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# Differential equations properties and Zeta function

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## Abstract

Using the differential equations properties, we prove that  $\frac{\zeta(s)}{s} \neq -\frac{\zeta(1-s)}{1-s}$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. Thanks to the Riemann functional equation, we deduce that any *non-trivial zero* of  $\zeta$  is of real part equal to  $\frac{1}{2}$ .

**Keywords:** Zeta function, Abel summation formula, Differential equations, Riemann hypothesis.

## 1 Main result

We study the function  $\psi : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{R}$  defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi(x, \tau) := \int_0^{+\infty} \frac{\{t\}}{t^{1+x}} \sin(\tau \ln(t)) dt, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . The following Main result I is proved in Sections 3 and 4.

**Main Result I.** *Consider the function  $\psi$  defined by Equation (1). Then,*

$$\forall \tau \in \mathbb{R}^*, \forall x \in (0, \frac{1}{2}) : \quad \psi(x, \tau) \neq \psi(1-x, \tau).$$

## 2 Riemann hypothesis

In [[1], page 14 Equation 2.1.5] the function  $\psi$  is the imaginary part of the function  $s := x + i\tau \mapsto \frac{\zeta(s)}{s}$  where  $\zeta$  is the representation of the Riemann

Zeta function defined by the Abel summation formula as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*. \quad (2)$$

However, there is no contradiction between the Main result I and the functional equation of the function Zeta because  $\psi(x, \tau) = \Im\left[\frac{\zeta(s)}{s}\right]$  where  $s := x + i\tau$ . We focus on the *non-trivial zeros* of the function  $\zeta$ , defined by Equation (2), in the following sense

**Definition 1.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non-trivial zero* of the function  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

The following Main result affirms the Riemann hypothesis.

**Main Result II.** Consider the function  $\zeta$  defined by Equation (2). Suppose that  $s \in \mathbb{C}$  is a non-trivial zero of the function  $\zeta$  then  $\Re(s) = \frac{1}{2}$ .

*Proof.* By the Riemann's functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . For every  $\tau \in \mathbb{R}^*$  and every  $x \in (0, 1)$  we have

$$\psi(x, \tau) = \Im\left[\frac{\zeta(s)}{s}\right] \quad \text{and} \quad \psi(1-x, \tau) = -\Im\left[\frac{\zeta(1-s)}{1-s}\right], \quad s := x + i\tau.$$

By the Main result I, the non-trivial zero  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .  $\square$

### 3 Main proposition

In this Section we study only the function  $\psi$ , defined in Equation (1), by transforming the problem to some linear scalar differential equation. In order to simplify the notation, we denote

$$A := \{t \geq 0, t \neq \ln(n), \quad \forall n \geq 2\},$$

and,

$$\forall r \in \left(0, \frac{1}{2}\right): \quad x_r := \frac{1}{2} - r.$$

**Proposition 2.** For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  let  $t \mapsto \theta_\tau(r, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \theta_\tau(r, t) &= x_r \theta_\tau(r, t) - 2 \exp(-rt) [1 + \{\exp(t)\}] \sinh(rt) \sin(\tau t), \\ \theta_\tau(r, 0) &= 0, \quad t \in A. \end{aligned} \quad (3)$$

Then

$$\forall \tau \in \mathbb{R}^*, \quad \forall r \in (0, \frac{1}{2}) : \quad \sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty.$$

**Lemma 3.** For every  $\tau \in \mathbb{R}^*$  and  $z > 0$ , there exists  $\beta_{z, \tau} > 0$  such that

$$\left| \int_v^s \exp(-zt) \left[ \{\exp(t)\} - \frac{1}{2} \right] \sin(\tau t) dt \right| \leq h_\tau(z, \beta_{z, \tau}, v, s), \quad \forall s > v \geq 0,$$

where

$$h_\tau(z, \beta_{z, \tau}, v, s) := \beta_{z, \tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \quad \forall s, v \in \mathbb{R}.$$

*Proof.* In order to simplify the notation, denote

$$I_\tau(z, s, v) := \int_v^s \exp(-zt) \{\exp(t)\} \sin(\tau t) dt.$$

Use the change of variable  $\exp(t) = u$  we get

$$I_\tau(z, s, v) = \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \{t\} \sin(\tau \ln(t)) dt.$$

Since the function  $u \mapsto \{u\}$  is 1-periodic function and since

$$\int_0^1 t dt = \frac{1}{2},$$

Then there exists a 1-periodic function  $t \mapsto p(t)$  such that

$$\int_s^t \{u\} du = \frac{1}{2}(t-s) + p(t) - p(s), \quad \forall t \geq s \geq 0.$$

