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Differential equations properties and ψ function

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Abstract

Using the differential equations properties, we study the function ψ which is a function defined by an integral. The particular case of this function coincide with the function $s \mapsto \zeta(s)/s$, where ζ is the Riemann Zeta function. In the end of the paper, we find that $\frac{\zeta(s)}{s} \neq -\frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) > 0$. Thanks to the Riemann functional equation, we deduce that any *non-trivial zero* of ζ is of real part equal to $\frac{1}{2}$.

Keywords: Linear differential equation, Zeta function, Riemann hypothesis.

1 Main result

Let be $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0, 1) \times \mathbb{R}_+^* \rightarrow \mathbb{C}$ defined as

$$\forall x \in (0, 1), \forall \tau > 0 : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \quad (1)$$

We focus about the function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following hypotheses

H_1 : $\exists \omega > 0$: $0 \leq \eta(t) < \omega$ for all $t \geq 0$,

H_2 : there exists an at most countable set $A \subset \mathbb{R}_+$ such that η is C^1 on \mathbb{R}_+/A .

We prove in this paper the following Main result

Main Result. Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfies the hypotheses H_1 and H_2 we have

$$\|\psi[\eta](x, \tau) - \psi[\eta](1 - x, \tau)\| \neq 0, \quad \forall x \in (0, \frac{1}{2}), \forall \tau > 0.$$

2 Main proposition

In this Section we study the function ψ , defined in Equation (1), by transforming the problem to some second order differential equation.

Notation 1. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . For every $r \in \mathbb{R}$ and $\tau > 0$ define the functions $t \mapsto h_r(\eta, t)$ as

$$h_r(\eta, t) := 2 \exp(-\frac{1}{2}t)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

In order to simplify the notation we denote $h_r(t) := h_r(\eta, t)$ for all $t \geq 0$.

Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we remark that

$$\begin{aligned} \|\psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau)\| &= \left\| \int_0^{+\infty} h_r(s) \exp(i\tau s) ds \right\|, \\ &\forall r \in (0, \frac{1}{2}), \forall \tau > 0. \end{aligned}$$

The aim is to study the behavior of the function

$$t \mapsto \int_0^t h_r(s) \exp(i\tau s) ds, \quad t \geq 0,$$

and prove that its limit is not equal to zero. In the following Proposition we study the differential equation of solution the function

$$t \mapsto \tau \sin(\tau t) \int_0^t \sin(\tau u) h_r(u) du + \tau \cos(\tau t) \int_0^t \cos(\tau u) h_r(u) du.$$

In order to simplify the notation, denote

$$\alpha_r := \lim_{t \rightarrow +\infty} \int_0^t h_r(s) ds.$$

Proposition 2. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . Then for every $r \in (0, \frac{1}{2})$ and every $\tau > 0$ the unique solution $t \mapsto \delta_{\tau,r}(t)$ of the following differential equation

$$\begin{aligned} \frac{d}{dt} \delta_{\tau,r}(t) &= -\tau^2 \int_0^t \delta_{\tau,r}(u) du + \tau h_r(t), \\ t &\in \mathbb{R}/\ln(A), \quad \delta_{\tau,r}(0) = 0, \end{aligned} \quad (2)$$

satisfies

$$\liminf_{t \rightarrow +\infty} \delta_{\tau,r}(t) < 0.$$

Lemma 3. For every $\epsilon > 0$, $r \in (0, \frac{1}{2})$ and every $\tau \in \mathbb{R}^*$ let $t \mapsto \Delta_{\tau,r}(\epsilon, t)$ be the function defined as

$$\Delta_{\tau,r}(\epsilon, t) = \frac{1}{\tau} \left[\alpha_r \left(1 - \cos(\tau t) \right) - \epsilon \sin(\tau t) \right], \quad \forall t \geq 0.$$

Then $\Delta_{\tau,r}(\epsilon, t)$ is the unique C^∞ solution of the following differential equation

$$\begin{aligned} \frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon, t) &= -\tau^2 \Delta_{\tau,r}(\epsilon, t) + \tau \alpha_r, \\ \Delta_{\tau,r}(\epsilon, 0) &= 0, \quad \frac{d}{dt} \Delta_{\tau,r}(\epsilon, T) = -\epsilon, \quad t \geq 0. \end{aligned} \quad (3)$$

In addition,

$$\forall \epsilon > 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau > 0 : \quad \liminf_{t \rightarrow +\infty} \frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon, t) < 0.$$

Proof of the Proposition 2. Let be $r \in (0, \frac{1}{2})$ and $\tau > 0$. The strategy to prove the present Proposition is to shows, for some $\epsilon > 0$, that

$$\frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon, t) > \delta_{\tau,r}(t), \quad \forall t \geq 0.$$

Using the hypothesis H_2 , the fact $\delta_{\tau,r}$ is the continuous solution and the fact $\delta_{\tau,r}(0) = 0$, by Equation (2), we get

$$\delta_{\tau,r}(t) = -\tau^2 \int_0^t \int_0^s \delta_{\tau,r}(u) du ds + \tau \int_0^t h_r(u) du.$$

Then

$$\frac{d}{dt} \delta_{\tau,r}(t) = \tau^4 \int_0^t \int_0^v \int_0^s \delta_{\tau,r}(u) du ds dv - \tau^3 \int_0^t \int_0^s h_r(u) du ds + \tau h_r(t).$$

