of G . The edges of this structure are defined as

$$\overline{\mathcal{E}} = \{(\overline{x}, \overline{y}) : \exists(x, y) \in \overline{x} \times \overline{y} \text{ with } A(x, y) > 0\}. \quad (2.21)$$

The corresponding weighted adjacency matrix is given by

$$\overline{A}(\overline{x}, \overline{y}) = \sum_{y \in \overline{y}} A(x, y), \quad (2.22)$$

and this value does not depend on the choice of $x \in \overline{x}$. Furthermore, if G is irreducible, then \overline{G} is also irreducible. This graph is termed the (weighted) reduced graph. In [38], it is also referred to as the factor graph $\mathcal{T} \setminus G$.

Note that if, for some $\lambda \in \mathbb{C}$ and $\overline{\psi} : \overline{G} \rightarrow \mathbb{C}$, we have $\overline{A}\overline{\psi} = \lambda\overline{\psi}$, then $A\psi = \lambda\psi$, where ψ is defined by $\psi(x) = \overline{\psi}(\overline{x})$ for all $x \in G$. Conversely, if $A\psi = \lambda\psi$ and ψ is \mathcal{T} -invariant, i.e., $\psi(\tau x) = \psi(x)$ for all $\tau \in \mathcal{T}$ and $x \in G$, then $\overline{A}\overline{\psi} = \lambda\overline{\psi}$, where $\overline{\psi}(\overline{x}) = \psi(x)$ for all $x \in G$. Besides, we denote by $\overline{\rho}$ the combinatorial spectral radius of the reduced graph. Clearly, $\rho \leq \overline{\rho}$, because for any $x, y \in G$ we have

$$\overline{A}^n(\overline{x}, \overline{y}) = \sum_{y \in \overline{y}} A^n(x, y). \quad (2.23)$$

Below, we shall provide conditions ensuring that $\overline{\rho} = \rho$. For a given $x_0 \in G$ and $n \geq 0$, set

$$B(x_0, n) = \left\{ x \in G : \exists 0 \leq k \leq n, A^k(x_0, x) > 0 \right\}. \quad (2.24)$$

For $L \subset G$, we define ∂L as the set of vertices $y \in G \setminus L$ such that there exists $x \in L$ with (x, y) being an edge of G . We use $|L|$ to represent the cardinality of L . Remember that \mathcal{T} is termed quasi-transitive if \overline{G} is finite.

Proposition 2.2. We have $\rho = \overline{\rho}$ if any of the following conditions is met:

- i) there exists an $x \in G$ such that \overline{x} is finite;
- ii) G is locally finite, A is symmetric, and there exist $x_0, x \in G$ such that

$$\lim_{n \rightarrow \infty} \frac{\ln(|B(x_0, n) \cap \overline{x}|)}{n} = 0; \quad (2.25)$$

- iii) there exists a positive \mathcal{T} -invariant function ψ such that $A\psi \leq \rho\psi$;

- iv) G is locally finite, \mathcal{T} is quasi-transitive, and either

- (a) G does not satisfy a strong isoperimetric inequality, i.e., $\inf_{L \subset G, L \neq \emptyset} \frac{|\partial L|}{|L|} = 0$;
- (b) or \mathcal{T} is amenable.

Proof. i) The power series $\sum_n A^n(x, y)z^n$, for $x, y \in G$, has a common radius of convergence $R = 1/\rho$ and possesses non-negative coefficients. From (2.23), we conclude that $\overline{\rho} \leq \rho$. ii) When A is symmetric and locally finite, it can be viewed as a bounded linear operator on $\ell^2(G)$. Besides, it comes from [38, Chap. II.10.] that $\|A\|_2 = \rho$ and $\lim_{n \rightarrow \infty} \|A^n\|_2^{1/n} = \rho$. Specifically, considering $f_n(\cdot) = \mathbb{1}_{B(x_0, n) \cap \overline{x}}$, we infer $\overline{\rho} \leq \rho$ from

$$\limsup_{n \rightarrow \infty} (\overline{A}^n(x_0, \overline{x}))^{1/n} \leq \limsup_{n \rightarrow \infty} \|A^n f_n\|_2^{1/n} \leq \limsup_{n \rightarrow \infty} |B(x_0, n) \cap \overline{x}|^{1/2n} \|A^n\|_2^{1/n}. \quad (2.26)$$

iii) By setting $\bar{\psi}(\bar{x}) = \psi(x)$, we get a positive function with the property $\overline{A\bar{\psi}} \leq \rho \bar{\psi}$ and thus we deduce that $\bar{\rho} \leq \rho$ by using [27]. iv) Given that \overline{G} is finite, there exists a positive function $\bar{\psi}$ such that $\overline{A\bar{\psi}} = \bar{\rho} \bar{\psi}$. Let ψ represent the corresponding symmetric function on G (a lift), and consider the random walk on G with transition probabilities defined as

$$p(x, y) = A(x, y) \frac{\psi(y)}{\bar{\rho} \psi(x)}. \quad (2.27)$$

Given the symmetry of A , ψ^2 is a reversible measure. Moreover, ψ^2 is bounded both above and below by positive constants. Referring to Corollary 10.12 in [38], we find

$$\limsup_{n \rightarrow \infty} p^n(x, y)^{1/n} = 1. \quad (2.28)$$

This implies $\rho = \bar{\rho}$ whenever either condition (a) or (b) holds. \square

Remark 2.2. *The preceding proposition can be applied to the infinite periodic lattices examined in [7], lending further rigor to their computation of the spectral radius.*

3 Spider MERWs

3.1 Model and Settings

The model we consider is a star graph with N half-lines perturbed at the origin. We refer to Figure 1. It can be parameterized as

$$G = \{(n, k) : n \in \mathbb{N}_0, k = 1, \dots, N\} \cup \{\mathbf{0}\}. \quad (3.1)$$

For convenience, we make the identification $\mathbf{0} = (0, 1) = \dots = (0, N)$. We will denote by $\partial G = \{\infty_1, \dots, \infty_N\}$ the geometric boundary of G . Moreover, given any kernel $P(x, y)$ or function $\pi(x)$ on G , we denote by P_k and π_k their restrictions to the k -th leg G_k , and we write $P_k(n, m) = P((n, k), (m, k))$ and $\pi_k(n) = \pi((n, k))$. The weighted adjacency matrix A we consider is defined for all $n \geq 1$ and $1 \leq k \leq N$ as

$$A(\mathbf{0}, \mathbf{0}) = \gamma_1, \quad A_k(0, 1) = \gamma_2^{(k)}, \quad \text{and} \quad A_k(n, n \pm 1) = \gamma_3. \quad (3.2)$$

We assume that $\gamma_1 \geq 0$ and $\gamma_2^{(k)}, \gamma_3 > 0$. Hereafter, we set

$$\mathcal{S}_2 = \gamma_2^{(1)} + \dots + \gamma_2^{(N)} \quad \text{and} \quad \Lambda = 2\gamma_3 - \gamma_1 - \mathcal{S}_2. \quad (3.3)$$

Definition 3.1. *The case when $\Lambda = 0$, $\Lambda < 0$, or $\Lambda > 0$ will be referred to as regular, attractive, or repulsive, respectively.*

It is noteworthy that τ , defined by $\tau((n, k)) = (n, (k + 1) \bmod N)$ when $n \geq 2$ and the identity otherwise, generates a subgroup \mathcal{T} of $\text{Aut}(G)$. By additionally assuming that $\gamma_2^{(k)} = \gamma_2$ is constant on each ray, one can verify that the automorphism group of the model is isomorphic to the cyclic group of order N . It can be interpreted as rotations of the plane given a proper embedding. The reduced model is the original one with a single leg and $\gamma_2^{(1)} = N\gamma_2$.

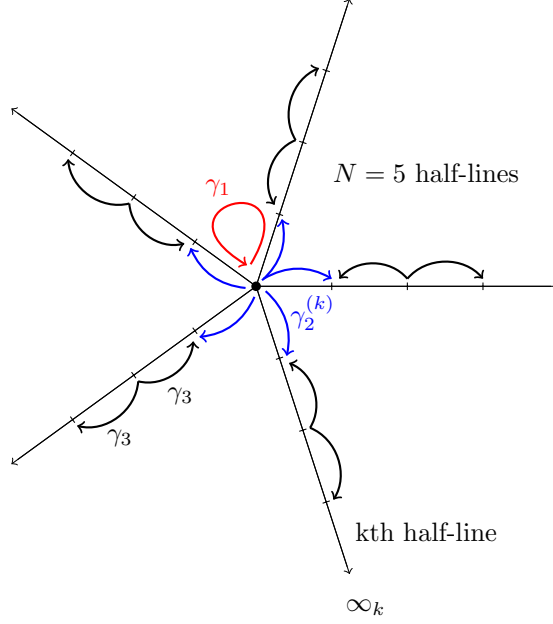


Figure 1: The spider network

3.2 Spectral Radius

We first observe that the spectral radius exhibits a phase transition phenomenon.

