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Abstract

Using the differential equations properties, we study the function ψ which is a function defined by an integral. The particular case of this function coincide with the function $s \mapsto \zeta(s)/s$, where ζ is the Riemann Zeta function. In the end of the paper, we find that $\frac{\zeta(s)}{s} \neq -\frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$. Thanks to the Riemann functional equation, we deduce that any *non-trivial zero* of ζ is of real part equal to $\frac{1}{2}$.

Keywords: Linear differential equation, Zeta function, Riemann hypothesis.

1 Main result

Let be $\eta : \mathbb{R}_+ \to \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0,1) \times \mathbb{R}^* \to \mathbb{C}$ defined as

$$\forall x \in (0,1),\ \forall \tau \in \mathbb{R}^* : \quad \psi[\eta](x,\tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \tag{1}$$

We focus about the function $\eta : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following hypotheses

H_1 : $\exists \omega > 0$: $0 \leq \eta(t) < \omega$ for all $t \geq 0$,

H_2 : there exists an at most countable set $A \subset \mathbb{R}_+$ such that η is C^1 on $\mathbb{R}_+/A$.

We prove in this paper the following Main result

Main Result. *Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \to \mathbb{R}$ that satisfies the hypotheses H_1 and H_2 we have*

$$\forall \tau \in \mathbb{R}^*,\ \forall x \in (0, \frac{1}{2}) : \quad \|\psi[\eta](x,\tau) - \psi[\eta](1-x,\tau)\| \neq 0.$$

2 Main proposition

In this Section we study the function ψ, defined in Equation (1), by transforming the problem to some non-homogeneous linear differential equation.

Proposition 1. *Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2. For every $r \in (0, \frac{1}{2})$ and $\tau > 0$ define the $h_{\tau,r} : \mathbb{R}_+ \to \mathbb{R}_+$ as*

$$h_{\tau,r}(t) := \tau \exp(-\frac{1}{2}t)[\eta(\exp(t)) + 1]\sinh(rt), \quad \forall t \geq 0.$$

Then for every $r \in (0, \frac{1}{2})$ and every $\tau > 0$, the unique continuous solution $t \mapsto \delta_{\tau,r}(t)$ of the following linear differential equation

$$\frac{d}{dt}\delta_{\tau,r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \delta_{\tau,r}(t) + \begin{pmatrix} 0 \\ h_{\tau,r}(t) \end{pmatrix},\ t \in \mathbb{R}/\ln(A),\ \delta_{\tau,r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2}$$

satisfies

$$\lim_{t \to +\infty} \|\delta_{\tau,r}(t)\| \neq 0.$$

Lemma 2. *For every $t \geq 0$ we have*

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t \right) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau}\sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}.$$

Proof. We have

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}^2 = -\tau^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k \frac{(\tau t)^{2k}}{(2k)!} \\ + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \sum_{k=0}^{+\infty} (-1)^k (\tau)^{2k} \frac{t^{2k+1}}{(2k+1)!}. \end{aligned}$$

Implies

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} t\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\tau t) + \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \frac{1}{\tau} \sin(\tau t)$$
$$= \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau}\sin(\tau t) \\ -\tau \sin(\tau t) & \cos(\tau t) \end{pmatrix}.$$

□

Proof of the Proposition 1. By contradiction, suppose that there exist $r \in (0, \frac{1}{2})$ and $\tau > 0$ such that

$$\lim_{t \to +\infty} \|\delta_{\tau,r}(t)\| = 0. \tag{3}$$

By the hypotheses H_1, integrate Equation (2) using the fact $\delta_{\tau,r}$ is the continuous solution and the fact $\delta_{\tau,r}(0) = 0$, we get

$$\delta_{\tau,r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \int_0^t \delta_{\tau,r}(u)du + \begin{pmatrix} 0 \\ \int_0^t h_{\tau,r}(u)du \end{pmatrix}, \quad \forall t \geq 0. \tag{4}$$

Since

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}^2 = -\tau^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\int_0^t \delta_{\tau,r}(u)du = \frac{-1}{\tau^2} \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \Big[\delta_{\tau,r}(t) - \begin{pmatrix} 0 \\ \int_0^t h_{\tau,r}(u)du \end{pmatrix}\Big], \quad \forall t \geq 0. \tag{5}$$

By hypothesis H_1 the function η is bounded and positive. Since $r \in (0, \frac{1}{2})$ then there exists $\alpha_r > 0$ such that

$$\lim_{t \to +\infty} \int_0^t h_{\tau,r}(u)du = \tau\alpha_r.$$

By Equation (5) and the hypothesis (3), we get

$$\lim_{t \to +\infty} \int_0^t \delta_{\tau,r}(u)du = \frac{1}{\tau^2} \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \tau\alpha_r \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}\alpha_r \\ 0 \end{pmatrix} \tag{6}$$

Equation (4), can be written as

$$\delta_{\tau,r}(t) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix} \Big[\int_0^t \delta_{\tau,r}(u)du - \begin{pmatrix} \frac{1}{\tau}\alpha_r \\ 0 \end{pmatrix}\Big] + \begin{pmatrix} 0 \\ -\int_t^{+\infty} h_{\tau,r}(u)du \end{pmatrix}, \quad \forall t \geq 0.$$