Since the function  $u \mapsto \{u\}$  is continuous on  $\mathbb{R}/\mathbb{Z}$  then the function  $t \mapsto p(t)$  is continuous on  $\mathbb{R}_+$  and  $C^1$  on  $\mathbb{R}/\mathbb{Z}$ . Use the integration part formula,

$$\begin{aligned} I_\tau(z, s, v) &= \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \\ &\quad + \exp(-(1+z)s) \sin(\tau s) (p(\exp(s)) - p(\exp(v))) \\ &\quad - \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} f_\tau(z, t) (p(t) - p(\exp(v))) dt, \\ f_\tau(z, t) &:= -(1+z) \sin(\tau \ln(t)) + \tau \cos(\tau \ln(t)). \end{aligned} \quad (4)$$

Since the function  $t \mapsto p(t)$  is 1-periodic, there exists  $\alpha > 0$  such that

$$\sup_{t \geq s \geq 0} |p(t) - p(s)| < \alpha.$$

By Equation (4), we get

$$\begin{aligned} & \left| I_\tau(z, s, v) - \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \right| \\ & \leq \alpha \left[ \exp(-(1+z)s) + [1+z+|\tau|] \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} dt \right] \\ & \leq \beta_{r,\tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \end{aligned}$$

where

$$\beta_{r,\tau} := \alpha \left[ 1 + \frac{|\tau|}{1+z} \right].$$

□

In order to simplify the notation, denote

$$\forall r \in (0, \frac{1}{2}) : \quad \tilde{x}_r := \frac{1+2r}{1-2r}. \quad (5)$$

**Lemma 4.** *Let  $(\epsilon_n)_n \subset \mathbb{R}$  and  $(v_n)_n \subset \mathbb{R}_+$  be a sequences such that*

- $\lim_{n \rightarrow +\infty} v_n = +\infty$ ,
- $\lim_{n \rightarrow +\infty} [\epsilon_n + \frac{3}{2} \Im[\frac{1}{-x_r + i\tau} \exp(i\tau v_n)]] \neq 0$ .

For every  $\tau \in \mathbb{R}^*$ ,  $r \in (0, \frac{1}{2})$  and every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \epsilon_n, \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (6)$$

Then,

$$\forall \tau \in \mathbb{R}^*, \quad \forall r \in (0, \frac{1}{2}) : \quad \sup_{n \geq 0} |\Delta_\tau(r, n, \tilde{x}_r v_n)| = +\infty.$$

*Proof.* In order to simplify the notation, denote

$$s_n := \tilde{x}_r v_n. \quad (7)$$

Integrate Equation (6) and use the fact  $\Delta_\tau(r, n, v_n) = \epsilon_n$ , we get

$$\begin{aligned} \Delta_\tau(r, n, s_n) &= \exp(x_r(s_n - v_n))\epsilon_n \\ &\quad - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) [1 + \{\exp(t)\}] \sin(\tau t) dt \\ &= \exp(x_r(s_n - v_n))\epsilon_n - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \sin(\tau t) dt \\ &\quad - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \{\exp(t)\} \sin(\tau t) dt. \end{aligned} \quad (8)$$

By Lemma 3,

$$\left| \int_{v_n}^{s_n} \exp(-x_r t) \left[ \{\exp(t)\} - \frac{1}{2} \right] \sin(\tau t) dt \right| \leq h_\tau(x_r, \beta_{\tau, r}, v_n, s_n),$$

where

$$h_\tau(x_r, \beta_{\tau, r}, v, s) := \beta_{\tau, r} \left[ \exp(-(1 + x_r)s) + \exp(-(1 + x_r)v) \right].$$

By hypothesis of the present Lemma we have  $\lim_{n \rightarrow +\infty} v_n = +\infty$  and by the Notation 5 we obtain

$$\lim_{n \rightarrow +\infty} \exp(x_r(s_n - v_n)) h_\tau(x_r, \beta_{\tau, r}, v_n, s_n) = 0.$$

By Equation (8), we get

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \Delta_\tau(r, n, s_n) \\ &= \lim_{n \rightarrow +\infty} \left[ \epsilon_n - \frac{3}{2} \int_{v_n}^{s_n} \exp(x_r(v_n - t)) \sin(\tau t) dt \right]. \end{aligned} \quad (9)$$