In order to simplify the notation, denote

$$\theta_{\tau,r}(t) := \int_0^t \int_0^s \delta_{\tau,r}(u) du ds, \quad \forall t \geq 0.$$

Then for every $t \in \ln(A)$ we get

$$\frac{d^3}{dt^3} \theta_{\tau,r}(t) = \tau^4 \int_0^t \theta_{\tau,r}(u) du - \tau^3 \int_0^t \int_0^s h_r(u) du ds + \tau h_r(t).$$

Let be $t \mapsto \Delta_{\tau,r}(\epsilon, t)$ as defined in Lemma 3. We have

$$\frac{d^3}{dt^3} \Delta_{\tau,r}(\epsilon, t) = \tau^4 \int_0^t \Delta_{\tau,r}(\epsilon, u) du - \tau^3 \alpha_r t + [\tau^2 \epsilon + \tau \alpha_r]$$

Implies

$$\begin{aligned} \frac{d^3}{dt^3} [\Delta_{\tau,r}(\epsilon, t) - \theta_{\tau,r}(t)] &= \tau^4 \int_0^t [\Delta_{\tau,r}(\epsilon, u) - \theta_{\tau,r}(u)] du \\ &\quad - \tau^3 \int_0^t [\alpha_r - \int_0^s h_r(u) du] ds \\ &\quad + \tau[\tau \epsilon + \alpha_r - h_r(t)] \end{aligned} \quad (4)$$

Choose $\epsilon_{\tau,r} > 0$ such that

$$\epsilon_{\tau,r} > \max_{t \geq 0} \left[\tau \int_0^t \left[\alpha_r - \int_0^s h_r(u) du \right] ds - \frac{1}{\tau} [\alpha_r - h_r(t)] \right]$$

Since $\frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon_{\tau,r}, 0) = \tau \alpha_r > 0$ and $\frac{d^2}{dt^2} \theta_{\tau,r}(0) = 0$, then there exists $\eta > 0$ such that

$$\frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon_{\tau,r}, t) > \frac{d^2}{dt^2} \theta_{\tau,r}(t), \quad \forall t \in [0, \eta].$$

Define

$$T := \sup \left\{ t > 0 : \frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon_{\tau,r}, s) > \frac{d^2}{dt^2} \theta_{\tau,r}(s), \quad \forall s \in [0, t] \right\}.$$

Prove that $T = +\infty$. By contradiction, suppose that $T < +\infty$. Then

$$\frac{d^2}{dt^2} \Delta_{\tau,r}(\epsilon_{\tau,r}, T) = \frac{d^2}{dt^2} \theta_{\tau,r}(T). \quad (5)$$

By Equation (4), we get

$$\begin{aligned} \frac{d^3}{dt^3} [\Delta_{\tau,r}(\epsilon_{\tau,r}, T) - \theta_{\tau,r}(T)] &> -\tau^3 \int_0^T \left[\alpha_r - \int_0^s h_r(u) du \right] ds \\ &\quad + \tau[\tau \epsilon_{\tau,r} + \alpha_r - h_r(T)] > 0. \end{aligned}$$

There exists $s < T$ close enough to T such that $\frac{d^2}{dt^2}\Delta_{\tau,r}(\epsilon_{\tau,r}, s) < \frac{d^2}{dt^2}\theta_{\tau,r}(s)$. Contradiction with Equation (5). Then $T = +\infty$ and

$$\frac{d^2}{dt^2}\Delta_{\tau,r}(\epsilon_{\tau,r}, t) > \frac{d^2}{dt^2}\theta_{\tau,r}(t) = \delta_{\tau,r}(t), \quad \forall t \geq 0.$$

By Lemma 3, we deduce that

$$\liminf_{t \rightarrow +\infty} \delta_{\tau,r}(t) < 0.$$

□

3 Proof of the Main results

Proof of the Main result. Let be $r \in (0, \frac{1}{2})$ and $\tau > 0$. Consider the function $\delta_{\tau,r}$ solution of Equation (2). We have,

$$\begin{aligned} \delta_{\tau,r}(t) &= \tau \sin(\tau t) \int_0^t \sin(\tau u) h_r(u) du \\ &\quad + \tau \cos(\tau t) \int_0^t \cos(\tau u) h_r(u) du. \end{aligned}$$

Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we obtain

$$\begin{aligned} \delta_{\tau,r}(t) &= \tau \sin(\tau t) \Im \left[\psi[\eta] \left(\frac{1}{2} - r, \tau \right) - \psi[\eta] \left(\frac{1}{2} + r, \tau \right) \right] \\ &\quad + \tau \cos(\tau t) \Re \left[\psi[\eta] \left(\frac{1}{2} - r, \tau \right) - \psi[\eta] \left(\frac{1}{2} + r, \tau \right) \right]. \end{aligned}$$

By the Main Proposition 2 we have $\|\psi[\eta] \left(\frac{1}{2} - r, \tau \right) - \psi[\eta] \left(\frac{1}{2} + r, \tau \right)\| \neq 0$ for every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$. □

4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (6)$$

and where $\{t\}$ is the fractional part of the real t . Define the *non-trivial zeros* of the function ζ in the following sense

Definition 4. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero of the function ζ* if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) > 0.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$ we have

$$\psi[\eta_*(x, \tau)] = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*(1-x, \tau)] = -\frac{\zeta(1-s)}{1-s}, \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1 . The function η_* is C^1 on \mathbb{R}_+/\mathbb{N} . Then η_* satisfies the hypotheses H_2 . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).