Proposition 3.1. *The combinatorial spectral radius is given by*

$$\rho = \begin{cases} 2\gamma_3, & \text{if } \Lambda \geq 0, \\ \frac{2\gamma_3(\gamma_1^2 + \mathcal{S}_2^2)}{\gamma_1(2\gamma_3 - \mathcal{S}_2) + \mathcal{S}_2\sqrt{\gamma_1^2 + 4\gamma_3(\mathcal{S}_2 - \gamma_3)}}, & \text{if } \Lambda < 0. \end{cases} \quad (3.4)$$

Proof. Let C_n be the n -th Catalan number. It is well-known that

$$S(z) = \sum_{n=0}^{\infty} C_n \gamma_3^{2n} z^{2n} = \frac{1 - \sqrt{1 - 4z^2\gamma_3^2}}{2z^2\gamma_3^2}. \quad (3.5)$$

The radius of convergence of S is $R_0 = (2\gamma_3)^{-1}$. Let $R = \rho^{-1}$ be the radius of convergence of $F(z) = \sum_{n \geq 0} A^n(\mathbf{0}, \mathbf{0})z^n$. Using classical tools of algebraic combinatorics, as illustrated in [39–41], we can express for all $|z| < R$,

$$F(z) = \frac{1}{1 - (\gamma_1 z + \mathcal{S}_2 \gamma_3 z^2 S(z))}. \quad (3.6)$$

Note that the function $x \mapsto \gamma_1 x + \mathcal{S}_2 \gamma_3 x^2 S(x)$ increases on $[0, R_0]$. Moreover, it can be verified that

$$\gamma_1 R_0 + \mathcal{S}_2 \gamma_3 R_0^2 S(R_0) = \frac{\gamma_1 + \mathcal{S}_2}{2\gamma_3}. \quad (3.7)$$

It follows that $R = R_0$ when $\Lambda \geq 0$. If not, R is the positive solution of $\gamma_1 R + \mathcal{S}_2 \gamma_3 R^2 S(R) = 1$, which is given by

$$R = \frac{\gamma_1(2\gamma_3 - \mathcal{S}_2) + \mathcal{S}_2\sqrt{\gamma_1^2 + 4\gamma_3(\mathcal{S}_2 - \gamma_3)}}{2\gamma_3(\gamma_1^2 + \mathcal{S}_2^2)}. \quad (3.8)$$

This concludes the proof. \square

Remark 3.1. The spectral radius ρ is identical to the model with a single leg where $\gamma_2^{(1)} = \mathcal{S}_2$. When $\gamma_2^{(k)} \equiv \gamma_2$ is constant, this is a direct result of Proposition 2.2.

3.3 Markov Kernels

Proposition 3.2 (regular case $\Lambda = 0$). There exists a unique MERW. The positive right eigenfunction is given by $\psi \equiv 1$. For all $1 \leq k \leq N$ and $n \geq 1$, the transition probabilities are

$$P_k(n, n+1) = \frac{1}{2}, \quad P_k(n, n-1) = \frac{1}{2}, \quad P_k(0, 1) = \frac{\gamma_2^{(k)}}{2\gamma_3}, \quad \text{and} \quad P(\mathbf{0}, \mathbf{0}) = \frac{\gamma_1}{2\gamma_3}. \quad (3.9)$$

The process is null-recurrent with an invariant measure given by $\pi_k(n) = \gamma_2^{(k)}$ and $\pi(\mathbf{0}) = \gamma_3$.

Proof. It can be readily verified that ψ is a positive eigenfunction associated with $\rho = 2\gamma_3$. Referring to the proof of Proposition 3.1, it is apparent that A is \mathcal{R} -recurrent. Hence, Proposition 2.1 implies the uniqueness of the MERW. The remainder of the proof follows directly. \square

Proposition 3.3 (attractive case $\Lambda < 0$). There exists a unique MERW. The positive right eigenfunction is $\psi_k(n) = \Gamma^n$ for all $n \geq 0$ and $1 \leq k \leq N$. The factor Γ is defined as

$$\Gamma = \frac{\rho - \gamma_1}{\mathcal{S}_2}. \quad (3.10)$$

The transition probabilities for all $1 \leq k \leq n$ and $n \geq 1$ are given by

$$P_k(n, n \pm 1) = \frac{\gamma_3}{\rho} \Gamma^{\pm 1}, \quad P_k(0, 1) = \frac{\gamma_2^{(k)} \Gamma}{\rho}, \quad \text{and} \quad P(\mathbf{0}, \mathbf{0}) = \frac{\gamma_1}{\rho}. \quad (3.11)$$

Moreover, the MERW is positive recurrent. Its invariant probability measure is

$$\pi_k(n) = \frac{\gamma_2^{(k)} (1 - \Gamma^2) \Gamma^{2n}}{\mathcal{S}_2 \Gamma^2 + \gamma_3 (1 - \Gamma^2)} \quad \text{and} \quad \pi(\mathbf{0}) = \frac{\gamma_3 (1 - \Gamma^2)}{\mathcal{S}_2 \Gamma^2 + \gamma_3 (1 - \Gamma^2)}. \quad (3.12)$$

Proof. From Proposition 3.1, we deduce the \mathcal{R} -recurrence, leading us via Proposition 2.1 to the existence of a unique MERW on G . For all $n \geq 1$, consider $\gamma_3 \psi_k(n+1) + \gamma_3 \psi_k(n-1) = \rho \psi_k(n)$ subject to

$$\sum_{k=1}^N \gamma_2^{(k)} \psi_k(1) = \rho - \gamma_1 \quad \text{and} \quad \psi(\mathbf{0}) = 1. \quad (3.13)$$

Let β be the root of $\gamma_3 X^2 - \rho X + \gamma_3 = 0$ in the interval $(0, 1)$. We can express $\psi_k(n)$ as

$$\psi_k(n) = a_k \beta^n + b_k \beta^{-n}, \quad (3.14)$$

with constants $a_k, b_k \in \mathbb{R}$. Using Proposition 3.1, we find that ρ satisfies

$$\gamma_1 + \mathcal{S}_2 \left[\gamma_3 \rho^{-1} S(\rho^{-1}) \right] = \rho, \quad \text{where} \quad S(z) = \frac{1 - \sqrt{1 - 4z^2 \gamma_3^2}}{2z^2 \gamma_3^2}. \quad (3.15)$$

From (3.15), we deduce

$$\beta = \frac{\rho - \sqrt{\rho^2 - 4\gamma_3^2}}{2\gamma_3} = \gamma_3 \rho^{-1} S(\rho^{-1}) = \Gamma. \quad (3.16)$$

We can verify that the function $\psi_k(n) = \Gamma^n$ is indeed the unique solution. The invariant probability measure of the MERW is found by analyzing the left eigenvector of the system. For this, we solve the equation $\gamma_3\varphi_k(n+1) + \gamma_3\varphi_k(n-1) = \rho\varphi_k(n)$ for all $n \geq 2$ and $1 \leq k \leq N$ subject to

$$\varphi_k(2) = \frac{\rho\varphi_k(1) - \gamma_2^{(k)}}{\gamma_3}, \quad \sum_{k=1}^N \varphi_k(1) = \frac{\rho - \gamma_1}{\gamma_3}, \quad \text{and } \varphi(\mathbf{0}) = 1. \quad (3.17)$$

We confirm that the function defined by $\varphi(\mathbf{0}) = 1$ and $\varphi_k(n) = \frac{\gamma_2^{(k)}\Gamma^n}{\gamma_3}$ meets these conditions. Hence, it is the unique solution. The invariant probability measure is then obtained using standard computations. \square

In the following, we set $\delta_{x,y}$ to be 1 if $x = y$ and 0 otherwise.

Proposition 3.4 (repulsive case $\Lambda > 0$). *There exists an infinite collection of MERWs generated by a finite number N of linearly independent eigenfunctions $\{\psi^{(i)} : 1 \leq i \leq N\}$. For all $1 \leq k \leq N$ and $n \geq 0$, these are given by*

$$\psi_k^{(i)}(n) = 1 + \delta_{i,k} \frac{\Lambda}{\gamma_2^{(k)}}. \quad (3.18)$$

More precisely, there exists a one-to-one correspondence between MERWs and probability distributions $(\mu_i)_{1 \leq i \leq N}$, through

$$\psi^{(\mu)} = \sum_{i=1}^N \mu_i \psi^{(i)}. \quad (3.19)$$

The associated transition probabilities, for all $1 \leq k \leq N$ and $n \geq 1$, are

$$P_k^{(\mu)}(n, n \pm 1) = \frac{1}{2} \frac{\gamma_2^{(k)} + \mu_k \Lambda (n \pm 1)}{\gamma_2^{(k)} + \mu_k \Lambda n}, \quad P_k^{(\mu)}(0, 1) = \frac{\gamma_2^{(k)} + \mu_k \Lambda}{2\gamma_3}, \quad P^{(\mu)}(\mathbf{0}, \mathbf{0}) = \frac{\gamma_1}{2\gamma_3}. \quad (3.20)$$

Furthermore, let \mathbb{P}_x^μ denote the distribution of the MERW associated with μ , starting from $x \in G$. Then, for all $1 \leq k \leq N$, we have

$$\mathbb{P}_0^\mu \left(\lim_{n \rightarrow \infty} X_n = \infty_k \right) = \mu_k. \quad (3.21)$$

Proof. We aim to solve $\psi_k(n+1) + \psi_k(n-1) = 2\psi_k(n)$ for all $1 \leq k \leq N$ and $n \geq 1$, under the boundary conditions

$$\psi(\mathbf{0}) = 1 \quad \text{and} \quad \gamma_1 + \sum_{k=1}^N \gamma_2^{(k)} \psi_k(1) = 2\gamma_3. \quad (3.22)$$

It immediately follows that $\psi_k(n) = 1 + c_k n$ for some constants $c_k \geq 0$, which leads to the relation

$$\sum_{k=1}^N \gamma_2^{(k)} c_k = \Lambda. \quad (3.23)$$

This yields equations (3.18), (3.19), and (3.20). Expanding upon this, we observe that A is \mathcal{R} -transient, meaning all the MERWs are transient. Let $(X_n)_{n \geq 0}$ be the MERW associated with the probability measure μ . An harmonic function h satisfies $\mathbb{E}_x[h(X_n)] = h(x)$ for all $x \in G$ and

$n \geq 0$ if and only if $h = \frac{\psi^{(\nu)}}{\psi^{(\mu)}}$ for some other probability distribution ν . The Martin boundary is thus represented by $\{1, \dots, N\}$ and the Martin kernel is given by

$$K((n, k); i) = \frac{\psi_k^{(i)}(n)}{\psi_k^{(\mu)}(n)}. \quad (3.24)$$

Standard results on the Martin boundary of random walks assert that if X_n starts from $\mathbf{0}$, it almost surely converges within the Martin compactification to i with probability μ_i . The MERW corresponding to $\psi^{(i)}$ is a classical symmetric nearest neighbor random walk in $\{(1, k), \dots\}$ for all $k \neq i$. Given its transient nature, $\lim_{n \rightarrow \infty} X_n = \infty_i$ almost surely. Hence, we identify the Martin boundary with $\{\infty_1, \dots, \infty_N\}$. \square