Use the fact

$$\begin{aligned} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt &= \Im \left[ \int_{\exp(v)}^{\exp(s)} t^{-1-z+i\tau} dt \right] \\ &= \exp(-zs) \Im \left[ \frac{1}{-z+i\tau} \exp(i\tau s) \right] \\ &\quad - \exp(-zv) \Im \left[ \frac{1}{-z+i\tau} \exp(i\tau v) \right]. \end{aligned}$$

Then

$$\begin{aligned} \int_{v_n}^{s_n} \exp(x_r(v_n - t)) \sin(\tau t) dt &= -\Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right] \\ &\quad + \exp(-x_r(s_n - v_n)) \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau s_n) \right]. \end{aligned} \quad (10)$$

By hypothesis of the present Lemma we have  $\lim_{n \rightarrow +\infty} v_n = +\infty$  and

$$\lim_{n \rightarrow +\infty} \left[ \epsilon_n + \frac{3}{2} \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right] \right] \neq 0,$$

thanks to Equation (9) and (10), we get

$$\lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \Delta_\tau(r, n, s_n) \neq 0,$$

since  $\lim_{n \rightarrow +\infty} v_n = +\infty$ , then  $\sup_{n \geq 0} |\Delta_\tau(r, n, s_n)| = +\infty$ .  $\square$

**Lemma 5.** *Let be  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$ . Consider the function  $t \mapsto \theta_\tau(r, t)$  solution of Equation (3) of the Proposition 2. Suppose that*

$$\sup_{t \geq 0} |\theta_\tau(r, t)| < +\infty.$$

Then

$$\lim_{n \rightarrow +\infty} \tau \theta_\tau(r, \frac{2n\pi}{\tau}) \leq 0.$$

*Proof.* Let be  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$ . Integrate Equation (3), use the fact  $t \mapsto \theta_\tau(r, t)$  is a continuous function and the fact  $\theta_\tau(r, 0) = 0$ , we get

$$\theta_\tau(r, t) = -\frac{2}{\tau} \exp(x_r t) \int_0^{\tau t} \exp(-\frac{1}{2\tau} u) \sinh(\frac{r}{\tau} u) [1 + \{\exp(\frac{u}{\tau})\}] \sin(u) du.$$

Use the integration part formula

$$\begin{aligned} \frac{\tau}{2} \exp(-\frac{x_r}{\tau} t) \theta_\tau(r, \frac{t}{\tau}) &= -\sin(t) \int_0^t \exp(-\frac{1}{2\tau} v) \sinh(\frac{r}{\tau} v) [1 + \{\exp(\frac{v}{\tau})\}] dv \\ &+ \int_0^t \cos(u) \int_0^u \exp(-\frac{1}{2\tau} v) \sinh(\frac{r}{\tau} v) [1 + \{\exp(\frac{v}{\tau})\}] dv du. \end{aligned}$$

There exists  $\alpha_{\tau, r} > 0$  such that

$$\alpha_{\tau, r} = \int_0^{+\infty} \exp(-\frac{1}{2\tau} v) \sinh(\frac{r}{\tau} v) [1 + \{\exp(\frac{v}{\tau})\}] dv.$$

In order to simplify the notation, denote

$$h_\tau(r, u) := -\alpha_{\tau, r} + 2 \int_0^u \exp(-\frac{1}{2\tau} v) \sinh(\frac{r}{\tau} v) [1 + \{\exp(\frac{v}{\tau})\}] dv, \quad \forall u \geq 0.$$

Then

$$\tau \exp\left(-\frac{x_r}{\tau}t\right)\theta_\tau\left(r, \frac{t}{\tau}\right) = -\sin(t)h_\tau(r, t) + \int_0^t h_\tau(r, u) \cos(u)du.$$

Repeat the process again. The integration part formula gives

$$\begin{aligned} \tau \exp\left(-\frac{x_r}{\tau}t\right)\theta_\tau\left(r, \frac{t}{\tau}\right) &= -\sin(t)h_\tau(r, t) + \cos(t) \int_0^t h_\tau(r, v)dv \\ &\quad + \int_0^t \int_0^u h_\tau(r, v)dv \sin(u)du. \end{aligned} \quad (11)$$

