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Differential equations properties and ψ function

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Abstract

Using the differential equations properties, we study the function ψ which is a function defined by an integral. The particular case of this function coincide with the function $s \mapsto \zeta(s)/s$, where ζ is the Riemann Zeta function. In the end of the paper, we find that $\frac{\zeta(s)}{s} \neq -\frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$. Thanks to the Riemann functional equation, we deduce that any *non-trivial zero* of ζ is of real part equal to $\frac{1}{2}$.

Keywords: Linear differential equation, Zeta function, Riemann hypothesis.

1 Main result

Let be $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{C}$ defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \quad (1)$$

We focus about the function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following hypotheses

H_1 : $\exists \omega > 0$: $0 \leq \eta(t) < \omega$ for all $t \geq 0$,

H_2 : there exists an at most countable set $A \subset \mathbb{R}_+$ such that η is C^1 on \mathbb{R}_+/A .

We prove in this paper the following Main result

Main Result. Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfies the hypotheses H_1 and H_2 we have

$$\forall \tau \in \mathbb{R}^*, \forall x \in (0, \frac{1}{2}) : \quad \|\psi[\eta](x, \tau) - \psi[\eta](1-x, \tau)\| \neq 0.$$

2 Main proposition

In this Section we study the function ψ , defined in Equation (1), by transforming the problem to some non-homogeneous linear differential equation.

Proposition 1. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . For every $r \in \mathbb{R}$ define the function $h_r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as

$$h_r(t) := 2 \exp(-\frac{1}{2}t)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

Then for every $r \in (0, \frac{1}{2})$ and every $\tau > 0$ the unique continuous solution $t \mapsto \delta_{\tau,r}(t)$ of the following linear differential equation

$$\frac{d}{dt} \delta_{\tau,r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \delta_{\tau,r}(t) + \begin{pmatrix} 0 \\ \tau h_r(t) \end{pmatrix}, \quad t \in \mathbb{R}/\ln(A), \quad \delta_{\tau,r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2)$$

satisfies

$$\lim_{t \rightarrow +\infty} \|\delta_{\tau,r}(t)\| \neq 0.$$

Lemma 2. For every $t \geq 0$ we have

$$\exp \left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t \right) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}.$$

Proof. We have

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}^2 = -\tau^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \exp \left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t \right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k \frac{(\tau t)^{2k}}{(2k)!} \\ &\quad + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k (\tau)^{2k} \frac{t^{2k+1}}{(2k+1)!}. \end{aligned}$$

Implies

$$\begin{aligned} \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\tau t) + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \frac{1}{\tau} \sin(\tau t) \\ &= \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}. \end{aligned}$$

□

Proof of the Proposition 1. By contradiction, suppose that there exist $r \in (0, \frac{1}{2})$ and $\tau > 0$ such that

$$\lim_{t \rightarrow +\infty} \|\delta_{\tau,r}(t)\| = 0. \quad (3)$$

Integrate Equation (2) using the fact $\delta_{\tau,r}$ is the continuous solution and the fact $\delta_{\tau,r}(0) = 0$, we get

$$\delta_{\tau,r}(t) = \int_0^t \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} (t-u)\right) \begin{pmatrix} 0 \\ h_{\tau,r}(u) \end{pmatrix} du.$$

By Lemma 2, we obtain,

$$\delta_{\tau,r}(t) = \begin{pmatrix} \delta_{1,\tau,r}(t) \\ \delta_{2,\tau,r}(t) \end{pmatrix} = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix} \begin{pmatrix} -\int_0^t \frac{1}{\tau} \sin(\tau u) h_{\tau,r}(u) du \\ \int_0^t \cos(\tau u) h_{\tau,r}(u) du \end{pmatrix}.$$

Then

$$\begin{pmatrix} \int_0^{+\infty} \frac{1}{\tau} \sin(\tau u) h_{\tau,r}(u) du \\ \int_0^{+\infty} \cos(\tau u) h_{\tau,r}(u) du \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Implies

$$\delta_{\tau,r}(t) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix} \begin{pmatrix} \int_t^{+\infty} \frac{1}{\tau} \sin(\tau u) h_{\tau,r}(u) du \\ -\int_t^{+\infty} \cos(\tau u) h_{\tau,r}(u) du \end{pmatrix}. \quad (4)$$

By Equation (2)

$$\frac{d}{dt} \left[[\tau \delta_{1,\tau,r}(t)]^2 + [\delta_{2,\tau,r}(t)]^2 \right] = 2\tau \delta_{2,\tau,r}(t) h_r(t). \quad (5)$$

Integrate, for every $s > 0$ we get

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left[[\tau \delta_{1,\tau,r}(t)]^2 + [\delta_{2,\tau,r}(t)]^2 \right] &= [\tau \delta_{1,\tau,r}(s)]^2 + [\delta_{2,\tau,r}(s)]^2 \\ &\quad + 2\tau \lim_{t \rightarrow +\infty} \int_s^t \delta_{2,\tau,r}(t) h_r(t) du \\ &\quad + 2\tau \lim_{t \rightarrow +\infty} \int_s^t \delta_{2,\tau,r}(t) h_r(t) du. \end{aligned}$$