Remark 3.2. *Standard results on hitting times indicate that the distribution of a simple symmetric random walk on \mathbb{Z} , conditioned to remain in \mathbb{N}_0 , corresponds to the MERW on the spider lattice when $N = 1$, $\gamma_1 = \gamma_2^{(1)} = 0$, and $\gamma_3 = 1$.*

4 Functional Scaling Limits

Let us introduce the space

$$\mathcal{G} = \{x = (\bar{x}, k) : \bar{x} \in [0, \infty), k = 1, \dots, N\} \cup \{\mathbf{0}\}. \quad (4.1)$$

Note that G is canonically embedded in \mathcal{G} . We identify $\mathbf{0} = (0, 1) = \dots = (0, N)$ and denote by $\mathcal{G}_k = \{(\bar{x}, k) : \bar{x} \geq 0\}$ the k -th leg. Furthermore, we equip \mathcal{G} with the usual railway distance defined by

$$d((x, i), (y, j)) = |x - y|\delta_{i,j} + (x + y)(1 - \delta_{i,j}). \quad (4.2)$$

Restricted to G , this becomes the standard graph distance. For all $x = (\bar{x}, k) \in \mathcal{G}$ and $\alpha \geq 0$, we set $\alpha x := (\alpha\bar{x}, k)$. For a proper planar embedding, the metric d is equivalent to the usual Euclidean metric, and αx corresponds to the conventional scalar multiplication. Let $(\mathbf{C}, \mathcal{U})$ denote the space of continuous functions from $[0, \infty)$ to \mathcal{G} , equipped with the topology of uniform convergence on compact sets. We use \implies to signify the convergence in distribution of stochastic processes in $(\mathbf{C}, \mathcal{U})$ with the associated Borel σ -field. Let \mathcal{F}_t , $t \geq 0$, represent the canonical filtration on \mathbf{C} . For any sequence of real numbers $(X_n)_{n \geq 0}$, we define for all $t \geq 0$,

$$X_t = X_{[t]} + (t - [t])(X_{[t]+1} - X_{[t]}). \quad (4.3)$$

Here, $[x]$ denotes the largest integer less than or equal to x .

Remark 4.1. *It is possible to extend all the functional convergences discussed below to the space of càdlàg functions, either endowed with the usual Skorokhod topology or the uniform topology as described above. For more details, we refer to [42, Chap. 18].*

4.1 Regular Case

We direct the reader to [43] for the definition of the Walsh Brownian motion and to [44] for the excursion theory of Brownian motion. Let $\{\mathbf{W}_t^{(\mu, x)} = (W_t, K_t) : t \geq 0\}$ be the standard Walsh Brownian motion on \mathcal{G} starting from $x = (\bar{x}, k)$ with spinning measure $\mu = (\mu_1, \dots, \mu_N)$. Notably, when $W_t = 0$, the value of $K_t \in \{1, \dots, N\}$ is inconsequential. This process can be roughly described as follows. It is a continuous stochastic process on \mathcal{G} where W is a standard one-dimensional reflected Brownian motion starting from \bar{x} . It is noteworthy that W_t can be

expressed as $W_t = |B_t| = \mathcal{B}_t + L_t$, where B and \mathcal{B} are two standard one-dimensional Brownian motions starting from \bar{x} , and L denotes the local time at 0 of W . To elaborate further, let τ represent the right-continuous inverse of L . Each excursion interval of W away from zero can be expressed as $I_0 = [0, \tau_0)$ or $I_s = (\tau_{s-}, \tau_s)$ for some $s > 0$. The set difference of the union of these intervals is $\{t \geq 0 : W_t = 0\}$ and has Lebesgue measure zero. Moreover, K remains constant, say α_{I_s} , over each I_s . We have $\alpha_{I_0} = k$ and, conditionally to W , $\{\alpha_{I_s} : s > 0, I_s \neq \emptyset\}$ constitutes an independent collection of μ -distributed random variables.

Theorem 4.1. *Let $\{X_n\}_{n \geq 0}$ be the MERW presented in Proposition 3.2 and define*

$$\mu = \left(\frac{\gamma_2^{(1)}}{\mathcal{S}_2}, \dots, \frac{\gamma_2^{(N)}}{\mathcal{S}_2} \right). \quad (4.4)$$

If $X_0 = x_0$ is deterministic and depends on $L > 0$ in such a way that for some $x \in \mathcal{G}$,

$$\frac{x_0}{\sqrt{L}} \xrightarrow{L \rightarrow \infty} x, \quad (4.5)$$

then the following functional scaling limit holds:

$$\left\{ \frac{X_{Lt}}{\sqrt{L}} \right\}_{t \geq 0} \xrightarrow{n \rightarrow \infty} \{\mathbf{W}_t^{(\mu, x)}\}_{t \geq 0}. \quad (4.6)$$

4.2 Attractive Case

The construction of the Walsh Brownian motion has been extended to various contexts. For Walsh diffusions, we direct the reader to [45] for a functional analysis approach on graphs and to [46–50] for semimartingale characterizations on rays. We allude to [51] for the general Itô's theory of excursions. Fix $\lambda > 0$ and $\bar{x} \geq 0$ and let Z be the solution of the reflecting stochastic differential equation

$$dZ_t = dB_t - \lambda dt + dL_t, \quad Z_t \geq 0, \quad Z_0 = \bar{x}, \quad (4.7)$$

with B being a standard one-dimensional Brownian motion and L a B -adapted, non-decreasing, continuous stochastic process which satisfies

$$\int_0^\infty \mathbb{1}_{\{Z_t > 0\}} dL_t = 0 \quad \text{and} \quad \int_0^\infty \mathbb{1}_{\{Z_s = 0\}} ds = 0 \quad \text{a.s.} \quad (4.8)$$

Introduce the Walsh diffusion $\{\mathbf{Z}_t^{(\mu, x)} = (Z_t, K_t) : t \geq 0\}$ on \mathcal{G} starting from $x = (\bar{x}, k)$ with the spinning measure μ . Similar to the Walsh Brownian motion, when $Z_t = 0$, the specific value of $K_t \in \{1, \dots, N\}$ is irrelevant. Moreover, we have $Z_t = |\mathcal{Z}_t|$, where \mathcal{Z} is a (weak) solution of $d\mathcal{Z}_t = d\mathcal{B}_t - \lambda \operatorname{sgn}(\mathcal{Z}_t) dt$, with $\mathcal{Z}_0 = \bar{x}$ and \mathcal{B} a standard one-dimensional Brownian motion. Here L represents the local time at zero for Z . The spinning measure μ is subject to the condition:

$$\forall i \in \{1, \dots, N\}, \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{0 < Z_s < \varepsilon\}} \mathbb{1}_{\{K_s = i\}} ds = \mu_i L_t \quad \text{a.s.} \quad (4.9)$$

As before, K remains constant across each excursion interval $(I_s)_{s \geq 0}$ of Z . We have $K_t = k$ for I_0 , and the values for $(I_s)_{s > 0}$ are independent and distributed according to μ , conditionally to Z . It is worth noting that \mathcal{Z} is an ergodic diffusion with its reversible probability measure and one can check that the invariant probability measure of Z is the exponential distribution of parameter 2λ .

Theorem 4.2. Let $\{X_n\}_{n \geq 0}$ be the MERW described in Proposition 3.3 with parameters γ dependent on $L > 0$. Assume there exists $\zeta \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ such that $\mathfrak{Z}_2 = \sum_{i=1}^N \zeta_2^{(i)} > 0$ and a positive constant λ satisfying

$$\frac{\Lambda}{\mathfrak{S}_2} \underset{L \rightarrow \infty}{\sim} -\frac{\lambda}{\sqrt{L}} \quad \text{and} \quad \gamma - \zeta = \mathcal{O}\left(\frac{1}{\sqrt{L}}\right). \quad (4.10)$$

Further assume that $X_0 = x_0$ is deterministic and that

$$\frac{x_0}{\sqrt{L}} \xrightarrow{L \rightarrow \infty} x. \quad (4.11)$$

Then, for the spinning measure defined as $\mu = \left(\frac{\zeta_2^{(1)}}{\mathfrak{Z}_2}, \dots, \frac{\zeta_2^{(N)}}{\mathfrak{Z}_2}\right)$, the following functional scaling limit holds:

$$\left\{ \frac{X_{Lt}}{\sqrt{L}} \right\}_{t \geq 0} \xrightarrow[n \rightarrow \infty]{} \{\mathbf{Z}_t^{(\mu, x)}\}_{t \geq 0}. \quad (4.12)$$

4.3 Repulsive Case

Firstly, introduce the well-known three-dimensional Bessel process $\{Y_t^y\}_{t \geq 0}$ starting from $y \geq 0$. This is the non-negative solution to the stochastic differential equation

$$dY_t^y = dB_t + \frac{1}{Y_t^y} dt, \quad Y_0 = y, \quad (4.13)$$

where B denotes a standard Brownian motion. For further details, we refer to [52]. In essence, this is a transient Markov process satisfying $Y_t^y > 0$ for all $t > 0$, even when it starts at zero. Subsequently, for any $\bar{x} \geq 0$ and $1 \leq k \leq N$, we consider the stochastic process $\{\mathbf{Y}_t^{(\bar{x}, k)}\}_{t \geq 0}$ on \mathcal{G} defined by

$$\mathbb{P}\left(\forall t \geq 0, \mathbf{Y}_t^{(\bar{x}, k)} = (Y_t^{\bar{x}}, k)\right) = 1. \quad (4.14)$$

This corresponds to the three-dimensional Bessel process on the k th leg. Furthermore, let $\mu = (\mu_k)_{1 \leq k \leq N}$ be a probability distribution. We define the process $\{\mathbf{Y}_t^{(\mu, x)}\}_{t \geq 0}$ as

1. For $x = \mathbf{0}$: $\mathbb{P}\left(\forall t \geq 0, \mathbf{Y}_t^{(\mu, \mathbf{0})} = \mathbf{Y}_t^{(0, k)}\right) = \mu_k$ for all $1 \leq k \leq N$.
2. For $x = (\bar{x}, k) \neq \mathbf{0}$:

(a) If $\mu_k \neq 0$: $\mathbb{P}\left(\forall t \geq 0, \mathbf{Y}_t^{(\mu, x)} = \mathbf{Y}_t^{(\bar{x}, k)}\right) = 1$.

