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# Differential equations properties and $\psi$ function

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## Abstract

Using the differential equations properties, we study the function  $\psi$  which is a function defined by an integral. The particular case of this function coincide with the function  $s \mapsto \zeta(s)/s$ , where  $\zeta$  is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Differential equation, Zeta function, Riemann hypothesis.

## 1 Main result

Let be  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a bounded function. We study the function  $\psi[\eta] : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{C}$  defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \quad (1)$$

We focus about the function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying the following hypotheses

$H_1$  :  $0 \leq \eta(t) \leq 1$  for all  $t \geq 0$ ,

$H_2$  : there exists an at most countable set  $A \subset \mathbb{R}_+$  such that  $\eta$  is  $C^1$  on  $\mathbb{R}_+/A$ .

We prove in this paper the following Main result

**Main Result.** *Consider the function  $\psi$  defined by Equation (1). Then, for every function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  that satisfies the hypotheses  $H_1$  and  $H_2$  we have*

$$\forall \tau \in \mathbb{R}^*, \forall x \in (0, \frac{1}{2}) : \quad \|\psi[\eta](x, \tau) - \psi[\eta](1-x, \tau)\| \neq 0.$$

## 2 Main proposition

In this Section we study the function  $\psi$ , defined in Equation (1), by transforming the problem to some non-homogeneous linear differential equation.

**Notation 1.** Let  $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$  be a function that satisfies  $H_1$  and  $H_2$ . For every  $r \in (0, \frac{1}{2})$  and  $\tau > 0$  define the functions  $t \mapsto h_r(\eta, t)$  as

$$h_r(\eta, t) := 2 \exp\left(-\frac{1}{2}t\right)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

In order to simplify the notation we denote  $h_r(t) := h_r(\eta, t)$  for all  $t \geq 0$ .

Use the change of variable,  $t \rightarrow \exp(u)$ , in Equation (1), we remark that

$$\begin{aligned} \|\psi[\eta]\left(\frac{1}{2} - r, \tau\right) - \psi[\eta]\left(\frac{1}{2} + r, \tau\right)\| &= \left\| \int_0^{+\infty} h_r(s) \exp(i\tau s) ds \right\|, \\ \forall r \in (0, \frac{1}{2}), \forall \tau > 0. \end{aligned}$$

The aim is to study the behavior of the function

$$t \mapsto \int_0^t h_r(s) \exp(i\tau s) ds, \quad t \geq 0,$$

and prove that its limit is not equal to zero. In the following Proposition we study the differential equation of solution the function

$$t \mapsto \sin(\tau t) \int_0^t \sin(\tau u) h_r(u) du + \cos(\tau t) \int_0^t \cos(\tau u) h_r(u) du.$$

**Proposition 2.** Let  $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$  be a function that satisfies  $H_1$  and  $H_2$ . Then for every  $r \in (0, \frac{1}{2})$  and every  $\tau > 0$  the unique continuous solution  $t \mapsto \delta_{\tau, r}(t)$  of the following linear differential equation

$$\frac{d}{dt} \delta_{\tau, r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \delta_{\tau, r}(t) + \begin{pmatrix} 0 \\ \tau h_r(t) \end{pmatrix}, \quad t \in \mathbb{R}/\ln(A), \quad \delta_{\tau, r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2)$$

satisfies

$$\lim_{t \rightarrow +\infty} \|\delta_{\tau, r}(t)\| \neq 0.$$

**Lemma 3.** For every  $t \geq 0$  we have

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t\right) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}.$$

*Proof.* We have

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}^2 = -\tau^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k \frac{(\tau t)^{2k}}{(2k)!} \\ &\quad + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k (\tau)^{2k} \frac{t^{2k+1}}{(2k+1)!}. \end{aligned}$$

Implies

$$\begin{aligned} \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\tau t) + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \frac{1}{\tau} \sin(\tau t) \\ &= \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}. \end{aligned}$$