Thanks to Equation (4)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left[[\tau \delta_{1,\tau,r}(t)]^2 + [\delta_{2,\tau,r}(t)]^2 \right] &= [\tau \delta_{1,\tau,r}(s)]^2 + [\delta_{2,\tau,r}(s)]^2 \\ &- 2\tau \lim_{t \rightarrow +\infty} \int_s^t h_r(u) \sin(\tau u) \int_u^{+\infty} \sin(\tau v) h_r(v) dv du \\ &- 2\tau \lim_{t \rightarrow +\infty} \int_s^t h_r(u) \cos(\tau u) \int_u^{+\infty} \cos(\tau v) h_r(v) dv du. \end{aligned}$$

Implies

$$\lim_{t \rightarrow +\infty} \left[[\tau \delta_{1,\tau,r}(t)]^2 + [\delta_{2,\tau,r}(t)]^2 \right] = \theta_{\tau,r}(s), \quad \forall s > 0, \quad (6)$$

where

$$\begin{aligned} \theta_{\tau,r}(s) &:= [\tau \delta_{1,\tau,r}(s)]^2 + [\delta_{2,\tau,r}(s)]^2 - 2\tau \left[\int_s^{+\infty} \sin(\tau v) h_r(v) dv \right]^2 \\ &- 2\tau \left[\int_s^{+\infty} \cos(\tau v) h_r(v) dv \right]^2, \quad \forall s > 0. \end{aligned} \quad (7)$$

We have $\theta_{\tau,r}(0) = 0$ and

$$\begin{aligned} \frac{d}{ds} \theta_{\tau,r}(s) &= \frac{d}{ds} \left[[\tau \delta_{1,\tau,r}(s)]^2 + [\delta_{2,\tau,r}(s)]^2 \right] \\ &+ 2\tau \sin(\tau s) h_r(s) \int_s^{+\infty} \sin(\tau v) h_r(v) dv \\ &+ 2\tau \cos(\tau s) h_r(s) \int_s^{+\infty} \cos(\tau v) h_r(v) dv. \end{aligned}$$

Then $\frac{d}{ds} \theta_{\tau,r}(0) = 0$. By Equation (5), we have

$$\begin{aligned} \frac{1}{2\tau} \frac{d^2}{ds^2} \theta_{\tau,r}(0) &= h_r(0) \frac{d}{ds} \delta_{2,\tau,r}(0) + \delta_{2,\tau,r}(0) \frac{d}{ds} h_r(0) \\ &- [\sin(0) h_r(0)]^2 - [\cos(0) h_r(0)]^2 = 0. \end{aligned}$$

Finally, we have

$$\frac{d^3}{ds^3} \theta_{\tau,r}(0) = 6\tau^2 \left[\frac{d}{ds} h_r(0) \right]^2 > 0.$$

Then there exists $\epsilon_{\tau,r} > 0$ such that

$$\theta_{\tau,r}(\epsilon_{\tau,r}) > 0.$$

By Equation (6), we obtain

$$\lim_{t \rightarrow +\infty} \left[\tau \delta_{1,\tau,r}(t) \right]^2 + [\delta_{2,\tau,r}(t)]^2 = \theta_{\tau,r}(\epsilon_{\tau,r}) > 0.$$

Contradiction with Equation (3). □

3 Proof of the Main results

Proof of the Main result. We have,

$$\delta_{\tau,r}(t) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix} \begin{pmatrix} -\int_0^t \frac{1}{\tau} \sin(\tau u) h_{\tau,r}(t) du \\ \int_0^t \cos(\tau u) h_{\tau,r}(u) du \end{pmatrix}.$$

Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we obtain

$$\begin{pmatrix} \frac{1}{\tau} \Im[\psi(\frac{1}{2} - r, \tau) - \psi(\frac{1}{2} + r, \tau)] \\ \Re[\psi(\frac{1}{2} - r, \tau) - \psi(\frac{1}{2} + r, \tau)] \end{pmatrix} = \lim_{t \rightarrow +\infty} \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau} \sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}^{-1} \delta_{\tau,r}(t).$$

By the Main Proposition 1 we have $\|\psi(\frac{1}{2} - r, \tau) - \psi(\frac{1}{2} + r, \tau)\| \neq 0$ for every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$. □

4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (8)$$

and where $\{t\}$ is the fractional part of the real t . Define the *non-trivial zeros* of the function ζ in the following sense

Definition 3. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero* of the function ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$ we have

$$\psi[\eta_*](x, \tau) = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*](1-x, \tau) = -\frac{\zeta(1-s)}{1-s}, \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1 . The function η_* is C^1 on \mathbb{R}_+/\mathbb{N} . Then η_* satisfies the hypotheses H_2 . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).