(b) If $\mu_k = 0$:

$$\mathbf{Y}_t^{(\mu, x)} = \begin{cases} (\bar{x} + W_t, k), & \text{for all } 0 \leq t \leq \tau_0, \\ \mathbf{Y}_{t-\tau_0}^{(\mu, \mathbf{0})}, & \text{for all } t \geq \tau_0, \end{cases} \quad (4.15)$$

where $\tau_0 = \inf\{t \geq 0 \mid \bar{x} + W_t = 0\}$ and W is a standard one-dimensional Brownian motion, independent of $\mathbf{Y}^{(\mu, \mathbf{0})}$.

Theorem 4.3. Let $\{X_n\}_{n \geq 0}$ be the MERW as specified in Proposition 3.4, associated with the probability distribution μ . If $X_0 = x_0$ is deterministic and relates to $L > 0$ such that

$$\frac{x_0}{\sqrt{L}} \xrightarrow{L \rightarrow \infty} x, \quad (4.16)$$

then the following functional scaling limits holds:

$$\left\{ \frac{X_{Lt}}{\sqrt{L}} \right\}_{t \geq 0} \xrightarrow[n \rightarrow \infty]{} \{\mathbf{Y}_t^{(\mu, x)}\}_{t \geq 0}. \quad (4.17)$$

Remark 4.2. In a manner similar to Remark 3.2, we can state that a Brownian motion conditioned to remain positive is a three-dimensional Bessel process (see, for instance, [53]). Functional scaling limits of random walks conditioned to remain positive have been explored in [54, 55].

Remark 4.3. The aforementioned results can be extended to the exclusion process. Consider two particles on \mathbb{Z} which can only jump to the right or to the left but cannot occupy the same site. This configuration corresponds to the lattice structure of the standard exclusion process. We direct the reader to [56, 57] for comprehensive reviews on this topic. Through symmetry, $\psi(x, y) = 1 + (y - x)$ emerges as a positive eigenfunction related to the spectral radius $\rho = 4$. In the scaling limit, the equation

$$d(Y_t - X_t) = dW_t + \frac{dt}{Y_t - X_t}, \quad (4.18)$$

is derived, where $X_t < Y_t$ signifies the positions of the particles at time t . The interesting point here is that the standard electrostatic force appears when assuming only a maximum entropy constraint.

4.4 Continuous-Time counterparts of the MERWs

In light of the described scaling limits, one may wonder if the limit processes can be interpreted as maximal entropy stochastic processes without involving MERWs. We will concentrate on the case where $N = 1$ and explore the possibility of interpreting the three-dimensional Bessel process and (4.7) as Maximal Entropy Stochastic Processes, the specifics of which will be outlined.

Kullback–Leibler Divergence (KLD). Let γ_n denote the (uniform) distribution of the first n -steps of the simple random walk on the regular graph \mathbb{Z} . It is noteworthy that maximizing the entropy on the right-hand side of (1.2) is equivalent to minimizing the Kullback–Leibler divergence (or the relative entropy) $D_{\text{KL}}(\nu_n || \gamma_n)$. Given two probability measures ν, γ where ν is absolutely continuous with respect to γ , the divergence is defined as

$$D_{\text{KL}}(\nu || \gamma) = \int \ln \left(\frac{d\nu}{d\gamma} \right) d\nu. \quad (4.19)$$

To adapt this definition for continuous stochastic processes, replace γ with W a $(\mathcal{F}_t)_{t \geq 0}$ -adapted Brownian motion. The selection of stochastic processes that are absolutely continuous relative to W will be made as follows. Let ψ be an absolutely continuous non-negative function on $[0, \infty)$ with the property that $U = \{\psi > 0\}$ is an open set in $[0, \infty)$. Define $\tau = \inf\{s \geq 0 : W_s \in U\}$ and for all $t \geq 0$,

$$M_t = \exp \left(\int_0^t \frac{\psi'(W_s)}{\psi(W_s)} dB_s - \frac{1}{2} \int_0^t \left(\frac{\psi'(W_s)}{\psi(W_s)} \right)^2 ds \right) \mathbf{1}_{\{t < \tau\}}. \quad (4.20)$$

Observe that $\tau = \infty$ when ψ is positive. Leveraging the results from [58, Chap. 6.3], it is clear that $\{M_t\}_{t \geq 0}$ is a \mathcal{F}_t -martingale under \mathbb{P}_x for every $x \in U$. Let \mathbb{Q}_x denote the distribution on $C([0, \infty), \mathbb{R})$ expressed by

$$d\mathbb{Q}_x^{(t)} = M_t d\mathbb{P}_x^{(t)}, \quad (4.21)$$

where $\mathbb{Q}_x^{(t)}$ and $\mathbb{P}_x^{(t)}$ represent the restrictions of \mathbb{Q}_x and \mathbb{P}_x to \mathcal{F}_t respectively. Under \mathbb{Q}_x , the law of W becomes a $\psi^2(x)dx$ -symmetric Markov process, ensuring that it never reaches ∂U when it starts at $x \in U$. By invoking the Girsanov theorem (as found in [52]), one has under the probability distribution \mathbb{Q}_x :

$$\tilde{B}_t = B_t - \int_0^t \left(\frac{\psi'(W_s)}{\psi(W_s)} \right)^2 ds, \quad (4.22)$$

is a Brownian motion. Thus, the stochastic process W satisfies under \mathbb{Q}_x the reflected stochastic differential equation

$$d\widetilde{W}_t = d\widetilde{B}_t + \frac{\psi'(\widetilde{W}_t)}{\psi(\widetilde{W}_t)} dt + d\widetilde{L}_t, \quad (4.23)$$

with the initial condition $\widetilde{W}_0 = x$ and \widetilde{L} is the local time at 0 of \widetilde{W} . Consequently, we get

$$D_{\text{KL}}\left(\mathbb{Q}_x^{(t)} \parallel \mathbb{P}_x^{(t)}\right) = \mathbb{E}_x \left[\int_0^t \left(\frac{\psi'(\widetilde{W}_s)}{\psi(\widetilde{W}_s)} \right)^2 ds \right]. \quad (4.24)$$

Nevertheless, when $\psi^2(x)dx$ is not a finite measure, it is possible that

$$h = \lim_{t \rightarrow \infty} \frac{1}{t} D_{\text{KL}}\left(\mathbb{Q}_x^{(t)} \parallel \mathbb{P}_x^{(t)}\right) = \infty. \quad (4.25)$$

We can state the following lemma whose proof is left to the reader.

Lemma 4.1. *Assume that $\pi(dx) = \psi^2(x)dx$ is a probability measure on U . Then, for any $x \in U$ and for $s > 0$, the relative rate entropy h is given by*

$$h = \lim_{t \rightarrow \infty} \frac{1}{t} D_{\text{KL}}\left(\mathbb{Q}_x^{(t)} \parallel \mathbb{P}_x^{(t)}\right) = \int_U (\psi'(x))^2 dx = \frac{1}{s} D_{\text{KL}}\left(\mathbb{Q}_\pi^{(s)} \parallel \mathbb{P}_\pi^{(s)}\right). \quad (4.26)$$

Repulsive case. Assume that $\psi(x) > 0$ on $]0, L[$ and $\psi(x) = 0$ otherwise for some $L > 0$. If we are looking for such a function which minimizes (4.26), we obtain that

$$h = \left(\frac{\pi}{L}\right)^2 \quad \text{and} \quad \psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right). \quad (4.27)$$

Thereafter, we retrieve the three-dimensional Bessel process by letting L go to infinity since for all $x > 0$,

$$\frac{\psi'(x)}{\psi(x)} \underset{L \rightarrow \infty}{\sim} \frac{1}{x}. \quad (4.28)$$

Attractive case. Regarding the case when $\gamma_1 = 1 + \frac{\lambda}{\sqrt{L}}$, $\gamma_2^{(1)} = 0$ and $\gamma_3 = 1$, we need to add constraints on ψ . To be clear, we assume that $\psi^2(x)dx$ is a probability distribution on $[0, \infty)$ but we also require that

$$\int_0^\infty x\psi^2(x)dx = \frac{1}{2\lambda}. \quad (4.29)$$

Let ψ be this minimizer and consider δ , a compactly supported smooth function where $\delta(0) = 0$ and $\int_0^\infty \delta(x)dx = 0$. Then by considering $\psi + \varepsilon\delta$ for sufficiently small $\varepsilon > 0$ and looking at the first and second order terms next to ε and ε^2 , we obtain that necessarily

$$-\psi''(x) + \beta\psi(x) = -\alpha\psi(x) \quad (4.30)$$

and

$$\int_0^\infty (\delta'(x))^2 dx + \beta \int_0^\infty x(\delta(x))^2 dx + \alpha \int_0^\infty (\delta(x))^2 dx \geq 0. \quad (4.31)$$

Here α, β are some Lagrange multipliers and the latter ordinary differential equation has to be understood in a weak sense when ψ is not twice differentiable. Equation (4.30) is nothing but a Schrödinger equation in a linear potential. When $\beta \neq 0$, solutions can be written as

$AAi(z) + BBi(z)$ with $z = -\beta^{-1/3}(x + \alpha)$ and $A, B \in \mathbb{R}$ where Ai and Bi are the Airy functions of first and second kinds. This can be proved by power series expansions or Fourier transform, for instance. However, (4.31) implies that $\alpha, \beta \geq 0$ since δ is an arbitrary perturbation. It is observed that $Ai(z)$ and $Bi(z)$ oscillate around zero as z approaches $-\infty$. This behavior implies that no non-negative solutions exist when $\beta > 0$. Thus $\beta = 0$ and then the unique non-negative normalized square integrable solution satisfying (4.29) is $\psi(x) = \sqrt{2\lambda}e^{-\lambda x}$. Using this, we obtain the relationship

$$\frac{\psi'(x)}{\psi(x)} = -\lambda. \quad (4.32)$$

Subsequently, we recover the reflected diffusion as given in (4.7).

