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# Differential equations properties and Zeta function

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## Abstract

Using the differential equations properties, we prove that any *non-trivial zero* of Riemann's zeta function is of real part equal to  $\frac{1}{2}$ . In other words, we prove that the Riemann hypothesis is true.

**Keywords:** Riemann hypothesis, Zeta function, Abel summation formula, Differential equations.

## 1 Introduction

In this paper, we study the function  $\psi : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{R}$  defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi(x, \tau) := \int_0^{+\infty} \frac{\{t\}}{t^{1+x}} \sin(\tau \ln(t)) dt, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . In [[1], page 14 Equation 2.1.5] the function  $\psi$  is the imaginary part of the function  $s := x + i\tau \mapsto \frac{\zeta(s)}{s}$  where  $\zeta$  is the representation of the Riemann Zeta function defined by the Abel summation formula as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*. \quad (2)$$

**Main Result I.** Consider the function  $\psi$  defined by Equation (1). Then,

$$\forall x \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^* : \quad \psi(x, \tau) \neq \psi(1-x, \tau).$$

We focus on the *non-trivial zeros* of the function  $\zeta$ , defined by Equation (2), in the following sense

**Definition 1.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non-trivial zero of the function  $\zeta$*  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

The following theorem affirms the Riemann hypothesis.

**Main Result II.** *Consider the function  $\zeta$  defined by Equation (2). Suppose that  $s \in \mathbb{C}$  is a non-trivial zero of the function  $\zeta$  then  $\Re(s) = \frac{1}{2}$ .*

## 2 Main proposition

In this Section we study the behavior of an linear scalar differential equation without discuss about the Zeta function. In order to simplify the notation, we denote on all this Section

$$A := \{t \geq 0, t \neq \ln(n), \quad \forall n \geq 2\},$$

and we denote,

$$\forall r \in (0, \frac{1}{2}) : \quad x_r := \frac{1}{2} - r.$$

**Proposition 2.** *For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  let  $t \mapsto \theta_\tau(r, t)$  be the continuous function satisfying the following differential equation,*

$$\begin{aligned} \frac{d}{dt} \theta_\tau(r, t) &= x_r \theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \\ \theta_\tau(r, 0) &= 0, \quad t \in A. \end{aligned} \tag{3}$$

Then,

$$\forall \tau \in \mathbb{R}^*, \quad \forall r \in (0, \frac{1}{2}) : \quad \sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty.$$

**Lemma 3.** *For every  $\tau \in \mathbb{R}^*$  and  $z > 0$ , there exists  $\beta_{z, \tau} > 0$  such that for all  $s > v \geq 0$  we have*

$$\left| \int_v^s \exp(-zt) \left[ \{\exp(t)\} - \frac{1}{2} \right] \sin(\tau t) dt \right| \leq h_\tau(z, \beta_{z, \tau}, v, s),$$

where

$$h_\tau(z, \beta_{z, \tau}, v, s) := \beta_{z, \tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \quad \forall s, v \in \mathbb{R}.$$

*Proof.* In order to simplify the notation, denote

$$I_\tau(z, s, v) := \int_v^s \exp(-zt) \{\exp(t)\} \sin(\tau t) dt.$$

Use the change of variable  $\exp(t) = u$  we get

$$I_\tau(z, s, v) := \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \{t\} \sin(\tau \ln(t)) dt.$$

Since the function  $u \mapsto \{u\}$  is 1-periodic function and since

$$\int_0^1 t dt = \frac{1}{2},$$

Then there exists a 1-periodic function  $t \mapsto p(t)$  such that

$$\int_s^t \{u\} du = \frac{1}{2}(t - s) + p(t) - p(s), \quad \forall t \geq s \geq 0.$$

Since the function  $u \mapsto \{u\}$  is continuous on  $\mathbb{R}/\mathbb{Z}$  then the function  $t \mapsto p(t)$  is continuous on  $\mathbb{R}_+$  and  $C^1$  on  $\mathbb{R}/\mathbb{Z}$ . Use the integration part formula,

$$\begin{aligned} I_\tau(z, s, v) &= \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \\ &\quad + \exp(-(1+z)s) \sin(\tau s) (p(\exp(s)) - p(\exp(v))) \\ &\quad - \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} f_\tau(z, t) (p(t) - p(\exp(v))) dt, \\ f_\tau(z, t) &:= -(1+z) \sin(\tau \ln(t)) + \tau \cos(\tau \ln(t)). \end{aligned} \tag{4}$$