□

*Proof of the Proposition 2.* By contradiction, suppose that there exist  $r \in (0, \frac{1}{2})$  and  $\tau > 0$  such that

$$\lim_{t \rightarrow +\infty} \|\delta_{\tau,r}(t)\| = 0. \quad (3)$$

Integrate Equation (2) using the fact  $\delta_{\tau,r}$  is the continuous solution and the fact  $\|\delta_{\tau,r}(0)\| = 0$ , we get

$$\frac{d}{dt} Y_{\tau,r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} Y_{\tau,r}(t) + \tau \int_0^t h_r(u) du \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad t \geq 0.$$

where

$$Y_{\tau,r}(t) := \int_0^t \delta_{\tau,r}(u) du.$$

Using Lemma 3, integrate to obtain

$$Y_{\tau,r}(t) = \tau \int_0^t \begin{pmatrix} \tau \sin(\tau(t-v)) \\ \cos(\tau(t-v)) \end{pmatrix} \int_0^v h_r(u) dudv, \quad \forall t \geq 0.$$

By Equations (3) and (2) the function  $\|\delta_{\tau,r}(t)\|$  is exponentially decreasing to zero. There exists a constant vector  $\omega_{\tau,r} \in \mathbb{R}^2$  such that

$$\lim_{t \rightarrow +\infty} \|Y_{\tau,r}(t) - \omega_{\tau,r}\| = 0.$$

Since  $r \in (0, \frac{1}{2})$ , then the function  $t \mapsto \int_0^v h_r(u) du dv$  is positive, increasing and converge. Then the limit

$$\lim_{t \rightarrow +\infty} \int_0^t \begin{pmatrix} \tau \sin(\tau(t-v)) \\ \cos(\tau(t-v)) \end{pmatrix} \int_0^v h_r(u) du dv,$$

does not exist. We have obtained a contradiction.  $\square$

### 3 Proof of the Main result

*Proof of the Main result.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau > 0$ . Consider the function  $\delta_{\tau,r}$  solution of Equation (2). Using Lemma 3, we integrate to obtain

$$\delta_{\tau,r}(t) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix} \int_0^t \begin{pmatrix} \cos(\tau v) & \frac{-1}{\tau} \sin(\tau v) \\ \tau \sin(\tau v) & \cos(\tau v) \end{pmatrix} \begin{pmatrix} 0 \\ \tau h_r(v) \end{pmatrix} dv.$$

Use the change of variable,  $t \rightarrow \exp(u)$ , in Equation (1), we obtain

$$\lim_{t \rightarrow +\infty} \begin{pmatrix} \cos(\tau t) & \frac{-1}{\tau} \sin(\tau t) \\ \tau \sin(\tau t) & \cos(\tau t) \end{pmatrix} \delta_{\tau,r}(t) = \begin{pmatrix} \Im \left[ \psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau) \right] \\ \tau \Re \left[ \psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau) \right] \end{pmatrix}.$$

By the Main Proposition 2 we have  $\|\psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau)\| \neq 0$  for every  $r \in (0, \frac{1}{2})$  and  $\tau > 0$ . For  $\tau < 0$  use the same proof.  $\square$

### 4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (4)$$

and where  $\{t\}$  is the fractional part of the real  $t$ . Define the *non-trivial zeros* of the function  $\zeta$  in the following sense

**Definition 4.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non-trivial zero* of the function  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote  $\eta_*(t) := \{t\}$ . For every  $\tau \in \mathbb{R}^*$  and every  $x \in (0, 1)$  we have

$$\psi[\eta_*](x, \tau) = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*](1-x, \tau) = -\frac{\zeta(1-s)}{1-s}, \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then  $\eta_*$  satisfies the hypotheses  $H_1$ . The function  $\eta_*$  is  $C^1$  on  $\mathbb{R}_+/\mathbb{N}$ . Then  $\eta_*$  satisfies the hypotheses  $H_2$ . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non-trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).