5 Proofs of the Functional Scaling Limits

We will prove these theorems by following the standard approach: first establishing tightness and then identifying the limit. In what follows, let \mathbb{P}^L represent the law on \mathbb{C} of the scaled MERW such that under the probability distribution $\mathbb{P}^L(d\omega)$, the sequence $\{\omega(t)\}_{t \geq 0}$ is distributed as the left-hand side of (4.6), (4.12), or (4.17), depending on the assumptions. The expectation under this probability distribution will be denoted by \mathbb{E}^L .

5.1 Tightness

The submartingale argument. To prove tightness, we will employ a submartingale argument as found in [59, chap. 1.4.]. While this method is applicable to continuous stochastic processes that take their values in \mathbb{R}^d , it can be readily extended to metric spaces, as indicated in [45, Theorem 2.1.]. More specifically, we will make use of the following result.

Theorem 5.1. *Let $\{\mathbb{P}^L : L > 0\}$ be a family of probability distributions on \mathbf{C} satisfying*

$$\mathbb{P}^L \left(\left\{ \omega \in \mathbf{C} : \forall n \geq 0, \omega \text{ is linear over } \left[\frac{n}{L}, \frac{n+1}{L} \right] \right\} \right) = 1. \quad (5.1)$$

Further, assume that for every $\varepsilon > 0$, there exist A_ε and $L_\varepsilon > 0$ such that for any $y \in \mathcal{G}$, there is a function f_ε^y on \mathcal{G} satisfying

$$(i) f_\varepsilon^y(y) = 1, \quad (ii) f_\varepsilon^y(x) = 0 \text{ if } d(x, y) \geq \varepsilon, \quad (iii) 0 \leq f_\varepsilon^y(x) \leq 1, \quad (5.2)$$

and for all $L \geq L_\varepsilon$,

$$\left\{ f_\varepsilon^y \left(\omega \left(\frac{n}{L} \right) \right) + A_\varepsilon \frac{n}{L} \right\}_{n \geq 0} \text{ is a } \mathbb{P}^L(d\omega)\text{-submartingale.} \quad (5.3)$$

Then, as L goes to infinity, the family of probability measures $\{\mathbb{P}^L : L > 0\}$ is tight.

Ito's formula for regular and attractive cases. Let $\{X_n\}_{n \geq 0}$ be the MERW satisfying the assumptions of Theorem 4.1 or 4.2, and let \bar{X}_n denote its first component. Let \mathbf{f} be a smooth function on \mathcal{G} with bounded derivatives. For all $x = (\bar{x}, i) \in \mathcal{G}$ with $\bar{x} \geq 1$, we define

$$\begin{aligned} \Delta_L \mathbf{f}(x) &= \Delta_L \mathbf{f}_i(\bar{x}) = \mathbf{f}_i \left(\frac{\bar{x} + 1}{\sqrt{L}} \right) - \mathbf{f}_i \left(\frac{\bar{x} - 1}{\sqrt{L}} \right) \\ \text{and } \Delta_L^2 \mathbf{f}(x) &= \Delta_L^2 \mathbf{f}_i(\bar{x}) = \mathbf{f}_i \left(\frac{\bar{x} + 1}{\sqrt{L}} \right) - 2\mathbf{f}_i \left(\frac{\bar{x}}{\sqrt{L}} \right) + \mathbf{f}_i \left(\frac{\bar{x} - 1}{\sqrt{L}} \right). \end{aligned} \quad (5.4)$$

It is noteworthy that

$$|\Delta_L \mathbf{f}(x)| \leq \frac{2\|\mathbf{f}'\|_\infty}{\sqrt{L}} \quad \text{and} \quad |\Delta_L^2 \mathbf{f}(x)| \leq \frac{2\|\mathbf{f}''\|_\infty}{L}. \quad (5.5)$$

The drift is defined for all $x = (\bar{x}, i) \neq \mathbf{0}$ by

$$D(x) = D_i(\bar{x}) = \mathbb{E}[\bar{X}_{n+1} - \bar{X}_n | X_n = x]. \quad (5.6)$$

Utilizing the classical discrete-time version of Ito's formula, as found in [60, p. 180] or [61, p. 132] for instance, we can express for all $0 \leq m \leq n$,

$$\begin{aligned} \mathbf{f}\left(\frac{X_n}{\sqrt{L}}\right) &= \mathbf{f}\left(\frac{X_m}{\sqrt{L}}\right) + \frac{1}{2} \sum_{i=1}^N \sum_{k=m}^{n-1} (\Delta_L^2 \mathbf{f}_i(\bar{X}_k) + D_i(\bar{X}_k) \Delta_L \mathbf{f}_i(\bar{X}_k)) \mathbf{1}_{\{\bar{X}_k \neq 0\}} \\ &\quad + \#\{m \leq k \leq n-1 : X_k = \mathbf{0}\} \sum_{i=1}^N P_i(0, 1) \left(\mathbf{f}_i\left(\frac{1}{\sqrt{L}}\right) - \mathbf{f}(\mathbf{0}) \right) + M_n - M_m, \end{aligned} \quad (5.7)$$

where $\{M_n\}_{n \geq 0}$ is a square integrable (\mathcal{F}_n) -martingale. Here, $\#E$ denotes the cardinality of a set E and $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

Taylor expansions with respect to γ . To go further, introduce

$$\mathcal{H} = \left\{ (x, y_1, \dots, y_N, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} : x + \sum_{k=1}^N y_k - 2z = 0 \right\}. \quad (5.8)$$

For a given $u \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, let $\pi(u) = (\pi_1(u), \pi_2(u), \pi_3(u))$ be the orthogonal projection of u onto \mathcal{H} , and let $\delta(u) = u - \pi(u) = (\delta_1(u), \delta_2(u), \delta_3(u))$. The i -th component of $\pi_2(u)$ or $\delta_2(u)$ is denoted by $\pi_2^{(i)}(u)$ or $\delta_2^{(i)}(u)$ for any $1 \leq i \leq N$. Observe that for all $1 \leq i \leq N$,

$$\delta_1(\gamma) = \delta_2^{(i)}(\gamma) = -\frac{\Lambda}{N+5} \quad \text{and} \quad \delta_3(\gamma) = \frac{2\Lambda}{N+5}. \quad (5.9)$$

Let us set

$$G(x, y, z) = \frac{z(x^2 + y^2)}{x(z - y) + y\sqrt{x^2 + 2yz - z^2}}. \quad (5.10)$$

Note that $\rho = G(\gamma_1, \mathcal{S}_2, 2\gamma_3)$. Standard computation shows that $\nabla G(x, y, z) = (0, 0, 1)$ when $z = x + y$. Furthermore, let $\Gamma(\gamma) \equiv \Gamma$ to highlight the dependence of Γ in Theorem 4.2 with respect to the parameters $\gamma \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. We get

$$\nabla \Gamma(\pi(\gamma)) = \left(\sum_{i=1}^N \pi_2^{(i)}(\gamma) \right)^{-1} (-1, \dots, -1, 2). \quad (5.11)$$

The first order Taylor expansion of $\Gamma(\gamma)$ at $\pi(\gamma)$ in the attractive case becomes

$$\Gamma = 1 + \frac{\Lambda}{\mathcal{S}_2 + \frac{N\Lambda}{N+5}} + \mathcal{O}(\Lambda^2) = 1 - \frac{\lambda}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right). \quad (5.12)$$

Besides, still in the attractive case, it holds that

$$D_i(\bar{x}) = \frac{\gamma_3(\Gamma^2 - 1)}{\rho\Gamma} = \frac{\Gamma^2 - 1}{\Gamma^2 + 1}. \quad (5.13)$$

Consequently, for $\bar{x} \geq 1$, one has

$$D_i(\bar{x}) = -\frac{\lambda}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right). \quad (5.14)$$

In the regular case, we have $D_i(\bar{x}) = 0$. In both scenarios, we obtain uniformly on $G \setminus \{\mathbf{0}\}$:

$$\left| \frac{1}{2} (\Delta_L^2 \mathbf{f}(x) + D_i(\bar{x}) \Delta_L \mathbf{f}(x)) \right| = \mathcal{O}\left(\frac{\|\mathbf{f}'\|_\infty + \|\mathbf{f}''\|_\infty}{L}\right). \quad (5.15)$$

The Lyapunov functions. Let $\varepsilon > 0$ and let f_ε be a smooth even function on \mathbb{R} satisfying the following properties: $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon(x) = 0$ for $|x| \geq \varepsilon$, $f_\varepsilon(x) = 1$ for $|x| \leq \varepsilon/2$, and f_ε is non-increasing on $[0, \infty)$. Given $y = (\bar{y}, i) \in \mathcal{G}$, we define for all $x = (\bar{x}, j) \in \mathcal{G}$,

$$\mathbf{f}_\varepsilon^y(x) = \begin{cases} f_\varepsilon(\bar{x} - \bar{y}) & \text{if } \bar{y} \geq 2\varepsilon \text{ and } j = i, \\ f_\varepsilon(\bar{x} - 2\varepsilon) & \text{if } \bar{y} < 2\varepsilon \text{ and } \bar{x} \geq 2\varepsilon, \\ 1 & \text{if } \bar{y} < 2\varepsilon \text{ and } \bar{x} < 2\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \quad (5.16)$$

It is evident that $\mathbf{f}_\varepsilon^y(y) = 1$, $0 < \mathbf{f}_\varepsilon^y \leq 1$, and $\mathbf{f}_\varepsilon^y(x) = 0$ whenever $d(x, y) \geq 5\varepsilon$. An illustration of this can be found in Figure 2. Furthermore, we have $\mathbf{f}_{\varepsilon, i}^y(1/\sqrt{L}) - \mathbf{f}_{\varepsilon, i}^y(\mathbf{0}) = 0$ for every L satisfying $1/\sqrt{L} \leq \varepsilon$, given that \mathbf{f}_ε^y is flat over the set $\{x \in \mathcal{G} : d(\mathbf{0}, x) \leq \varepsilon\}$. From equations (5.15) and (5.7), we deduce that

$$\mathbb{E} \left[\mathbf{f}_\varepsilon^y \left(\frac{X_n}{\sqrt{L}} \right) - \mathbf{f}_\varepsilon^y \left(\frac{X_m}{\sqrt{L}} \right) \middle| \mathcal{F}_m \right] \geq -A_\varepsilon \left(\frac{n}{L} - \frac{m}{M} \right), \quad (5.17)$$

where A_ε depends only on the parameters, $\|f'_\varepsilon\|_\infty$, $\|f''_\varepsilon\|_\infty$, and ε . Consequently, we can establish the tightness in Theorems 4.1 and 4.2 using Theorem 5.1.