We have

$$\begin{aligned} |h_\tau(r, u)| &= \left| \int_u^{+\infty} \exp\left(-\frac{1}{2\tau}v\right) \sinh\left(\frac{r}{v}\tau\right) [1 + \{\exp(\frac{v}{\tau})\}] dv \right| \\ &\leq \frac{2}{\frac{1}{2}-r} \exp\left(-\left(\frac{1}{2}-r\right)u\right) + \frac{2}{\frac{1}{2}+r} \exp\left(-\left(\frac{1}{2}+r\right)u\right), \quad \forall u \geq 0. \end{aligned}$$

There exists  $\tilde{\alpha}_{\tau,r} > 0$  such that

$$-\tilde{\alpha}_{\tau,r} = \int_0^{+\infty} h_\tau(r, v)dv.$$

In order to simplify the notation, denote

$$\tilde{h}_\tau(r, u) := \int_0^u h_\tau(r, v)dv + \tilde{\alpha}_{\tau,r}, \quad \forall u \geq 0.$$

Then

$$\begin{aligned} \tau \exp\left(-\frac{x_r}{\tau}t\right)\theta_\tau\left(r, \frac{t}{\tau}\right) &= -\sin(t)h_\tau(r, t) + \cos(t)\tilde{h}_\tau(r, t) \\ &\quad - \tilde{\alpha}_{\tau,r} + \int_0^t \tilde{h}_\tau(r, u) \sin(u)du. \end{aligned} \quad (12)$$

By hypothesis of the present Lemma

$$\lim_{t \rightarrow +\infty} \exp\left(-\frac{x_r}{\tau}t\right)\theta_\tau\left(r, \frac{t}{\tau}\right) = 0.$$

Since

$$\lim_{t \rightarrow +\infty} [-\sin(t)h_\tau(r, t) + \cos(t)\tilde{h}_\tau(r, t)] = 0, \quad (13)$$

then

$$\lim_{t \rightarrow +\infty} [-\tilde{\alpha}_{\tau,r} + \int_0^t \tilde{h}_\tau(r, u) \sin(u)du] = 0$$



The function  $u \mapsto \tilde{h}_\tau(r, u)$  is positive and decreasing. Using the symmetry of the function  $\sin(u)$  on  $[0, 2\pi]$  about the point  $\pi$  we get,

$$\forall n \in \mathbb{N} : \int_{2n\pi}^{2(n+1)\pi} \tilde{h}_\tau(r, u) \sin(u) du \geq$$

Then the sequence

$$n \mapsto \int_0^{2n\pi} \tilde{h}_\tau(r, u) \sin(u) du$$

is increasing and converge to  $\tilde{\alpha}_{\tau, r}$ . Implies

$$\forall n \in \mathbb{N} : -\tilde{\alpha}_{\tau, r} + \int_0^{2n\pi} \tilde{h}_\tau(r, u) \sin(u) du \leq 0.$$

By Equations (12) and (13), we deduce that

$$\lim_{n \rightarrow +\infty} \tau \theta_\tau(r, \frac{2n\pi}{\tau}) \leq 0.$$

□

*Proof of Proposition 2 .* By contradiction, suppose that there exist  $\tau \in \mathbb{R}^*$  and  $r \in (0, 1)$  such that

$$\sup_{t \geq 0} |\theta_\tau(r, t)| < +\infty. \quad (14)$$

Denote  $v_n = \frac{2n\pi}{\tau}$  for all  $n \in \mathbb{N}$ . Then

$$\lim_{n \rightarrow +\infty} v_n = +\infty. \quad (15)$$

By Lemma 5,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \tau \left[ \theta_\tau(r, v_n) + \frac{3}{2} \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right] \right] \\ = \lim_{n \rightarrow +\infty} \left[ \tau \theta_\tau(r, \frac{2n\pi}{\tau}) + \frac{3}{2} \frac{-\tau^2}{x_r^2 + \tau^2} \right] \\ \leq \frac{3}{2} \frac{-\tau^2}{x_r^2 + \tau^2} < 0. \end{aligned} \quad (16)$$

For every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \theta_\tau(r, v_n), \quad t \in A, t \geq v_n. \end{aligned} \quad (17)$$

Denote  $\eta_\tau(r, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ . By Equations (3) and (17), we have

$$\begin{aligned} \frac{d}{dt}\eta_\tau(r, n, t) &= x_r\eta_\tau(r, n, t) + \exp(-2rt)[1 + \{\exp(t)\}]\sin(\tau t), \\ \eta_\tau(r, n, v_n) &= 0, \quad t \in A, \quad t \geq v_n. \end{aligned}$$

