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# Asymmetry of Generalized $\zeta$ Functions under the Rotation Number Hypothesis

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## Abstract

In this manuscript, we show that the Riemann zeta function satisfies  $(\zeta(s), \zeta(1 - \bar{s})) \neq (0, 0)$  for any  $s$  in the critical strip, except on the critical line. This still holds even when the fractional part function is replaced by a function satisfying a specific *Rotation Number Hypothesis*.

**Keywords:** Riemann zeta function, Rotation number, Critical strip.

**MSC subject classifications:** 11M06, 11M26

## 1 Introduction

Denote by  $\mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$  the set of bounded and locally integrable functions from  $[1, +\infty)$  to  $\mathbb{C}$ . We consider the following *Rotation Number Hypothesis*:

$$\exists \rho \in \mathbb{C}^* : \sup_{t \geq 1} \left| \int_1^t (\eta(u) - \rho) du \right| < +\infty. \quad (\mathbf{H})$$

The complex number  $\rho$  is called the *rotation number* of  $\eta$ . Notice that if  $\rho_\eta$  exists, then it is unique; this can be proved by contradiction. Denote by  $\mathcal{L}[\eta]$  the following transformation

$$\mathcal{L}[\eta](t) := \int_1^t (\eta(u) - \rho_\eta) du, \quad \forall t \geq 1.$$

**Notation 1.** Let  $\mathbb{C}_+$  denote the right half-plane  $\{w \in \mathbb{C} : \Re(w) > 0\}$ . For every  $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ , let  $\mu_\eta : \mathbb{C}_+ \rightarrow \mathbb{C}$  be defined as:

$$\mu_\eta(w) = -1 - (1-w) \int_1^{+\infty} u^{-1-w} \eta(u) du, \quad \forall w \in \mathbb{C}_+.$$

The function  $\mu_\eta$  is well-defined on  $\mathbb{C}_+$  since  $\eta$  is bounded and the integral converges absolutely for  $\Re(w) > 0$ . The Rotation number hypothesis implies that  $\mathcal{L}[\eta] \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$ . Hence, the function  $\mu_{\mathcal{L}[\eta]}$  is defined on  $\mathbb{C}_+$ .

Denote by  $B$  the critical strip, excluding the critical line and the real axis:

$$B := \left\{ w \in \mathbb{C} : \Re(w) \in (0, 1), \Re(w) \neq \frac{1}{2}, \Im(w) \neq 0 \right\}.$$

In [3] we have the following results

**Theorem 2.** *Let  $\eta \in \mathcal{B}_{\text{loc}}([1, \infty), \mathbb{C})$  satisfy the Rotation Number Hypothesis (H). Suppose that the functions  $\mu_\eta$  and  $\mathcal{L}[\eta]$  satisfies:*

$$\forall x \in \mathbb{R}^* : \quad \mu_\eta(1+ix) \neq 0 \quad \text{and} \quad \mu_{\mathcal{L}[\eta]}(1+ix) \neq \frac{\rho_\eta - ix}{1+ix} - \frac{1}{|1+ix|^2}.$$

Then, for every  $s \in B$ , we have

$$(\mu_\eta(s), \mu_\eta(1-\bar{s})) \neq (0, 0).$$

**Remark 3.** Note that

$$\frac{|1+ix|^2}{ix} \left[ \frac{\rho_\eta - ix}{1+ix} - \mu_{\mathcal{L}[\eta]}(1+ix) - \frac{1}{|1+ix|^2} \right] = \lim_{r \rightarrow > 0} \mu_\eta(r+ix),$$

is the analytic extension of the function  $\mu_\eta$  to the line  $\Re(z) = 0$ . Consequently, the condition of the previous theorem is equivalent to

$$\forall x \in \mathbb{R}^* : \quad \mu_\eta(1+ix) \neq 0 \quad \text{and} \quad \mu_\eta(ix) \neq 0.$$

## 2 Application to the Riemann $\zeta$ Function

The fractional part function  $\eta_*(t) = \{t\}$  is bounded and locally integrable.

**Proposition 4.** *The function  $\eta_*(t) = \{t\}$  satisfies hypothesis (H) with  $\rho = \frac{1}{2}$ .*

*Proof.* The function  $p(t) = \int_1^t (1/2 - \{u\}) du$  is 1-periodic and vanishes at integer values. Thus, it is bounded for all  $t \geq 1$ .  $\square$

From the integral representation of the Riemann zeta function (Titchmarsh, [2], page 14, Equation 2.1.5):

$$\forall w \in \mathbb{C}_+ \setminus \{1\} : \quad \frac{1-w}{w} \zeta(w) = \mu_{\eta_*}(w). \quad (1)$$

We have  $\zeta(1+i\tau) \neq 0$  for  $\tau \in \mathbb{R}$  (Hadamard, 1896, [1]), then  $\mu_{\eta_*}(1+i\tau) \neq 0$ . Use the Riemann functional equation and Remark 3,

$$\frac{1-i\tau}{i\tau} \zeta(i\tau) := \mu_{\mathcal{L}[\eta_*]}(1+i\tau) - \frac{\frac{1}{2}-i\tau}{1+i\tau} + \frac{1}{|1+i\tau|^2} \neq 0, \quad \forall \tau \in \mathbb{R}^*.$$

In fact,

$$\zeta(i\tau) = i 2^{i\tau} \pi^{-1+i\tau} \sinh\left(\frac{\pi\tau}{2}\right) \Gamma(1-i\tau) \zeta(1-i\tau) \neq 0, \quad \forall \tau \in \mathbb{R}^*.$$

By Theorem 2, for  $s \in B$  we have  $(\mu_{\eta_*}(s), \mu_{\eta_*}(1-\bar{s})) \neq (0, 0)$ . Implies  $(\zeta(s), \zeta(1-\bar{s})) \neq (0, 0)$ .

## References

- [1] J. Hadamard, Sur la distribution des zéros de la fonction  $\zeta(s)$  et ses conséquences arithmétiques, Bull. Soc. Math. France, 24, 199–220. (1896)
- [2] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986)
- [3] W. Oukil, Rigidity and Structural Asymmetry of Bounded Solutions. Version 6, HAL Id : hal-05552822. (2026)