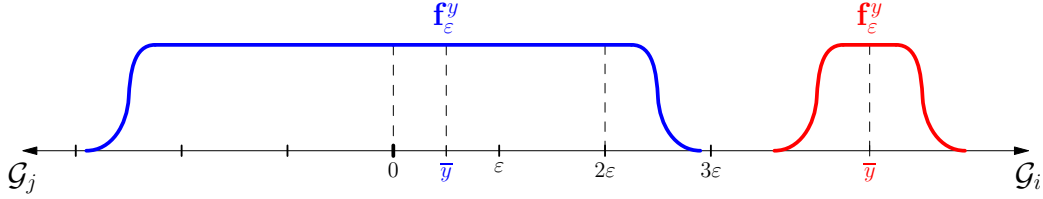


Figure 2: Lyapunov functions

Focus on the repulsive case. We only consider the primary distinctions compared to the previous cases. For every $x = (\bar{x}, i) \neq \mathbf{0}$, we have

$$D(x) = D_i(\bar{x}) = \mathbb{E} [\bar{X}_{n+1} - \bar{X}_n | X_n = x] = \frac{\mu_i \Lambda}{\gamma_2^{(i)} + \mu_i \Lambda \bar{x}}. \quad (5.18)$$

It can also be verified that

$$D_i(\bar{x}) \Delta_L \mathbf{f}_{\varepsilon, i}^y(\bar{x}) = \mathcal{O}\left(\frac{\|f'_\varepsilon\|}{\varepsilon L}\right) \mathbb{1}_{\{\bar{x} \geq \varepsilon \sqrt{L}\}}. \quad (5.19)$$

Again, we get from the Ito's formula 5.7:

$$\mathbb{E} \left[\mathbf{f}_\varepsilon^y \left(\frac{X_n}{\sqrt{L}} \right) - \mathbf{f}_\varepsilon^y \left(\frac{X_m}{\sqrt{L}} \right) \middle| \mathcal{F}_m \right] \geq -A_\varepsilon \left(\frac{n}{L} - \frac{m}{M} \right), \quad (5.20)$$

here for all $L \geq \varepsilon^{-2}$. The constant A_ε solely depends on the parameters, $\|f'_\varepsilon\|_\infty$, $\|f''_\varepsilon\|_\infty$, and ε . Using again Theorem 5.1, we conclude the proof for tightness.

5.2 Limit Processes

Let \mathbb{P}^* be a limit point of \mathbb{P}^L as L approaches infinity, and let \mathbb{E}^* represent the associated expectation. We aim to demonstrate that \mathbb{P}^* corresponds to the distribution of $\mathbf{W}^{(\mu,x)}$, $\mathbf{Z}^{(\mu,x)}$, or $\mathbf{Y}^{(\mu,x)}$ as specified in Theorems 4.1, 4.2, or 4.3, depending on the underlying assumptions. To achieve this, we characterize these stochastic processes in terms of local martingale and/or submartingale problems. Given a continuous function \mathbf{f} on \mathcal{G} , which is smooth on every ray, we define for all $x = (\bar{x}, i) \neq \mathbf{0}$:

$$\mathcal{L}_{\mathbf{W}}\mathbf{f}(x) = \frac{1}{2}\mathbf{f}_i''(\bar{x}), \quad \mathcal{L}_{\mathbf{Z}}\mathbf{f}(x) = \frac{1}{2}\mathbf{f}_i''(\bar{x}) - \lambda\mathbf{f}'_i(\bar{x}) \quad (5.21)$$

and

$$\mathcal{L}_{\mathbf{Y}}\mathbf{f}(x) = \frac{1}{2}\mathbf{f}_i''(\bar{x}) + \frac{1 - \delta_{\mu_i,0}}{\bar{x}}\mathbf{f}'_i(\bar{x}). \quad (5.22)$$

Martingale and submartingale problems. The following result is referenced in [46] and [62].

Theorem 5.2. *Let us define \mathcal{L} as either $\mathcal{L}_{\mathbf{Z}}$ or $\mathcal{L}_{\mathbf{W}}$. In the context where $\mathcal{L} = \mathcal{L}_{\mathbf{Z}}$ (resp. $\mathcal{L} = \mathcal{L}_{\mathbf{W}}$), the Walsh diffusion $\mathbf{Z}^{(\mu,x)}$ (resp. $\mathbf{W}^{(\mu,x)}$) is the unique solution \mathbb{Q} on \mathbf{C} to the martingale and submartingale problem described by*

$$\omega_0 = x, \quad \int_0^\infty \mathbf{1}_{\{\omega_s = \mathbf{0}\}} ds = 0 \quad \text{almost surely with respect to } \mathbb{Q}(d\omega), \quad (5.23)$$

and for every sufficiently smooth and bounded function \mathbf{f} on \mathcal{G} satisfying either

$$i) \sum_{i=1}^N \mu_i \mathbf{f}'_i(0) = 0 \quad \text{or} \quad ii) \sum_{i=1}^N \mu_i \mathbf{f}'_i(0) \geq 0, \quad (5.24)$$

the stochastic process $\{M_t^{\mathbf{f}}\}_{t \geq 0}$ is either *i) a martingale or ii) a submartingale under the distribution $\mathbb{Q}(d\omega)$ where*

$$M_t^{\mathbf{f}} = \mathbf{f}(\omega_t) - \int_0^t \mathcal{L}\mathbf{f}(\omega_s) \mathbf{1}_{\{\omega_s \neq \mathbf{0}\}} ds \quad (5.25)$$

Remark 5.1. *As a matter of fact, the class of functions \mathbf{f} considered in [46, 62] consists of continuous functions that are twice continuously differentiable on each ray. It is only required that $M^{\mathbf{f}}$ is a local martingale (resp. local submartingale). By using classical localization and density arguments, one can restrict the domain of functions \mathbf{f} as stated above and require that $M^{\mathbf{f}}$ be a martingale (resp. submartingale).*

Regarding $\mathbf{Y}^{(\mu,x)}$, the results in [46, 62] do not apply. Additionally, the drift of the three-dimensional Bessel process in (4.13) is singular at the origin, presenting an additional challenge. However, one can state the following result, the proof of which is provided below.

Theorem 5.3. *The diffusion $\mathbf{Y}^{(\mu,0)}$ is the unique solution \mathbb{Q} on \mathbf{C} of the following martingale problem, given by*

$$\omega_0 = \mathbf{0}, \quad \int_0^\infty \mathbf{1}_{\{\omega_s = \mathbf{0}\}} ds = 0 \quad \mathbb{Q}(d\omega)\text{-a.s.}, \quad (5.26)$$

for all $0 \leq i \leq N$,

$$\mathbb{Q}(\{\omega \in \mathbf{C} : \forall t \geq 0, \omega_t \in \mathcal{G}_i\}) = \mu_i, \quad (5.27)$$

and for all sufficiently smooth and bounded functions \mathbf{f} with compact support included in $\mathcal{G} \setminus \{\mathbf{0}\}$,

$$M^{\mathbf{f}} = \left\{ \mathbf{f}(\omega_t) - \int_0^t \mathcal{L}_{\mathbf{Y}}\mathbf{f}(\omega_s) ds \right\}_{t \geq 0} \text{ is a } \mathbb{Q}(d\omega)\text{-martingale.} \quad (5.28)$$

Proof. It is clear that the distribution of $\mathbf{Y}^{(\mu, \mathbf{0})}$ solves the martingale problem. The challenge remains to show that it is the unique solution. Initially, we assume that $N = 1$. In this case, $\mathbf{Y}^{(\mathbf{0}, \mu)} = Y^0$ is simply the three-dimensional Bessel process starting from 0. We denote its distribution by Q_0 and its infinitesimal generator by \mathcal{L}_Y . For $\eta > \varepsilon > 0$, introduce

$$\sigma_\eta(\omega) = \inf\{t \geq 0 : \omega_t = \eta\} \quad \text{and} \quad \tau_\varepsilon(\omega) = \inf\{t \geq 0 : \omega_{\sigma_\eta+t} = \varepsilon\}. \quad (5.29)$$

From our assumptions and using standard localization and approximation arguments, we conclude that

$$\left\{ f(\omega_{\sigma_\eta+t \wedge \tau_\varepsilon}) - \int_0^{t \wedge \tau_\varepsilon} \mathcal{L}_Y f(\omega_{\sigma_\eta+s}) ds \right\}_{t \geq 0} \text{ is a } \mathbb{Q}(\cdot | \sigma_\eta < \infty)\text{-martingale,} \quad (5.30)$$

for all sufficiently smooth functions f on $[0, \infty)$. Letting $\varepsilon \rightarrow 0$ and applying standard results on martingale problems and stochastic differential equations, we deduce that $\mathbb{Q}(\cdot | \sigma_\eta < \infty)$ equals the distribution Q_η of a three-dimensional Bessel process starting from η . Moreover, from (5.26), we find $\mathbb{Q}(\cdot | \sigma_\eta < \infty) \implies \mathbb{Q}$ as $\eta \downarrow 0$ because

$$\mathbb{P} \left(\bigcup_{\eta \downarrow 0} \{\sigma_\eta < \infty\} \right) = 1. \quad (5.31)$$

Since the three-dimensional Bessel process is a Feller Markov process, we have $Q_\eta \implies Q_0$ as $\eta \downarrow 0$. Thus, we obtain $\mathbb{Q} = Q_0$. Finally, assuming N is arbitrary, for any $1 \leq i \leq N$, introduce the stopping time $\tau_i(\omega) = \inf\{t \geq 0 : \omega_t \notin \mathcal{G}_i\}$. From (5.27) and (5.26), we deduce that $\tau_i \sim \mu_i \delta_\infty + (1 - \mu_i) \delta_0$. Specifically, $\{\tau_i = \infty\} \in \bigcap_{s>0} \mathcal{F}_s$. In the sequel, we denote the expectation with respect to the conditional probability $\mathbb{Q}(\cdot | \tau_i = \infty)$ by $\mathbb{E}^{(i)}$. For fixed $t \geq s > 0$, let Υ be a bounded \mathcal{F}_s -measurable random variable. We obtain

$$\mathbb{E} \left[(M_{t \wedge \tau_i}^{\mathbf{f}} - M_{s \wedge \tau_i}^{\mathbf{f}}) \mathbb{1}_{\{\tau_i = \infty\}} \Upsilon \right] = \mu_i \mathbb{E}^{(i)} \left[(M_t^{\mathbf{f}} - M_s^{\mathbf{f}}) \Upsilon \right] = 0. \quad (5.32)$$

Note that the latter equality also holds for $s = 0$ due to continuity. Hence, $M^{\mathbf{f}}$ is a $\mathbb{Q}(\cdot | \tau_i = \infty)$ -martingale. Utilizing the result for $N = 1$, we conclude that \mathbb{Q} is the distribution of $\mathbf{Y}^{(\mu, \mathbf{0})}$. \square

Bound on the local time at zero. To ensure that conditions (5.23) or (5.26) hold for any limit point \mathbb{Q}^* , we require the following lemma.