Since the function  $t \mapsto p(t)$  is 1-periodic, there exists  $\alpha > 0$  such that

$$\sup_{t \geq s \geq 0} |p(t) - p(s)| < \alpha.$$

By Equation (4), we get

$$\begin{aligned} &\left| I_\tau(z, s, v) - \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \right| \\ &\leq \alpha \left[ \exp(-(1+z)s) + [1+z + |\tau|] \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} dt \right] \\ &\leq \beta_{r,\tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \end{aligned}$$

where

$$\beta_{r,\tau} := \alpha \left[ 1 + \frac{|\tau|}{1+z} \right].$$

□

In order to simplify the notation, denote

$$\forall r \in (0, \frac{1}{2}) : \quad \tilde{x}_r := \frac{\frac{1}{2} + r}{\frac{1}{2} - r}. \quad (5)$$

**Lemma 4.** *Let  $(\epsilon_n)_n \subset \mathbb{R}$  and  $(v_n)_n \subset \mathbb{R}_+$  be a sequences such that*

- $\lim_{n \rightarrow +\infty} v_n = +\infty$ ,
- $\lim_{n \rightarrow +\infty} [\epsilon_n - \frac{3}{2} \Im[\frac{1}{-x_r + i\tau} \exp(i\tau v_n)]] \neq 0$ .

*For every  $\tau \in \mathbb{R}^*$ ,  $r \in (0, \frac{1}{2})$  and every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,*

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \epsilon_n, \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (6)$$

*Then,*

$$\forall \tau \in \mathbb{R}^*, \quad \forall r \in (0, \frac{1}{2}) : \quad \sup_{n \geq 0} |\Delta_\tau(r, n, \tilde{x}_r v_n)| = +\infty.$$

*Proof.* In order to simplify the notation, denote

$$s_n := \tilde{x}_r v_n. \quad (7)$$

Integrate Equation (6) and use the fact  $\Delta_\tau(r, n, v_n) = \epsilon_n$ , we get

$$\begin{aligned} \Delta_\tau(r, n, s_n) &= \exp(x_r(s_n - v_n)) \epsilon_n \\ &\quad - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) [1 + \{\exp(t)\}] \sin(\tau t) dt \\ &= \exp(x_r(s_n - v_n)) \epsilon_n - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \sin(\tau t) dt \\ &\quad - \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \{\exp(t)\} \sin(\tau t) dt. \end{aligned} \quad (8)$$

By Lemma 3,

$$\left| \int_{v_n}^{s_n} \exp(-x_r t) \left[ \{\exp(t)\} - \frac{1}{2} \right] \sin(\tau t) dt \right| \leq h_\tau(x_r, \beta_{\tau,r}, v_n, s_n),$$

where

$$h_\tau(x_r, \beta_{\tau,r}, v, s) := \beta_{\tau,r} \left[ \exp(-(1+x_r)s) + \exp(-(1+x_r)v) \right].$$

By hypothesis of the present Lemma we have  $\lim_{n \rightarrow +\infty} v_n = +\infty$  and by the notation 5 we have  $\tilde{x}_r > 1$  then

$$\lim_{n \rightarrow +\infty} \exp(x_r(s_n - v_n)) h_\tau(x_r, \beta_{\tau,r}, v_n, s_n) = 0.$$

Then by Equation (8), we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \Delta_\tau(r, n, s_n) \\ &= \lim_{n \rightarrow +\infty} \left[ \epsilon_n + \frac{3}{2} \int_{v_n}^{s_n} \exp(x_r(v_n - t)) \sin(\tau t) dt \right]. \end{aligned} \quad (9)$$

Use the fact

$$\begin{aligned} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt &= \Im \left[ \int_{\exp(v)}^{\exp(s)} t^{-1-z+i\tau} dt \right] \\ &= \exp(-zs) \Im \left[ \frac{1}{-z+i\tau} \exp(i\tau s) \right] \\ &\quad - \exp(-zv) \Im \left[ \frac{1}{-z+i\tau} \exp(i\tau v) \right]. \end{aligned}$$

Then

$$\begin{aligned} \int_{v_n}^{s_n} \exp(x_r(v_n - t)) \sin(\tau t) dt &= \exp(-x_r(s_n - v_n)) \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau s_n) \right] \\ &\quad - \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right]. \end{aligned} \quad (10)$$