Since  $t \mapsto \eta_\tau(r, n, t)$  is a continuous function, integrate. Use the notation of  $\tilde{x}_r$  in Equation (5) and the fact since  $(v_n)_n \in \mathbb{R}_+$ , we obtain

$$|\eta_\tau(r, n, \tilde{x}_r v_n)| \leq 2 \exp(-2rv_n) \int_{v_n}^{\tilde{x}_r v_n} \exp(x_r(\tilde{x}_r v_n - s)) ds \leq \frac{2}{x_r}.$$

Then

$$\sup_{n \geq 0} |\eta_\tau(r, n, \tilde{x}_r v_n)| < +\infty. \quad (18)$$

By Equations (16), (17) and by Lemma 4, we have

$$\sup_{n \geq 0} |\Delta_\tau(r, n, \tilde{x}_r v_n)| = +\infty.$$

Since  $\eta_\tau(r, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ , thanks to Equation (18) we find,

$$\sup_{n \geq 0} |\theta_\tau(r, \tilde{x}_r v_n)| = +\infty.$$

Contradiction with the hypothesis (14).  $\square$

## 4 Proof of the Main results

*Proof of the Main result I.* For every  $\tau \in \mathbb{R}^*$ , define the function  $\phi_\tau : (0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  as

$$\phi_\tau(x, t) := - \int_0^t \left[ \exp(-(1-x)u) - \exp(-xu)\{\exp(u)\} \right] \sin(\tau u) du. \quad (19)$$

Use the change of variable,  $t \rightarrow \exp(u)$ , in Equation (1), we obtain

$$\forall x \in (0, 1) : \quad \psi(x, \tau) = \lim_{t \rightarrow +\infty} \phi_\tau(x, t).$$

Prove the present Main result by contradiction. Suppose that there exists  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  such that

$$\psi\left(\frac{1}{2} - r, \tau\right) = \psi\left(\frac{1}{2} + r, \tau\right). \quad (20)$$

By Equation (19), for all  $t \geq 0$  we have

$$\begin{aligned}\phi_\tau\left(\frac{1}{2} + r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(ru) - \exp(-ru)\{\exp(u)\}] \sin(\tau u) du, \\ \phi_\tau\left(\frac{1}{2} - r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(-ru) - \exp(ru)\{\exp(u)\}] \sin(\tau u) du.\end{aligned}$$

Denote

$$y_\tau(r, t) := \exp\left(\frac{1}{2}t\right) \left[ \phi_\tau\left(\frac{1}{2} + r, t\right) - \phi_\tau\left(\frac{1}{2} - r, t\right) \right], \quad \forall t \geq 0.$$

Then

$$y_\tau(r, t) = -2 \int_0^t \exp\left(\frac{1}{2}(t-u)\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \quad (21)$$

The function  $t \mapsto y_\tau(r, t)$  is a continuous function and satisfies the following differential equation

$$\begin{aligned}\frac{d}{dt} y_\tau(r, t) &= \frac{1}{2} y_\tau(r, t) - 2 \sinh(rt) [1 + \{\exp(t)\}] \sin(\tau t), \\ y_\tau(r, 0) &= 0, \quad \forall t \in A.\end{aligned} \quad (22)$$

By Equation (20), we have

$$\int_0^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du = 0.$$

implies

$$\begin{aligned}& \int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \\ &= - \int_t^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du.\end{aligned}$$

Then

$$\begin{aligned}& 2 \left| \int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \right| \\ &\leq \frac{2}{\frac{1}{2} - r} \exp\left(-\left(\frac{1}{2} - r\right)t\right) + \frac{2}{\frac{1}{2} + r} \exp\left(-\left(\frac{1}{2} + r\right)t\right), \quad \forall t \geq 0.\end{aligned}$$

Denote  $\theta_\tau(r, t) := \exp(-rt)y_\tau(r, t)$ . By the last Inequality and Equation (21), we get

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = \sup_{t \geq 0} |\exp(-rt)y_\tau(r, t)| \leq \frac{2}{\frac{1}{4} - r^2} \quad (23)$$

By Equation (22), The function  $t \mapsto \theta_\tau(r, t)$  is a continuous function satisfies  $\theta_\tau(r, 0) = 0$  and satisfies the following differential equation

$$\frac{d}{dt}\theta_\tau(r, t) = \left(\frac{1}{2} - r\right)\theta_\tau(r, t) - 2\exp(-rt)[1 + \{\exp(t)\}] \sinh(rt) \sin(\tau t), \quad t \in A.$$

By the Proposition 2 we have  $\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty$ . Contradiction with Equation (23).  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).