Lemma 5.1. *Let $\{X_n\}_{n \geq 0}$ be the MERW specified in Theorems 4.1, 4.2, or 4.3. Remember that $\lambda > 0$ is provided in Theorem 4.2. For any $u, v, \eta > 0$, there exists a positive constant $C_{v, \lambda}$ such that*

$$\limsup_{L \rightarrow \infty} \mathbb{E} \left[\frac{\#\{0 \leq k \leq \lceil Lu \rceil : d(X_k, \mathbf{0}) \leq \eta \sqrt{L}\}}{L} \right] \leq \frac{(u+v)(1 - e^{-2\lambda\eta})}{C_{v, \lambda}}. \quad (5.33)$$

Proof. Firstly, using a simple coupling argument, we can reduce the problem to proving this lemma for a MERW starting from the origin under the assumptions of Theorem 4.2. Indeed, let Q denote the Markov kernel associated with a regular or repulsive MERW. We represent the corresponding parameters by $g \in \mathbb{R}^{N+2}$. Choose $\zeta \in \mathbb{R}^{N+2}$ such that $\zeta_1 \geq 0$, $\zeta_2^{(k)} \geq 0$ for all $1 \leq k \leq N$ with $\sum_k \zeta_2^{(k)} > 0$, $\zeta_3 > 0$, and $\zeta_1 + \sum_k \zeta_2^{(k)} = 2\zeta_3$ subject to the condition

$$\frac{\zeta_1}{2\zeta_3} > \frac{g_1}{2g_3}. \quad (5.34)$$

Let P be the transition kernel associated with the MERW with parameters γ as in Theorem 4.2, converging to ζ . We then have

$$P(\mathbf{0}, \mathbf{0}) = \frac{\zeta_1}{2\zeta_3} + \mathcal{O}\left(\frac{1}{\sqrt{L}}\right). \quad (5.35)$$

From this, we get $P(\mathbf{0}, \mathbf{0}) \geq Q(\mathbf{0}, \mathbf{0})$ for large L . Also, noting that $P_k(n, n-1) \geq Q_k(n, n-1)$ for all $n \geq 1$ and $1 \leq k \leq N$, constructing a coupling for which $X^P \leq X^Q$ becomes feasible, where X^P and X^Q are the MERW associated with P and Q .

Secondly, let π be the invariant probability distribution of the MERW $\{X_n\}_{n \geq 0}$ in Theorem 4.2. Using (5.12) and (3.12), we deduce

$$\sum_{k=1}^N \pi_k((\eta\sqrt{L}, \infty)) = \frac{\mathcal{S}_2 \Gamma^{2\lfloor \eta\sqrt{L} \rfloor}}{\gamma_3(1 - \Gamma^2) + \mathcal{S}_2 \Gamma^2} = e^{-2\lambda\eta} + o(1), \quad (5.36)$$

leading to

$$\frac{\mathbb{E}_\pi \left[\#\{0 \leq k < \lceil L(u+v) \rceil : \bar{X}_k \leq \eta\sqrt{L}\} \right]}{L} \underset{L \rightarrow \infty}{\sim} (u+v)(1 - e^{-2\lambda\eta}). \quad (5.37)$$

Define $T = \inf\{k \geq 0 : \bar{X}_k = 0\}$. Employing the strong Markov property, we have

$$\mathbb{E}_\pi \left[\sum_{k=0}^{\lceil L(u+v) \rceil - 1} \mathbb{1}_{\{\bar{X}_k \leq \eta\sqrt{L}\}} \right] \geq \sum_{i=0}^{\lceil L(u+v) \rceil - 1} \mathbb{P}_\pi(T = i) \mathbb{E}_0 \left[\sum_{k=0}^{\lceil L(u+v) \rceil - 1 - i} \mathbb{1}_{\{\bar{X}_k \leq \eta\sqrt{L}\}} \right] \quad (5.38)$$

$$\geq \mathbb{P}_\pi(T < \lceil Lv \rceil) \mathbb{E}_0 \left[\sum_{k=0}^{\lceil Lu \rceil - 1} \mathbb{1}_{\{\bar{X}_k \leq \eta\sqrt{L}\}} \right]. \quad (5.39)$$

Let Ξ_k be a random variable distributed as π conditionally on $\{\bar{x}, k : \bar{x} \geq 1\}$ and let $(\xi_i)_{i \geq 1}$ be a sequence of i.i.d. Rademacher random variables with parameter $p = \gamma_3 \Gamma \rho^{-1}$ independent of Ξ_k . Set $S_n = \xi_1 + \dots + \xi_n$. Through a simple coupling argument, we can infer

$$\mathbb{P}_\pi(T < \lceil Lv \rceil) \geq \pi(\mathbf{0}) + \sum_{k=1}^N \pi_k((0, \infty)) \mathbb{P}(S_{\lceil Lv \rceil - 1} \leq -\Xi_k). \quad (5.40)$$

Additionally, we have

$$\mathbb{P}(S_{\lceil Lv \rceil - 1} \leq -\Xi_k) = \mathbb{P}\left(\frac{S_{\lceil Lv \rceil - 1} + \lambda\sqrt{Lv}}{\sqrt{Lv}} \leq \frac{-\Xi_k}{\sqrt{Lv}} + \lambda\sqrt{v}\right). \quad (5.41)$$

From (5.14), we observe that $\mathbb{E}[S_{\lceil Lv \rceil - 1}] = -\lambda\sqrt{Lv} + o(\sqrt{L})$ as L approaches infinity. Then, analogously to (5.36) and with the aid of the central limit theorem, we deduce

$$\frac{S_{\lceil Lv \rceil - 1} + \lambda\sqrt{Lv}}{\sqrt{Lv}} \otimes \frac{\Xi_k}{\sqrt{L}} \underset{L \rightarrow \infty}{\xrightarrow{(\cdot)}} U, V \sim \mathcal{N}(0, 1) \otimes \mathcal{E}(2\lambda). \quad (5.42)$$

Given that $\pi(\mathbf{0})$ converges to 0, we conclude

$$\liminf_{L \rightarrow \infty} \mathbb{P}_\pi(T < \lceil Lv \rceil) \geq \mathbb{P}(U + \sqrt{2}V \leq \lambda\sqrt{v}). \quad (5.43)$$

Lastly, combining this with the aforementioned equations, the proof is completed. \square

Identification of the limit. To go further, one can express

$$\mathbf{f}\left(\omega\left(\frac{n}{L}\right)\right) = \mathbf{f}\left(\omega\left(\frac{m}{L}\right)\right) + \frac{1}{L} \sum_{k=m}^{n-1} \mathcal{L}_L \mathbf{f}\left(\omega\left(\frac{k}{L}\right)\right) + M_n - M_m, \quad (5.44)$$

where M is a square-integrable \mathbb{P}^L -martingale. For every $x = (\bar{x}, i) \in \frac{1}{\sqrt{L}} \cdot G \setminus \{\mathbf{0}\}$,

$$\begin{aligned} \mathcal{L}_L \mathbf{f}(x) &= \frac{L}{2} \left(\mathbf{f}_i\left(\bar{x} + \frac{1}{\sqrt{L}}\right) + \mathbf{f}_i\left(\bar{x} - \frac{1}{\sqrt{L}}\right) - 2\mathbf{f}_i(\bar{x}) \right) \\ &\quad + \frac{LD_i(\bar{x}\sqrt{L})}{2} \left(\mathbf{f}_i\left(\bar{x} + \frac{1}{\sqrt{L}}\right) - \mathbf{f}_i\left(\bar{x} - \frac{1}{\sqrt{L}}\right) \right), \end{aligned} \quad (5.45)$$

and

$$\mathcal{L}_L \mathbf{f}(\mathbf{0}) = L \sum_{i=1}^N P_i(0, 1) \left(\mathbf{f}_i\left(\frac{1}{\sqrt{L}}\right) - \mathbf{f}(\mathbf{0}) \right). \quad (5.46)$$

Subsequently, we assume that the test functions \mathbf{f} are sufficiently smooth and bounded, along with their derivatives, on each ray. Importantly, uniformly on \mathbb{R} , we find that

$$\frac{L}{2} \left(\mathbf{f}_i\left(\bar{x} + \frac{1}{\sqrt{L}}\right) + \mathbf{f}_i\left(\bar{x} - \frac{1}{\sqrt{L}}\right) - 2\mathbf{f}_i(\bar{x}) \right) = \frac{1}{2} \mathbf{f}_i''(\bar{x}) + \mathcal{O}\left(\frac{1}{\sqrt{L}}\right). \quad (5.47)$$

It is noteworthy that the constant in the big \mathcal{O} depends exclusively on $\|\mathbf{f}_i'''\|_\infty$. Our analysis will center on the remaining terms in (5.45) and (5.46).