By hypothesis of the present Lemma we have  $\lim_{n \rightarrow +\infty} v_n = +\infty$  and

$$\lim_{n \rightarrow +\infty} \left[ \epsilon_n - \frac{3}{2} \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right] \right] \neq 0,$$

thanks to Equation (9) and (10), we get

$$\lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \Delta_\tau(r, n, s_n) \neq 0,$$

since  $\lim_{n \rightarrow +\infty} v_n = +\infty$ , then  $\sup_{n \geq 0} |\Delta_\tau(r, n, s_n)| = +\infty$ .  $\square$

*Proof of Proposition 2 .* First, prove that there exists a sequence  $(v_n)_n \subset \mathbb{R}_+$  such that

$$\lim_{n \rightarrow +\infty} v_n = +\infty, \quad (11)$$

and such that

$$\lim_{n \rightarrow +\infty} \left[ \theta_\tau(r, v_n) - \frac{3}{2} \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau v_n) \right] \right] \neq 0. \quad (12)$$

By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} \left[ \theta_\tau(r, t) - \frac{3}{2} \Im \left[ \frac{1}{-x_r + i\tau} \exp(i\tau t) \right] \right] = 0. \quad (13)$$

Since  $t \mapsto \theta_\tau(r, t)$  is a continuous function, integrate Equation (3), we obtain for all  $s \geq t \geq 0$  the following equality

$$\begin{aligned} \theta_\tau(r, t) - \theta_\tau(r, s) &= x_r \int_s^t \theta_\tau(r, u) du \\ &\quad - \int_s^t [1 - \exp(-2ru)] [1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned} \quad (14)$$

By the hypothesis (13), for every sequences  $(u_n)_n, (\tilde{u}_n)_n \subset \mathbb{R}_+$  such that

$$\sup_{n \geq 0} |u_n - \tilde{u}_n| < +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_n = +\infty,$$

we find

$$\lim_{n \rightarrow +\infty} [g_\tau(r, u_n) - g_\tau(r, \tilde{u}_n)] = \lim_{n \rightarrow +\infty} \int_{\tilde{u}_n}^{u_n} [1 + \{\exp(u)\}] \sin(\tau u) du.$$

where the  $C^\infty$  and  $\frac{2\pi}{\tau}$ -periodic function  $s \mapsto g_\tau(r, s)$  is defined as

$$g_\tau(r, s) := \frac{3}{2\tau} \Im \left[ \frac{\tau + ix_r}{-x_r + i\tau} \exp(i\tau s) \right], \quad \forall s \geq 0.$$

Contradiction. We deduce then Equation (11) and (12).

Now, prove the present Lemma by contradiction. Suppose that there exists  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  such that

$$\sup_{t \geq 0} |\theta_\tau(r, t)| < +\infty. \quad (15)$$

For every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \theta_\tau(r, v_n), \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (16)$$

Denote  $\eta_\tau(r, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ . By Equations (3) and (16), we have

$$\begin{aligned} \frac{d}{dt} \eta_\tau(r, n, t) &= x_r \eta_\tau(r, n, t) + \exp(-2rt)[1 + \{\exp(t)\}] \sin(\tau t), \\ \eta_\tau(r, n, v_n) &= 0, \quad t \in A, \quad t \geq v_n. \end{aligned}$$

Since  $t \mapsto \eta_\tau(r, n, t)$  is a continuous function, integrate. Use the notation of  $\tilde{x}_r$  in Equation (5) and the fact since  $(v_n)_n \in \mathbb{R}_+$ , we obtain

$$|\eta_\tau(r, n, \tilde{x}_r v_n)| \leq 2 \exp(-2rv_n) \int_{v_n}^{\tilde{x}_r v_n} \exp(x_r(\tilde{x}_r v_n - s)) ds \leq \frac{2}{x_r}.$$