Integrate,

$$\int_0^t \delta_{\tau,r}(u)du = \begin{pmatrix} 1-\cos(\tau t) & \frac{1}{\tau}-\sin(\tau t) \\ \tau\sin(\tau t) & 1-\cos(\tau t) \end{pmatrix} \begin{pmatrix} \frac{1}{\tau}\alpha_r \\ 0 \end{pmatrix} \tag{7}$$

$$+ \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau}\sin(\tau t) \\ -\tau\sin(\tau t) & \cos(\tau t) \end{pmatrix} \begin{pmatrix} \int_0^t \frac{1}{\tau}\sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu \\ -\int_0^t \cos(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu \end{pmatrix}. \tag{8}$$

By definition of the function $h_{\tau,r}$ and by the hypothesis H_1, the function $u \mapsto \int_u^{+\infty} h_{\tau,r}(v)dv$ is positive and increasing. Using the symmetry of the function $\sin(\tau u)$ on $[0, \frac{2\pi}{\tau}]$ about the point $\frac{\pi}{2\tau}$ we get,

$$\int_0^t \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu > 0, \quad \forall t > 0.$$

In particular,

$$\int_0^t \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu > \int_0^t \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu, \ \forall t > \frac{2\pi}{\tau}. \tag{9}$$

Then the function

$$t \mapsto= \frac{\int_0^t \cos(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu}{\int_0^t \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu}$$

is continuous and uniformly bounded on $(\frac{2\pi}{\tau}, +\infty)$. In addition, the limit

$$\lim_{t\to+\infty} \frac{\int_0^t \cos(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu}{\int_0^t \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu}$$

exists. Then there exists sequence $t_n \to +\infty$ such that

$$\frac{\cos(\tau t_n)}{\sin(\tau t_n)} := \frac{\int_0^{t_n} \cos(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu}{\int_0^{t_n} \sin(\tau u)\int_u^{+\infty} h_{\tau,r}(v)dvdu},$$

and such that

$$\lim_{n\to+\infty}[t_n - (\frac{n\pi}{\tau} + \frac{\pi}{2\tau})] \neq 0, \tag{10}$$

Implies that, there exists a sub-sequence $(t_{n_k})_k \subset (t_n)_n$ such that $\lim_{k\to+\infty}\cos(\tau t_{n_k}) := \alpha \neq 0$. By Equation (7), we get

$$\lim_{k\to+\infty}\int_0^{t_{n_k}} \delta_{2,\tau,r}(u)du = \lim_{k\to+\infty}\frac{1}{\tau}\alpha_r\Big(1-\cos(\tau t_{n_k})\Big) \neq \frac{1}{\tau}\alpha_r,$$

where $(\delta_{1,\tau,r}, \delta_{2,\tau,r}) := \delta_{\tau,r}$. Contradiction with Equation (6).

□

3 Proof of the Main results

Proof of the Main result. Integrate Equation (2) using the fact $\delta_{\tau,r}$ is the continuous solution and the fact $\delta_{\tau,r}(0) = 0$, we get

$$\delta_{\tau,r}(t) = \int_0^t \exp\left(\begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}(t-u)\right)\begin{pmatrix} 0 \\ h_{\tau,r}(u) \end{pmatrix} du.$$

By Lemma 2, we obtain,

$$\delta_{\tau,r}(t) = \begin{pmatrix} \cos(\tau t) & \frac{1}{\tau}\sin(\tau t) \\ -\tau\sin(\tau t) & \cos(\tau t) \end{pmatrix}\begin{pmatrix} -\int_0^t \frac{1}{\tau}\sin(\tau u)h_{\tau,r}(t)du \\ \int_0^t \cos(\tau u)h_{\tau,r}(u)du. \end{pmatrix}.$$

Use the change of variable, $t \to \exp(u)$, in Equation (1), we obtain

$$\begin{pmatrix} \frac{1}{\tau}\Im[\psi(\frac{1}{2}-r,\tau)-\psi(\frac{1}{2}+r,\tau)] \\ \Re[\psi(\frac{1}{2}-r,\tau)-\psi(\frac{1}{2}+r,\tau)] \end{pmatrix} = \lim_{t\to+\infty}\begin{pmatrix} \cos(\tau t) & \frac{1}{\tau}\sin(\tau t) \\ -\tau\sin(\tau t) & \cos(\tau t) \end{pmatrix}^{-1}\delta_{\tau,r}(t).$$

By the Main Proposition 1 we have $\|\psi(\frac{1}{2}-r,\tau)-\psi(\frac{1}{2}+r,\tau)\| \neq 0$ for every $r \in (0,\frac{1}{2})$ and $\tau \in \mathbb{R}^*$. □

4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s\int_0^{+\infty}\frac{\{t\}}{t^{1+s}}dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \tag{11}$$

and where $\{t\}$ is the fractional part of the real t. Define the *non-trivial zeros* of the function ζ in the following sense

Definition 3. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau \in \mathbb{R}^*$ and every $x \in (0,1)$ we have

$$\psi[\eta_*](x,\tau) = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*](1-x,\tau) = -\frac{\zeta(1-s)}{1-s}, \quad s := x+i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1. The function η_* is C^1 on $\mathbb{R}_+/\mathbb{N}$. Then η_* satisfies the hypotheses H_2. By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).