Focus on the regular and attractive cases. We shall prove that \mathbb{P}^* is the solution of the well-posed martingale/submartingale problem in Theorem 5.2. First, the assumptions of Theorems 4.1 and 4.2 allow us to see that for some $\alpha > 0$, one has

$$\mathcal{L}_L \mathbf{f}(\mathbf{0}) = \alpha \sum_{i=1}^N \mu_i \mathbf{f}_i'(0) \sqrt{L} + \mathcal{O}(1). \quad (5.48)$$

Here we use $P_i(0, 1) = \frac{\zeta_2^{(i)}}{2\zeta_3} + \mathcal{O}\left(\frac{1}{\sqrt{L}}\right)$ for the attractive case. Recall that $D_i(\bar{x}\sqrt{L}) = 0$ as soon as $\bar{x} \in \frac{1}{\sqrt{L}} \cdot \mathbb{N}$ in the regular case whereas in the attractive case we obtain from (5.14) that

$$\frac{LD_i(\bar{x}\sqrt{L})}{2} \left(\mathbf{f}_i\left(\bar{x} + \frac{1}{\sqrt{L}}\right) - \mathbf{f}_i\left(\bar{x} - \frac{1}{\sqrt{L}}\right) \right) = -\lambda \mathbf{f}_i'(\bar{x}) + o(1), \quad (5.49)$$

By using (5.47) we deduce for $\mathcal{L} \in \{\mathcal{L}_{\mathbf{W}}, \mathcal{L}_{\mathbf{Z}}\}$ according to the assumptions that

$$\mathcal{L}_L \mathbf{f}(x) = (\mathcal{L} \mathbf{f}(x) + o(1)) \mathbf{1}_{\{x \neq \mathbf{0}\}} + (A_L \mathbf{f} + \mathcal{O}(1)) \mathbf{1}_{\{x = \mathbf{0}\}}, \quad (5.50)$$

where $A_L \mathbf{f}$ satisfies

$$\sum_{i=1}^N \mu_i \mathbf{f}_i'(0) = 0 \quad \left(\text{resp.} \quad \sum_{i=1}^N \mu_i \mathbf{f}_i'(0) \geq 0 \right) \implies A_L \mathbf{f} = 0 \quad (\text{resp.} \quad A_L \mathbf{f} \geq 0). \quad (5.51)$$

Let $T, \varepsilon, R \geq 0$ be given and set for all $\omega \in \mathbf{C}$,

$$\delta_\omega(T, \varepsilon) = 1 \wedge \sup\{d(\omega(t), \omega(s)) : |t - s| \leq \varepsilon, 0 \leq s, t \leq T\}. \quad (5.52)$$

Here we denote $a \wedge b = \min(a, b)$. By using (5.44) one can write for all $0 \leq s \leq t \leq T$,

$$\mathbf{f}(\omega(t)) = \mathbf{f}(\omega(s)) + \frac{1}{L} \sum_{k=\lfloor Ls \rfloor}^{\lfloor Lt \rfloor - 1} \mathcal{L}_L \mathbf{f} \left(\omega \left(\frac{k}{L} \right) \right) + M_{\lfloor Lt \rfloor} - M_{\lfloor Ls \rfloor} + \mathcal{O} \left(\delta_\omega \left(T, \frac{1}{L} \right) \right), \quad (5.53)$$

where the constant in the big \mathcal{O} depends only on the $\|\mathbf{f}'_i\|_\infty$ for $1 \leq i \leq N$. Furthermore, we get from (5.50) that for all $0 \leq s \leq t \leq T$,

$$\begin{aligned} \frac{1}{L} \sum_{k=\lfloor Ls \rfloor}^{\lfloor Lt \rfloor - 1} \mathcal{L}_L \mathbf{f} \left(\omega \left(\frac{k}{L} \right) \right) \mathbf{1}_{\{\omega(\frac{k}{L}) \neq \mathbf{0}\}} &= \int_{\frac{\lfloor Ls \rfloor}{L}}^{\frac{\lfloor Lt \rfloor}{L}} \mathcal{L}_L \mathbf{f} \left(\omega \left(\frac{\lfloor Lu \rfloor}{L} \right) \right) \mathbf{1}_{\{\omega(\frac{\lfloor Lu \rfloor}{L}) \neq \mathbf{0}\}} du \\ &= \int_s^t \mathcal{L} \mathbf{f}(\omega(u)) \mathbf{1}_{\{\omega(u) \neq \mathbf{0}\}} du + o(1) + \mathcal{O} \left(\delta_\omega \left(T, \frac{1}{L} \right) + \frac{1}{L} \right). \end{aligned} \quad (5.54)$$

At this point we need to note that if we assume that $\lfloor Lt \rfloor \geq \lfloor Ls \rfloor + 1$ and $Ls \neq \lfloor Ls \rfloor$ then by using (5.48) and (5.44) one has

$$\mathbb{E}^L [M_{\lfloor Lt \rfloor} - M_{\lfloor Ls \rfloor} | \mathcal{F}_s](\omega) = M_{\lfloor Ls \rfloor + 1}(\omega) - M_{\lfloor Ls \rfloor}(\omega) = \mathcal{O} \left(\delta_\omega \left(T, \frac{1}{L} \right) + \frac{1}{\sqrt{L}} \right). \quad (5.55)$$

Otherwise $\mathbb{E}^L [M_{\lfloor Lt \rfloor} - M_{\lfloor Ls \rfloor} | \mathcal{F}_s] = 0$. Finally, we deduce that for all \mathbf{f} sufficiently smooth with bounded derivatives on each ray satisfying the left-hand-side of (5.51) one has

$$\begin{aligned} \mathbb{E}^L \left[\mathbf{f}(\omega(t)) - \mathbf{f}(\omega(s)) - \int_s^t \mathcal{L} \mathbf{f}(\omega(u)) \mathbf{1}_{\{\omega(u) \neq \mathbf{0}\}} du \middle| \mathcal{F}_s \right] &\geq (\text{resp. } =) \\ o(1) + \mathcal{O} \left(\mathbb{E}^L \left[\delta_\omega \left(T, \frac{1}{L} \right) \right] \right) &+ \mathcal{O} \left(\frac{\mathbb{E}^L [\#\{0 \leq k \leq \lfloor LT \rfloor : X_k = \mathbf{0}\}]}{L} \right). \end{aligned} \quad (5.56)$$

Note that the functional into the expectation of the left-hand-side of (5.56) is continuous and bounded with respect to $\omega \in \mathbf{C}$. Besides, it follows from the tightness and Lemma 5.1 (by letting $\eta \rightarrow 0$) that the second and the third term in the right-hand-side of (5.56) goes to 0 as L goes to infinity. We deduce (5.25). To conclude, it remains to prove (5.23). To this end, one can write for all $\eta > 0$,

$$\mathbb{E}^* \left[\int_0^T \mathbf{1}_{\{d(\omega(s), \mathbf{0}) < \eta\}} ds \right] \leq \liminf_{L \rightarrow \infty} \frac{\mathbb{E}^L [\#\{0 \leq k \leq \lfloor LT \rfloor : d(X_k, \mathbf{0}) < \eta \sqrt{L}\}]}{L}. \quad (5.57)$$

Here we use $\{\omega \in \mathbf{C} : d(\omega(s), \mathbf{0}) < \eta\}$ is open and the Fatou's Lemma. Applying again Lemma 5.1 and letting $\eta \rightarrow 0$ we obtain (5.23). This completes the proof of Theorems 4.1 and 4.2.

Focus on the repulsive case. The proof follows the main lines as in the regular and attractive cases above and most of the previous notations are kept. For instance, as for the regular and attractive cases, we deduce from Lemma 5.1 that

$$\mathbb{E}^* \left[\int_0^\infty \mathbf{1}_{\{\omega(s) = \mathbf{0}\}} ds \right] = 0. \quad (5.58)$$

The test functions \mathbf{f} we consider are supposed to have a compact support in $\mathcal{G} \setminus \{\mathbf{0}\}$ as in Theorem 5.3. In particular, assuming $\mu_i \neq 0$, the asymptotic (5.49) becomes

$$\frac{LD_i(\bar{x}\sqrt{L})}{2} \left(\mathbf{f}_i \left(\bar{x} + \frac{1}{\sqrt{L}} \right) - \mathbf{f}_i \left(\bar{x} - \frac{1}{\sqrt{L}} \right) \right) = \frac{\mathbf{f}'_i(\bar{x})}{\bar{x}} + o(1). \quad (5.59)$$

Here we use (5.18). Furthermore, we need to distinguish whether or not $x = \mathbf{0}$ and when $x = (\bar{x}, i) \neq \mathbf{0}$ whether or not $\mu_i = 0$.

1) Assume that $x = (\bar{x}, i) \neq \mathbf{0}$ and $\mu_i \neq 0$. One can prove as previously that

$$\left\{ \mathbf{f}(\omega_t) - \int_0^t \mathcal{L}_{\mathbf{Y}} \mathbf{f}(\omega_s) ds \right\}_{t \geq 0} \text{ is a } \mathbb{P}^*(d\omega)\text{-martingale.} \quad (5.60)$$

As in the proof of Theorem 5.3, we deduce that $\mathbb{P}^* \sim \mathbf{Y}^{(\mu, x)}$ since the hitting time of 0 of a three-dimensional Bessel process starting from $\bar{x} > 0$ is infinite almost-surely.

2) Assume that $x = (\bar{x}, i) \neq \mathbf{0}$ and $\mu_i = 0$. Again one has (5.60) and we deduce that the restriction of \mathbb{P}^* to the σ -algebra \mathcal{F}_{τ_0} is a standard Brownian motion on the i th ray starting to x up to the hitting time of $\mathbf{0}$. Then by using the Markov property, we deduce that $\mathbb{P}^* \sim \mathbf{Y}^{(\mu, x)}$ provided the result is proved assuming $x = \mathbf{0}$.

3) Assume that $x = \mathbf{0}$. Once again (5.60) still holds and in order to apply Theorem 5.3 we only need to show that \mathbb{P}^* satisfies (5.27). As a matter of facts, it is a simple consequence of (3.21) and thus $\mathbb{P}^* \sim \mathbf{Y}^{(\mu, x)}$.

This ends the proof the scaling limits. □

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