Then

$$\sup_{n \geq 0} |\eta_\tau(r, n, \tilde{x}_r v_n)| < +\infty. \quad (17)$$

By Equations (12), (16) and by Lemma 4, we have

$$\sup_{n \geq 0} |\Delta_\tau(r, n, \tilde{x}_r v_n)| = +\infty.$$

Since  $\eta_\tau(r, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ , thanks to Equation (17) we find,

$$\sup_{n \geq 0} |\theta_\tau(r, \tilde{x}_r v_n)| = +\infty.$$

Contradiction with the hypothesis (15).  $\square$

### 3 Proof of the Main results

*Proof of the Main result I.* For every  $\tau \in \mathbb{R}^*$ , define the function  $\phi_\tau : (0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  as

$$\phi_\tau(x, t) := - \int_0^t \left[ \exp(-(1-x)u) - \exp(-xu)\{\exp(u)\} \right] \sin(\tau u) du. \quad (18)$$



Use the change of variable,  $t \rightarrow \exp(u)$ , in Equation (1), we obtain

$$\forall x \in (0, 1) : \quad \psi(x, \tau) = \lim_{t \rightarrow +\infty} \phi_\tau(x, t).$$

Prove the present Main result by contradiction. Suppose that there exists  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  such that

$$\psi\left(\frac{1}{2} - r, \tau\right) = \psi\left(\frac{1}{2} + r, \tau\right). \quad (19)$$

By Equation (18), for all  $t \geq 0$  we have

$$\begin{aligned} \phi_\tau\left(\frac{1}{2} + r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(ru) - \exp(-ru)\{\exp(u)\}] \sin(\tau u) du, \\ \phi_\tau\left(\frac{1}{2} - r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(-ru) - \exp(ru)\{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

Denote

$$y_\tau(r, t) := \exp\left(\frac{1}{2}t\right) \left[ \phi_\tau\left(\frac{1}{2} + r, t\right) - \phi_\tau\left(\frac{1}{2} - r, t\right) \right], \quad \forall t \geq 0.$$

Then

$$y_\tau(r, t) = -2 \int_0^t \exp\left(\frac{1}{2}(t-u)\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \quad (20)$$

The function  $t \mapsto y_\tau(r, t)$  is a continuous function and satisfies the following differential equation

$$\begin{aligned} \frac{d}{dt} y_\tau(r, t) &= \frac{1}{2} y_\tau(r, t) - 2 \sinh(rt) [1 + \{\exp(t)\}] \sin(\tau t), \\ y_\tau(r, 0) &= 0, \quad \forall t \in A. \end{aligned} \quad (21)$$

By Equation (19), we have

$$\int_0^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du = 0,$$

implies

$$\begin{aligned} &\int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \\ &= - \int_t^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

Then

$$\begin{aligned} & 2 \left| \int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \right| \\ & \leq \frac{2}{\frac{1}{2} - r} \exp\left(-\left(\frac{1}{2} - r\right)t\right) + \frac{2}{\frac{1}{2} + r} \exp\left(-\left(\frac{1}{2} + r\right)t\right), \quad \forall t \geq 0. \end{aligned}$$

Denote  $\theta_\tau(r, t) := \exp(-rt)y_\tau(r, t)$ . By the last Inequality and Equation (20), we get

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = \sup_{t \geq 0} |\exp(-rt)y_\tau(r, t)| \leq \frac{2}{\frac{1}{4} - r^2} \quad (22)$$

By Equation (21), The function  $t \mapsto \theta_\tau(r, t)$  is a continuous function satisfies  $\theta_\tau(r, 0) = 0$  and satisfies the following differential equation

$$\frac{d}{dt} \theta_\tau(r, t) = \left(\frac{1}{2} - r\right) \theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \quad t \in A.$$

By the Proposition 2 we have  $\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty$ . Contradiction with Equation (22).  $\square$

*Proof of the Main result II.* Prove the Main result by contradiction. Suppose that there exist  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  such that

$$\zeta\left(\frac{1}{2} - r + i\tau\right) = 0,$$

By the Riemann's functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(s) = \frac{1}{2}$ . We get

$$\zeta\left(\frac{1}{2} + r + i\tau\right) = 0.$$

By identification, for every  $v \in \mathbb{R}^*$  and every  $x \in (0, 1)$ , we have

$$\psi(x, v) = \Im \left[ \frac{\zeta(s)}{s} \right] \quad \text{and} \quad \psi(1-x, v) = -\Im \left[ \frac{\zeta(1-s)}{1-s} \right], \quad s := x + iv.$$

Then  $\psi(\frac{1}{2} - r, \tau) = \psi(\frac{1}{2} + r, \tau) = 0$ . Contradiction with the Main result I.  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).